BESSEL AND AIRY MOMENTS: ARITHMETIC, PERIODS AND HODGE THEORY

(3) EXPONENTIAL HODGE THEORY

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by

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Abstract. Broadhurst and Roberts computed experimentally Hodge numbers related to some symmetric powers of Bessel differential equations and conjectured a general formula for these numbers. One of the main question was to construct a pure motive that would produce these numbers. We will explain how the exponential Hodge theory of Kontsevich and Soibelman makes easier such a computation and how it can be adapted to the case of symmetric powers of Airy differential equations, where irregular Hodge numbers appear. (Joint work with J. Fresán and J.-D. Yu.)

1. Exponential mixed Hodge structures (after Kontsevich-Soibelman)

1.a. The category EMHS. The notion of *exponential mixed Hodge structure* (EMHS) introduced by Kontsevich and Soibelman happens to be a convenient framework for computing usual Hodge numbers in our situation. I will first explains some general results in this direction.

Let $\mathsf{MHM}(\mathbb{A}^1)$ be the Saito category of \mathbb{Q} -mixed Hodge modules on the affine line \mathbb{A}^1 (with coordinate θ , say), whose objects consist of tuples $M = ((\mathcal{M}, F_{\bullet}\mathcal{M}), (\mathcal{F}_{\mathbb{Q}}, W_{\bullet}\mathcal{F}_{\mathbb{Q}}), \operatorname{comp})$. The full subcategory EMHS of $\mathsf{MHM}(\mathbb{A}^1)$ consists of those objects for which the global cohomology of the underlying perverse sheaf \mathcal{F} is zero. I recall the fundamental properties of this category. Let $j : \mathbb{A}^1 \smallsetminus \{0\} = \mathbb{G}_{\mathrm{m}} \hookrightarrow \mathbb{A}^1$ denote the open inclusion and $s : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ denote the sum map. The convolution $M_1 \star M_2 = s_*(M_1 \boxtimes M_2)$ of two objects of $\mathsf{MHM}(\mathbb{A}^1)$ takes in general values in $\mathsf{D}^{\mathrm{b}}(\mathsf{MHM}(\mathbb{A}^1))$. However, if we consider the case $M_2 = j_!^{\mathsf{P}} \mathbb{Q}^{\mathsf{H}}_{\mathbb{G}_{\mathrm{m}}}$, we have

$$\Pi(M) := M \star j_! \mathbb{Q}^{\mathsf{H}}_{\mathbb{G}_{\mathsf{m}}} \in \mathsf{EMHS} \quad \text{and} \quad \Pi(M) = M \text{ if } M \in \mathsf{EMHS}$$

The functor Π is exact and kills any subquotient of M whose associated perverse sheaf is constant on \mathbb{A}^1 . There is a Betti fiber functor and a de Rham fiber functor. The latter is defined as $M \mapsto H^1_{\mathrm{dR}}(\mathbb{A}^1, \mathcal{M} \otimes \mathcal{E}^{\theta})$, with $\mathcal{M} \otimes \mathcal{E}^{\theta}$ is \mathcal{M} equipped with the connection $\nabla + \mathrm{d}\theta$.

Proposition. The category EMHS is an abelian category and each morphism is strict with respect to the filtration $W_{\bullet}^{\text{EMHS}}(M) = \Pi(W_{\bullet}M)$ for $M \in \text{EMHS}$. Furthermore, C. SABBAH

the convolution product makes it a neutral Tannakian category with (faithful) fiber functor being the de Rham fiber functor.

1.b. The categories MHS and $MHS^{\hat{\mu}}$ as subcategories of EMHS. The category MHS := MHM(pt) is equivalent to the full subcategory EMHS^{cst} of EMHS whose objects M satisfy the following property: the restriction $\mathcal{F}|_{\mathbb{G}_m}$ is constant. This is equivalent to asking that $M|_{\mathbb{G}_m}$ is isomorphic to the constant Hodge module $({}^{P}\mathbb{Q}_{\mathbb{G}_m}^{H})^r$ of some rank r. The equivalence is obtained via the following functors:

 $\mathsf{EMHS}^{\mathrm{cst}} \ni M \longmapsto \phi_{\theta,1} M \in \mathsf{MHS}, \qquad \mathsf{MHS} \ni H \longmapsto \Pi(i_*H), \quad i: \{0\} \longleftrightarrow \mathbb{A}^1.$

This equivalence preserves the tensor structure.

