HYPERGEOMETRIC
DIFFERENTIAL AND q-DIFFERENCE EQUATIONS

by

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Introduction

These lectures are based on the articles [**11, 10**] and, for the q-difference part, on [**13**]. We consider algebraic differential equations on the torus of dimension p from an algebraic point of view. Mellin transform is introduced in §1. It converts such a system into a system of finite difference equations. A similar transformation exists for q-differences, but is more symmetric. The Mellin transform is an analogue, over the torus, of the Laplace transform for systems of algebraic differential equations on the affine space.

Lectures given at the workshop "$\mathscr{D}$-Modules and Hypergeometric Functions" (Lisbon, 11-14 july 2005). The part on q-hypergeometric equations is not written in these notes and the reader is referred to [**13**], which is also available at `http://www.math.polytechnique.fr/~sabbah/articles.html`.

We mainly consider *holonomic* systems of such equations. Bernstein's theory applies to the difference case or q-difference case. For the difference equations, this follows from Bernstein's theory of differential equations on the torus after Mellin transform. For q-difference equations, we apply methods analogous to that in the differential case.

For holonomic systems, we then get the classical finiteness theorems (Theorem 1.10 in the difference case, Proposition 2.3.4 and Theorem 2.7.2 of [**13**] in the q-difference case).

The second part of the lectures is devoted to hypergeometric systems. After having recalled the classical theorem of Ore (Proposition 1.7 in the differential/difference case, Proposition 3.1.1 in [**13**] in the q-difference case), we describe the irreducible hypergeometric modules. The differential/difference case in one variable follows [**7**], although we give some simpler proofs [(1)]. For $p \geqslant 2$ variables, we follow [**10**], and we also give some complementary results due to Gabber.

Some hypergeometric systems of differential equations can be obtained in a geometrical way, as attached to a set of p algebraic functions on a smooth affine manifold. One defines the *Aomoto complex* associated to these functions. In interesting cases, this complex has nonzero cohomology in a single degree at most. The *determinant* of this complex gives rise to a hypergeometric system of differential equations. A geometric interpretation of the decomposition of such a system as a product of Γ factors is given in [**11**], following previous work of Varchenko in the case where the functions define an arrangement of hyperplanes.

In [**7**], one finds arithmetic analogues of hypergeometric $\mathscr{D}$-modules on the one-dimensional torus. In [**5, 9**], one finds arithmetic analogues on higher dimensional tori.

1. Preliminaries

1.a. Algebraic Mellin transform and systems of finite difference equations

Let

$$T^p \simeq (\mathbb{C}^*)^p \overset{\text{def}}{=} \operatorname{Spec} \mathbb{C}[t_1, \dots, t_p, t_1^{-1}, \dots, t_p^{-1}]$$

be the p-dimensional complex torus (it is also denoted by $(\mathbb{G}_m)^p$) and denote by $\mathbb{C}[t, t^{-1}]\langle t\partial_t \rangle$ the algebra of algebraic differential operators on T^p, where we put $t = (t_1, \dots, t_p)$ and $t\partial_t = (t_1\partial_{t_1}, \dots, t_p\partial_{t_p})$. Recall that $\mathbb{C}[t, t^{-1}]\langle t\partial_t \rangle$ is the quotient algebra of the free algebra generated by $\mathbb{C}[t, t^{-1}]$ and $\mathbb{C}[t\partial_t]$ by the relations $[t_i\partial_{t_i}, t_j] = \delta_{ij} t_i$, where δ_{ij} denotes the Kronecker symbol.

The correspondence $t_i = t_i$ and $s_i = -t_i\partial_{t_i}$ identifies this algebra to the algebra $\mathbb{C}[s]\langle t, t^{-1} \rangle$ of finite difference operators, that is, the quotient algebra of the free algebra

1. Notice also that Katz gives a detailed analysis of the Galois theory of such differential hypergeometric systems.

generated by $\mathbb{C}[s]$ and $\mathbb{C}[t,t^{-1}]$ by the relations

$$\begin{aligned} t_i\cdot s_j &= s_j\cdot t_i && \text{if } i\neq j\\ t_i\cdot s_i &= (s_i+1)\cdot t_i && \forall\, i=1,\dots,p. \end{aligned}\tag{1.1}$$

Let M be a (left) holonomic $\mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle$-module. We call *algebraic Mellin transform of* M, that we also denote by M, the module M seen as a $\mathbb{C}[s]\langle t,t^{-1}\rangle$-module. We say that M is an algebraic *holonomic* system of finite difference equations (FDEs) if M is $\mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle$-holonomic (that is, if $\operatorname{Ext}^k_{\mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle}(M,\mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle)=0$ for $k\neq p$, see [**3**]).

To any $\mathbb{C}[s]\langle t,t^{-1}\rangle$-module M one associates the $\mathbb{C}(s)$-vector space $M(s)$ defined by

$$M(s)\overset{\text{def}}{=}\mathbb{C}(s)\otimes_{\mathbb{C}[s]}M.$$

Notice that $M(s)$ comes equipped with an invertible action of t_i $(i=1,\dots,p)$ satisfying the relations (1.1). We say that $M(s)$ is a *rational* system of finite difference equation. One also says that the field $\mathbb{C}(s)$ is a difference field, and $M(s)$ is a difference vector space over this difference field.

If moreover $\dim_{\mathbb{C}(s)}M(s)<\infty$, we say $M(s)$ is *rational holonomic*.

Remark 1.2. Starting from M as above, one usually considers its localization $M(t)=\mathbb{C}(t)\otimes_{\mathbb{C}[t,t^{-1}]}M$, which is a differential vector space over the differential field $(\mathbb{C}(t),\partial_t)$. Although M can be seen from two distinct points of view ($\mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle$-module or $\mathbb{C}[s]\langle t,t^{-1}\rangle$-module), the relation between both localizations $M(t)$ and $M(s)$ is less direct. In these notes, we will focus on $M(s)$.

1.b. Rational holonomic systems. As defined above, a *rational holonomic system of FDEs* is a $\mathbb{C}(s)$-vector space of finite dimension equipped with $\mathbb{C}$-linear automorphisms $t_1,\dots,t_p$ which commute pairwise and which satisfy Relations (1.1).

Let $M(s)$ be such a system (of dimension r) and let us choose a $\mathbb{C}(s)$-basis $\boldsymbol{m}$ of $M(s)$. Let us denote by $A_i(s)$ the $r\times r$-matrix of t_i in this basis. It has entries in $\mathbb{C}(s)$. The relations $[t_i,t_j]=0$ mean that the matrices A_i satisfy the relations

$$A_i(s+\mathbf{1}_j)\cdot A_j(s)=A_j(s+\mathbf{1}_i)\cdot A_i(s)$$

for all $i,j=1,\dots,p$, where $\mathbf{1}_i$ denotes the i-th basis vector of the natural basis of $\mathbb{C}^p$. That t_i is invertible means that A_i is invertible (hence belongs to $\mathrm{GL}(r,\mathbb{C}(s))$) and the matrix of t_i^{-1} in the basis $\boldsymbol{m}$ is equal to

$$A_i(s-\mathbf{1}_i)^{-1}.$$

On the other hand, after a base change with matrix $B(s)\in\mathrm{GL}(r,\mathbb{C}(s))$, the matrix $A_i'(s)$ of t_i is given by

$$A_i'(s)=B(s+\mathbf{1}_i)\cdot A_i(s)\cdot B(s)^{-1}.$$

Operations on rational systems of FDEs. If $M(s)$ and $M'(s)$ are two holonomic systems of FDEs, so are $M(s) \otimes_{\mathbb{C}(s)} M'(s)$, $\mathrm{Hom}_{\mathbb{C}(s)}(M(s), M'(s))$, and

$$\det M(s) \overset{\mathrm{def}}{=} \bigwedge^r M(s)$$

where $r = \dim_{\mathbb{C}(s)} M(s)$. The action of t is defined by the following formulas:

$$t \cdot (m \otimes m') = (tm) \otimes (tm'), \quad (t \cdot \varphi)(m) = t[\varphi(t^{-1}m)],$$
$$t \cdot (m_1 \wedge \cdots \wedge m_r) = (tm_1) \wedge \cdots \wedge (tm_r).$$

[Such a definition can be better understood by saying that "translating a tensor product consists in translating each factor"; this is exactly what one does when translating the product of two functions of s.]

Exercise 1.3. Show that these formulas define a left $\mathbb{C}[s]\langle t, t^{-1}\rangle$-structure.

1.c. Hypergeometric systems and the hypergeometric group

***Definition 1.4* (Hypergeometric systems).** We say that a holonomic $\mathbb{C}[t, t^{-1}]\langle t\partial_t\rangle$-module M is *hypergeometric* if $\dim_{\mathbb{C}(s)} M(s) = 1$.

We also say that a rational holonomic system $M(s)$ is hypergeometric if it has dimension 1 over $\mathbb{C}(s)$.

The set of isomorphism classes of rational hypergeometric systems form a group (under the tensor product), that we call the *hypergeometric group*.

Example 1.5. Assume that $p = 1$, and put $s = s_1$, $t = t_1$. An isomorphism class of rational hypergeometric systems consists of the datum of $\varphi \in \mathbb{C}(s)^*$ modulo the action base changes, which take the form

$$\psi(s) = \frac{h(s+1)}{h(s)} \cdot \varphi(s)$$

with $h \in \mathbb{C}(s)^*$. Hence, the hypergeometric group is equal to $\mathbb{C}(s)^*/{\sim}$, where $\sim$ denotes the equivalence relation defined by the base changes above. An element of the hypergeometric group can therefore be written in a unique way

$$c \cdot \prod_{\alpha \in \mathbb{C}/\mathbb{Z}} (s - \alpha)^{\gamma_\alpha} \tag{1.6}$$

with $\gamma_\alpha \in \mathbb{Z}$, $\gamma_\alpha = 0$ except for a finite number of $\alpha \in \mathbb{C}/\mathbb{Z}$ and, moreover, $c \in \mathbb{C}^*$. We can also express by saying that the elements of the hypergeometric group are in one-to-one correspondence with the equations satisfied by the functions

$$c^s \cdot \prod_{\alpha \in \mathbb{C}/\mathbb{Z}} \Gamma(s - \alpha)^{\gamma_\alpha}.$$

The structure of the hypergeometric group for $p > 1$ is given by Proposition 1.7. The hypergeometric group is obtained as follows: let $HG(p) \subset (\mathbb{C}(s)^*)^p$ be the set of $(\varphi_1, \dots, \varphi_p)$ satisfying the integrability condition

$$\frac{\varphi_i(s+\mathbf{1}_j)}{\varphi_i(s)} = \frac{\varphi_j(s+\mathbf{1}_i)}{\varphi_j(s)}$$

for all $i, j = 1, \dots, p$, with the group structure defined by the termwise multiplication; Let $\sim$ be the equivalence

$$(\varphi_1, \dots, \varphi_p) \sim (\psi_1, \dots, \psi_p)$$

iff there exists $h \in \mathbb{C}(s)^*$ such that, for all $i = 1, \dots, p$, one has

$$\psi_i(s) = \frac{h(s+\mathbf{1}_i)}{h(s)} \cdot \varphi_i(s).$$

Then the hypergeometric group $\mathscr{H}G(p)$ is the quotient $HG(p)/\sim$.

