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FILTERED HOLONOMIC D-MODULES IN DIMENSION ONE

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Abstract. Holonomic D-modules on the affine complex line offer a simple prototype of various properties also occurring in higher dimensions. The interesting holonomic D-modules occurring in Algebraic Geometry often come equipped with a natural filtration. In the first lecture, we will focus on filtered holonomic D-modules from various points of view.

In the second lecture, we focus on rigid irreducible holonomic D-modules on the affine line. Generic (possibly confluent) hypergeometric differential equations give naturally rise to examples of such objects. After having explained the Katz algorithm (respectively the Arinkin-Deligne algorithm) for reducing the regular (respectively possibly irregular) such D-modules to ones having generic rank one, we will consider the behaviour of Hodge (respectively Deligne) filtrations along the algorithm (joint work with M. Dettweiler) and we will state a result in the irregular case, like a confluent hypergeometric differential equation.

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LECTURE 1

THE DELIGNE FILTRATION

Summary. In the first lecture, we start from a filtered regular holonomic D-module M underlying a Hodge module, and we explain the construction an some properties of the associated Deligne filtration on the D-module obtained from M by applying an exponential twist. This is strongly related to considering the Laplace transform of M. We will give motivations for considering such a filtration.

Notation. Let t be a coordinate on the affine line $\mathbb{A}^1$, with ring $\mathbb{C}[t] = \Gamma(\mathbb{A}^1, \mathscr{O}_{\mathbb{A}^1})$. Let $\Sigma \subset \mathbb{A}^1$ be a finite set, and set $j : U = \mathbb{A}^1 \smallsetminus \Sigma \hookrightarrow \mathbb{A}^1$. We identify $\mathscr{O}(U)$ with $\mathbb{C}[t, *\Sigma]$ of rational functions of t with poles in Σ. Given an $\mathscr{O}(U)$-module V, we denote by j_*V the module V regarded as a $\mathbb{C}[t]$-module.

Let $\mathbb{C}[t]\langle\partial_t\rangle$ be the ring of differential operators with polynomial coefficients. A left $\mathbb{C}[t]\langle\partial_t\rangle$-module is nothing but an $\mathbb{C}[t]$-module with a connection. We have a similar property for $\mathscr{O}(U)\langle\partial_t\rangle$-modules. For such a module V, j_*V as defined above is a $\mathbb{C}[t]\langle\partial_t\rangle$-module. For an operator $P = \sum_{i=1}^d a_i(t)\partial_t^i \in \mathbb{C}[t]\langle\partial_t\rangle$ with $a_d \neq 0$, we set $\operatorname{ord} P = -d$ and $F^p\mathbb{C}[t]\langle\partial_t\rangle = \{P \mid \operatorname{ord} P \geqslant p\}$ (convention: $\operatorname{ord}(0) = -\infty$). This defines a *decreasing* filtration.

The singularities of such a P are the roots of a_d. We say that P is *Fuchsian at infinity* if $\deg a_i - i \leqslant \deg a_d - d$ for every i, and that P is *Fuchsian at finite distance* if $i - v_c(a_i) \leqslant d - v_c(a_d)$ for every i and every singularity c of P.

1.0. Motivations

Due to a theorem of Griffiths, classical Hodge theory and irregular singularities are incompatible. Deligne (1984, see [**Del07**]) has proposed a framework enlarging Hodge theory which is compatible with some irregular singularities. The goal of this lecture is to explain this framework revisited.

1.1. A review of holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-modules in one complex variable

A $\mathbb{C}[t]\langle\partial_t\rangle$-module M of finite type is *holonomic* if any element m is annihilated by some *nonzero* differential operator $P \in \mathbb{C}[t]\langle\partial_t\rangle$, i.e., $Pm = 0$. It is said to have a regular singularity at infinity (resp. at finite distance) if for any m, P can be chosen to be Fuchsian at infinity (resp. at finite distance).

Some properties.