It is natural to consider the larger full subcategory $\mathsf{EMHS}^{\hat{\mu}}$ consisting of objects of EMHS whose objects M satisfy the following property: the restriction $\mathcal{F}|_{\mathbb{G}_{\mathrm{m}}}$ becomes constant after some finite cyclic morphism $\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$. It is equivalent to the category $\mathsf{MHS}(\hat{\mu})$ of mixed Hodge structures equipped with an automorphism of finite order (i.e., an action of $\mathbb{Z}/m\mathbb{Z}$ for some m). However, this equivalence is not compatible with the tensor structure. The objects of $\mathsf{MHS}^{\hat{\mu}}$ take the form $((\mathcal{H}_{\mathbb{Q}}, T), F^{\bullet}_{\hat{\mu}}\mathcal{H}_{\mathbb{C}}, W^{\hat{\mu}}_{\bullet}\mathcal{H}_{\mathbb{Q}})$, where T is of finite order on $\mathcal{H}_{\mathbb{Q}}$ and

• $\mathcal{H}_{\mathbb{Q}} = \mathcal{H}_{\mathbb{Q},1} \oplus \mathcal{H}_{\mathbb{Q},\neq 1}$ and $\mathcal{H}_{\mathbb{C}} = \bigoplus_{\zeta \in \mu_m} \mathcal{H}_{\zeta};$

• $F_{\hat{\mu}}^{\mathsf{p}} \mathcal{H}_{\mathbb{C}} = \bigoplus_{\zeta} F_{\hat{\mu}}^{\mathsf{p}} \mathcal{H}_{\zeta}$ is an exhaustive decreasing filtration indexed by $\mathsf{p} \in \mathbb{Q}$, and for each $\zeta = \exp(-2\pi i a)$ with $a \in (-1, 0] \cap \mathbb{Q}$, the filtration $F_{\hat{\mu}}^{\mathsf{p}} \mathcal{H}_{\zeta}$ jumps at most at $\mathbb{Z} - a$;

• $W^{\widehat{\mu}} \mathcal{H}_{\mathbb{Q}} = W^{\widehat{\mu}} \mathcal{H}_1 \oplus W^{\widehat{\mu}} \mathcal{H}_{\neq 1}$ is an exhaustive increasing filtration indexed by \mathbb{Z} , satisfying the following property (Scherk-Steenbrink): setting

(1.1)
$$F^{p}\mathcal{H}_{\mathbb{C}} = \bigoplus_{\zeta \in \mu_{m}} F_{\widehat{\mu}}^{p-a}\mathcal{H}_{\zeta}, \quad p \in \mathbb{Z}$$
$$W_{\ell}\mathcal{H}_{\mathbb{Q}} = W_{\ell}^{\widehat{\mu}}\mathcal{H}_{1} \oplus W_{\ell+1}^{\widehat{\mu}}\mathcal{H}_{\neq 1},$$

we impose that $(\mathcal{H}_{\mathbb{Q}}, F^{\bullet}\mathcal{H}_{\mathbb{C}}, W_{\bullet}\mathcal{H}_{\mathbb{Q}})$ is a (graded polarizable) mixed Hodge structure. Morphisms are the natural ones. With the induced automorphism T, the latter becomes an object of $\mathsf{MHS}(\hat{\mu})$.

