BESSEL AND AIRY MOMENTS: ARITHMETIC, PERIODS AND HODGE THEORY

(1) GENERAL INTRODUCTION

KYOTO, JANUARY 29–31, 2024

by

Claude Sabbah

Abstract. After presenting the general framework envisioned by the physicists Broadhurst and Roberts concerning the moments of Kloosterman sums, we will summarize the known results within their program and the methods that lead to such results. We will also consider moments of cubic sums, for which similar arithmetic results are less advanced. We will finally emphasize open questions. (Joint work with J. Fresán and J.-D. Yu.)

1. The program of Broadhurst and Roberts

Our work is intended to achieve results in the direction of the general ambitious program of D. Broadhurst and D. P. Roberts. It relies on the many ideas they have developed for this topic. I will freely quote some of their papers or slides.

"*We are inspired by the theory of motives, but actually working experimentally. Our main modus operandi is to formulate motivically plausible conjectures which are supported by high precision numerical computation.*"

Let p be a prime number, $\mathbb{F}_p, \mathbb{F}_q$ as usual, where q is some power of p and let $\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p} \colon \mathbb{F}_q \to \mathbb{F}_p$ be its trace map. Set $\zeta_p = \exp(2\pi\mathtt{i}/p)$. Let $f(x,z) \in \mathbb{Z}[x,z]$ or in $\mathbb{Z}[x,x^{-1},z]$. For each $z \in \mathbb{F}_q^\times$, the f-exponential sum is the real number

$$S_f(z;q) = \sum_{x\in X(\mathbb{F}_q)} \zeta_p^{\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p} f(x,z)}, \quad X(\mathbb{F}_q) = \mathbb{A}^1(\mathbb{F}_q) = \mathbb{F}_q \text{ or } \mathbb{G}_\mathrm{m}(\mathbb{F}_q) = \mathbb{F}_q^\times.$$

1.a. Moments of Kloosterman sums. We consider the case with $f(x,z) = x + z/x \in \mathbb{Z}[x,x^{-1},z]$, and we obtain the *Kloosterman sum*

$$\mathrm{Kl}_2(z;q) = \sum_{x\in\mathbb{F}_q^\times} \exp\bigl((2\pi\mathtt{i}/p)\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p}(x+z/x)\bigr).$$

Weil: $\exists\, \alpha_z \in \overline{\mathbb{Q}}^*$, $|\alpha_z| = \sqrt{q}$ s.t. $\mathrm{Kl}_2(z;q) = -(\alpha_z + q/\alpha_z)$. For each integer $k \geqslant 1$, define

$$\mathrm{Kl}_2^{\mathrm{Sym}^k}(z;q) = \sum_{i=0}^{k} \alpha_z^i (q/\alpha_z)^{k-i}.$$

The *moments*

$$m_2^k(q) = \sum_{z \in \mathbb{F}_q^\times} \mathrm{Kl}_2^{\mathrm{Sym}^k}(z;q)$$

generate the zeta series

$$Z_k(p;T) = \exp\Bigl(\sum_{n=1}^\infty m_2^k(p^n)\frac{T^n}{n}\Bigr).$$

It is known to be a polynomial with integer coefficients. For example,

$$Z_4(p;T) = \begin{cases} 1-T & \text{if } p=2,\\ (1-T)(1-p^2T) & \text{if } p>2, \end{cases}$$

The polynomial $Z_k(p;T)$ is always divisible by $1-T$. Other so-called "trivial factors" appear when k is a multiple of 4 or when k is even and p is small compared with k. By removing these trivial factors (i.e., considering the pure part of the putative underlying motive) one obtains the polynomial $M_k(p;T)$, from which one forms the L-function as an Euler product: $L_k(s) = \prod_{p\text{ prime}} M_k(p;p^{-s})^{-1}$

1.b. The functional equation. The standard theory also defines from the putative underlying motive a function $L_k(\infty;s)$ which is a suitable product of Γ factors and a conductor N such that the completed L-function

$$\Lambda(s) = N^{s/2}L_k(\infty;s)L_k(s)$$

should satisfy a functional equation $\Lambda(k+2-s) = \varepsilon_k\Lambda(s)$ for some sign $\varepsilon_k = \pm 1$.