Let $\mathscr{L}$ be a subset of nonzero linear forms on $\mathbb{Q}^p$ with relatively prime integers as coefficients, such that, for any such form L, either $L \in \mathscr{L}$ or $-L \in \mathscr{L}$. One can for instance choose $\mathscr{L}$ as follows: $L(s) = \sum \lambda_i s_i$ belongs to $\mathscr{L}$ iff $\lambda_1 > 0$ when $\lambda_1 \neq 0$, $\lambda_2 > 0$ when $\lambda_1 = 0$ and $\lambda_2 \neq 0$, etc. Let $\mathbb{Z}^{[\mathscr{L}\times\mathbb{C}/\mathbb{Z}]}$ be the set of maps $\mathscr{L} \times \mathbb{C}/\mathbb{Z} \to \mathbb{Z}$ with finite support, endowed with its natural group structure.

Proposition 1.7. *Let $\sigma : \mathbb{C}/\mathbb{Z} \to \mathbb{C}$ be a section of the projection $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$. Then the map*

$$(\mathbb{C}^*)^p \times \mathbb{Z}^{[\mathscr{L}\times\mathbb{C}/\mathbb{Z}]} \longrightarrow \mathscr{H}G(p)$$

which associates to $[(c_1, \dots, c_p); \gamma]$ the isomorphism class of the system satisfied by

$$c_1^{s_1} \cdots c_p^{s_p} \prod_{L\in\mathscr{L}} \prod_{\alpha\in\mathbb{C}/\mathbb{Z}} \Gamma(L(s) - \sigma(\alpha))^{\gamma_{L,\alpha}}$$

does not depend on the chosen section σ and is a group isomorphism.

Remarks 1.8

(1) In the following, we will write $\Gamma(L(s) - \alpha)$. Moreover, we will denote

$$\mathrm{FDE}\left[c_1^{s_1} \cdots c_p^{s_p} \prod_{L\in\mathscr{L}} \prod_{\alpha\in\mathbb{C}/\mathbb{Z}} \Gamma(L(s) - \alpha)^{\gamma_{L,\alpha}}\right]$$

the isomorphism class of the system of FDEs satisfied by the function between brackets.

(2) Proposition 1.7 and its proof when $p=2$ are contained in [**12**]. In [**1**], K. Aomoto attributes it to M. Sato (where the hypergeometric group appears as the cohomology group $H^1(\mathbb{Z}^p, \mathbb{C}(s_1, \dots, s_p))$). The proof of M. Sato is published in [**15**].

Proof of Proposition 1.7. The independence with respect to σ is immediate (we are mainly reduced to showing to the systems satisfied by $\Gamma(L(s)-\beta)$ and $\Gamma(L(s)-\beta+1)$ for $\beta \in \mathbb{C}$ are isomorphic).

Lemma 1.9. *Let $(\varphi_1,\dots,\varphi_p)\in HG(p)$. There exists $(\psi_1,\dots,\psi_p)\in HG(p)$ equivalent to $(\varphi_1,\dots,\varphi_p)$ such that, for any $i=1,\dots,p$, ψ_i is a product like*

$$c_i\cdot\prod_{L\in\mathscr{L}}\prod_{\alpha\in\mathbb{C}/\mathbb{Z}}\prod_{\lambda\in\mathbb{Z}}(L(s)-\alpha+\lambda)^{n_i(L,\alpha,\lambda)}$$

where $c_i\in\mathbb{C}^*$ and $n_i(L,\alpha,\lambda)\in\mathbb{Z}$ is zero except on a finite set.

Proof. In the following, we assume that $p\geqslant 2$. Let $P(s)$ be an irreducible polynomial and, for any $i=1,\dots,p$, let us denote by

$$\prod_{\sigma\in\mathbb{Z}^p}P(s+\sigma)^{n_i(P,\sigma)}$$

the product of integral translates of P which appear in φ_i, with $n_i(P,\sigma)\in\mathbb{Z}$. This decomposition is unique if P is not invariant (up to a multiplicative constant) by any integral translation of $\mathbb{C}^p$. Let us begin with this case. The integrability relation is equivalent to

$$n_i(P,\sigma-\mathbf{1}_j)-n_i(P,\sigma)=n_j(P,\sigma-\mathbf{1}_i)-n_j(P,\sigma)\overset{\mathrm{def}}{=}n_{i,j}(\sigma)$$

for all $i,j=1,\dots,p$, and one tries to write the product as $h(s+\mathbf{1}_i)/h(s)$, that is, one searches for a function $m(\sigma)$ (exponent of $P(s+\sigma)$ in h) having finite support on $\mathbb{Z}^p$, with values in $\mathbb{Z}$, such that, for any $i=1,\dots,p$ one has

$$m(\sigma-\mathbf{1}_i)-m(\sigma)=n_i(P,\sigma).$$

Such a function, if it exists, is unique and is given by the formula

$$m(\sigma)=-\sum_{k\geqslant 0}n_i(P,\sigma-k\mathbf{1}_i).$$

We have to show that

(1) $m(\sigma)$ does not depend on i
(2) $m(\sigma)$ has finite support.

For $i\neq j$ one can write

$$m(\sigma)=\sum_{\ell\geqslant 0}\sum_{k\geqslant 0}n_{i,j}(\sigma-k\mathbf{1}_i-\ell\mathbf{1}_j)$$

and the sum is finite, as $n_{i,j}$ has finite support. Therefore, $m(\sigma)$ does not depend on i.

In order to show that $m(\sigma)$ has finite support, it is enough to verify that $m(\sigma)=0$ as soon as σ_i is large enough or small enough. The second case does not cause trouble, as n_i has finite support. For the first case, it amounts to verifying that

$$-\sum_{k\in\mathbb{Z}}n_i(P,\sigma-k\mathbf{1}_i)=0$$

for the same reason. But, because of the integrability relation, one has, for $i\neq j$,

$$\sum_{k\in\mathbb{Z}}n_i(P,\sigma-k\mathbf{1}_i)=\sum_{k\in\mathbb{Z}}n_i(P,\sigma-\mathbf{1}_j-k\mathbf{1}_i)$$

and, for $\sigma_j\ll 0$, one has $n_i(P,\sigma)=0$. Therefore, By iterating this process, all terms in the sum vanish.

More generally, let $C \in \mathbb{C}^p$ be an irreducible hypersurface with equation $P = 0$ and let $\mathsf{R} \subset \mathbb{Z}^p$ the maximal sub-lattice stabilizing C. Then C is also stable by $\mathbb{C} \otimes_{\mathbb{Z}} \mathsf{R}$ and, more precisely, C is the inverse image by the projection $\pi : \mathbb{C}^p \to \mathbb{C}^p/(\mathbb{C} \otimes_{\mathbb{Z}} \mathsf{R}) = \mathbb{C}^q$ of a hypersurface C'. If $q = 1$ we are in the situation of Example 1.5. Let us then consider the case where $q \geqslant 2$. Let s' be a coordinate system on $\mathbb{C}^q$ and let $Q(s')$ be a reduced equation of C'. One must have, if

$$\prod_{\sigma' \in \pi(\mathbb{Z}^p)} Q(s' + \sigma')^{n_i(Q,\sigma')}$$

is the contribution of the integral translates of C to φ_i, the relation

$$n_i(Q, \sigma' - \pi(\mathbf{1}_j)) - n_i(Q, \sigma') = n_j(Q, \sigma' - \pi(\mathbf{1}_i)) - n_j(Q, \sigma') \overset{\mathrm{def}}{=} n_{i,j}(\sigma')$$

for all i, j. One searches for a function $m(\sigma')$ as above, that one finds in an analogous way, when $q \geqslant 2$. □

Let us come back to the proof of Proposition 1.7. So, let $L \in \mathscr{L}$, $\alpha \in \mathbb{C}/\mathbb{Z}$ and let us consider the contribution of integral translates of the hyperplane $L(s) - \alpha$ to φ_i in the following form:

$$\prod_{\lambda \in \mathbb{Z}} (L(s) - \alpha + \lambda)^{n_i(\lambda)}$$

where $n_i : \mathbb{Z} \to \mathbb{Z}$ has finite support and satisfies, for all $\lambda \in \mathbb{Z}$,

$$n_i(\lambda - \lambda_j) - n_i(\lambda) = n_j(\lambda - \lambda_i) - n_j(\lambda) \overset{\mathrm{def}}{=} n_{i,j}(\lambda)$$

for all i, j, by putting $L(s) = \sum_{i=1}^p \lambda_i \cdot s_i$. In particular, if $\lambda_i = 0$ one has $n_i \equiv 0$. Set

$$m_{L,\alpha,i}(\lambda) = \begin{cases} -\sum_{k \geqslant 0} n_i(\lambda - k\lambda_i) & \text{if } \lambda_i > 0 \\ \sum_{k \geqslant 1} n_i(\lambda + k\lambda_i) & \text{if } \lambda_i < 0 \\ 0 & \text{if } \lambda_i = 0 \end{cases}$$

Expressing as above $m_{L,\alpha,i}(\lambda)$ in terms of $n_{i,j}$ allows one to verify that $m_{L,\alpha,i}$ does not depend on i and that $m_{L,\alpha}(\lambda) = 0$ for $\lambda \ll 0$. Moreover, one can, up to equivalence, eliminate in $(\varphi_1, \ldots, \varphi_p)$ the terms where L and α appear if and only if $m_{L,\alpha}(\lambda) = 0$ for $\lambda \gg 0$ (*i.e.* if $m_{L,\alpha}$ has finite support).