Then the functor

$$(\mathcal{M}, F^{\bullet}\mathcal{M}, W_{\bullet}\mathcal{M}) \longmapsto \phi_{\theta,1}(\mathcal{M}, F^{\bullet}, W_{\bullet}) \oplus \psi_{\theta,\neq 1}(\mathcal{M}, F^{\bullet}, W_{\bullet})$$

$$\downarrow$$

$$\phi_{\theta,1}^{\hat{\mu}}(\mathcal{M}, F^{\bullet}, W_{\bullet}) \oplus \psi_{\theta,\neq 1}^{\hat{\mu}}(\mathcal{M}, F^{\bullet}, W_{\bullet})$$

is an equivalence of categories compatible with the tensor structure.

1.c. Examples of exponential mixed Hodge structures. Let $f : U \to \mathbb{C}$ be a regular function on a smooth complex quasi-projective variety U of dimension d. Let ${}^{\mathbb{P}}\mathbb{Q}_{U}^{\mathbb{H}}$ be the constant pure Hodge module (of weight d) on U. Then for each $r \ge 0$ we define

 $H^r(U, f), H^r_c(U, f) \in \mathsf{EMHS}$ and $H^r_{\mathrm{mid}}(U, f) := \mathrm{image}[H^r_c(U, f) \to H^r(U, f)]$ by

$$H^{r}(U,f) = \Pi(\mathcal{H}^{r-d}f_{*}^{\mathbf{P}}\mathbb{Q}_{U}^{\mathbf{H}}) \quad \text{and} \quad H^{r}_{\mathbf{c}}(U,f) = \Pi(\mathcal{H}^{r-d}f_{!}^{\mathbf{P}}\mathbb{Q}_{U}^{\mathbf{H}}).$$

The de Rham fiber of these objects of EMHS are respectively

$$H^r(U, (\Omega^{\bullet}_U, \mathbf{d} + \mathbf{d}f))$$
 and $H^r_{\mathbf{c}}(U, (\Omega^{\bullet}_U, \mathbf{d} + \mathbf{d}f)).$

Theorem. Assume that $U = \mathbb{A}^1_t \times V$ and $f(t, y) = t^m g(y)$ for some regular function $g: V \to \mathbb{C}$ and $m \ge 1$. Then $H^r(U, f)$ and $H^r_c(U, f)$ belong to $\mathsf{EMHS}^{\widehat{\mu}}$ and

• are identified respectively with the mixed Hodge structures $H^r_{g^{-1}(0)}(V)$ and $H^{r-2}_c(g^{-1}(0))(-1)$ if m = 1,

• and if $m \ge 2$, decomposing $H_?^r$ as $(H_?^r)_1 \oplus (H_?^r)_{\ne 1}$, the component $(H_?^r)_1$ is as above, and the component $(H_?^r)_{\ne 1}$ is identified with

$$\left[H^1(\mathbb{A}^1,[m])\otimes H^{r-1}_?(V_m^*)\right]^{\mu_m}\in \mathsf{EMHS}^{\widehat{\mu}}_{\neq 1},$$

where V_m^* is defined by the Cartesian diagram (with $V^* = V \setminus \{0\}$)

$$\begin{array}{c} V_m^* \longrightarrow V^* \\ g_m \downarrow \qquad \qquad \downarrow g \\ \mathbb{G}_m \xrightarrow{[m]} \mathbb{G}_m \end{array}$$

and [m] is the degree m cyclic covering.

Idea of proof. I will give the idea in the case m = 1. The proof essentially relies to the following simple case. Assume $V = \mathbb{A}^1$ with coordinate τ , that g is the identity $\tau \mapsto \tau$, so that $f(t,\tau) = t\tau$. Let \mathcal{M} be a holonomic $\mathcal{D}_{\mathbb{A}^1\tau}$ -module and consider the $\mathcal{D}_{\mathbb{A}^1 \times \mathbb{A}^1}$ -module $(\mathcal{O}_{\mathbb{A}^1_t} \boxtimes \mathcal{M}(*0), \nabla + \mathbf{d}(t\tau))$. Then it is a matter of proving that its global de Rham cohomology vanishes. By integrating first along \mathbb{A}^1_t , this is the global de Rham cohomology of the Fourier transform of $\mathcal{M}(*0)$, equivalently the complex of $\mathbb{C}[\tau]$ -modules $\mathcal{M}(*0) \xrightarrow{\tau} \mathcal{M}(*0)$ with $\mathcal{M} = \Gamma(\mathbb{A}^1, \mathcal{M})$, so the result is clear. \Box

