An overview on the relative Riemann-Hilbert correspondence

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Setting.
X, S: smooth connected cplx mflds, $p : X\times S \to S =$ projection.

Definition ((Coherent) S-local system F).
Loc. const. sheaf of (coherent) $p^{-1}\mathscr{O}_S$-modules.

- Given F, $\exists\, G$ (coh.) $\mathscr{O}_S$-module s.t.

$$\text{locally on } X\times S, \quad F \simeq p^{-1}G.$$

  Can choose $G = i_x^{-1}F$, any $x\in X$, $\quad i_x : \{x\}\times S \hookrightarrow X\times S$.

- $\boxed{F \longleftrightarrow \rho : \pi_1(X, x_o) \to \mathrm{Aut}_{\mathscr{O}_S}(i_{x_o}^{-1}F)}$

Definition (Flat relative connection).
Coherent $\mathscr{O}_{X\times S}$-mod. $\mathscr{M}$ with

$$\nabla : \mathscr{M} \longrightarrow \Omega^1_{X\times S/S}\otimes\mathscr{M}, \quad \nabla^2 = 0.$$

Theorem (Deligne (1970): RH correspondence by $\mathrm{DR}_{X\times S/S}$).

$$\boxed{\textit{Flat rel. connections} \longleftrightarrow \textit{coh. } S\textit{-local systems}}$$

- Integrable case:
  - $F = p^{-1}\mathscr{O}_S \otimes_{\mathbb{C}} q^{-1}L$, L loc. syst. of $\mathbb{C}$-vect. spaces on $X\times S$,
  - ∇ lifts to $\nabla : \mathscr{M} \to \Omega^1_{X\times S}\otimes\mathscr{M}$ with $\nabla^2 = 0$.
- Non-abelian Hodge theory:
  - X projective, $S = \mathbb{C}^*$, (M, ∇) ***simple*** vect. bdle with flat conn. on X
  - Simpson-Corlette: $\exists\,(\mathscr{M}, \nabla)$ flat on $X\times\mathbb{C}^*/\mathbb{C}^*$, $\mathscr{M}_1 = M$ and $\lim_{s\to 0}\mathscr{M}_s$ is a stable Higgs bundle with vanishing Chern classes.
  - Integrable case $\longleftrightarrow$ (M, ∇) var. of pol. cplx Hodge struct.
- Kernel of integral transforms - Fourier-Mukai:
  - $X = A$: abelian var., $S = A^\sharp$: moduli space of bdles with flat conn. on A,
  - $\mathscr{M} = \mathscr{P}$: Poincaré bdle on $A\times A^\sharp$ with rel. flat conn.
- Kernel of integral transforms - Mellin:
  - $S = \mathbb{C}^d \ni (s_1, \dots, s_d)$, $f_1, \dots, f_d : Y \to \mathbb{C}$ holom. functs., $X = Y \smallsetminus \bigcup_i f_i^{-1}(0)$
  - $\mathscr{M} = (\mathscr{O}_{X\times S}, f^{-s}\circ \mathrm{d}_{X\times S/S}\circ f^s)$, $\quad F = p^{-1}\mathscr{O}_S\cdot f^{-s}$.

Motivations/Questions

- ¿Category of $\mathbb{C}$-constructible sheaves on X of $p^{-1}\mathscr{O}_S$-modules? (S-loc. cst on each $X_\alpha \times S$, w.r.t. a stratif. (X_α) of X)
- ¿Corresponding notion of perversity?
- Possible application: construction of moduli spaces of perverse sheaves on X w.r.t. a stratif. (X_α) of X (work of Nitsure-CS (1996), Nitsure (1999, 2004)).
- ¿Corresponding notion of ***regular holonomic*** relative $\mathscr{D}$-module?
- ¿Riemann-Hilbert correspondence?
  PhD thesis of Liqing Wang (Chicago, 2008) Adv. H. Gillet.
- Possible applications:
  - Properties of integral transform of a reg. hol. $\mathscr{D}$-module by means of ***local properties*** on $X \times S$
  (analogous to what is done for Fourier transform).
  - Realize the RH corresp. as a ***morphism*** between moduli spaces.