(i) If V is a free $\mathscr{O}(U)$-module of finite rank with connection, then j_*V is holonomic.

(ii) For any holonomic M there exists a finite set Σ such that $M_{|U} := \mathscr{O}(U) \otimes M$ is free of finite rank, but M may be different of $j_*M_{|U}$.

(iii) There is a duality (contravariant) functor $M \mapsto M^\vee$, so that $(M^\vee)_{|U}$ is the dual free $\mathscr{O}(U)$-module $\mathrm{Hom}_{\mathscr{O}(U)}(M_{|U}, \mathscr{O}(U))$ with the dual connection.

(iv) Given a free $\mathscr{O}(U)$-module with connection V, there is a unique $\mathbb{C}[t]\langle\partial_t\rangle$-submodule of j_*V which has no quotient supported on a discrete set (in fact, on Σ). We denote it by $j_{!*}V$. It is called the *intermediate extension* of V. It satisfies $(j_{!*}V)^\vee \simeq j_{!*}(V^\vee)$.

(v) There is a one-to-one correspondence between *irreducible* free $\mathscr{O}(U)$-modules with connection V and irreducible $\mathbb{C}[t]\langle\partial_t\rangle$-modules with singular set contained in Σ, and not of the form $\mathbb{C}[t]\langle\partial_t\rangle/\mathbb{C}[t]\langle\partial_t\rangle \cdot (t-c)$ for some $c \in \mathbb{C}$. It is given by $V \mapsto j_{!*}V$.

(vi) Given M holonomic, the *exponential twist* $M \otimes E^t$ is the $\mathbb{C}[t]$-module M with the action of ∂_t changed as follows: $\partial_t(m \otimes \text{“e}^t\text{”}) = (\partial_t + 1)m$. It is also holonomic. It satisfies $(M \otimes E^t)^\vee \simeq M^\vee \otimes E^{-t}$.

Examples.

(GM) (Gauss-Manin systems) X: smooth quasi-projective variety, $f : X \to \mathbb{A}^1$, complex $\boldsymbol{R}f_*(\Omega_X^\bullet[\tau], \mathrm{d} + \tau \mathrm{d}f)[\dim X]$. The cohomology sheaves M_f^k are known to be holonomic with regular singularity. The $\mathbb{C}[t]\langle\partial_t\rangle$-module structure is defined as follows: the action of ∂_t is induced by multiplication by $-\tau$; the action of t is induced by $t \cdot \omega_k \tau^k = f\omega_k\tau^k + k\omega_k\tau^{k-1}$.

- If X is *affine*, then $M_f^k = H^{k+\dim X}(\Omega^\bullet(X)[\tau], \mathrm{d} + \tau\mathrm{d}f)$.
- If f is *projective*, each M_f^k is known to be semi-simple (i.e., direct sum of irreducible $\mathbb{C}[t]\langle\partial_t\rangle$-modules): this is provided by the *decomposition theorem* (M. Saito).

(Hyp) (Hypergeometric systems) Fix $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n \in \mathbb{C}$, consider the hypergeometric differential equation

$$P_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \prod_{i=1}^m (t\partial_t - \alpha_i) - t \prod_{j=1}^n (t\partial_t - \beta_j),$$

and set $H_{\boldsymbol{\alpha},\boldsymbol{\beta}} := \mathbb{C}[t]\langle\partial_t\rangle/\mathbb{C}[t]\langle\partial_t\rangle \cdot P_{\boldsymbol{\alpha},\boldsymbol{\beta}}$.

- If $m = n$, $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ has regular singularities at $0, 1, \infty$.
- If $n < m$, $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ has a regular singularity at ∞ and an irregular singularity at 0, and vice-versa if $n > m$.

Assume that
$$\begin{cases} \alpha_i - \alpha_{i'} \in \mathbb{Z} \Longrightarrow \alpha_i = \alpha_{i'}, \\ \beta_j - \beta_{j'} \in \mathbb{Z} \Longrightarrow \beta_j = \beta_{j'}, \\ \qquad \alpha_i - \beta_j \notin \mathbb{Z} \quad \forall\, i, j. \end{cases}$$

Then (Beukers-Heckman, Katz) $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is irreducible.