In conclusion, let us write $(\varphi_1, \ldots, \varphi_p)$ as

$$\varphi_i = c_i \cdot \prod_{L \in \mathscr{L}} \prod_{\alpha \in \mathbb{C}/\mathbb{Z}} \prod_{\lambda \in \mathbb{Z}} (L(s) - \alpha + \lambda)^{n_i(L,\alpha,\lambda)}.$$

Then $(\varphi_1, \ldots, \varphi_p)$ is equivalent to the image, by the map of the proposition, of

$$\big[(c_1, \ldots, c_p); (m_{L,\alpha}(+\infty))_{L \in \mathscr{L}, \alpha \in \mathbb{C}/\mathbb{Z}}\big]$$

where $m_{L,\alpha}(+\infty)$ denotes the asymptotic value of $m_{L,\alpha}(\lambda)$. Moreover, the argument above also shows that this map is injective. □

1.d. Algebraic holonomic *versus* rational holonomic.

The link between "algebraic holonomic" and "rational holonomic" is given by the following theorem.

Theorem 1.10

(1) *Let M be an algebraic holonomic system of FDEs. Then $M(s)$ is rational holonomic.*

(2) *Conversely, if $M(s)$ is rational holonomic then, for any sub-$\mathbb{C}[s]\langle t,t^{-1}\rangle$-module $M\subset M(s)$ such that $M(s)=\mathbb{C}(s)\otimes_{\mathbb{C}[s]}M$, there exists an algebraic holonomic system $M'\subset M$ such that $M(s)=\mathbb{C}(s)\otimes_{\mathbb{C}[s]}M'$.*

Remark 1.11. There is an analogous statement for differential modules, namely, if M is holonomic, then $M(t)$ is finite dimensional over $\mathbb{C}(t)$ and, conversely, in any such $M(t)$ there is a holonomic M. The proof of these statements directly follows from Bernstein's theory of holonomic modules. The proof that we give below for $M(s)$ is a simple adaptation of it.

Proof of 1.10(1). Let M be an holonomic algebraic system of FDEs, *i.e.* M $\mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle$-holonomic. Let us forget for a while the previous correspondence and let us introduce new variables $s=(s_1,\dots,s_p)$. Put $T^p_{\mathbb{C}(s)}=\operatorname{Spec}\mathbb{C}(s)[t,t^{-1}]$ and let $\mathbb{C}(s)[t,t^{-1}]\langle t\partial_t\rangle$ be the algebra of differential operators on this torus (*i.e.* like $\mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle$, but with base field $\mathbb{C}(s)$ instead of $\mathbb{C}$).

Denote by $\mathscr{T}^s$ the module $\mathbb{C}(s)[t,t^{-1}]$ equipped with the twisted action of $\mathbb{C}(s)[t,t^{-1}]\langle t\partial_t\rangle$ defined as being the standard action of $\mathbb{C}(s)[t,t^{-1}]$ and the twisted action of $t_i\partial_{t_i}$ defined by

$$t_i\partial_{t_i}\cdot 1=s_i.$$

[This action can also formally be written as $t^{-s}\circ t_i\partial_{t_i}\circ t^s$, with $t^s=t_1^{s_1}\cdots t_p^{s_p}$.]

If M is any $\mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle$-module, it defines a $\mathbb{C}(s)[t,t^{-1}]\langle t\partial_t\rangle$-module $M(s)=\mathbb{C}(s)\otimes_{\mathbb{C}}M$, and we consider the $\mathbb{C}(s)[t,t^{-1}]\langle t\partial_t\rangle$-module $M(s)\otimes_{\mathbb{C}(s)[t,t^{-1}]}\mathscr{T}^s$, that we denote for short by $M(s)t^s$. We have $M(s)t^s=\mathbb{C}(s)\otimes_{\mathbb{C}}M$, where the action on the right-hand term is given by the following rule:

$$t_i\partial_{t_i}(\varphi(s)\otimes m)=\varphi(s)\otimes(t_i\partial_{t_i}m)+s_i\varphi(s)\otimes m.$$

One shows, as in [**2**], that $M(s)t^s$ is $\mathbb{C}(s)[t,t^{-1}]\langle t\partial_t\rangle$-holonomic. Let $\pi:T^p_{\mathbb{C}(s)}\to\operatorname{Spec}\mathbb{C}(s)$ be the constant map. It follows from [**2**] (see [**4**]) that the cohomology groups of the direct image $\pi_+M(s)t^s$ are finite dimensional over $\mathbb{C}(s)$. On the other hand, the module $M(s)t^s$ comes equipped with an action of translation operators τ_i $(i=1,\dots,p)$:

$$\tau_i\cdot(\varphi(s)\otimes m)=\varphi(s+\mathbf{1}_i)\otimes t_im.$$

Moreover, this action commutes with that of $\mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle$. Consequently, the cohomology modules of the complex $\pi_+M(s)t^s$ are endowed with a left $\mathbb{C}(s)\langle\tau,\tau^{-1}\rangle$-module structure, and therefore are rational holonomic systems of FDEs. The first par of Theorem 1.10 follows now from the lemma below.

Lemma 1.12. *For all $i\neq 0$ we have $H^i\pi_+M(s)t^s=0$ and*

$$H^0\pi_+M(s)t^s=M(s).$$

One decomposes π in projections along coordinate axes. By induction, one is reduced to showing the result for the projection ϖ on the first $p-1$ coordinates:

$$\varpi:T^p_{\mathbb{C}(s)}\longrightarrow T^{p-1}_{\mathbb{C}(s)}.$$

The complex $\varpi_+ M(s)t^s$ is the relative algebraic de Rham complex of the holonomic module $M(s)t^s$ with respect to ϖ, that is,

$$0 \longrightarrow M(s)t^s \xrightarrow{\partial_{t_p}} M(s)t^s \longrightarrow 0$$

where the right-hand term (corresponding to Ω^1) has degree 0 in the complex. This complex is quasi-isomorphic to the complex

$$0 \longrightarrow M(s)t^s \xrightarrow{t_p\partial_{t_p}} M(s)t^s \longrightarrow 0$$

as t_p in an invertible way, and this last complex can also be written as

$$0 \longrightarrow \mathbb{C}(s) \otimes_{\mathbb{C}} M \xrightarrow{t_p\partial_{t_p} + s_p} \mathbb{C}(s) \otimes_{\mathbb{C}} M \longrightarrow 0.$$

Put $s' = (s_1, \dots, s_{p-1})$ and $K' = \mathbb{C}(s')$. The K'-linear morphism

$$t_p\partial_{t_p} + s_p : K'[s_p] \otimes_{\mathbb{C}} M \longrightarrow K'[s_p] \otimes_{\mathbb{C}} M$$

is injective (immediate) hence it remains so after tensoring with $K'(s_p)$ (flatness of $K'(s_p)$ on $K'[s_p]$). Therefore, $H^{-1}\varpi_+ M(s)t^s = 0$. Moreover, the map

$$\begin{aligned} K'[s_p] \otimes_{\mathbb{C}} M &\longrightarrow M(s')t'^{s'} \\ \sum_{i\geqslant 0} s_p^i \otimes m_i &\longmapsto \sum_{i\geqslant 0} (-t_p\partial_{t_p})^i \cdot m_i \end{aligned}$$

induces an isomorphism between $\operatorname{Coker}(t_p\partial_{t_p}+s_p)$ and $M(s')t'^{s'}$, under which multiplication by s_p on $\operatorname{Coker}(t_p\partial_{t_p}+s_p)$ corresponds to the action de $-t_p\partial_{t_p}$ on $M(s')t'^{s'}$ and the action of t_p to the multiplication by t_p. The lemma follows. □

Sketch of proof of 1.10(2). There is a unique finite increasing filtration $M_\bullet$ of M by sub $\mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle$-modules such that each graded piece M_k/M_{k-1} has dimension k exactly. By Bernstein's inequality, this filtration can be written as

$$0 \subset M_p \subset M_{p+1} \subset \cdots \subset M_{2p} = M.$$

Tensoring this filtration with $\mathbb{C}(s)$ over $\mathbb{C}[s]$ gives a filtration satisfying analogous properties over $\mathbb{C}(s)\langle t, t^{-1}\rangle$. But $M(s)$ has pure dimension p over $\mathbb{C}(s)\langle t, t^{-1}\rangle$. This means that $M_p(s) = M(s)$. Therefore, $M' \overset{\text{def}}{=} M_p$ satisfies the requirement. □

Exercise 1.13. Show the partial analogue of Theorem 1.10(1): divide the variables $t = (t_1, \dots, t_p)$ in two sets $t = (t', t'')$ and denote by $s = (s', s'')$ the corresponding s-variables. Let M be a holonomic $\mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle$-module. Show that $M(s') \overset{\text{def}}{=} \mathbb{C}(s') \otimes_{\mathbb{C}[s]} M$ is holonomic over the ring $k'[t'', t''^{-1}]\langle s''\rangle$, with $k' = \mathbb{C}(s')$. [*Hint:* in the proof of 1.10(1), introduce only the new variables s'.]

The dimension of $M(s)$ can be computed in term of the algebraic de Rham complex of M on the torus. Recall that the de Rham complex ${}^p\mathrm{DR}\, M$ is the complex $(\Omega_T^{p+\bullet} \otimes_{\mathbb{C}[t,t^{-1}]} M, \nabla)$. We use the convention of $\mathscr{D}$-module theory for the degrees. In particular, de degrees in the de Rham complex are non positive. In coordinates, this complex is the single complex associated to the multiple complex with edges

$$0 \longrightarrow M \xrightarrow{t_i\partial_{t_i}} M \longrightarrow 0,$$

and, in the corresponding cube, the abutment of all arrows $t_i\partial_{t_i}$ has multi-degree $(0,\dots,0)$.

If M is holonomic, the cohomology spaces of ${}^p\mathrm{DR}\, M$ are finite dimensional, and we denote by $\chi({}^p\mathrm{DR}\, M)$ the Euler characteristics of this complex.

Proposition 1.14. *If M is holonomic, then* $\dim_{\mathbb{C}(s)} M(s) = \chi({}^p\mathrm{DR}\, M)$.