Relative $\mathbb{C}$-constructible/perverse sheaves

- $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p^{-1}\mathscr{O}_S) \Leftrightarrow \exists\,(X_\alpha)$ Whitney strat. of X s.t. $\forall\,\alpha, j$, $\mathscr{H}^j F_{|X_\alpha\times S}$ is ***coherent S-locally constant***.
- t-structure: $(\mathrm{D}^{\mathrm{b},\leqslant 0}_{\mathbb{C}\text{-c}}(p^{-1}\mathscr{O}_S), \mathrm{D}^{\mathrm{b},\geqslant 0}_{\mathbb{C}\text{-c}}(p^{-1}\mathscr{O}_S))$
  Support and cosupport cond. use $i_x : \{x\}\times S \hookrightarrow X\times S$.
- $\rightsquigarrow \mathrm{Perv}(p^{-1}\mathscr{O}_S)$.
- ***Caveat***: not stable by duality. $\rightsquigarrow$ dual perverse t-struct. (Fiorot & Monteiro Fernandes)
- F ***strictly perverse***: if F and DF are perverse.

$$\text{(Cosupp+)}\qquad \mathscr{H}^k i^!_{\{x\}\times\Sigma} F = 0,\ \begin{cases}\forall\, x\in X_\alpha,\ \forall\,\Sigma\subset S,\\ \forall\, k<\operatorname{codim}_S\Sigma+\dim X_\alpha.\end{cases}$$

Example. L coherent S-loc. cst sheaf on $X\times S$.

$$L[\dim X] \text{ strictly perverse} \iff L \text{ is } p^{-1}\mathscr{O}_S\text{-loc. free.}$$

Definition.
$\mathscr{M}$ coh. $\mathscr{D}_{X\times S/S}$-module. $\mathscr{M}$ is ***rel. holonomic*** if $\exists\,\Lambda \subset T^*X$ conic Lagrangian s.t. $\mathrm{Char}\,\mathscr{M} \subset \Lambda \times S$.

Example.
$S = \mathbb{C}^d$, $f_1, \dots, f_d : X \to \mathbb{C}$, M holonomic $\mathscr{D}_X$-mod. such that $M = M[f^{-1}]$, m a local section of M,

$$\mathscr{M} = \mathscr{D}_X[s_1, \dots, s_d] \cdot mf^s \subset M[s]f^s.$$

Theorem (Ph. Maisonobe).
$\mathscr{M}$ is $\mathscr{D}_X[s_1, \dots, s_d]$-holonomic.

Theorem.

- *$\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{X\times S/S})$. Then ${}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}$, ${}^{\mathrm{p}}\mathrm{Sol}\,\mathscr{M}$ are S-$\mathbb{C}$-construct.*
- *$\mathscr{M} \in \mathsf{Mod}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{X\times S/S})$. Then ${}^{\mathrm{p}}\mathrm{Sol}\,\mathscr{M}$ is perverse and ${}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}$ is dual perverse.*

Theorem. *$\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{X\times S/S})$. Equiv:*

- *$\mathscr{M} = \mathscr{H}^0\mathscr{M}$ and is **strict** (i.e., $\mathscr{O}_S$-flat).*
- *$D\mathscr{M} = \mathscr{H}^0 D\mathscr{M}$ and is **strict**.*
- *${}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}$ is **strictly perverse**.*

Proper pushforward and pullback w.r.t. X.

Theorem (Kashiwara's estimate, Schapira-Schneiders).

- *$\mathscr{M}$ rel. holonomic on $X \times S$,*
- *$f : X \to Y$ proper, $\rightsquigarrow f : X \times S \to Y \times S$.*

Assume that $\mathscr{M}$ is f-good. Then ${}_{\mathrm{D}}f_*\mathscr{M}$ is rel. holonomic.

