INTRODUCTION TO GOOD FORMAL STRUCTURES
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Contents

1. The main problems

The goal of this introductory talk is to explain the main problems one encounters when trying to extend to higher dimension (e.g., dimension 2) the Levelt-Turrittin theorem for finite-dimensional differential vector spaces over the differential field $(K(\!(x)\!), d)$, where K is any algebraically closed field of characteristic zero. One can first think of $K = \mathbb{C}$.

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1.1. Dimension one

Theorem 1.1.1 **(Levelt-Turrittin).** *Let $\widehat{V}$ be a finite-dimensional differential vector spaces over $(K(\!(x)\!), d)$. Then there exists $e \in \mathbb{N}^*$ and a finite set $\Phi \subset K(\!(y)\!)/K[\![y]\!]$ such that, through the ramification $y \mapsto x = y^e$, $K(\!(y)\!) \otimes_{K(\!(x)\!)} \widehat{V} \simeq \bigoplus_{\varphi \in \Phi} (E(\varphi) \otimes R_\varphi)$, where $E(\varphi) = (K(\!(y)\!), d + d\widetilde{\varphi})$, with $\widetilde{\varphi} \in K(\!(y)\!)$ (a representative of φ) and R_φ has a regular singularity.*

In matrix terms, this means that, given a matrix $A(x)dx/x$ with entries in $K(\!(x)\!)$, there exists an invertible matrix $P(y) \in \mathrm{GL}_d(K(\!(y)\!))$ such that $B(y)dy/y := eP^{-1}A(y^e)Pdy/y + P^{-1}dP$ is block-diagonal, with blocks $d\widetilde{\varphi}\,\mathrm{Id} + C_\varphi dy/y$ for some constant matrix C_φ.

We notice that $E(\widetilde{\varphi}) \xrightarrow{\sim} E(\widetilde{\psi})$ if $\widetilde{\varphi} - \widetilde{\psi} \in K[\![y]\!]$. This is why one can assume that the sum is indexed by a finite set of $\varphi \in K(\!(y)\!)/K[\![y]\!] = y^{-1}K[y^{-1}]$.

***Remark 1.1.2*.** There is no nonzero morphism $E(\varphi) \otimes R_\varphi \to E(\psi) \otimes R_\psi$ if $\varphi \neq \psi$. Hence the decomposition is unique. Moreover, by considering invariants under the Galois action of $\mathbb{Z}/e\mathbb{Z}$, one finds a decomposition defined over $K(\!(x)\!)$, and the classification of irreducible objects and indecomposable objects is known and there are irreducible objects of arbitrary rank.

In practice, one starts with a smooth curve X (in the algebraic or complex analytic setting), which is equipped with a reduced divisor D (locally finite set of points), and a locally free $\mathscr{O}_X(*D)$-module $\mathscr{M}$ of finite rank equipped with a connection ∇. For each $x_o \in D$, we set $\widehat{V} = \widehat{\mathscr{O}}_{x_o} \otimes_{\mathscr{O}_{X,x_o}} \mathscr{M}_{x_o}$.

1.2. Dimension one with parameter. Assume now that $K = \mathbb{C}(\!(z)\!)$. This field is not algebraically closed, but the Levelt-Turrittin theorem extends to non algebraically closed fields K, by working with a finite extension K' of K in the conclusion. Here, a finite extension would be $\mathbb{C}(\!(z^{1/e})\!)$.

Let us now replace K with a Noetherian ring A, e.g., $A = \mathbb{C}[z]$ or $A = \mathbb{C}[\![z]\!]$. Assume that $\widehat{V}$ is a free $A(\!(x)\!)$-module of finite rank with a connection. Then the conclusion of the Levelt-Turrittin holds if one replaces, in the conclusion, the ring A with the integral closure of a suitable localization. For instance, one replaces $A = \mathbb{C}[z]$ with $A' = \mathbb{C}[z, t, q(z)^{-1}]/(p(z,t))$, where $q(z)$ is the resultant of $p(z,t)$ and $p'_t(z,t)$. On the other hand, if $A = \mathbb{C}[\![z]\!]$, one replaces A with $\mathbb{C}(\!(z^{1/e'})\!)$ for a suitable e'.