Proof. For simplicity, let us begin with the one-dimensional case, so $p = 1$. By definition, $\chi(T, M)$ is the characteristic number of the complex

$$0 \longrightarrow M \xrightarrow{t\partial_t} M \longrightarrow 0,$$

that is, according to the shifting assumption, $\dim \operatorname{Coker} t\partial_t - \dim \operatorname{Ker} t\partial_t$. Working on the s-side, it is also the characteristic number of the complex

$$\boldsymbol{L}i^*M = \{0 \longrightarrow M \xrightarrow{s} M \longrightarrow 0\},$$

if $i : \{0\} \hookrightarrow \operatorname{Spec} \mathbb{C}[s]$ denotes the inclusion. On the other hand, if N is any finite type $\mathbb{C}[s]$-module, the characteristic number $\chi\left(\boldsymbol{L}i^*N\right)$ is equal to the dimension of the generic fibre $N(s)$. As M is not of finite type over $\mathbb{C}[s]$ in general, we cannot conclude immediately. However, choose a finite type $\mathbb{C}[s]$-module $N \subset M$ such that $N(s) = M(s)$. It is thus enough to show that, there exists such a submodule such that $\chi\left(\boldsymbol{L}i^*N\right) = \chi\left(\boldsymbol{L}i^*M\right)$.

Let us put $M_0 = N$ and, for any $k > 0$, define $M_k = \sum_{|j|\leqslant k} t^j N$. We get in that way an increasing filtration of M such that, for any $k \geqslant 0$, M_{k+1}/M_k consists of $\mathbb{C}[s]$-torsion. More precisely, $M_{k+1}/M_k = (t^{k+1}N + t^{-(k+1)}N + M_k)/M_k$ and, if $b(s)\cdot M_1/M_0 = \{0\}$, then $b(s+k)b(s-k)\cdot M_{k+1}/M_k = \{0\}$. If k is large enough, s is prime to $b(s+k)b(s-k)$ and therefore

$$s : M_{k+1}/M_k \longrightarrow M_{k+1}/M_k$$

is bijective. It follows that the complex $0 \to M \xrightarrow{s} M \to 0$ is quasi-isomorphic to the complex $0 \to M_k \xrightarrow{s} M_k \to 0$, hence the result.

In general, the proof is by induction on p. So, assume it is true for $p-1$. We will use Exercise 1.13, with $t' = (t_1,\dots,t_{p-1})$. Consider the exact sequence

$$0 \longrightarrow M_{-1} \longrightarrow M \xrightarrow{t_p\partial_{t_p}} M \longrightarrow M_0 \longrightarrow 0.$$

Then, by Bernstein's theorem, M_0 and M_{-1} are holonomic as modules over $\mathbb{C}[t', t'^{-1}]\langle t'\partial_{t'}\rangle$ (holonomy of the direct images with respect to the projection $t \mapsto t'$). We also have, by tensoring with $\mathbb{C}(s')$, an exact sequence

$$0 \longrightarrow M_{-1}(s') \longrightarrow M(s') \xrightarrow{t_p\partial_{t_p}} M(s') \longrightarrow M_0(s') \longrightarrow 0. \tag{1.15}$$

We now have

$$\begin{aligned}\chi(T^p, {}^p\mathrm{DR}\, M) &= \chi(T^{p-1}, {}^p\mathrm{DR}\, M_0) - \chi(T^{p-1}, {}^p\mathrm{DR}\, M_{-1})\\ &= \dim_{\mathbb{C}(s')} M_0(s') - \dim_{\mathbb{C}(s')} M_{-1}(s') \quad \text{(induction hypothesis)}\\ &= \chi(T, {}^p\mathrm{DR}\, M(s')) \quad \text{(from (1.15))}\\ &= \dim_{\mathbb{C}(s)} M(s) \quad \text{(first part of the proof } (p=1) \text{ over the field } \mathbb{C}(s')),\end{aligned}$$

as $\mathbb{C}(s')(s_p) = \mathbb{C}(s)$. □

1.e. The Aomoto complex. Let U be a smooth complex affine algebraic variety and let

$$f = (f_1, \dots, f_p) : U \longrightarrow T^p = (\mathbb{C}^*)^p$$

be an algebraic morphism. Let M be a holonomic $\mathscr{D}(U)$-module, where $\mathscr{D}(U)$ denotes the ring of differential operators on U with coefficients in $\mathscr{O}(U)$. Let us put $\mathscr{D}(U)(s) = \mathbb{C}(s) \otimes_{\mathbb{C}} \mathscr{D}(U)$, where $s = (s_1, \dots, s_p)$ are new variables and let $M(s)f^s$ be the $\mathscr{D}(U)(s)$-module obtained by twisting M with f^s (define $\mathscr{T}_f^s$ as in the proof of Theorem 1.10(1) and put $M(s)f^s = M \otimes_{\mathscr{O}(U)(s)} \mathscr{T}_f^s$). Bernstein's theory shows that it is a holonomic $\mathscr{D}(U)(s)$-module. Let p be the constant map from $U \times \operatorname{Spec} \mathbb{C}(s)$ to $\operatorname{Spec} \mathbb{C}(s)$. One deduces from Bernstein's Theorem (see *loc. cit*) that the complex $p_+ M(s)f^s$ has finite dimensional cohomology over $\mathbb{C}(s)$. As U is affine, this complex is:

$$\Omega(U)^{\bullet + \dim U} \otimes_{\mathscr{O}(U)} M(s)f^s. \tag{1.16}$$

If for instance $M = \mathscr{O}(U)$, the complex $\Omega(U)^{\bullet + \dim U} \otimes_{\mathscr{O}(U)} M(s)f^s$ is nothing but the complex $\mathbb{C}(s) \otimes_{\mathbb{C}} \Omega(U)^{\bullet + \dim U}$ with differential d_s given by

$$d_s(\varphi(s) \otimes \omega) = \varphi(s) \otimes d\omega + \sum_{i=1}^{p} s_i \varphi(s) \otimes \frac{df_i}{f_i} \wedge \omega.$$

The $\mathscr{D}(U)(s)$-module $M(s)f^s$ comes equipped with invertible translation operators τ_i defined by the formula:

$$\tau_i \cdot [\varphi(s) m f^s] = \varphi(s + \mathbf{1}_i)(f_i m) f^s.$$

These operators extend in a natural way to each term of the complex(1.16) and commute to the differential d_s. It follows that they act on the cohomology.

The *Aomoto complex* $p_+ M(s)f^s$ is denoted by $\mathscr{A}_{f_1, \dots, f_p}(M)(s)$. If $U = T^p$ and $f = \mathrm{Id}$, one has $\mathscr{A}_{t_1, \dots, t_p}(M)(s) = M(s)$. In general, let $f_+ M$ be the Gauss-Manin complex of the $\mathscr{D}(U)$-module M. The good behaviour of direct images under composition (see for instance [**4**]) shows that one has an isomorphism in the derived category of complexes of left $\mathbb{C}(s)\langle t, t^{-1}\rangle$-modules:

$$\mathscr{A}_{f_1, \dots, f_p}(M)(s) \simeq \mathscr{A}_{t_1, \dots, t_p}(f_+ M)(s). \tag{1.17}$$

Indeed, it is enough to verify that

$$(f_+ M)(s) t^s = f_+(M(s) f^s),$$

which is immediate. □

We also call Aomoto complex the complex $\mathscr{A}_{f_1,\dots,f_p}(M) = p_+M[s]f^s$, the cohomology of which has finite type over $\mathbb{C}[s]\langle t, t^{-1}\rangle$ and to which we can apply the same reasoning as above.

Determinant. We have defined in §1.b the determinant of a rational holonomic system of FDEs as well as it class in the hypergeometric group $\mathscr{H}G(p)$. In the same way, one associates to the Aomoto complex a determinant by the formula

$$\det \mathscr{A}_{f_1,\dots,f_p}(M)(s) = \prod_i \big[\det H^i\big(\mathscr{A}_{f_1,\dots,f_p}(M)(s)\big)\big]^{(-1)^i}$$

which also belong to the hypergeometric group.

Isolated singularities. In various examples, the Aomoto complex has nonzero cohomology in a single degree. To prove that such a property holds in a given example, one can use a topological approach and the comparison theorem for regular holonomic $\mathscr{D}$-modules. This relies on the following two facts.

(1) Assume that for any $\alpha \in \mathbb{C}^p$ general enough, if $i_\alpha : \{\alpha\} \hookrightarrow \mathbb{C}^p$ denotes the inclusion, the restriction $\mathbf{L}i_\alpha^* \mathscr{A}_{f_1,\dots,f_p}(M)$ of the Aomoto complex has cohomology in degree d. Then the same holds for its restriction to the generic point $\mathscr{A}_{f_1,\dots,f_p}(M)(s)$.

(2) Assume that M has only regular singularities (even at infinity). Then the Aomoto complex $\mathscr{A}_{f_1,\dots,f_p}(M)(s)$ has cohomology in degree d only if and only if

$$H^i(U, {}^p\mathrm{DR}(M) \otimes f^{-1}\mathscr{L}_\mu) = 0 \qquad \text{for } i \neq d,$$

where $\mathscr{L}_\mu$ is the local system of rank one on the torus $(\mathbb{C}^*)^p$ with monodromy μ_i^{-1} around $t_i = 0$ and μ is such that $\mu_i = \exp 2i\pi\alpha_i$ with α general enough as above.

Example 1.18. $f_1, \dots, f_p$ are affine linear form on $U = \mathbb{C}^n$. Denote by $A_1, \dots, A_p$ the hyperplanes they define and A_∞ the hyperplane at infinity in $\mathbb{P}^n$. If $\mu = (\mu_1, \dots, \mu_p) \in (\mathbb{C}^*)^p$, set $\mu_\infty = -1/\mu_1 \cdots \mu_p$. Let $\mathscr{L}_\mu$ the local system on $\mathbb{C}^n \smallsetminus \bigcup_{i=1}^p A_i$ with monodromy μ_i^{-1} around A_i (hence with monodromy μ_∞^{-1} around A_∞). Put $I = \{1, \dots, p, \infty\}$. One has (see also [**8, 6**])

(SI$_1$) *If for any subset $J \subset I$ such that $\bigcap_{i\in J} A_i \neq \varnothing$ one has $\prod_{i\in J} \mu_i \neq 1$ then $H^i(\mathbb{C}^n - \bigcup_{i=1}^p A_i, \mathscr{L}_\mu) = 0$ except for $i = n$.*

Example 1.19. Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial. Let $G \subset \mathbb{P}^n \times \mathbb{C}$ be the closure of the graph of f and $F : G \to \mathbb{C}$ the map induced by the second projection. The critical locus of F is, by definition, the union of the singular locus of G and of the closure in G of the critical locus of the restriction of F to the smooth part of G. We say that the polynomial f has only isolated singularities (even at infinity) if the restriction of F to its critical locus is finite onto its image.