"*The standard way to prove these analytic properties is to identify L as also being an automorphic L-function.*"

B-R conjecture a precise form for N and $L_k(\infty;s)$.

"*By working with $k \leqslant 24$ and extrapolating, we found experimentally the expression for N. We also found experimentally a formula for the sign ε_k of the functional equation.*"

For example,

$$L_k(\infty;s) \overset{?}{=} \prod_{j=1}^m \Gamma\Bigl(\frac{s-j}{2}\Bigr), \quad m = \begin{cases} (k-1)/2 & \text{if } k \equiv 1,3 \bmod 4,\\ (k-2)/2 & \text{if } k \equiv 2 \bmod 4,\\ (k-4)/2 & \text{if } k \equiv 0 \bmod 4. \end{cases}$$

1.c. The motive. "*We didn't realize it at the beginning of our project, but Zhiwei Yun has actually constructed the motives M_k which have been guiding us. By design, they underlie L_k.*"

After a general construction of a motive attached to moments of Kloosterman sums relative to any finite-dimensional representation of a reductive group, Yun has considered the specific case of $\mathrm{Sym}^k\,\mathrm{Kl}_2$ and has given an explicit locally closed affine subvariety of $\mathbb{A}^{2k}$ defined over $\mathbb{Z}$, of dimension $k+1$. However, the geometry and singularities of this variety are difficult to analyze.

"*For $k \geqslant 9$, the underlying motive* M_k *is more complicated than motives related to classical modular forms. The L-function* L_k *we have is not known to have good analytic properties. So if we want to do anything at all, we are forced to be experimental. We compute with* L_k *using Tim Dokchitser's algorithms, as implemented either in Pari or Magma.*"

"*We view our computational approach as a good thing, a helpful complement to the literature on motives, which tends to be very theoretical.*"

Definition of the motive M_k**.** In our work, we find another way of expressing a motive M_k related to $\mathrm{Sym}^k \mathrm{Kl}_2$, as a toric hypersurface equipped with the action of a symmetric group. The analysis of this geometric object is easier.

There are various meanings for Kl_2 and $\mathrm{Sym}^k \mathrm{Kl}_2$ (ℓ-adic sheaf on $\mathbb{G}_\mathrm{m}$, complex vector bundle with connection...). Let us consider the complex aspect.

The Kloosterman connection Kl_2 on $\mathbb{G}_{\mathrm{m}z}$ is the pushforward by $\pi : (x, z) \mapsto z$ of the connection $(\mathcal{O}_{\mathbb{G}_\mathrm{m}^2}, \mathrm{d} + \mathrm{d}f)$. Then $\mathrm{Sym}^k \mathrm{Kl}_2$ is the k-th symmetric product, which is a rank $k+1$-vector bundle with a connection having a regular singular singularity at $z = 0$ and an irregular one at $z = \infty$. A similar definition can be given for the Kloosterman ℓ-adic sheaf.

Our objective is to understand that $H^1_{\mathrm{dR},?}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2)$ admits a natural $\mathbb{Q}$-Hodge structure and that $H^1_{\mathrm{dR,mid}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2) := \mathrm{image}[H^1_{\mathrm{dR,c}} \to H^1_{\mathrm{dR}}]$ is a pure Hodge substructure. This will be the Hodge realization of our motive M_k.

It is easier to consider first the pullback $\widetilde{\mathrm{Kl}}_2$ of Kl_2 by the 2-to-1 map $t \mapsto t^2 = z$, so that $\mathrm{Kl}_2 = \widetilde{\mathrm{Kl}}_2^{\mu_2}$. Setting $y = x/t$, we write $f(x, t^2) = tg(y)$ with $g(y) = y + 1/y$.

Set $g_k(y_1, \dots, y_k) = g(y_1) + \cdots + g(y_k)$. Then $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$ is obtained by taking the $\mathfrak{S}_k$-invariants of the pushforward by $\pi_k : (y_1, \dots, y_k, t) \mapsto t$ of $(\mathcal{O}_{\mathbb{G}_\mathrm{m}^{k+1}}, \mathrm{d} + \mathrm{d}(tg_k))$, and $\mathrm{Sym}^k \mathrm{Kl}_2$ by taking furthermore its μ_2-invariants.