The proof of this result is obtained by following precisely the proof of the Levelt-Turrittin theorem (cf. e.g., [**BV85**]).

One can give a much more precise result (cf. [**BV85**], [**And07**, Th. 3.4.1]), but I will state and use it later. In fact, one cannot go much further and analyze precisely what happens, since in this very general setting one encounters confluence phenomena, which may be very difficult to understand. Moreover, the method of "changing coordinates by blowing-up", which will play an important role later, cannot be applied in this setting, since one variable is privileged among all variables.

1.3. Dimension two. Let now X be a smooth surface and let D be a reduced divisor of X. Let $\mathscr{M}$ be *locally free* $\mathscr{O}_X(*D)$-module of finite rank equipped with an *integrable* connection ∇.

Remark 1.3.1 **("Locally free" condition).** A priori, one should consider coherent $\mathscr{O}_X(*D)$-modules. However, each such object is locally a direct summand of a free $\mathscr{O}_X(*D)$-module, the other summand being also free. By using the trivial connection d on this other summand, one reduces the study to locally free $\mathscr{O}_X(*D)$-modules when we are interested in local properties.

Remark 1.3.2 **(Integrability condition).** The integrability condition $\nabla^2 = 0$ is essential in order to get generalizations of Levelt-Turrittin. Otherwise, one finds confluence phenomena which are very difficult to classify.

***Example 1.3.3*.** It is not easy to produce explicit examples of locally free $\mathscr{O}_X(*D)$-module of rank $\geqslant 2$ with an integrable connection and having interesting irregular singularities. One way to produce them is to start from a connection having regular singularity, twist it by an irregular connection, and apply various functors from the theory of $\mathscr{D}$-modules, like taking the Gauss-Manin connection (direct image) by an algebraic map to a surface.

An example of such a procedure is given by the Fourier transform, or the partial Fourier transform. Assume that $\mathscr{M}$ is presented as $\mathbb{C}[x_1, x_2]\langle\partial_{x_1}, \partial_{x_2}\rangle/(P_{i\in I})$, where $(P_{i\in I})$ is the left ideal generated by the operators P_i for $i \in I$, then the partial Fourier transform with respect to x_1 consists in composing with the automorphism $x_1 \mapsto -\partial_{x_1}$, $\partial_{x_1} \mapsto x_1$. The total Fourier transform uses the automorphism with respect to both variables. An example of such objects are the hypergeometric $\mathscr{D}$-modules of Gelfand, Kapranov and Zelevinski.

Other examples are confluent hypergeometric systems of many variables, like the following:

$$P_1 = x_1\partial_{x_1}^2 + x_2\partial_{x_1}\partial_{x_2} + (\gamma - x_1)\partial_{x_1} - \beta_1, \quad P_2 = x_2\partial_{x_2}^2 + x_1\partial_{x_1}\partial_{x_2} + (\gamma - x_2)\partial_{x_2} - \beta_2.$$

***Remark 1.3.4*.** There are various possibilities for defining the formalization $\widehat{\mathscr{O}}_{x_o}$ at $x_o \in D$. The most natural ones are $\mathscr{O}_{\widehat{x}_o}$ and $\mathscr{O}_{\widehat{D},x_o}$. The latter allows "moving the Levelt-Turrittin decomposition along D", but it is not the right one to use in general.

***Remark 1.3.5*.** It can be expected that the singularities of the divisor make things more complicated. One first reduces them by a sequence of point blowing-ups, in order to get a divisor with normal crossings. This reduction is relatively easy in dimension two, but becomes more complicated in higher dimension. From now on, I will assume that *D has only normal crossing singularities.* Then $\mathscr{O}_{\widehat{x}_o}(*D)$ is either $K[\![x_1, x_2]\!][x_1^{-1}]$ or $K[\![x_1, x_2]\!][(x_1x_2)^{-1}]$.