***Definition* (De Rham cohomology and de Rham complex).** If M is a $\mathbb{C}[t]\langle\partial_t\rangle$-module, $H^{-1}_{\mathrm{dR}}M$ and $H^0_{\mathrm{dR}}M$ are respectively the kernel and cokernel of the complex

$$ {}^{\mathrm{p}}\mathrm{DR}\, M := \Big\{0 \longrightarrow M \xrightarrow{\partial_t} \underset{\bullet}{M} \longrightarrow 0\Big\}. $$

If M is holonomic, $H^j_{\mathrm{dR}}M$ are finite dimensional vector spaces.

1.2. Laplace transformation

We consider the involution $\mathbb{C}[t]\langle\partial_t\rangle \xrightarrow{\sim} \mathbb{C}[t]\langle\partial_t\rangle$ given by $t \mapsto \partial_t$, $\partial_t \mapsto -t$. Iterating it twice gives the involution $\iota = -\operatorname{Id}$. It is clearer to change the name of the variables and regard it as an isomorphism $\mathbb{C}[\tau]\langle\partial_\tau\rangle \xrightarrow{\sim} \mathbb{C}[t]\langle\partial_t\rangle$, $\tau \mapsto -\partial_t$, $\partial_\tau \mapsto t$.

Given a $\mathbb{C}[t]\langle\partial_t\rangle$-module M, its *Laplace transform* ${}^{F}M$ is the $\mathbb{C}$-vector space M endowed with the $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-structure induced by the above correspondence. We thus have ${}^{F}({}^{F}M) = \iota^* M$.

Some properties. Assume that M is an holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module.

(1) Then, so is ${}^{F}M$.

(2) We have ${}^{F}(M^\vee) \simeq \iota^*({}^{F}M)^\vee$.

(3) ${}^{F}M$ can be recovered by an "integral formula". Set $M[\tau] = \mathbb{C}[\tau] \otimes_{\mathbb{C}} M$ with its natural structure of $\mathbb{C}[t,\tau]\langle\partial_t,\partial_\tau\rangle$-module, and denote by $M[\tau]\otimes E^{t\tau}$ the exponentially twisted module. Then the $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-linear morphism

$$ M[\tau] \otimes E^{t\tau} \xrightarrow{\partial_t} M[\tau] \otimes E^{t\tau} $$

is injective, and its cokernel is identified with ${}^{F}M$ as a $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module, by means of the morphism induced by $\sum_{k\geqslant 0} m_k\tau^k \mapsto \sum_k (-\partial_t)^k m_k$.

(4) Note also that the exponentially twisted module $M \otimes E^t$ is the cokernel of

$$ M[\tau] \otimes E^{t\tau} \xrightarrow{\tau - 1} M[\tau] \otimes E^{t\tau}. $$

Similarly, the restriction at $\tau = 1$ of ${}^{F}M$, that is, the complex

$$ {}^{F}M \xrightarrow{\tau - 1} \underset{\bullet}{{}^{F}M} $$

is identified with ${}^{\mathrm{p}}\mathrm{DR}(M \otimes E^t)$.

(5) If M has a regular singularity at ∞, then ${}^{F}M$ has a regular singularity at $\tau = 0$ and no other singularity at finite distance. In particular, $\tau = 1$ is not a singularity, and therefore $H^i_{\mathrm{dR}}M = 0$ for $j \neq 0$.

1.3. Filtered holonomic $\mathscr{D}$-modules

Most holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-modules coming from geometry (e.g. given by a Picard-Fuchs equation) come naturally equipped with a coherent (decreasing) filtration $F^\bullet M$.

