

Aspects of the Fourier-Laplace transform

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Introduction

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- Different branches of physics are distinguished by the range of the variable x and by the names used for $h(x)$, $g(x)$ and for the integral.
- Of course this is a joke, physics is not a part of mathematics. However, it is true that the main mathematical problem of physics is the calculation of integrals of the form $\int h(x)e^{g(x)}dx$.

Exp. twisted \mathcal{D} -modules

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Work of Céline Roucairol

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- Question: Describe the irregular singular points of $\int_f^k \mathcal{O}_U e^g$ in terms of the geometry of f and g .

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- \mathcal{M} has **regular singularities** and singular support $S \subset \mathbb{A}^2$.
- $S =$ **discriminant** of (f, g) .
- e.g. \mathcal{M} is a bundle with flat connection on $\mathbb{A}^2 \setminus S$.

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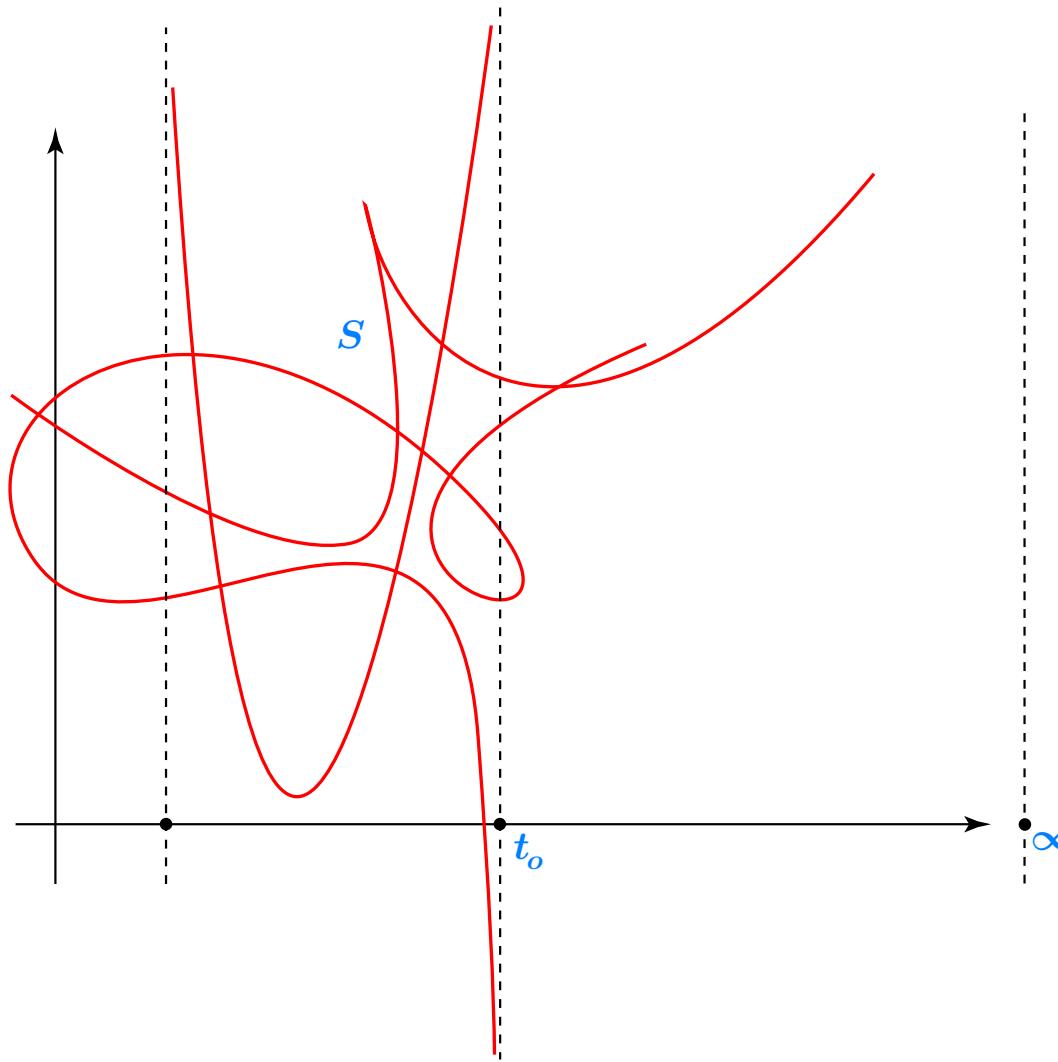
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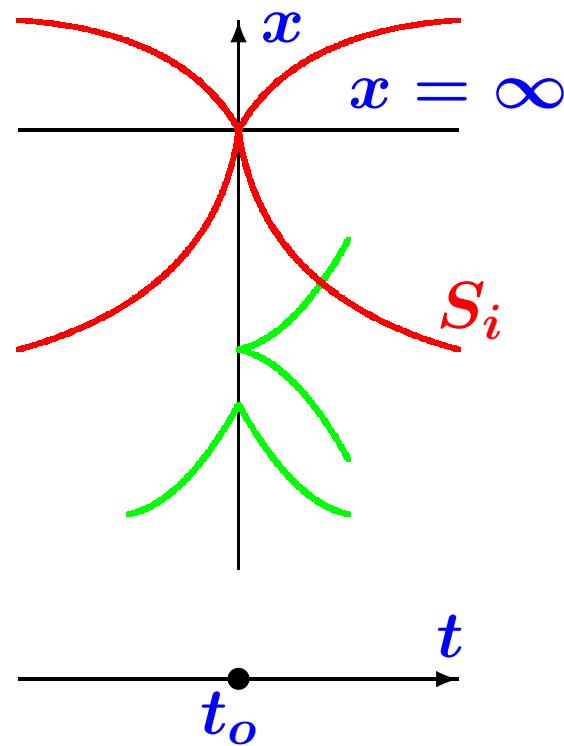
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Sketch of proof.