Let $\mathcal{K}$ be the toric hypersurface $g_k^{-1}(0) \subset \mathbb{G}_\mathrm{m}^k$.

Definition. The pure motive M_k is $\mathrm{gr}^W_{k+1}[H^{k-1}_\mathrm{c}(\mathcal{K})(-1)]^{\mathfrak{S}_k \times \mu_2, \chi}$, i.e., the χ-isotypic component of $\mathrm{gr}^W_{k+1} H^{k-1}_\mathrm{c}(\mathcal{K})(-1)$ with respect to the action of $\mathfrak{S}_k \times \mu_2$, where $\chi(\sigma, \varepsilon) = \mathrm{sgn}(\sigma)$.

1.d. Hodge numbers of the motive. "*The Hodge vectors* $(h^{w,0}, \dots, h^{0,w})$ *of the first cohort are definitely* $(1, 0, 1)$ *for* $k = 5$*,* $(1, 0, 0, 1)$ *for* $k = 6$*,* $(1, 0, 1, 0, 1)$ *for* $k = 7$*,* $(1, 0, 0, 0, 0, 1)$ *for* $k = 8$ *(after removal of the* ζ *part). Hodge vectors can be inferred from motivic L-functions via p-adic valuations of coefficients. We confidently conjecture that to pass from* k *to* $k + 4$ *one prepends* $(1, 0)$ *and appends* $(0, 1)$*.*"

Our first main result is the confirmation of the conjecture of Broadhurst and Roberts concerning the Hodge numbers of the motive M_k.

Theorem. *The Hodge numbers of* M_k *are as conjectured by Broadhurst and Roberts.*

We were not able to compute the Hodge numbers directly. Instead, we identify the de Rham realization of M_k with the cohomology of a dimension one object: the de Rham cohomology of the connection $\mathrm{Sym}^k \mathrm{Kl}_2$, which is a direct summand of that of the k-th symmetric power of the modified Bessel connection. As a main tool, we make full use of the notion of exponential mixed Hodge structure introduced by Kontsevich and Soibelman, and that of irregular Hodge filtration.

Theorem. *There exists an isomorphism* $\mathrm{M}_{k,\mathrm{dR}} \simeq H^1_{\mathrm{dR,mid}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2)$ *which is filtered with respect to the Hodge filtration of the LHS and the* irregular Hodge filtration *of the RHS.*

This will be explained in Lecture 3. The main observation is that the interesting de Rham cohomology $H^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2)$ underlies an exponential mixed Hodge structure which happen to be a mixed Hodge structure, which is indeed isomorphic to that of M_k.

1.e. Automorphicity of the L function. "*We are hopeful that automorphicity may be perhaps established soon as a consequence of general results. The fact that all Hodge numbers are* 1 *or* 0 *is essential for this optimism.*"

In order to prove the automorphicity of the L-function, we show that the motive M_k produces a family of Galois representations that forms a *weakly compatible system* satisfying, in particular due to the theorem on Hodge numbers, properties which allow one to apply a theorem of Patrikis and Taylor ensuring the existence of a functional equation for the completed L-function. This will be explained in Lecture 2.

Theorem.

- *If k is odd, the completed L-function, as conjectured by B-R with the sign ε_k equal to* 1*, satisfies the functional equation.*
- *If k is even, up to a missing determination of the sign ε_k and of the conductor for $p = 2$, the completed L-function, as conjectured by B-R satisfies the functional equation.*

1.f. Periods and the conjecture of Deligne. The *critical integers* are the integral values of s for which neither $L_k(\infty; s)$ and $L_k(\infty; k+2-s)$ has a pole, and the critical values are the values of $L_k(s)$ at critical integers. For example, if $k = 4r + 1$, the critical integers are $2r + 1, 2r + 2$. A conjecture of Deligne relates the critical values to periods of the motive. For that purpose, one needs to compute the period matrix of the motive. B-R define a matrix whose entries are *Bessel moments* of the form

$$\mathrm{BM}_k(i,j) = (-1)^{k-i} 2^{k-j} (\pi \mathtt{i})^i \int_0^\infty I_0(t)^i K_0(t)^{k-i} t^j \mathrm{d}t,$$

where $I_0(t), K_0(t)$ are the two solution of the modified Bessel differential operator $(t\partial_t)^2 - t^2$ given by

$$\begin{aligned} I_0(t) &= \frac{1}{2\pi \mathtt{i}} \oint \exp\Big(-\frac{t}{2}\big(y+1/y\big)\Big)\frac{\mathrm{d}y}{y},\\ K_0(t) &= \frac{1}{2}\int_0^\infty \exp\Big(-\frac{t}{2}\big(y+1/y\big)\Big)\frac{\mathrm{d}y}{y} \qquad (|\arg t| < \pi/2). \end{aligned}$$