Definition. A coherent filtration on M is an decreasing filtration $F^\bullet M$ bounded from below by $\mathbb{C}[t]$-modules of finite type, such that

$$ F^k\mathbb{C}[t]\langle\partial_t\rangle \cdot F^\ell M \subset F^{\ell+k}M, \quad \text{with equality for any } \ell \leqslant \ell_o. $$

Example.

(GM) We can filter the complex $(\Omega_X^\bullet[\tau], \mathrm{d}+\tau\mathrm{d}f)$ by setting

$$F^p(\Omega_X^\bullet[\tau], \mathrm{d}+\tau\mathrm{d}f) = (\Omega_X^\bullet[\tau]_{\leqslant -p+\bullet}, \mathrm{d}+\tau\mathrm{d}f).$$

For $p \geqslant 0$, $F^p(\Omega_X^\bullet[\tau], \mathrm{d}+\tau\mathrm{d}f)$ starts in degree p with Ω_X^p. If f is proper, this gives rise to a coherent filtration on each M_f^k.

1.4. Laplace transformation and coherent filtrations

Let $(M, F^\bullet M)$ be a holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module equipped with a coehernt filtration. By the identification of the $\mathbb{C}$-vector spaces $M = {}^F\!M$, we can regard F_kM as a $\mathbb{C}[\partial_\tau]$-submodule of ${}^F\!M$. In this way, $F^\bullet M$ does not form, however, a coherent filtration of ${}^F\!M$ as a $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module. We will associate to $F^\bullet M$ another object, called the *Brieskorn lattice.*[(1)]

We denote by G the holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module $\mathbb{C}[\partial_t, \partial_t^{-1}] \otimes_{\mathbb{C}[\partial_t]} M$. If we identify $\mathbb{C}[t]\langle\partial_t\rangle$ with $\mathbb{C}[\tau]\langle\partial_\tau\rangle$ by the Laplace correspondence $t \mapsto \partial_\tau$, $\partial_t \mapsto -\tau$, we also regard G as a holonomic $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module on which the multiplication by τ is bijective. It is therefore also a $\mathbb{C}[\tau,\tau^{-1}]$-module. We will denote by $\widehat{\mathrm{loc}}$ the natural morphism $M \to G$.

The Brieskorn lattice $G_0^{(F)}$ of the filtration $F^\bullet M$ is defined as the saturation of the filtration by the operator ∂_t^{-1}, that is,

$$G_0^{(F)} := \sum_j \partial_t^j \widehat{\mathrm{loc}}(F^jM) \subset G. \tag{$*$}$$

It is naturally a $\mathbb{C}[\partial_t^{-1}]$-module (equivalently, a $\mathbb{C}[\theta]$-module, with $\theta = \tau^{-1}$). Moreover, we have

$$t\cdot\partial_t^j\widehat{\mathrm{loc}}(F^jM) \subset \partial_t^j\widehat{\mathrm{loc}}(tF^jM) + \partial_t^{j-1}\widehat{\mathrm{loc}}(F^jM) \subset \partial_t^j\widehat{\mathrm{loc}}(F^jM) + \partial_t^{j-1}\widehat{\mathrm{loc}}(F^{j-1}M),$$

hence $tG_0^{(F)} \subset G_0^{(F)}$ and so $\theta^2\partial_\theta G_0^{(F)} \subset G_0^{(F)}$. In other words, the meromorphic connection on G induces a connection with a pole of order two on $G_0^{(F)}$.