(SI$_2$) *If $\mu \in \mathbb{C}^*$ is general enough, one has, when f has only isolated singularities, $H^i(\mathbb{C}^n \smallsetminus f^{-1}(0), f^{-1}\mathscr{L}_\mu) = 0$ for $i \neq n$.*

Example 1.20. Let $f_1,\dots,f_p:\mathbb{C}^n\to\mathbb{C}$ be polynomials. Let $G\subset\mathbb{P}^n\times\mathbb{C}^p$ the closure of the graph of $f=(f_1,\dots,f_p)$ and $F:G\to\mathbb{C}^p$ the projection. If one defines the critical locus of F as in the previous example, we say that f has only isolated singularities (even at infinity) if the restriction of F to its critical locus is finite onto its image.

(SI$_3$) *If $\mu\in(\mathbb{C}^*)^p$ is general enough, one has, when f has only isolated singularities,*
$$H^i(\mathbb{C}^n\smallsetminus\textstyle\bigcup_i f_i^{-1}(0),f^{-1}\mathscr{L}_\mu)=0 \textit{ for } i\neq n.$$

2. Irreducible hypergeometric systems on the torus

In this section, we will classify the irreducible hypergeometric systems on the torus. We come back to the side of differential equations. By Proposition 1.14, a holonomic $\mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle$-module M is hypergeometric iff $\chi({}^p\mathrm{DR}\,M)=1$.

2.a. Hypergeometric systems in dimension one.

In this subsection, $\mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle$ denotes the one-variable algebra of differential operators on $T=\mathbb{C}^*=\mathbb{G}_m$.

Basic properties of hypergeometric systems. Let $P,Q\in\mathbb{C}[s]$ be two nonzero polynomials. Consider the differential equation $P(-t\partial_t)-tQ(-t\partial_t)$. We denote by $\mathscr{H}_{P,Q}$ the quotient of $\mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle$ by the left ideal generated by $P(-t\partial_t)-tQ(-t\partial_t)$. It is a holonomic $\mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle$-module [(2)]. The corresponding system of FDEs is $\mathbb{C}[s]\langle t,t^{-1}\rangle/(P(s)-tQ(s))$. After tensoring with $\mathbb{C}(s)$ one obtains the rational system $\mathbb{C}(s)\langle t,t^{-1}\rangle/(t-P/Q)$, which is a one dimensional vector space over $\mathbb{C}(s)$ (notice that, in the quotient, t can be replaced with P/Q, t^2 by $t\cdot(P/Q)(s)=(P/Q)(s+1)\cdot t\simeq(P/Q)(s+1)(P/Q)(s)$, *etc.*). Hence, $\mathscr{H}_{P,Q}$ is hypergeometric.

Write $P(s)=p\prod(s-a_i)$ and $Q(s)=q\prod(s-b_j)$ with $p,q,a_i,b_j\in\mathbb{C}^*$. Notice that, in the coordinate $t'=1/t$ centered at ∞, the ideal defining $\mathscr{H}_{P,Q}$ is generated by $Q(t'\partial_{t'})-t'P(t'\partial_{t'})$.

If $\deg P=\deg Q$, then the singularities of $\mathscr{H}_{P,Q}$ are $0,\infty,\lambda$, with $\lambda=p/q$, and they are regular singular.

If $\deg P<\deg Q$, the module $\mathscr{H}_{P,Q}$ has a regular singularity at infinity and an irregular one at 0. If $\deg P>\deg Q$, the roles of 0 and ∞ are exchanged. In such a case, $\mathscr{H}_{P,Q}$ is free over $\mathbb{C}[t,t^{-1}]$ of rank equal to $\max\deg P,\deg Q$.

The $\mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle$-module $\mathscr{H}^*_{P,Q}$ dual to $\mathscr{H}_{P,Q}$ is presented by the adjoint operator
$$(P(-t\partial_t)-tQ(-t\partial_t))^*=P(\partial_t t)-Q(\partial_t t)t=P(t\partial_t+1)-tQ(t\partial_t+2).$$
Hence, $\mathscr{H}^*_{P,Q}=\mathscr{H}_{P^*,Q^*}$ with $P^*(s)=P(-s+1)$ and $Q^*(s)=Q(-s+2)$.

Let us consider the effect of a translation by 1 in the roots of P or Q. So let us write
$$P(s)=(s-a)R(s),\quad P_1(s)=(s-a-1)R(s),\quad Q_1(s)=Q(s).$$

2. In dimension one, any quotient of $\mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle$ by a nonzero ideal is holonomic, see *e.g.* [**14**].

Notice that we have the commutation relation

$$[P(s) - tQ(s)] \cdot (s - a - 1) = (s - a) \cdot [P_1(s) - tQ_1(s)],$$

so the right multiplication $\cdot (s-a-1) : \mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle \to \mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle$ sends the left ideal $(P - tQ)$ into the left ideal $(P_1 - tQ_1)$, hence defines a morphism $\varphi : \mathscr{H}_{P,Q} \to \mathscr{H}_{P_1,Q_1}$. At the level of rational Mellin transforms, it reduces to multiplication by $(s - a - 1)$ on $\mathbb{C}(s)$, so it is nonzero.

Proposition 2.1 **(Translation of the roots).** *The following properties are equivalent:*

(1) *the morphism φ is onto,*
(2) *the morphism φ is injective,*
(3) *the morphism φ is an isomorphism,*
(4) $Q(a + 1) \neq 0$.

Proof

(1) $\Leftrightarrow$ (4): the cokernel of φ is isomorphic to the quotient of $\mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle$ by the sum of the left ideals generated by $P_1(s) - tQ_1(s)$ and $s - a - 1$. It is therefore isomorphic to $\mathbb{C}[t,t^{-1}]/(P_1(a+1) - tQ_1(a+1))$. By definition, $P_1(a+1) = 0$, so the cokernel is $\mathbb{C}[t,t^{-1}]/(Q_1(a+1))$. Hence φ is onto iff $Q_1(a+1) = Q(a+1) \neq 0$.

(2) $\Leftrightarrow$ (4): φ is injective if and only if its transpose $\varphi^* : \mathscr{H}^*_{P_1,Q_1} \to \mathscr{H}^*_{P,Q}$ is onto. We have $\mathscr{H}^*_{P,Q} = \mathscr{H}_{P^*,Q^*}$ with $P^*(s) = P(-s+1) = (-s+1-a)R^*(s)$ and $Q^*(s) = Q(-s+2)$, and similarly for P_1^*, Q_1^*. We have $P_1^*(s) = (s - a^*)(-R^*(s))$ and $P^*(s) = (s - a^* - 1)(-R^*(s))$, with $a^* = -a$; so the previous reasoning shows that the transpose φ^* is onto iff $Q^*(a^*+1) \neq 0$, that is, iff $Q(-(-a+1)+2) \neq 0$. □

Corollary 2.2. *If P and Q have no common root mod $\mathbb{Z}$, then the modules $\mathscr{H}_{P_1,Q_1}$ obtained by translating roots of P and Q by integers are all isomorphic.*

Proof. Immediate by induction, from Proposition 2.1. □

Characterization of irreducible hypergeometric $\mathscr{D}$-modules. We will first characterize the modules $\mathscr{H}_{P,Q}$ which are irreducible.

Proposition 2.3. *A module $\mathscr{H}_{P,Q}$ is irreducible if and only if P and Q have no common root mod $\mathbb{Z}$.*

Proof. Assume that $\mathscr{H}_{P,Q}$ is irreducible. If $\mathscr{H}_{P_1,Q_1}$ is obtained by translating some root of P by 1, the corresponding morphism φ is nonzero, hence, as $\mathscr{H}_{P,Q}$ is irreducible, it is injective, hence an isomorphism by Proposition 2.1. Therefore, $\mathscr{H}_{P_1,Q_1}$ is also irreducible, and we can continue to translate the roots of P by $\mathbb{N}$, without meeting a situation where a is a root of P and $a + 1$ a root of Q. To translate by $-\mathbb{N}$, use the dual modules.

In order to show the converse for any P, Q having no common root mod $\mathbb{Z}$, it is enough to show that $\mathscr{H}_{P,Q}$ has no $\mathbb{C}[s]$-torsion for any such P, Q. Indeed, by duality, this would say that $\mathscr{H}_{P,Q}$ has neither torsion submodule, nor torsion quotient; as $\mathscr{H}_{P,Q}$ is holonomic, it has a Jordan-Hölder sequence, each quotient being a irreducible holonomic $\mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle$-module and, as $\dim_{\mathbb{C}(s)}$ is additive in such a sequence, there is one and only one quotient in the sequence which is irreducible and hypergeometric; the first term in the sequence cannot be torsion, as $\mathscr{H}_{P,Q}$ has no torsion; hence it is hypergeometric; but the next quotient is also zero by the dual reason, so $\mathscr{H}_{P,Q}$ is irreducible.

Let us then show that $\mathscr{H}_{P,Q}$ has no $\mathbb{C}[s]$-torsion. This is equivalent to showing that, if $A(s,t),B(s,t)\in\mathbb{C}[s]\langle t,t^{-1}\rangle$ are such that $A(s,t)(P(s)-tQ(s))=(s-a)B(s,t)$ for some $a\in\mathbb{C}$, then $B(s,t)$ belongs to the left ideal generated by $P(s)-tQ(s)$, or equivalently that $A(s,t)=(s-a)C(s,t)$ for some $C(s,t)\in\mathbb{C}[s]\langle t,t^{-1}\rangle$.