Question 1.3.6 **(Levelt-Turrittin in dimension two).** Let $\widehat{\mathscr{M}}$ be a free $\mathbb{C}[\![x_1, x_2]\!][x_1^{-1}]$ (resp. $\mathbb{C}[\![x_1, x_2]\!][(x_1x_2)^{-1}]$) module of finite rank, equipped with an integrable connection. Does there exist $e_1 \in \mathbb{N}^*$ (resp. $e_1, e_2 \in \mathbb{N}^*$) and a finite set $\widehat{\Phi} \subset \mathbb{C}[\![y_1, x_2]\!][1/y_1]/\mathbb{C}[\![y_1, x_2]\!]$ (resp. $\widehat{\Phi} \subset \mathbb{C}[\![y_1, y_2]\!][1/y_1y_2]/\mathbb{C}[\![y_1, y_2]\!]$) such that, through the ramification $y_1 \mapsto y_1^{e_1} = x_1$ (resp. and $y_2 \mapsto y_2^{e_2} = x_2$), $\mathbb{C}[\![y_1, x_2]\!][1/y_1] \otimes \widehat{\mathscr{M}}$ (resp. $\mathbb{C}[\![y_1, y_2]\!][1/y_1y_2] \otimes \widehat{\mathscr{M}}$) decomposes as $\bigoplus_{\varphi\in\widehat{\Phi}}(E(\varphi) \otimes R_\varphi)$ where R_φ has a regular singularity ?

In matrix terms, this is expressed as follows. Let $\Omega = A_1(x_1, x_2)dx_1 + A_2(x_1, x_2)dx_2$ be a matrix differential form, where A_1, A_2 have entries in $\mathbb{C}[\![x_1, x_2]\!][x_1^{-1}]$ (resp. in $\mathbb{C}[\![x_1, x_2]\!][(x_1x_2)^{-1}]$), satisfying the integrability condition

$$\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} = [A_1, A_2].$$

Does there exist e_1 and a matrix $P \in \mathrm{GL}_d(\mathbb{C}[\![y_1, y_2]\!][y_1^{-1}])$ (resp. e_1, e_2 and $P \in \mathrm{GL}_d(\mathbb{C}[\![y_1, y_2]\!][(y_1y_2)^{-1}]))$ such that, setting $\Omega' = A_1(y_1^{e_1}, y_2^{e_2})dy_1^{e_1} + A_2(y_1^{e_1}, y_2^{e_2})dy_2^{e_2}$, the matrix $P^{-1}\Omega' P + P^{-1}dP$ is block-diagonal, with diagonal blocks of the form $d\widetilde{\varphi}\,\mathrm{Id} + \Omega'_\varphi$, and Ω'_φ having at most logarithmic poles along $y_1 = 0$ (resp. $y_1y_2 = 0$).

If it exists, the decomposition is unique, and the set $\widehat{\Phi}$ is uniquely determined. However, the answer is "no" in general.

***Example 1.3.7* (Counter-example to L.-T., cf. [And07]).** Let us set

$$A = \begin{pmatrix} x_2/x_1 & -1 \\ 0 & 0 \end{pmatrix}$$

and consider the connection $\nabla \boldsymbol{m} = \boldsymbol{m} \cdot A(dx_1/x_1 - dx_2/x_2)$ on the rank-two module $\widehat{\mathscr{M}} = \big(\mathbb{C}[\![x_1, x_2]\!][(x_1x_2)^{-1}]\big)^2$ with basis $\boldsymbol{m} = (m_1, m_2)$. The integrability condition to be checked is

$$x_1\partial_{x_1}A_2 - x_2\partial_{x_2}A_1 = [A_1, A_2], \quad A_1 = A,\ A_2 = -A,$$

that is, $(x_1\partial_{x_1} + x_2\partial_{x_2})A = 0$, which is obvious.