We can compute $G_0^{(F)}$ by only using one generating term of the filtration. Indeed, let p_o be an index of generation, so that $F^{p_o-\ell}M = F^{p_o}M + \cdots + \partial_t^\ell F^{p_o}M$ for any $\ell \geqslant 0$. Then we have

$$G_0^{(F)} = \partial_t^{p_o}\sum_{j\geqslant 0}\partial_t^{-j}\widehat{\mathrm{loc}}(F^{p_o}M). \tag{$**$}$$

Let us check that $(*) = (**)$. Let us write $(*)$ as

$$G_0^{(F)} = \partial_t^{p_o}\sum_j \partial_t^j\widehat{\mathrm{loc}}(F^{p_o+j}M).$$

Firstly, for $j \geqslant 0$, we have

$$\partial_t^j\widehat{\mathrm{loc}}(F^{p_o+j}M) = \widehat{\mathrm{loc}}(\partial_t^jF^{p_o+j}M) \subset \widehat{\mathrm{loc}}(F^{p_o}M),$$

(1) The name refers to an object considered by Brieskorn in [**Bri70**] in his study of isolated critical points of holomorphic functions. The terminology "lattice", meaning a free $\mathbb{C}[\tau^{-1}]$-module of finite rank which generates G, will be justified in the proposition at the end of this section.

so we can also write $(*)$ as

$$G_0^{(F)}=\partial_t^{p_o}\sum_{\ell\geqslant 0}\partial_t^{-\ell}\widehat{\mathrm{loc}}(F^{p_o-\ell}M)=\partial_t^{p_o}\sum_{\ell\geqslant 0}\partial_t^{-\ell}\big[\widehat{\mathrm{loc}}(F^{p_o}M)+\cdots+\partial_t^{\ell}\widehat{\mathrm{loc}}(F^{p_o}M)\big]$$
$$=\partial_t^{p_o}\sum_{j\geqslant 0}\partial_t^{-j}\widehat{\mathrm{loc}}(F^{p_o}M).$$

Proposition. *Assume M has a regular singularity at infinity and $F^\bullet M$ is a coherent filtration of M. Then the Brieskorn lattice $G_0=G_0^{(F)}$ is a lattice.*

Proof. One checks the assertion on M of the form $M=\mathbb{C}[t]\langle\partial_t\rangle/\mathbb{C}[t]\langle\partial_t\rangle\cdot P$ for some algebraic differential operator with a regular singularity at infinity and the coherent filtration on M naturally induced by $F^\bullet\mathbb{C}[t]\langle\partial_t\rangle$. Then one deduces the general case by comparing Brieskorn lattices for various filtrations. □

1.5. The irregular Hodge filtration on $H^0_{\mathrm{dR}}(M\otimes E^t)$

We assume that M has only regular singularities (at infinity is enough). For simplicity, we assume that the eigenvalues of the monodromy at infinity have absolute value equal to one. In particular, $G/(\theta-1)G=H^0_{\mathrm{dR}}(M\otimes E^t)$, and $H^i_{\mathrm{dR}}(M\otimes E^t)=0$ for $i\neq 0$. Since the connection on G has a regular singularity at $\tau=0$, we can consider, for every $\beta\in\mathbb{R}$, the Deligne extension $V^\beta G$, that is, the $\mathbb{C}[\tau]$-submodule of G which satisfies the following properties:

(1) $\mathbb{C}[\tau,\tau^{-1}]\otimes_{\mathbb{C}[\tau]}V^\beta G=G$,
(2) $\tau\partial_\tau V^\beta G\subset V^\beta G$,
(3) the residue of the connection on $V^\beta G$ has its eigenvalues in $(\beta-1,\beta]$, i.e., there exist $\nu_\gamma\in\mathbb{N}$ such that $\prod_{\gamma\in(\beta-1,\beta]}(\tau\partial_\tau-\gamma)^{\nu_\gamma}$ vanishes on $V^\beta G/\tau\cdot V^\beta G$.

This defines an decreasing filtration of G indexed by $\mathbb{R}$, with a finite mod$\mathbb{Z}$ index set. We have $V^{\beta+1}G=\tau V^\beta G$. We obtain a bundle $\mathscr{G}_0^\beta$ on $\mathbb{P}^1$ by gluing $V^\beta G$ and G_0 by means of the isomorphism

$$\mathbb{C}[\tau,\tau^{-1}]\otimes_{\mathbb{C}[\tau]}V^\beta G=G=\mathbb{C}[\theta,\theta^{-1}]G_0.$$

We have $\Gamma(\mathbb{P}^1,\mathscr{G}_0^\beta)=V^\beta G\cap G_0$, where the intersection is taken in G.