We write $A(s,t)=\sum_{k\in\mathbb{Z}}A_k(s)t^k$, so that

$$\begin{aligned}A(s,t)(P(s)-tQ(s))&=\sum_{k\in\mathbb{Z}}A_k(s)(P(s+k)-Q(s+k+1)t)t^k\\&=\sum_{k\in\mathbb{Z}}[A_k(s)P(s+k)-A_{k-1}(s)Q(s+k)]t^k.\end{aligned}$$

Therefore, $A(s,t)(P(s)-tQ(s))$ is divisible on the left by $s-a$ if and only if, for any $k\in\mathbb{Z}$, we have $A_k(a)P(a+k)-A_{k-1}(a)Q(a+k)=0$. If $A_k(a)\neq 0$ for some k, denote by k_- (resp. k_+) the minimum (resp. maximum) of the set of k with $A_k(a)\neq 0$. Then $P(a+k_-)=0$ and $Q(a+k_++1)=0$, in contradiction with the assumption on P,Q. This shows the absence of $\mathbb{C}[s]$-torsion. □

We can now extend the result to any irreducible $\mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle$ hypergeometric module. We will show the following result, due to N. Katz [**7**, th. 3.7.1]. The proof given here follows that of [**10**].

Theorem 2.4. *Let M be a hypergeometric $\mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle$-module. Then the following conditions are equivalent:*

(1) *M is irreducible*
(2) *M has neither sub-module nor quotient module having $\mathbb{C}[s]$-torsion*
(3) *M is isomorphic to some $\mathscr{H}_{P,Q}$ where P and Q have no common zero mod $\mathbb{Z}$.*

Proof of Theorem 2.4. Clearly, (1) ⇒ (2). Moreover, if M is not irreducible, let M' be a proper sub-module. Then, either M' is a $\mathbb{C}[s]$-torsion submodule, or $\dim_{\mathbb{C}(s)}M'(s)=1$ and thus M/M' is a $\mathbb{C}[s]$-torsion module. Hence (2) ⇒ (1).

That (3) implies (1) follows from Proposition 2.3.

(1) ⇒ (3): If M is irreducible, then M is contained in $M(s)$, as it has no $\mathbb{C}[s]$-torsion. By Example 1.5, there exists P,Q having no common root mod $\mathbb{Z}$ such that $M(s)\simeq\mathscr{H}_{P,Q}(s)$. As such a $\mathscr{H}_{P,Q}$ is irreducible (*cf.* Proposition 2.3), we also have $\mathscr{H}_{P,Q}\subset M(s)$. The intersection $M\cap\mathscr{H}_{P,Q}$ in $M(s)$ cannot be zero, hence, by irreducibility, it is equal to M and to $\mathscr{H}_{P,Q}$, so that $M=\mathscr{H}_{P,Q}$. □

Remark 2.5. Let us describe the irreducible hypergeometric subquotient of $\mathscr{H}_{P,Q}$ when P and Q may have common roots mod $\mathbb{Z}$. Choose P' dividing P, Q' dividing Q, each one of maximal degree such that P' and Q' have no common root mod $\mathbb{Z}$. Then one can show that the unique hypergeometric irreducible quotient $\mathscr{H}_{P,Q}^{\mathrm{irred}}$ in the Jordan-Hölder sequence of $\mathscr{H}_{P,Q}$ is isomorphic to $\mathscr{H}_{P',Q'}$.

On the other hand, I do not know if any hypergeometric $\mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle$-module is isomorphic to some $\mathscr{H}_{P,Q}$.

Convolution of irreducible hypergeometric systems. Recall that, if M, M' are $\mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle$-modules, the tensor product $M\otimes_{\mathbb{C}[s]} M'$ is naturally equipped with a left $\mathbb{C}[s]\langle t,t^{-1}\rangle$-structure by putting $t\cdot(m\otimes m') = (tm)\otimes(tm')$ (*cf.* Exercise 1.3 for the rational case). We denote by $M\star M'$ the corresponding $\mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle$-module, that we call, as in [**7**], the *convolution* of M and M'. Clearly, the rational system associated with $M\star M'$ is the tensor product $M(s)\otimes_{\mathbb{C}(s)} M'(s)$. In particular, if M and M' are hypergeometric, then so is their convolution $M\star M'$.

We will now prove that any irreducible hypergeometric system can be decomposed as the convolution of elementary such systems. For the associated rational hypergeometric system, this corresponds to the decomposition (1.6).

The elementary systems are the systems corresponding to the functions c^s, $\Gamma(s-\alpha)$ or $1/\Gamma(s-\alpha)$. They are the systems $\mathscr{H}_{c,1}$, $\mathscr{H}_{(s-\alpha),1}$ and $\mathscr{H}_{1,(s-\alpha)}$.

***Theorem 2.6** (**cf.** [**7**, 5.3.2.1]).* *The irreducible hypergeometric systems are exactly the systems obtained by convolution of elementary systems $\mathscr{H}_{c,1}$ ($c\in\mathbb{C}$), $\mathscr{H}_{(s-\alpha),1}$ and $\mathscr{H}_{1,(s-\alpha)}$ where α varies in a subset of $\mathbb{C}$ in which there is no pair of elements differing by a nonzero integer.*

Before proving the theorem, let us consider the effect of convolution with an elementary system.

Lemma 2.7. *Assume that P and Q have no common root mod $\mathbb{Z}$ and let $\alpha\in\mathbb{C}$ be such that $Q(\alpha+k)\neq 0$ for any $k\in\mathbb{Z}$. Then $\mathscr{H}_{P,Q}\star\mathscr{H}_{(s-\alpha),1}\simeq\mathscr{H}_{(s-\alpha)P,Q}$.*

Proof. Denote by e the class of 1 in $\mathscr{H}_{P,Q}$ and by ε the class of 1 in $\mathscr{H}_{(s-\alpha),1}$. We will first prove that, under the assumption of the lemma, the submodule $\mathscr{H}'$ of $\mathscr{H}_{P,Q}\star\mathscr{H}_{(s-\alpha),1}$ generated over $\mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle$ by $e\otimes\varepsilon$ is equal to $\mathscr{H}_{P,Q}\star\mathscr{H}_{(s-\alpha),1}$.

Notice that e satisfies, for any $\ell\geqslant 0$, the equation

$$P(s-1)\cdots P(s-\ell-1)t^{-\ell}e = Q(s)\cdots Q(s-\ell)e,$$

and that ε satisfies

$$\begin{aligned} t^k\varepsilon &= (s-\alpha)\cdots(s-\alpha+k-1)\varepsilon \quad \text{if } k\geqslant 0,\\ \varepsilon &= (s-\alpha-1)\cdots(s-\alpha-\ell)t^{-\ell}\varepsilon \quad \text{if } \ell\geqslant 0. \end{aligned}$$

It follows that $e\otimes t^k\varepsilon = (s-\alpha)\cdots(s-\alpha+k-1)(e\otimes\varepsilon)$ belongs to $\mathscr{H}'$ for any $k\geqslant 0$. Using the other relations, one finds that $e\otimes t^{-\ell}\varepsilon$ also belongs to $\mathscr{H}'$: for instance, if

$\ell = 1$, we can write, from the relation $(P(s-1)t^{-1} - Q(s))e = 0$,

$$Q(\alpha+1)e = P(\alpha)t^{-1}e + (s-\alpha-1)(a(s)t^{-1} + b(s)),$$

hence, using that $(s-\alpha-1)t^{-1}\varepsilon = \varepsilon$,

$$\begin{aligned} Q(\alpha+1)e \otimes t^{-1}\varepsilon &= P(\alpha)t^{-1}(e\otimes\varepsilon) + [(a(s)t^{-1} + b(s))e]\otimes\varepsilon \\ &= P(\alpha)t^{-1}(e\otimes\varepsilon) + a(s)t^{-1}(e\otimes t\varepsilon) + b(s)(e\otimes\varepsilon) \\ &= \big[P(\alpha)t^{-1} + a(s)t^{-1}(s-\alpha) + b(s)\big](e\otimes\varepsilon). \end{aligned}$$

Now that any $e\otimes t^k\varepsilon$ ($k\in\mathbb{Z}$) belongs to $\mathscr{H}'$, it is easy to conclude that $\mathscr{H}' = \mathscr{H}_{P,Q}\star\mathscr{H}_{(s-\alpha),1}$.

On the other hand, notice that $e\otimes\varepsilon$ satisfies $[(s-\alpha)P(s) - tQ(s)](e\otimes\varepsilon) = 0$. We therefore have a well defined morphism $\mathscr{H}_{(s-\alpha)P,Q} \to \mathscr{H}_{P,Q}\star\mathscr{H}_{(s-\alpha),1}$, which is onto by the previous argument. As $\mathscr{H}_{(s-\alpha)P,Q}$ is irreducible, by Proposition 2.3, this morphism is an isomorphism. □

By exchanging the roles of P and Q (and by changing t with $t' = 1/t$), we also obtain that, if P and Q have no common root mod $\mathbb{Z}$ and if $P(\alpha+k)\neq 0$ for any $k\in\mathbb{Z}$, then $\mathscr{H}_{P,Q}\star\mathscr{H}_{1,(s-\alpha)} \simeq \mathscr{H}_{P,(s-\alpha)Q}$. On the other hand, we have $\mathscr{H}_{P,Q}\star\mathscr{H}_{c,1}\simeq\mathscr{H}_{cP,Q}$. Now, the proof of the theorem follows easily by induction on the degrees of P and Q. □

Corollary 2.8. *Let $\pi:\mathbb{C}\to\mathbb{C}$ be given by $\pi(s) = ns+\beta$, $n\in\mathbb{Z}\smallsetminus\{0\}$ and $\beta\in\mathbb{C}$. If M is an irreducible hypergeometric $\mathbb{C}[s]\langle t,t^{-1}\rangle$-module, then so is π^*M.*

Proof. As translation does not cause any trouble, we only consider the case where $\beta = 0$ and we first treat the case where M is elementary. If m is a generator of M satisfying for instance $t\cdot m = (s-\alpha)m$, so that $M = \varinjlim_k \prod_{j=1}^k (s-\alpha-j)^{-1}\cdot\mathbb{C}[s]$, then

$$\pi^*M = \varinjlim_k \prod_{j=1}^k (ns-\alpha-j)^{-1}\cdot\mathbb{C}[s],$$

hence a surjective morphism $\mathscr{H}_{P,Q}\to\pi^*M$ with $Q=1$, $P=\prod_{\ell=0}^{n-1}(ns-\alpha-\ell)$. As P and Q have no common zero mod $\mathbb{Z}$, $\mathscr{H}_{P,Q}$ is irreducible and in particular has no $\mathbb{C}[s]$-torsion. Therefore, this morphism is an isomorphism.