There is an exact sequence

$$0 \longrightarrow \widehat{\mathscr{M}_1} \longrightarrow \widehat{\mathscr{M}} \longrightarrow \widehat{\mathscr{M}_2} \longrightarrow 0,$$

where $\mathscr{M}_1$ is generated by m_1 and $\mathscr{M}_2$ by the class of m_2. Then $\mathscr{M}_1 = E(x_2/x_1)$ and $\mathscr{M}_2 = (\mathbb{C}[\![x_1, x_2]\!][(x_1x_2)^{-1}], d)$ is regular. The exact sequence splits if one works over $K(\!(x_1)\!)$ with $K = \mathbb{C}(\!(x_2)\!)$, and this gives the Levelt-Turrittin decomposition of $\widehat{\mathscr{M}_K}$. On the other hand, the image of $[m_2]$ by a section compatible with connection of $\widehat{\mathscr{M}} \to \widehat{\mathscr{M}_2}$ must be of the form $m_2 + a(x_1, x_2)m_1$ with

$$\nabla_{x_1\partial_{x_1}}(m_2 + am_1) = 0, \quad \nabla_{x_2\partial_{x_2}}(m_2 + am_1) = 0,$$

that is

$$(x_1\partial_{x_1} + x_2\partial_{x_2})a = 0, \quad (x_1\partial_{x_1} + x_2/x_1)a = 1,$$

whose solutions are of the form

$$a(x_1, x_2) = ce^{x_2/x_1} + \sum_{\ell\geqslant 0}(-1)^\ell \ell!(x_1/x_2)^{\ell+1}, \quad c \in \mathbb{C},$$

none of them belong to $\mathbb{C}[\![x_1,x_2]\!][(x_1x_2)^{-1}]$, so the sequence does not split over $\mathbb{C}[\![x_1,x_2]\!][(x_1x_2)^{-1}]$.

Remark 1.3.8. Given any formal curve $\mathbb{C}[\![x_1,x_2]\!]\to\mathbb{C}[\![x]\!]$ expressed by formal power series $x_1(x),x_2(x)\in x\mathbb{C}[\![x]\!]$, the restriction $\mathbb{C}[\![x]\!]\otimes_{\mathbb{C}[\![x_1,x_2]\!]}\widehat{\mathscr{M}}$ is a differential vector space on $\mathbb{C}(\!(x)\!)$, and thus has a Levelt-Turrittin decomposition.

***Question 1.3.9* (Main finiteness question).** How does this Levelt-Turrittin decomposition vary with the choice of the curve? Can one recover the exponential factors $\varphi(x)$ for all possible curves from a *finite set* $\widehat{\Phi}$ of $\varphi(x_1,x_2)$?

In Example 1.3.7 one can take $\widehat{\Phi}=\{x_2/x_1,0\}$.

Question 1.3.10. Assume $\mathscr{M}$ is $\mathscr{O}_X(*D)$-locally free. If we have a two-dimensional Levelt-Turrittin decomposition at $x_o\in D$ (cf. Question 1.3.6), is $\widehat{\Phi}$ included in $\mathscr{O}_{X,x_o}(*D)/\mathscr{O}_{X,x_o}$ (better than in $\mathscr{O}_{\widehat{x}_o}(*D)/\mathscr{O}_{\widehat{x}_o}$)?

The answer to Question 1.3.10 is "yes" (cf. Theorem 2.2.1 below), and this is not much difficult. But one should be aware that the same question in dimension $\geqslant 3$ does not have a clear answer in general.

***Definition 1.3.11* (Semi-stable points, cf. [And07]).** Let $\mathscr{M}$ be a locally free $\mathscr{O}_X(*D)$-module with integrable connection. A point $x_o\in D$ is said to be *semi-stable* for $\mathscr{M}$ if Levelt-Turrittin holds for $\mathscr{O}_{\widehat{x}_o}\otimes_{\mathscr{O}_{X,x_o}}\mathscr{M}$.

***Definition 1.3.12* (Goodness).** Let us consider a finite set

$$\widehat{\Phi}\subset\mathbb{C}[\![x_1,x_2]\!][x_1^{-1}]/\mathbb{C}[\![x_1,x_2]\!]\quad(\text{resp. }\widehat{\Phi}\subset\mathbb{C}[\![x_1,x_2]\!][(x_1x_2)^{-1}]/\mathbb{C}[\![x_1,x_2]\!]).$$

We say that $\widehat{\Phi}$ is *good* if, given $\varphi,\psi\in\widehat{\Phi}$, with $\varphi\neq\psi$, for one choice (or any choice) of representatives $\widetilde{\varphi},\widetilde{\psi}$ in $\mathbb{C}[\![x_1,x_2]\!][x_1^{-1}]$ (resp. $\mathbb{C}[\![x_1,x_2]\!][(x_1x_2)^{-1}]$), the divisor of $\widetilde{\varphi}-\widetilde{\psi}$ is $\leqslant 0$.