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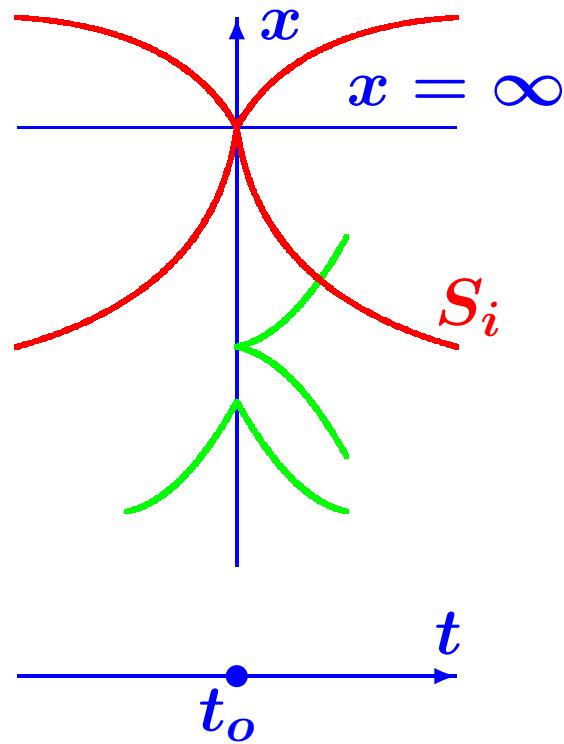
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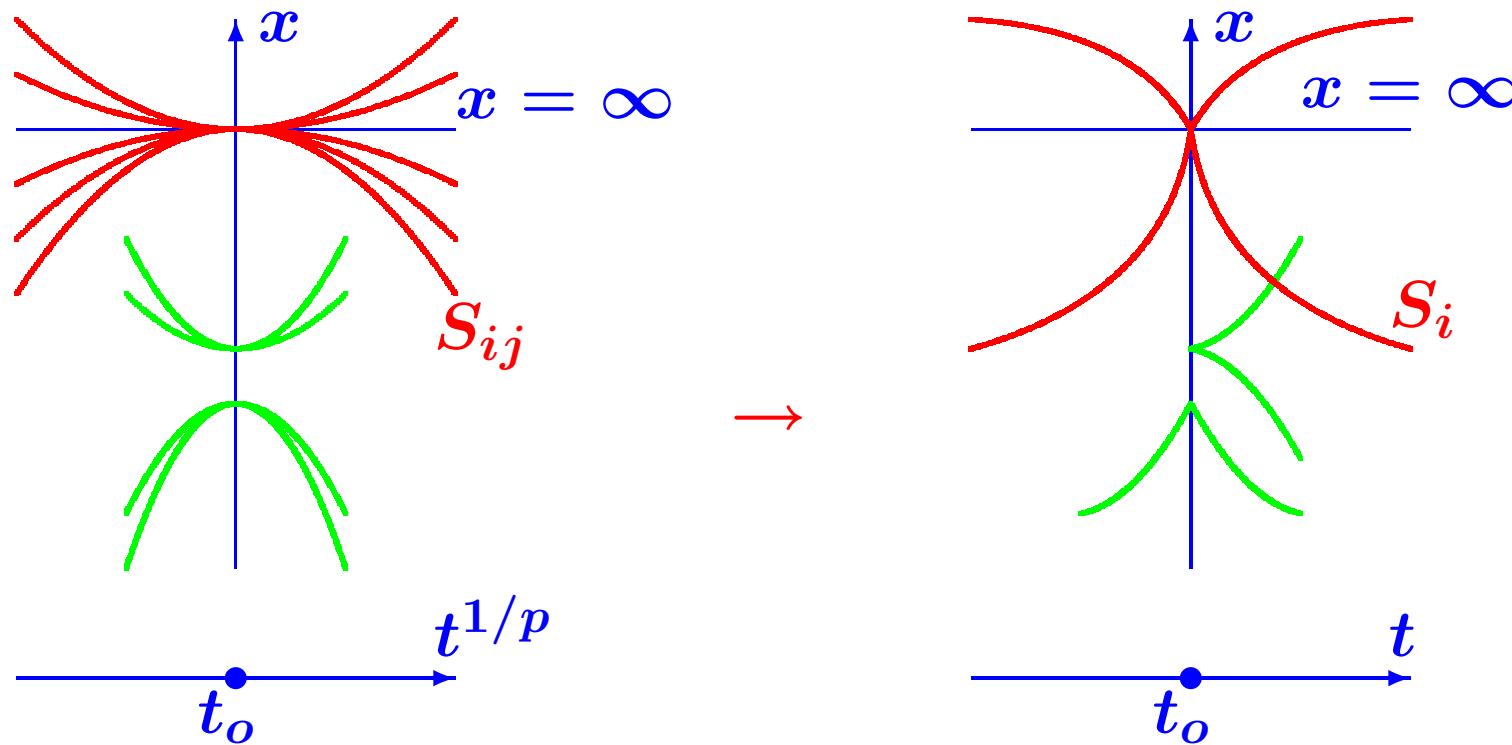
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 - (Laurent & Malgrange) $\text{gr}^V = \psi_t^{\text{mod}}$ compatible
with $\int_t \cdot$ → compute $\psi_t^{\text{mod}}(\mathcal{M} \otimes e^{-\varphi})$.
 - Simplify the computation by blowing-ups → S is a N.C.D.