According to Theorem 2.6, it remains to showing that, for any $\alpha,\beta\in\mathbb{C}$, if $\alpha-\beta\notin\mathbb{Z}$, then $(\alpha+a)/n - (\beta+b)/n\notin\mathbb{Z}$ for any $a,b = 0,\dots,n-1$, which is clear. □

Remarks 2.9

(1) The theorem can be restated by saying that one has $\mathscr{H}_{P,Q}\star\mathscr{H}_{P',Q'}\simeq\mathscr{H}_{PP',QQ'}$ as soon as $\mathscr{H}_{PP',QQ'}$ is irreducible, or equivalently, as soon as PP' and QQ' have no common root mod $\mathbb{Z}$.

(2) As a $\mathbb{C}[s]$-module, $\mathscr{H}_{c,1}$ is free of rank one. The $\mathbb{C}[s]$-modules $\mathscr{H}_{(s-\alpha),1}$ and $\mathscr{H}_{1,(s-\alpha)}$ are flat, and more precisely, inductive limit of free $\mathbb{C}[s]$-modules of rank one: for instance, $\mathscr{H}_{(s-\alpha),1}$ is the inductive limit (union):

$$(s-\alpha-1)^{-1}\mathbb{C}[s]\subset(s-\alpha-1)^{-1}(s-\alpha-2)^{-1}\mathbb{C}[s]\subset\cdots$$

It follows that any irreducible $\mathscr{H}_{P,Q}$ is $\mathbb{C}[s]$-flat.

(3) Let $s_0 \in \mathbb{C}$. Denote by $\mathbb{C}[s]_{(s_0)} \subset \mathbb{C}(s)$ the subring of rational fractions having no pole at s_0 (localization of $\mathbb{C}[s]$ at s_0). The localized inductive limit is equal to $\mathbb{C}[s]_{(s_0)}$ if $s_0 \notin \alpha+\mathbb{N}^*$, and is equal to $\frac{1}{s-s_0}\mathbb{C}[s]_{(s_0)}$ otherwise. In particular, for any s_0, $\mathbb{C}[s]_{(s_0)} \otimes_{\mathbb{C}[s]} \mathscr{H}_{(s-\alpha),1}$ is free of rank one over $\mathbb{C}[s]_{(s_0)}$ (but has lost the translation structure). As $\mathbb{C}[s]_{(s_0)}$ is flat over $\mathbb{C}[s]$, the same result holds for any tensor product of elementary modules, in particular for any irreducible $\mathscr{H}_{P,Q}$.

(4) Let us explain precisely why a tensor product of two irreducible modules can be reducible. Let us consider the example of $\mathscr{H}_{s,1} \otimes_{\mathbb{C}[s]} \mathscr{H}_{1,s-1}$ for simplicity.

In $\mathscr{H}_{s,1} = \mathbb{C}[s]\langle t, t^{-1}\rangle/(t-s)$, the class of 1 is a generator as a $\mathbb{C}[s]\langle t, t^{-1}\rangle$-module, and is also a generator of $\mathscr{H}_{s,1(s_0)}$ as a $\mathbb{C}[s]_{(s_0)}$-module at any $s_0 \notin \mathbb{N}^*$. At $s_0 \in \mathbb{N}^*$, it is not a generator, and a generator is given by the class of t^{-s_0}.

For any $s_0 \in \mathbb{C}$ and any $k \in \mathbb{Z}$, the class of t^k in $\mathscr{H}_{s,1(s_0)}$ is (up to sign)

– the class of $s(s+1)\cdot(s+k-1)$ in $\mathbb{C}[s]_{(s_0)}$ if $k \geqslant 1$,
– the class of $[(s-1)\cdots(s+k)]^{-1}$ in $\mathbb{C}[s]_{(s_0)}$ if $k \leqslant -1$.

Similarly, for $\mathscr{H}_{1,s-1}$, the class of t^k in $\mathscr{H}_{1,s-1(s_0)}$ is (up to sign)

– the class of $[s(s+1)\cdot(s+k-1)]^{-1}$ in $\mathbb{C}[s]_{(s_0)}$ if $k \geqslant 1$,
– the class of $(s-1)\cdots(s+k)$ in $\mathbb{C}[s]_{(s_0)}$ if $k \leqslant -1$.

Let us see why $1\otimes 1$ is not a $\mathbb{C}[s]\langle t, t^{-1}\rangle$-generator of $\mathscr{H}_{s,1} \otimes_{\mathbb{C}[s]} \mathscr{H}_{1,s-1}$. Let us consider the germs, at some $s_0 \in \mathbb{Z}$, that we obtain by translating $1\otimes 1$. Recall that $t^k(1\otimes 1) = (t^k \otimes t^k)$. The germ at s_0 of this element is, up to sign,

– the class of $s(s+1)\cdot(s+k-1)\cdot[s(s+1)\cdot(s+k-1)]^{-1} = 1$ in $\mathbb{C}[s]_{(s_0)}$ if $k \geqslant 1$,
– the class of $[(s-1)\cdots(s+k)]^{-1}\cdot(s-1)\cdots(s+k) = 1$ in $\mathbb{C}[s]_{(s_0)}$ if $k \leqslant -1$,
– the class of 1 if $k=0$,

so in all cases, the class of 1. Therefore, the sub $\mathbb{C}[s]\langle t, t^{-1}\rangle$-module generated by $1\otimes 1$ is strictly contained in $\mathscr{H}_{s,1} \otimes_{\mathbb{C}[s]} \mathscr{H}_{1,s-1}$. [Here, a generator of $\mathscr{H}_{s,1} \otimes_{\mathbb{C}[s]} \mathscr{H}_{1,s-1}$ would be $t\otimes 1$ for instance.]

2.b. Hypergeometric systems in dimension $p \geqslant 2$. Let us now assume that $p \geqslant 2$. Let $L : \mathbb{C}^p \to \mathbb{C}$ be a linear form with coefficients $(\ell_1, \dots, \ell_p)$ in $\mathbb{Z}$, relatively prime. Let $i_L : \mathbb{G}_m \hookrightarrow (\mathbb{G}_m)^p$ be defined by $t_i = \theta^{\ell_i}$ for $i = 1, \dots, p$. If M is a holonomic $\mathbb{C}[\theta, \theta^{-1}]\langle\theta\partial_\theta\rangle$-module, then $i_{L+}M$ is $\mathbb{C}[t, t^{-1}]\langle t\partial_t\rangle$-holonomic (as i_L is a closed inclusion, $i_{L+}M$ consists of a single module). The Mellin transform of $i_{L+}M$ is nothing but $L^*M = \mathbb{C}[s] \otimes_{\mathbb{C}[\sigma]} M$, if $\sigma = L(s)$, with the difference structure given by $t_i(1\otimes m) = 1\otimes\theta^{\ell_i}m$. In particular L^*M is holonomic.

In a similar way, let $\mathbb{C}^p \simeq \mathbb{C}^{p-1}\times\mathbb{C}$ be an rational linear isomorphism, let L be the projection on the factor $\mathbb{C}$ and let $i : \mathbb{C}^{p-1}\times\{0\} \hookrightarrow \mathbb{C}^p$ be the inclusion of the hyperplane $L(s) = 0$. Then, if M is holonomic on $\mathbb{C}^p$, $\boldsymbol{L}i^*M$ has holonomic cohomology on $\mathbb{C}^{p-1}$: if one considers the corresponding decomposition $(\mathbb{G}_m)^p \simeq (\mathbb{G}_m)^{p-1}\times\mathbb{G}_m$, the inverse Mellin transform of $\boldsymbol{L}i^*M$ is nothing but the direct image of M by the projection $(\mathbb{G}_m)^p \to (\mathbb{G}_m)^{p-1}$.

Last, if $T : \mathbb{C}^p \to \mathbb{C}^p$ is any translation, and if M is holonomic, T^*M comes naturally equipped with the structure of a $\mathbb{C}[s]\langle t, t^{-1}\rangle$-module and is holonomic.

Let $L : \mathbb{C}^p \to \mathbb{C}$ be as above and put $\mathscr{H}_{P,Q,L} = L^*\mathscr{H}_{P,Q}$. One deduces from the previous remarks that:

Lemma 2.10. *The module $\mathscr{H}_{P,Q,L}$ is holonomic on $\mathbb{C}[s]\langle t, t^{-1}\rangle$. Moreover, if P and Q do not have common root mod $\mathbb{Z}$, this module is $\mathbb{C}[s]$-flat (in fact inductive limit of free rank one $\mathbb{C}[s]$-modules). So is any tensor product over $\mathbb{C}[s]$ of a finite number of such modules.* $\square$

The holonomy property of the tensor product of two holonomic modules is obtained by expressing it as the restriction to the diagonal of the external tensor product.

Lemma 2.11. *Let M' be isomorphic to a tensor product over $\mathbb{C}[s]$ of some modules $\mathscr{H}_{P,Q,L}$ with P and Q having no common root mod $\mathbb{Z}$. Then M' contains a unique irreducible holonomic sub-$\mathbb{C}[s]\langle t, t^{-1}\rangle$-module M, and M is hypergeometric.*

Proof. By the previous remark, M' is holonomic and, by flatness (Lemma 2.10), we have $M' \subset M'(s)$ and $\dim M'(s) = 1$. Moreover, M' contains at least one non trivial irreducible submodule M (holonomic modules have finite length) and it has generic rank equal to one over $\mathbb{C}[s]$ (its generic rank is positive, otherwise M' would be a $\mathbb{C}[s]$-torsion module, hence would not be $\mathbb{C}[s]$-flat). Such a sub-module is unique: let M_1 and M_2 be two distinct such sub-modules; one then has $M_1 \cap M_2 = \{0\}$, hence an injective morphism $M_1 \to M'/M_2$; the latter module is torsion, so the morphism is zero, thus $M_1 = \{0\}$. $\square$

Theorem 2.12. *Let M be a hypergeometric $\mathbb{C}[t, t^{-1}]\langle t\partial_t\rangle$-module. Then the following conditions are equivalent:*

(1) *M is irreducible*

(2) *M has neither sub-module nor quotient module having $\mathbb{C}[s]$-torsion*

(3) *M is isomorphic to the unique irreducible hypergeometric sub-$\mathbb{C}[s]\langle t, t^{-1}\rangle$-module of a tensor product over $\mathbb{C}[s]$ of a finite number of modules $\mathscr{H}_{P,Q,L}$, where, for any term, P and Q have no common root mod $\mathbb{Z}$.*

Remark 2.13. The proof also gives that any rational hypergeometric system of FDEs contains exactly one irreducible hypergeometric $\mathbb{C}[s]\langle t, t^{-1}\rangle$-module. Hence, there is a bijective correspondence between the isomorphism classes of irreducible hypergeometric $\mathbb{C}[t, t^{-1}]\langle t\partial_t\rangle$-modules and the elements of the hypergeometric group $\mathscr{H}G(p)$.