Examples.

(1) If $\#\widehat{\Phi}=1$, then $\widehat{\Phi}$ is good.
(2) If $D=\{x_1=0\}$, the family $\widehat{\Phi}=\{0,x_2/x_1\}$ is not good.
(3) If $D=\{x_1x_2=0\}$, the family $\widehat{\Phi}=\{1/x_1,1/x_2\}$ is not good.
(4) Given η, $\widehat{\Phi}$ is good if and only if $\widehat{\Phi}+\eta$ is good. In particular, one can often reduce to the case where $0\in\widehat{\Phi}$.
(5) If $\widehat{\Phi}$ is good and contains 0, then the order $\operatorname{ord}(\varphi)\in\mathbb{Z}^2$ of $\varphi\in\widehat{\Phi}\smallsetminus\{0\}$ is well-defined and the family $\{\operatorname{ord}(\varphi)\mid\varphi\in\widehat{\Phi}\}$ is totally ordered.

Lemma 1.3.13. *Given a finite set $\widehat{\Phi}\subset\mathscr{O}_{\widehat{x}_o}(*D)$, there exists a finite sequence of point blowing-up such that the pull-back of $\widehat{\Phi}$ becomes good.*

Example 1.3.14. Let $\widehat{\Phi}=\{0,x_2/x_1\}$. The blowing-up of the origin produces two charts with respective coordinate systems (x'_1,x'_2) and (x''_1,x''_2), and the blowing-up map is given by $x_1=x'_1$, $x_2=x'_1x'_2$, resp. $x_1=x''_1x''_2$, $x_2=x''_2$. Then, in the first chart, the pull-back of $\widehat{\Phi}$ reduces to $\{0\}$ and, in the second chart, it reduces to $\{0,1/x''_1\}$, and both are good.

***Definition 1.3.15* (Good formal structure/stable points).** Let $\widehat{\mathscr{M}}$ be a free $\mathscr{O}_{\widehat{X}_o}(*D)$-module with integrable connection.

(1) We say that $\widehat{\mathscr{M}}$ has a *good decomposition* if it has a decomposition as in Question 1.3.6 with $e_1=1$ (resp. $e_1=e_2=1$) indexed by a *good set* $\widehat{\Phi}$.

(2) We say that $\widehat{\mathscr{M}}$ has a *good formal structure* or that x_o is a *stable point for* $\widehat{\mathscr{M}}$, if it has a good decomposition after some ramification around the components of D.

(3) If x_o is not a stable point for $\widehat{\mathscr{M}}$, it is called a *turning point for* $\widehat{\mathscr{M}}$.

***Theorem 1.3.16* (Kedlaya [Ked10], Mochizuki [Moc09a] (algebraic case))**

Let $\widehat{\mathscr{M}}$ be a free $\mathscr{O}_{\widehat{X}_o}(*D)$-module with integrable connection. There exists a finite sequence of point blowing-ups $e:(Y,E)\to(X,x_o)$ such that $e^*\widehat{\mathscr{M}}$ has a good formal structure at each point of E (equivalently, each point of E is stable for $e^*\widehat{\mathscr{M}}$).

In Example 1.3.7, a single blow-up of the origin is enough to realize stability of all points of the exceptional divisor.

2. Improvements and applications

In this talk, I will give, mainly without proof, some consequences of the theorem of Kedlaya and Mochizuki. Firstly, I will give some improvements of the main statement, which are due to Mochizuki. I will assume that D is a divisor with normal crossings in the algebraic (or complex analytic) surface X and that $\mathscr{M}$ is a locally free $\mathscr{O}_X(*D)$-module of finite rank, equipped with a integrable connection.