They conjecture that it is the period matrix of the motive M_k. The conjecture is supported by a conjectural relation, called *quadratic relation*, satisfied by this matrix and checked to high precision. We prove the following:

Theorem. *There exist suitable bases of* $\mathrm{M}_{k,\mathrm{dR}}$ *and* $\mathrm{M}_{k,\mathrm{Betti}}$ *such that the period matrix* P_k *in these bases is* $(\mathrm{BM}_k(i,j))_{i,j}$. *Furthermore, it satisfies the standard quadratic relations*

$$\mathsf{P}_k \cdot (\mathsf{S}_k)^{-1} \cdot {}^t\mathsf{P} = (-2\pi \boldsymbol{i})^{k+1}\mathsf{B}_k,$$

where S_k *(resp.* B*) is the de Rham (resp. Betti) intersection pairing in these bases.*

Remarque.

(1) B-R have found experimentally quadratic relations of the previous form with an inductive definition for $(\mathsf{S}_k)^{-1}$, but it is not clear that their $(\mathsf{S}_k)^{-1}$ as a geometric meaning.

(2) The interpretation of these pairings as geometric pairings related to the motive follows from the relation between M_k and $H^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$. We were not able to compute the period matrix and check the quadratic relations directly on M_k.

B-R define some subdeterminants of the matrix of Bessel moments and conjecture that they are the expected critical integers as conjectured by Deligne.

Theorem. *This conjecture of B-R holds true.*

2. Extension of the program to moments of cubic sums

We consider the case where $f(x,z) = x^3 - zx \in \mathbb{Z}[x,z]$, the corresponding cubic sum $\mathrm{Ai}(z;q)$, the moments $\mathrm{Ai}^{\mathrm{Sym}^k}(z;q)$ and the L-function.

For Airy moments, a strategy strictly similar to that for Kl_2 to understand the Galois theoretic properties of the corresponding motivic object M_k is not possible, basically because the corresponding M_k is not a Nori motive, but an *ulterior motive* in the sense of Anderson. The Airy connection Ai is a rank-two bundle on $\mathbb{A}^1$ with connection and we are interested in the de Rham cohomology $H^1(\mathbb{A}^1, \mathrm{Sym}^k \mathrm{Ai})$, which underlies an exponential mixed Hodge structure. The new phenomenon is that it is not a mixed Hodge structure but “almost”. Nevertheless, Hodge theory can be developed in this context (arithmetic Hodge structure in the sense of Anderson), with the caveat that the Hodge numbers $h^{\mathsf{p},\mathsf{q}}$ may occur for rational exponents p,q subject to $\mathsf{p}+\mathsf{q}\in\mathbb{Z}$. The notion of irregular Hodge filtration covers such a situation.

Theorem. *The nonzero irregular Hodge numbers are equal to one.*

(We will give an explicit formula). Although arithmetic results on the L-function of $\mathrm{Sym}^k\mathrm{Ai}$ exist (Haessig, 2009), they are not enough for the moment to construct a weakly compatible system of Galois representations that satisfy the conditions of the theorem of Patrikis and Taylor. It is thus not clear that the motive M_k is potentially automorphic.

Also, an expression of the period matrix of M_k in terms of Airy moments, that is, integrals of suitable powers of the Airy functions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$, can be obtained, but we are not able to distinguish the terms corresponding to the "classical part".