Proof of Theorem 2.12. It is now enough to show (1) $\Rightarrow$ (3).

If M is irreducible and hypergeometric, one has $M \subset M(s)$. There exists M' isomorphic to a tensor product of modules $\mathscr{H}_{P,Q,L}$ (with P and Q having no common root mod $\mathbb{Z}$) such that $M(s) = M'(s)$ (this follows from the description of the hypergeometric group $\mathscr{H}G(p$ in Proposition 1.7). Let us denote by M'_1 the unique irreducible sub-module of M'. Then M and M'_1 are irreducible hypergeometric modules contained in $M(s)$. We argue as in Lemma 2.11 to conclude that $M = M'_1$. $\square$

Following O. Gabber, we will try to give an explicit expression of this unique irreducible submodule in some cases.

Given a finite set of linear forms $L_k(s_1, \dots, s_p)$ with rational coefficients and a set of complex numbers α_k, let us consider the tensor product $M = \otimes_k \mathscr{H}_{\sigma-\alpha_k,1,L_k}$. Denote by $\mathbf{1}$ the class of $1 \otimes \cdots \otimes 1$ in M.

***Theorem 2.14* (O. Gabber).** *If there exists $s_0 \in \mathbb{R}^p$ such that $L_k(s_0) - \operatorname{Re}\alpha_k > 0$ for all k, then the submodule $\mathbb{C}[s]\langle t, t^{-1}\rangle \cdot \mathbf{1} \subset M$ is irreducible.*

Proof. Fix $\sigma_0 \in \mathbb{C}$. Consider the germ $\mathscr{H}_{\sigma-\alpha,1(\sigma_0)}$ of $\mathscr{H}_{\sigma-\alpha,1}$ at σ_0. Then, as in Remark 2.9(4), we see that the submodule $(\mathbb{C}[\sigma]\cdot 1)_{(\sigma_0)}$, where 1 denotes the class of 1 in $\mathscr{H}_{\sigma-\alpha,1}$, is equal to $\mathscr{H}_{\sigma-\alpha,1(\sigma_0)}$ if $\sigma_0 \notin \alpha + \mathbb{N}^*$, and is equal to $(\sigma - \sigma_0)\cdot \mathscr{H}_{\sigma-\alpha,1(\sigma_0)}$ otherwise.

Let us come back to M. For any $s_0 \in \mathbb{C}^p$, denote by K_{s_0} the set of k such that $L_k(s_0) - \alpha_k \in \mathbb{N}^*$. We similarly have

$$(\mathbb{C}[s]\cdot \mathbf{1})_{(s_0)} = \Big(\prod_{k\in K_{s_0}} L_k(s-s_0)\Big) M_{(s_0)},$$

where $M_{(s_0)}$ is a free $\mathbb{C}[s]_{(s_0)}$-module of rank one. Consider now the submodule $\mathbb{C}[s]\langle t, t^{-1}\rangle \cdot \mathbf{1}$. We similarly find

$$(\mathbb{C}[s]\langle t, t^{-1}\rangle \cdot \mathbf{1})_{(s_0)} = \sum_{n\in\mathbb{Z}^p} \Big[\Big(\prod_{k\in K_{s_0+n}} L_k(s-s_0)\Big) M_{(s_0)}\Big]. \tag{2.15}$$

Let M' be the unique irreducible submodule of M, given by Theorem 2.12. As $M' \cap (\mathbb{C}[s]\langle t, t^{-1}\rangle \cdot \mathbf{1}) \neq 0$ (because both modules give rise to the same rational holonomic module), we have $M' \subset (\mathbb{C}[s]\langle t, t^{-1}\rangle \cdot \mathbf{1})$ and there exists $f(s) \in \mathbb{C}[s]$ such that $f(s)\mathbf{1} \in M'$. In order to prove that $\mathbb{C}[s]\langle t, t^{-1}\rangle \cdot \mathbf{1}$ is irreducible, it is enough to show that $\mathbb{C}[s]\langle t, t^{-1}\rangle \cdot \mathbf{1} = \mathbb{C}[s]\langle t, t^{-1}\rangle \cdot f(s)\mathbf{1}$. We have, for any $s_0 \in \mathbb{C}^p$,

$$(\mathbb{C}[s]\langle t, t^{-1}\rangle \cdot f(s)\mathbf{1})_{(s_0)} = \sum_{n\in\mathbb{Z}^p} \Big[f(s+n)\Big(\prod_{k\in K_{s_0+n}} L_k(s-s_0)\Big) M_{(s_0)}\Big].$$

In order to show that the germs at s_0 of both modules are equal, it is enough to show, as $M_{(s_0)}$ is a free $\mathbb{C}[s]_{(s_0)}$-module of rank one, that the ideals generated in $\mathbb{C}[s]_{(s_0)}$ by the $\prod_{k\in K_{s_0+n}} L_k(s-s_0)$ $(n \in \mathbb{Z}^p)$ and the ideal generated by the $f(s+n)\prod_{k\in K_{s_0+n}} L_k(s-s_0)$ $(n \in \mathbb{Z}^p)$ are the same. For that purpose, it is enough to show that, for any $n \in \mathbb{Z}^p$, there exists $m \in \mathbb{Z}^p$ such that $f(s_0+m) \neq 0$ and $K_{s_0+m} \subset K_{s_0+n}$.

Assume that n is fixed. Denote by $K^-_{s_0+n}$ the set of k such that $L_k(s_0+n) - \alpha_k \in -\mathbb{N}$. Then, for $\nu \in \mathbb{Z}^p$, we will have $K_{s_0+n+\nu} \subset K_{s_0+n}$ as soon as $L_k(\nu) \leqslant 0$ for any $k \in K^-_{s_0+n}$.

In order to find such a ν satisfying moreover $f(s_0+n+\nu)\neq 0$, it is enough to prove that, for any s_0 and n, the set

$$\{\nu\in\mathbb{Z}^p\mid \forall\, k\in K^-_{s_0+n},\ L_k(\nu)\leqslant 0\}$$

is Zariski dense in $\mathbb{C}^p$. For that purpose, it is enough to prove (replacing s_0+n with s_0) that, for any $s_0\in\mathbb{C}^p$, the cone

$$C_{s_0}=\{s\in\mathbb{R}^p\mid \forall\, k\in K^-_{s_0},\ L_k(s)\leqslant 0\}$$

has a nonempty interior in $\mathbb{R}^p$.

On the one hand, we have, by definition, $L_k(s_0)-\alpha_k\in-\mathbb{N}$ for any $k\in K^-_{s_0}$. On the other hand, by assumption, there exists s'_0 such that $\operatorname{Re}(L_k(s'_0)-\alpha_k)>0$ for any k; hence this holds for any s' in a suitable open neighbourhood of s'_0 and, in particular, for any $k\in K^-_{s_0}$. Therefore, for any such k and any such s', we have $\operatorname{Re}L_k(s_0-s')<0$, hence $\operatorname{Re}(s_0-s')\in C_{s_0}$. Consequently, C_{s_0} has a nonempty interior. □

***Corollary 2.16* (O. Gabber).** *Any irreducible hypergeometric $\mathbb{C}[t,t^{-1}]\langle t\partial_t\rangle$-module M can be obtained as the convolution (i.e. tensor product over $\mathbb{C}[s]$) of well-chosen elementary systems of the form $\mathscr{H}_{\sigma-\alpha,1,L}$ ($(L,\alpha)\in\mathscr{L}\times\mathbb{C}$) and $L_i^*\mathscr{H}_{c_i,1}$, with $c_i\in\mathbb{C}^*$ and $L_i(s)=s_i$ ($i=1,\dots,p$).*

Proof. By Proposition 1.7, the rational system $M(s)$ is the tensor product, over $\mathbb{C}(s)$ of elementary systems as in Theorem 2.14, where α is determined modulo $\mathbb{Z}$. Moreover, M is the unique irreducible $\mathbb{C}[s]\langle t,t^{-1}\rangle$-module contained in $M(s)$. As the modules $L_i^*\mathscr{H}_{c_i,1}$ do not cause any trouble, I will assume that $c_i=1$ for $i=1,\dots,p$.

As the number of linear forms L entering in the decomposition is finite, and as we can replace a given L by $-L$, we can choose the decomposition in such a way that there exists a hyperplane in the space $(\mathbb{Q}^p)^*$ of linear forms with rational coefficients, such that all the linear forms of the decomposition are contained in a chosen open half-space bounded by this hyperplane.

I claim that, with such a choice, for any $s_0\in\mathbb{C}^p$ there exists $n\in\mathbb{Z}^p$ such that the set K_{s_0+n} is empty. Indeed, due to the choice of the set of linear forms, there exists a half-line in $\mathbb{Q}^p$ on which all L_k are <0, hence tend to $-\infty$ when the parameter tends to ∞. There also exists a sequence n_ℓ of integral points on this half-line such that, for any k, $L_k(n_\ell)\to-\infty$ when $\ell\to\infty$. Given $s_0\in\mathbb{C}^p$, one can find ℓ such that, for any k, $\operatorname{Re}(L_k(s_0+n_\ell)-\alpha_k)<0$, hence $K_{s_0+n_\ell}=\varnothing$.

Looking back to (2.15), we conclude that, with such a choice, the germ at any s_0 of the submodule $\mathbb{C}[s]\langle t,t^{-1}\rangle\cdot\mathbf{1}$ is equal to the germ $M_{(s_0)}$.

It remains to verify that we can translate each α by a suitable integer in such a way that the set of (L,α) fulfills the condition in Theorem 2.14.

Fix some s_0 in $\mathbb{R}^p$. For any $\mathscr{H}_{\sigma-\alpha,1,L}(s)$ entering in the decomposition of $M(s)$ (with the L's chosen as above), choose an integer $n_{L,\alpha}$ such that $L(s_0)-\operatorname{Re}\alpha+n_{L,\alpha}>0$. Then replace α with $\alpha-n_{L,\alpha}$. □

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