2.1. Genericity and turning points

Lemma 2.1.1. *There is a Zariski dense open set $U\in D$ such that each point of U is semi-stable with respect to $\mathscr{M}$.*

Proof. By reducing to an open set, one can assume that D is smooth, and one essentially can work with local coordinates x_1,x_2 such that $D=\{x_1=0\}$. Considering $\mathscr{M}$ with respect to $\nabla_{x_1\partial_{x_1}}$, one finds a Zariski open set of semi-stable points (cf. § 1.2) and one uses integrability to show that the corresponding Levelt-Turrittin decomposition is stable by $\nabla_{\partial_{x_2}}$. □

It follows from Lemma 1.3.13 that there is a Zariski dense open set of D consisting of stable points for $\mathscr{M}$. One can characterize the stable points on the smooth locus of D.

Theorem 2.1.2 **([BV85] for "if", [And07, Th. 3.4.1]).** *Set* $A = \mathbb{C}[x_1, x_2, p(x_2)^{-1}]$ *with* $p(0) \neq 0$. *Let* $\mathscr{M}$ *be a free* $A[x_1^{-1}]$*-module with an integrable connection. Then the origin is a stable point of* $\mathscr{M}$ *if and only if* $\mathbb{C}(\!(x_2)\!)(\!(x_1^{1/e_1})\!) \otimes_{A[x_1^{-1}]} \mathscr{M}$ *has a good Levelt -Turrittin decomposition.*

2.2. Openness of the good formal structure

Theorem 2.2.1. *Let* $\mathscr{M}$ *be a locally free* $\mathscr{O}_X(*D)$*-module with an integrable connection. Assume that* $\widehat{\mathscr{M}_{x_o}}$ *has a good decomposition. Then the subset* $\widehat{\Phi}_{x_o} \subset \mathscr{O}_{\widehat{x}_o}(*D)/\mathscr{O}_{\widehat{x}_o}$ *is in fact a subset* $\Phi_{x_o} \subset \mathscr{O}_{X,x_o}(*D)/\mathscr{O}_{X,x_o}$ *and, in some neighbourhood of* x_o*, each point* x *is stable and* Φ_x *is the restriction to* x *of a representative of* Φ_{x_o}*. Moreover, the good model as* x_o *is also a good model at each* x *near* x_o*.*

In dimension two, the proof can be done by comparing various formalizations at x_o (cf. [**Sab00**, § I.2.4]). In dimension $\geqslant 3$, the openness of the set of stable points (which is a direct consequence of the theorem) is already less obvious. It relies on the notion of a good lattice, which is explained below.

In any case, the global picture is as follows.

Corollary 2.2.2. *Let* $\mathscr{M}$ *be a locally free* $\mathscr{O}_X(*D)$*-module with an integrable connection. Let* D_i *be the irreducible components of* D *and* D_i^o *be the intersection of* D_i *with the smooth part of* D*. Lastly, let* C *denote the set of crossing points of* D*. Assume that all points of* D *are stable for* $\mathscr{M}$*. Then*

(1) *for each* i, $\mathscr{M}_{\widehat{D_i^o}}$ *admits locally on* D_i^o *a Levelt-Turrittin decomposition after a local finite ramification around* D_i^o*,*

(2) *for each* $c \in C$, $\mathscr{M}_{\widehat{c}}$ *admits a Levelt-Turrittin decomposition after a local finite ramification around the components of* D *going through* c*,*

(3) *and the good model at* c *is also a good model at each* $x \in D$ *in some neighbourhood of* c*.*

Definition 2.2.3 **(T. Mochizuki [Moc09b, Moc10]).** Let $E_{x_o} \subset \mathscr{M}_{x_o}$ be a free $\mathscr{O}_{X,x_o}$-module generating $\mathscr{M}_{x_o}$ over $\mathscr{O}_{X,x_o}(*D)$. We say that E_{x_o} is a non-ramified good lattice if $(E_{\widehat{x}_o}, \widehat{\nabla})$ decomposes as in the L.-T. decomposition, where R_φ is now a free $\mathscr{O}_{X,x_o}$-module with logarithmic connection.