Proposition (easy).

- *$\mathscr{M}$ rel. holonomic on $Y \times S$,*
- *$f : X \to Y$ **smooth**, $\rightsquigarrow f : X \times S \to Y \times S$.*

Then ${}_{\mathrm{D}}f^*\mathscr{M}$ is rel. holonomic.

Problem. What if f ***not smooth***?
Need a priori to restrict to some open subset of S depending on $\mathscr{M}$.
Lack of a good theory of the Bernstein-Sato polynomial.

Proper pushforward and pullback w.r.t. S.

- $\pi : S \to T$, $\rightsquigarrow \pi : X \times S \to X \times T$.
- $\mathscr{M}$ rel. holonomic on $X \times S$ and π-good. Then $R\pi_*\mathscr{M}$ is rel. holonomic.
- $\mathscr{M}$ rel. holonomic on $X \times T$ and π-good. Then $L\pi^*\mathscr{M}$ is rel. holonomic.

For each $s \in S$, $i_s : X \times \{s\} \hookrightarrow X \times S$.

Definition (Regularity). $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{X\times S/S})$.

$\mathscr{M}$ is regular if $\forall\, s \in S$, $Li_s^*\mathscr{M}$ has regular $\mathscr{D}_X$-cohomologies.

Examples.

- $\mathscr{N}$ a regular holonomic $\mathscr{D}_{X\times S}$-module with $\mathrm{Char}(\mathscr{N}) \subset \Lambda \times T^*S$, $\Lambda \subset T^*X$ ***Lagrangian***. Let $\mathscr{M} \subset \mathscr{N}$ be a coherent $\mathscr{D}_{X\times S/S}$-submodule of $\mathscr{N}$. Then $\mathscr{M}$ is rel. regular holonomic.
- If $\mathscr{M}$ comes from a regular twistor $\mathscr{D}$-module, then $\mathscr{M}$ is rel. regular holonomic. (Probably true, I did not check.)
- $f : X \to S = \mathbb{C}^d$, $D = \bigcup_i f_i^{-1}(0)$, $M = M(*D)$ regular holonomic $\mathscr{D}_X$-module

  $\rightsquigarrow$ then $M[s]f^s$ is rel. regular holonomic.

Proposition.

- $\mathsf{Mod}_{\mathrm{rhol}}(\mathscr{D}_{X\times S/S})$ *is stable by sub-quotient in* $\mathsf{Mod}_{\mathrm{coh}}(\mathscr{D}_{X\times S/S})$
- $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathscr{D}_{X\times S/S})$ *is a full triangulated subcategory of* $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_{X\times S/S})$ *which is stable by duality.*

- $f : X \to Y$, $\rightsquigarrow f : X \times S \to Y \times S$
- $\pi : S \to T$, $\rightsquigarrow \pi : X \times S \to X \times T$

- $\mathscr{M}$ reg. holonomic on $X \times S$ and f- or π-good,
  - f proper $\Longrightarrow {}_{\mathrm{D}}f_*\mathscr{M}$ reg. holonomic on $Y \times S$ (standard),
  - π projective $\Longrightarrow R\pi_*\mathscr{M}$ reg. holonomic on $X \times T$ (a little tricky).
- $\mathscr{M}$ reg. holonomic on $X \times T \Longrightarrow L\pi^*\mathscr{M}$ reg. holonomic on $X \times S$ (obvious).

Theorem.
$\mathscr{M}$ reg. holonomic on $Y \times S \Longrightarrow {}_{\mathrm{D}}f^*\mathscr{M}$ reg. holonomic on $X \times S$.

The proof relies on properties of Deligne's canonical meromorphic extension.