2. The irregular Hodge filtration

For a pair (U, f), the de Rham fiber $H^r_{dR,?}(U, f) = H^r_?(U, (\Omega^{\bullet}_U, d+df))$ is equipped with a filtration F^{\bullet}_{irr} indexed by \mathbb{Q} , called the irregular Hodge filtration, and defined e.g. by means of the Kontsevich-Yu complex. Let us fix a compactification $f: X \to \mathbb{P}^1$ of $f: U \to \mathbb{A}^1$ such that $D = X \setminus U$ is a ncd. Let P denote the (non-reduced) divisor $f^*(\infty)$ with support P_{red} . For any $a \in \mathbb{Q}$, we can consider the integral part $\lfloor aP \rfloor$,

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that is, if $P = \sum_i m_i P_i$ with P_i reduced, we set $\lfloor aP \rfloor = \sum_i \lfloor am_i \rfloor P_i$. The family of divisors $\lfloor aP \rfloor$ with $a \in \mathbb{Q}$ is increasing, and there exists a finite set \mathcal{A} of rational numbers in [0, 1) such that the jumps occur at most for $a \in \mathcal{A} + \mathbb{Z}$ (since the jumps occur at most when the denominator of a divides some m_i). Multiplication by fsends $\mathcal{O}_X(\lfloor aP \rfloor)$ to $\mathcal{O}_X(\lfloor (a+1)P \rfloor)$. On noting that $df = f \cdot df/f$ and that both d and df/f preserve logarithmic poles along D, we can consider for each $\alpha \in \mathcal{A}$ the subsheaf of $\Omega_X^k(\log D)(\lfloor (\alpha P \rfloor))$:

$$\Omega^k(\log D, f, \alpha) = \ker \Big[\mathrm{d}f: \Omega^k_X(\log D)(\lfloor (\alpha P \rfloor) \to \Omega^{k+1}_X(*D) / \Omega^{k+1}_X(\log D)(\lfloor \alpha P \rfloor) \Big]$$

and consider the filtered Kontsevich-Yu complex $(\Omega^{\bullet}(\log D, f, \alpha), d + df)$ with $F^{p}(\Omega^{\bullet}(\log D, f, \alpha), d + df)$ given by:

$$\{0 \to \dots \to 0 \to \Omega^p(\log D, f, \alpha) \to \dots \to \Omega^n(\log D, f, \alpha) \to 0\}.$$

We omit α in the notation if $\alpha = 0$.

Theorem.

(1) For each $\alpha \in \mathcal{A}$, the inclusion

$$(\Omega^{\bullet}(\log D, f), \mathrm{d} + \mathrm{d}f) \hookrightarrow (\Omega^{\bullet}(\log D, f, \alpha), \mathrm{d} + \mathrm{d}f)$$

is a quasi-isomorphism, leading to an identification

$$H^k\big(X, (\Omega^{\bullet}(\log D, f), \mathrm{d} + \mathrm{d}f)\big) = H^k\big(X, (\Omega^{\bullet}(\log D, f, \alpha), \mathrm{d} + \mathrm{d}f)\big).$$

(2) Furthermore, for each $\alpha \in A$, $p \ge 0$ and $k \ge 0$, the natural morphism

 $H^k(X, F^p(\Omega^{\bullet}(\log D, f, \alpha), d + df)) \longrightarrow H^k_{dR}(U, f)$

is injective, with image defining the decreasing filtration $F^{\bullet}_{\mathrm{irr},\alpha}H^k_{\mathrm{dR}}(U,f)$. For each k, we have a decomposition

$$H^k_{\mathrm{dR}}(U,f) \simeq \bigoplus_{p \ge 0} \operatorname{gr}^p_{F_{\mathrm{irr},\alpha}} H^k_{\mathrm{dR}}(U,f) \simeq \bigoplus_{p+q=k} H^q(X,\Omega^p(\log D, f, \alpha))$$