Theorem 2.2.4 **(T. Mochizuki [Moc10]).** *If* $\mathscr{M}$ *has a good formal decomposition at each* x *in some neighbourhood of* x_o*, it has a non-ramified good lattice at* x_o*.*

Theorem 2.2.5 **(T. Mochizuki [Moc08]).** *Let* (E, ∇) *be a free* $\mathscr{O}_{X,x_o}$*-module with flat meromorphic connection. If* (E, ∇) *is a (non-ramified) good lattice with formal exponential factors* $\widehat{\Phi}_{x_o}$*, then* $\widehat{\Phi}_{x_o} = \Phi_{x_o} \subset \mathscr{O}_{X,x_o}(*D)/\mathscr{O}_{X,x_o}$ *and* (E, ∇) *has a good* Φ_x*-decomposition at each* x *near* x_o *with the same formal model.*

Remark 2.2.6. Malgrange has shown (cf. [**Mal96**]) that any meromorphic connection $\mathscr{M}$ has a canonical lattice, that Mochizuki calls the *Deligne-Malgrange lattice*, which generalizes the notion of Deligne lattice in the case of regular singularities. If $\mathscr{M}$ is good along D, then Mochizuki has shown that the Deligne-Malgrange lattice is a good lattice. This is useful when considering global questions on X.

2.3. Malgrange's conjecture. Let X be a complex surface let Z be a divisor in X and let $x_o \in Z$. Let $(\mathscr{M}, \nabla)$ be a locally free $\mathscr{O}_X(*Z)$-module of finite rank equipped with an integrable connection. Let $\gamma : (\mathbb{C}, 0) \to (X, x_o)$ be a germ of analytic curve whose image is not contained in (Z, x_o). The following result was given as a corollary of the expected theorem 1.3.16:

***Corollary 2.3.1* (of Th. 1.3.16, cf. [Sab00, Th. I.3.2.3]).** *The irregularity number of the pull-back connection $\gamma^+(\mathscr{M}, \nabla)$ satisfies*

$$\mathrm{ir}_{x_o}\big(\gamma^+(\mathscr{M}, \nabla)\big) \leqslant \sum_i (\gamma, Z_i)_{x_o} \cdot \mathrm{ir}_{Z_i}(\mathscr{M}, \nabla),$$

where Z_i are the local irreducible components of Z at x_o, $(\gamma, Z_i)_{x_o}$ is the valuation of the ideal of Z_i through γ, and $\mathrm{ir}_{Z_i}(\mathscr{M}, \nabla)$ is the generic irregularity number of $(\mathscr{M}, \nabla)$ along Z_i, or equivalently the irregularity of the pull-back of $(\mathscr{M}, \nabla)$ to a curve transverse to the smooth part of Z_i.

However, this result can be proved *without using the full strength of Theorem* 1.3.16, as was done by Y. André [**And07**], who proved

***Theorem 2.3.2* (cf. [And07]).** *With the same assumptions,*

$$\mathrm{ir}_{x_o}\big(\gamma^+(\mathscr{M}, \nabla)\big) \leqslant \sum_i (\gamma, Z_i)_{x_o} \cdot \mathrm{ir}_{Z_i}(\mathscr{M}, \nabla),$$
$$\rho_{x_o}\big(\gamma^+(\mathscr{M}, \nabla)\big) \leqslant \sum_i (\gamma, Z_i)_{x_o} \cdot \rho_{Z_i}(\mathscr{M}, \nabla),$$

where ρ is the Katz rank of the connection (maximal slope of the Newton polygon).

2.4. Concluding remarks. The theorem of Kedlaya and Mochizuki has been generalized by the same authors in higher dimensions (Mochizuki proves the algebraic case, but with analytic methods). Complemented with Theorems 2.2.4 and 2.2.5, it allows to develop the methods of asymptotic analysis originated in the work of Sibuya and Majima, and produces a Riemann-Hilbert correspondence for good meromorphic connections.

A nice application, considered in [**Sab00**] for dimension two and due to Mochizuki in higher dimension [**Moc10**], is the solution of the following conjecture of Kashiwara concerning distribution solutions of holonomic systems of differential equations:

***Corollary 2.4.1*.** *Let $\mathscr{M}$ be a holonomic $\mathscr{D}$-module on a complex analytic variety X. Then $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathfrak{Db}_X)$ is a holonomic module on the complex conjugate manifold, and the higher $\mathscr{E}xt$ vanish.*

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