- (X, D): complex manifold with a ncd. $j : X \smallsetminus D \hookrightarrow X$.
- F: coherent S-locally constant sheaf on $(X \smallsetminus D) \times S$,
- $E_F = (\mathscr{O}_{(X\smallsetminus D)\times S} \otimes_{p^{-1}\mathscr{O}_S} F, \mathrm{d}_{(X\smallsetminus D)\times S/S} \otimes \mathrm{Id})$
- $F = \mathscr{H}^0 \mathrm{DR}\, E_F$
- notion of ***moderate growth*** (loc. uniformly w.r.t. S)
- Deligne's extension $\widetilde{E}_F \subset j_* E_F$: moderate growth condition in local poly-sectors.

Definition (D-type). A reg. holonomic $\mathscr{D}_{X\times S/S}$-module $\mathscr{M}$ is of D-type on $(X, D)\times S$ if $\mathscr{M} = \mathscr{M}(*D)$ and $\mathscr{M}$ smooth on $(X \smallsetminus D) \times S$ (i.e., char. variety contained in the zero section of $T^*(X \smallsetminus D) \times S$).

Theorem.

- *For any F, $\widetilde{E}_F$ is of D-type on $(X, D) \times S$.*
- *Conversely,*

$$\mathscr{M} \text{ of D-type } \Longrightarrow \mathscr{M} \simeq \widetilde{E}_F, \text{ with } F = \mathscr{H}^0 \mathrm{DR}\, \mathscr{M}_{|(X\smallsetminus D)\times S}.$$

Main argument obtained when F is $p^{-1}\mathscr{O}_S$-locally free. Reduction to this case by means of base changes (flattening theorem).

$\rightsquigarrow$ need preservation of regularity by $L\pi^*$ and $R\pi_*$.

Theorem (Fiorot, Monteiro Fernandes, CS).

(Assume $\dim S = 1$). The functor

$${}^{\mathrm{p}}\mathrm{Sol} : \mathrm{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathscr{D}_{X\times S/S}) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p^{-1}\mathscr{O}_S)$$

is an equivalence of categories, having

$$\mathrm{RH}^S : \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p^{-1}\mathscr{O}_S) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathscr{D}_{X\times S/S})$$

as a quasi-inverse functor.

- If F is locally constant,

$$\mathrm{RH}^S(F) = \mathrm{R}\mathscr{H}om_{p^{-1}\mathscr{O}_S}(F, \mathscr{O}_{X\times S})[\dim X] \simeq D'F \otimes_{p^{-1}\mathscr{O}_S} \mathscr{O}_{X\times S}[\dim X]$$

  satisfies ${}^{\mathrm{p}}\mathrm{Sol}(\mathrm{RH}^S(F)) \simeq F$.
- Near the singularities of $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p^{-1}\mathscr{O}_S)$, want to consider solutions with moderate growth along ***any*** closed subset of X.
- $\rightsquigarrow$ can only consider closed X-subanalytic subsets.
- $\rightsquigarrow$ work on the subanalytic site and define on it the "sheaf" of holomorphic functions with moderate growth along any closed X-subanalytic subset (Kashiwara-Schapira).

- Work on $X_{\mathrm{sa}} \times S$ (Monteiro Fernandes & Prelli)
- $\rho_S : X \times S \to X_{\mathrm{sa}} \times S$,
- $\mathcal{O}^{\mathrm{t},S}_{X\times S}$: X-tempered holomorphic "functions".

Definition (of RH^S, similar to Kashiwara's RH).

$$\mathrm{RH}^S(F) = \rho_S^{-1}\,\mathrm{R}\mathscr{H}om_{\rho_{S*}p^{-1}\mathcal{O}_S}(\rho_{S*}F, \mathcal{O}^{\mathrm{t},S}_{X\times S})[\dim X].$$

Main points for the theorem:

- Check good behaviour of RH^S w.r.t. pullback and proper push-forward by $f : X \to Y$ and $\pi : S \to T$.
- F local system on $(X \smallsetminus D) \times S$ ($D =$ ncd in X), then
  - $\mathrm{RH}^S(j_! D' F) \simeq \widetilde{E}_F$ (relative Deligne's merom. exension),
  - ${}^{\mathrm{p}}\mathrm{Sol}(\widetilde{E}_F) \simeq j_! D' F$.