(3) A similar result holds for the Kontsevich-Yu complex "with compact support" $(\Omega^{\bullet}(\log D, f, \alpha)(-D), d + df).$

Complement. The irregular Hodge filtration can also be defined from the meromorphic de Rham complex. In such a way, by looking at the order of the poles, one finds an inclusion, for $\alpha, \beta \in [0, 1)$:

$$p - \alpha \ge q - \beta \implies F^p_{\operatorname{irr},\alpha} H^k_{\operatorname{dR}}(U, f) \subset F^q_{\operatorname{irr},\beta} H^k_{\operatorname{dR}}(U, f).$$

One can thus consider the irregular Hodge filtration as indexed by $\mathbf{p} \in -\mathcal{A} + \mathbb{Z}$: if $\mathbf{p} = p - \alpha$, then one sets $F_{irr}^{\mathbf{p}} H_{dR}^{k}(U, f) = F_{irr,\alpha}^{p} H_{dR}^{k}(U, f)$.

3. Computation of the Hodge filtration with the irregular Hodge filtration

In case $f = t^m g(y)$, $H^r_{dR}(U, f)$ comes equipped with a priori two distinct filtrations (and similarly for compact support):

• The Hodge filtration of $H_{g^{-1}(0)}^r(V)$ (m = 1) or, if $m \ge 2$, the filtration obtained as tensor product of the Hodge filtration of $H^{r-1}(V_m^*)$ with the irregular Hodge filtration of $H^1(\mathbb{A}^1, [m])$ (and taking invariants by μ_m).

• The irregular Hodge filtration defined by the Kontsevich-Yu complex.

Theorem. Both filtrations coincide.

Idea of proof. It consists in few steps.

The first step prepares to a reduction of the problem to dimension one. One constructs, for each smooth algebraic variety X a category IrrMHM(X) (it is enough to consider as subcategory denoted by EMHM(X)) so that each object has an underlying holonomic \mathcal{D}_X -module equipped with a filtration F_{irr}^{\bullet} called its *irregular Hodge filtration*. The construction is done so that $EMHS = EMHM(pt) \subset IrrMHM(pt)$. It is a subcategory of the category of mixed twistor D-modules of T. Mochizuki. One shows that the pushforward functor with respect to projective morphisms exists in this category and that the irregular Hodge filtration behaves well with respect to this functor (i.e., degeneration at E_1 occurs).

In the second step, we consider an object M of EMHS, and we let $(\mathcal{M}, F_{irr}^{\bullet})$ be the associated filtered holonomic $\mathcal{D}_{\mathbb{A}^1}$ -module.

Lemma. Assume that M is an object of $\text{EMHS}^{\hat{\mu}}$. Then there exists an isomorphism of filtered vector spaces

$$(M_{\mathrm{dR}}, F^{\bullet}_{\mathrm{irr}}) := (H^1(\mathbb{A}^1, (\Omega^{\bullet}_{\mathbb{A}^1}, \mathrm{d} + \mathrm{d}\theta)), F^{\bullet}_{\mathrm{irr}}) \xrightarrow{\sim} (\phi_{\theta} \mathcal{M}, F^{\bullet}_{\widehat{\mu}}).$$

(This could also be obtained by using a result of Takahiro Saito, since M is monodromic.) We will apply this result to $\Pi(\mathcal{H}^{r-d}f_* {}^{\mathbb{P}}\mathbb{Q}_U^{\mathbb{H}})$ and $\Pi(\mathcal{H}^{r-d}f_! {}^{\mathbb{P}}\mathbb{Q}_U^{\mathbb{H}})$.