3. Open questions and extensions

3.a. Central vanishing. For $\mathrm{Sym}^k\mathrm{Kl}_2$, Broadhurst and Roberts conjecture that, for $k\geqslant 12$ a multiple of 4, the L-function $L(\mathrm{M}_k,s)$ always vanishes at the central point $s=(k+2)/2$, even when the sign ε_k of the functional equation is positive. According to Beilinson's conjecture, this vanishing should be explained by the existence of certain non-trivial extension of M_k by $\mathbb{Q}(-k/2)$. It seems possible to construct such an extension by considering the quotient of the cohomology with compact support of $\mathrm{Sym}^k\mathrm{Kl}_2$ by its weight zero piece and to relate its non-splitting to the shape of the full period matrix $\mathsf{P}_k^{\mathrm{mod},\mathrm{rd}}$. Concretely, the question amounts to see whether the moments $\mathrm{BM}_k(k/2,2j-1)$ and $\mathrm{BM}_k^{\mathrm{reg}}(k/2,k'+2j)$, for $1\leqslant j<k/4$, are not all in $(2\pi\mathtt{i})^{k/2}\mathbb{Q}$. Some numerical evidence has been given by Broadhurst and Roberts.

3.b. Extension of the results to $\mathrm{Sym}^k\mathrm{Kl}_{n+1}$ ($n\geqslant 2$). One can consider the exponential sum attached to $f(x_1,\dots,x_n,z)=x_1+\cdots+x_n+z/x_1\cdots x_n$, and the corresponding vector bundle with connection or ℓ-adic sheaf, denoted by Kl_{n+1}.

The general strategy developed for $\mathrm{Sym}^k\mathrm{Kl}_2$ can be applied to $\mathrm{Sym}^k\mathrm{Kl}_{n+1}$ for any $n\geqslant 2$. Also, Yun has developed the arithmetic part of the program for moments of Kloosterman sums attached to other irreducible representations of GL_n and of other algebraic groups.

However, the computation of the Hodge numbers is more complicated and does not always lead to the values 0 or 1, preventing in general the application of the theorem of Patrikis and Taylor.

***Theorem* (Y. Qin).**

(1) *Assume* $\gcd(k,n+1)=1$. *Then the Hodge number* $h^{p,nk+1-p}$ *of the motive attached to* $\mathrm{Sym}^k(\mathrm{Kl}_{n+1}$ *is the coefficient of* t^px^k *in the generating series*

$$\frac{(1-t)t^{n+1}}{(1-t^{n+1})(1-x)(1-tx)\cdots(1-t^nx)}.$$

(2) *If $n=2$, the missing Hodge numbers (i.e., if $3 \mid k$) are given by the formula, for $p \leqslant k$:*

$$h^{p,2k+1-p} = \begin{cases} \lfloor p/6 \rfloor - \delta_{p,k} & \text{if } p \equiv 0,1,2,4 \bmod 6, \\ \lfloor p/6 \rfloor + 1 & \text{if } p \equiv 3,5 \bmod 6. \end{cases}$$

***Theorem* (Y. Qin).** *If $n=2$ and $k \leqslant 9$ or $n=3,4$ and $k=2,3$ (and a few other cases), then the partial L-function of $\operatorname{Sym}^k \mathrm{Kl}_{n+1}$ satisfies a functional equation (whose sign ε_k is not determined). Furthermore, some of them arise from modular forms.*

3.c. Extension of the results to $\operatorname{Sym}^k \mathrm{Ai}_n$ $(n \geqslant 3)$. One can consider the exponential sum attached to $f(x,z) = x^{n+1} - zx$, and the corresponding vector bundle with connection or ℓ-adic sheaf, denoted by Ai_n.

***Theorem* (Y. Qin).** *Assume $\gcd(k,n)=1$. Then, the Hodge number $h^{\mathsf{p},\mathsf{q}}$ of the motive attached to $\operatorname{Sym}^k(\mathrm{Ai}_n)$ is zero if $\mathsf{p} \neq (p+n+k)/(n+1)$ with $0 \leqslant p \leqslant nk+1-(n+k)$, and otherwise $h^{\frac{p+n+k}{n+1}, \frac{nk+1-p}{n+1}}$ is the coefficient of $t^p x^k$ in the generating series*

$$\frac{(1-t)}{(1-t^{n+1})(1-x)(1-tx)\cdots(1-t^n x)}.$$

C. Sabbah, CMLS, CNRS, École polytechnique, Institut Polytechnique de Paris, 91128 Palaiseau cedex, France • *E-mail :* Claude.Sabbah@polytechnique.edu
Url : https://perso.pages.math.cnrs.fr/users/claude.sabbah