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- Compute $\psi_f^{\text{mod}}(\mathcal{R} \otimes e^{1/x^n})$.

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- Local Fourier-Laplace transform ($t = \text{new var.}$):

$$\mathcal{F}^{(0,\infty)} M = \widehat{M} = \int_t^1 e^{-\rho/t} \otimes M((t))$$

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- \pm classical formula, formulated by Laumon (1987) in the ℓ -adic case, proved by Fu in the ℓ -adic case (2007), proved by C.S./Fang (2007).

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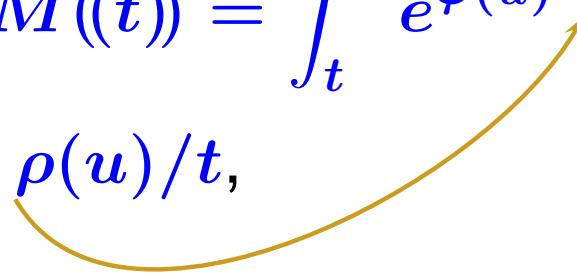
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Fourier-Laplace and Hodge

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Why? The jumps of the Hodge filtration (t fixed) **depend on** t .

Fourier-Laplace and Hodge

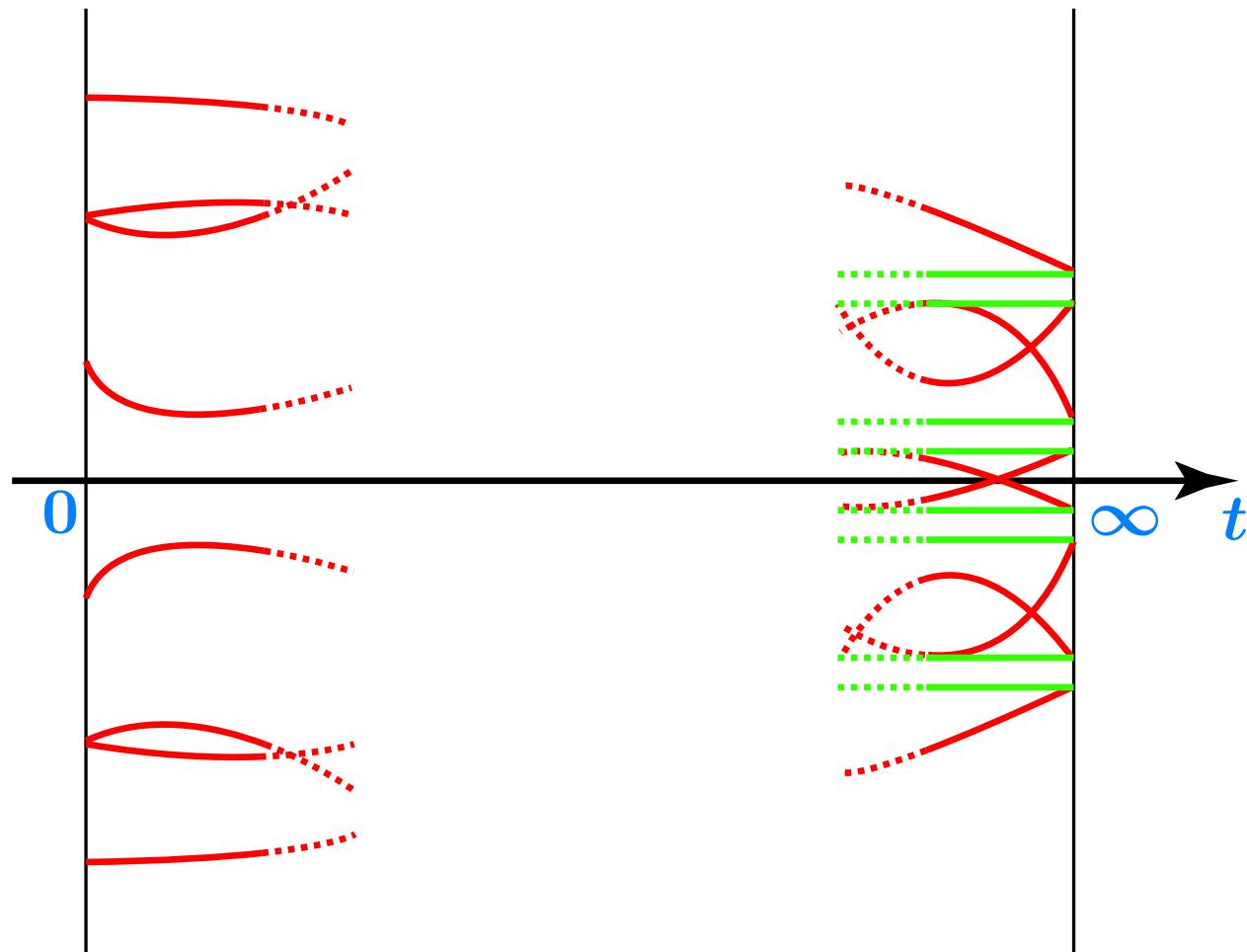
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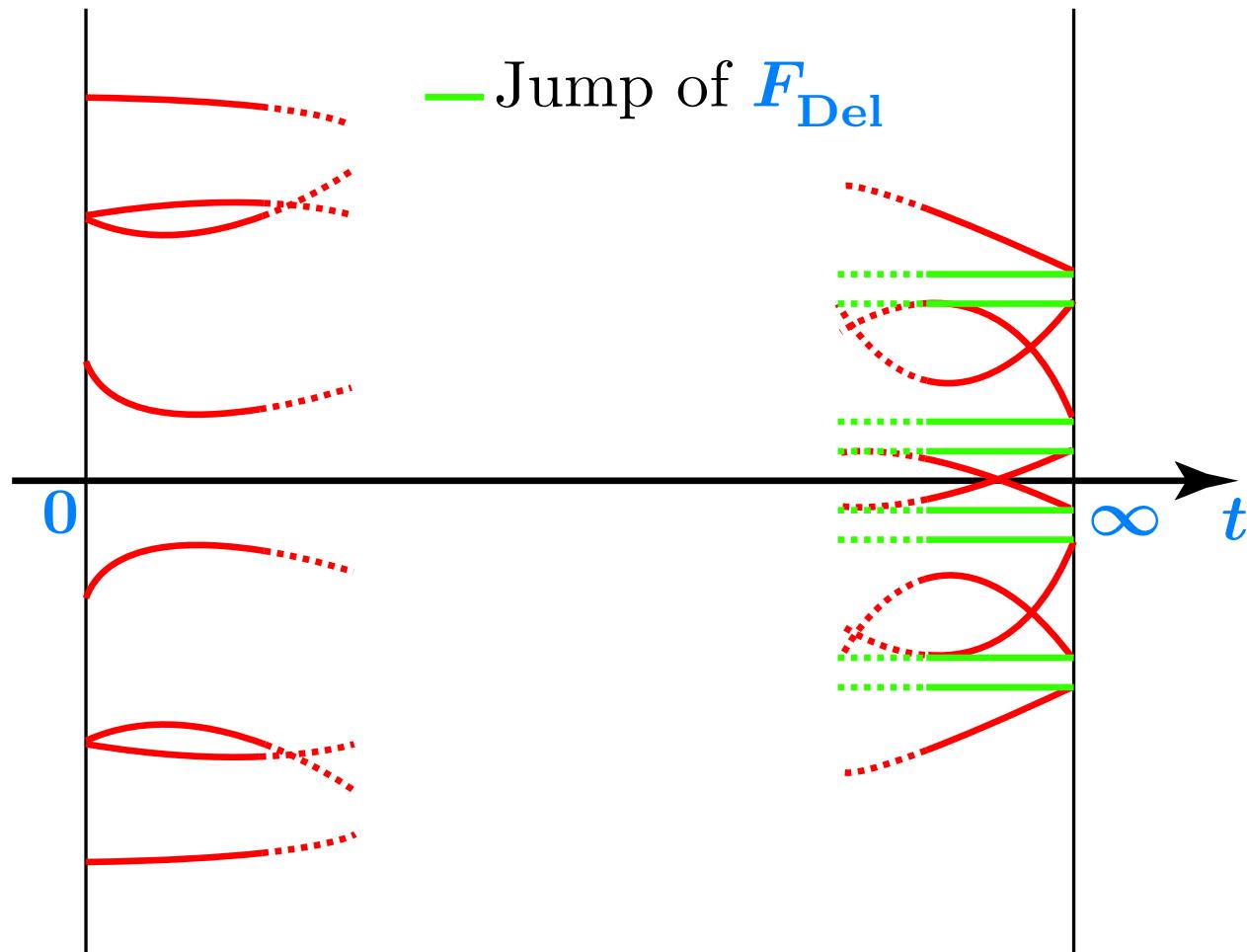
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