One shows that there exists an object denoted $\mathbb{T}^{f/z}$ in $\mathsf{Irr}\mathsf{MHM}(X)$ whose underlying holonomic \mathcal{D}_X -module is $\mathcal{E}^f_X := (\mathcal{O}_X(*D), d + df)$ such that its irregular Hodge filtration $F^{\bullet}_{irr}\mathcal{O}_X(*D)$ satisfies the property that, for each $\alpha \in [0, 1)$ and each $p \in \mathbb{Z}$,

$$F_{\rm irr}^{p-\alpha}$$
 DR($\mathcal{O}_X(*D)$, d + df) $\simeq F^p(\Omega^{\bullet}_X(\log D, f, \alpha))$, d + df).

From the compatibility of F_{irr}^{\bullet} with projective pushforward, one deduces that

$$F_{\mathrm{irr}}^{\bullet}H_{\mathrm{dR}}^{r}(U,f) = F_{\widehat{\mu}}^{\bullet}\phi_{\theta}(\Pi(\mathcal{H}^{r-d}f_{*}^{\mathbf{P}}\mathbb{Q}_{U}^{\mathrm{H}}))$$

and similarly with compact support. One has

$$F^{\bullet}_{\widehat{\mu}}\phi_{\theta}(\Pi(\mathcal{H}^{r-d}f_{*}^{\mathbf{P}}\mathbb{Q}^{\mathbf{H}}_{U})) = F^{\bullet}_{\widehat{\mu}}\phi_{\theta}(\mathcal{H}^{r-d}f_{*}^{\mathbf{P}}\mathbb{Q}^{\mathbf{H}}_{U})$$

and one uses compatibility with pushforward in MHM to deduce the result.

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4. Application to the Kloosterman and Airy moments

For Kloosterman, we consider the function $g: \mathbb{G}_m \to \mathbb{A}^1$ defined as $y \mapsto y + 1/y$ and its k-fold Thom-Sebastiani sum $g_k : \mathbb{G}_m^k \to \mathbb{A}^1$. For Airy, we consider the function $g: \mathbb{A}^1 \to \mathbb{A}^1$ defined as $y \mapsto \frac{1}{3}y^3 - y$ and k-fold Thom-Sebastiani sum $g_k: \mathbb{A}^k \to \mathbb{A}^1$. We have an identification of

$$\mathbf{M}_{k,\mathbf{H}}(\mathbf{Kl}) \in \mathsf{MHS} \quad \text{with} \quad H^{k+1}_{\mathbf{H},\mathrm{mid}}(\mathbb{G}^{k+1}_{\mathbf{m}},tg_k)^{\mathfrak{S}_k \times \mu_2,\chi} \in \mathsf{EMHS}^{\mathrm{cst}} \simeq \mathsf{MHS},$$

and of

$$M_{k,H}(Ai) \in \mathsf{MHS}$$
 with $H^{k+1}_{\mathrm{H mid}}(\mathbb{G}_{\mathrm{m}} \times \mathbb{A}^{k}, t^{3}g_{k})^{\mathfrak{S}_{k} \times \mu_{2}, \chi} \in \mathsf{EMHS}^{\widehat{\mu}} \simeq \mathsf{MHS}$.

Furthermore, the underlying de Rham fibers are isomorphic respectively to $H^1_{\text{mid}}(\mathbb{G}_m, \operatorname{Sym}^k \operatorname{Kl}_2)$ and $H^1_{\text{mid}}(\mathbb{A}^1, \operatorname{Sym}^k \operatorname{Ai})$.

The latter cohomologies are simpler to understand, and in particular one can make explicit a candidate filtration G^{\bullet} on them that should be the irregular Hodge filtration via these identifications. The corresponding "Hodge" numbers are as stated in the theorem, so the point is to prove the identification $G^{\bullet} = F_{irr}^{\bullet}$.

The inclusion $G^{\bullet} \subset F^{\bullet}_{irr}$ can be checked by analyzing an embedded resolution of the hypersurface $g_k^{-1}(0)$. When k is odd (Kloosterman) or $k \neq 0 \mod 3$ (Airy), a duality argument leads to the reverse inclusion. However, in the other cases, more tricky arguments are needed, relying on general properties of the irregular Hodge filtration by Fourier transformation.

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