***Definition* (Irregular Hodge filtration).** The *irregular Hodge filtration* is the decreasing filtration indexed by $\mathbb{R}$ naturally induced by $V^\lambda\cap G_0$ on $H:=G_0/(\theta-1)G_0$:

$$F^\lambda_{\mathrm{irr}}H:=(V^\lambda\cap G_0)\big/(V^\lambda\cap G_0)\cap(\theta-1)G_0.$$

A *trivializing lattice* for G_0 is a free $\mathbb{C}[\tau]$-lattice G^o of G (i.e., $\mathbb{C}[\tau]$-submodule of maximal rank) such that the bundle $\mathscr{G}_0^o$ on $\mathbb{P}^1$ obtained by gluing G_0 and G^o is trivializable. Such lattices do exist (any basis of G_0 as a $\mathbb{C}[\theta]$-module generates such a $\mathbb{C}[\tau]$-module). Then $\Gamma(\mathbb{P}^1,\mathscr{G}_0^o)=G^o\cap G_0$ is a finite-dimensional $\mathbb{C}$-vector space of dimension $\operatorname{rk}G_0$ and we have

$$G_0=\bigoplus_{j\geqslant 0}\theta^j(G^o\cap G_0),\quad G^o=\bigoplus_{j\geqslant 0}\tau^j(G^o\cap G_0),\quad G=\bigoplus_{j\in\mathbb{Z}}\theta^j(G^o\cap G_0).$$

***Definition* (V-adapted trivializing lattice).** We say that G^o is a *V-adapted trivializing lattice for G_0* if it moreover satisfies the following property. For every $\beta \in \mathbb{R}$,

$$V^\beta G \cap G_0 = \bigoplus_{j \geqslant 0} \theta^j (V^{\beta+j} G \cap G^o \cap G_0) = \bigoplus_{j \geqslant 0} \tau^{-j} (V^{\beta+j} G \cap G^o \cap G_0).$$

***Proposition* (J.-D. Yu).**

(1) *There exists a V-adapted trivializing lattice for G_0.*

(2) *For a trivializing lattice G^o, we have natural identifications of filtered vector spaces (induced by $G_0 \to G_0/(\theta - 1)G_0$)*

$$(G^o \cap G_0, V^\bullet G \cap G^o \cap G_0) \xrightarrow{\sim} (H, F^\bullet_{\mathrm{irr}} H).$$

Some properties. By using V-adapted trivializing lattices, one can show the good behaviour of $F^\bullet_{\mathrm{irr}}$ with respect to tensor product and duality of G_0. Such properties go back to the work of Varchenko for isolated hypersurface singularities.

1.6. The Deligne filtration

We consider the inclusion $i : \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$. We now consider $\mathscr{D}_{\mathbb{P}^1}$-modules. If M is a $\mathbb{C}[t]\langle\partial_t\rangle$-module, it corresponds to a sheaf $\mathcal{M}$ of $\mathscr{D}_{\mathbb{A}^1}$-modules, and one can define $\widetilde{\mathcal{M}} := i_*\mathcal{M}$, which is a sheaf of $\mathscr{D}_{\mathbb{P}^1}$-modules. We can also compute $H^0_{\mathrm{dR}}(M \otimes E^t)$ is a sheaf-theoretical way: we have

$$H = H^0_{\mathrm{dR}}(M \otimes E^t) = \boldsymbol{H}^0(\mathbb{P}^1, {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}} \otimes \mathcal{E}^t)).$$

We are now interested in finding a filtration $F^\bullet_{\mathrm{Del}}(\widetilde{\mathcal{M}} \otimes \mathcal{E}^t)$ which coincides with $F^\bullet\mathcal{M}$ on $\mathbb{C}$, such that $F^\bullet_{\mathrm{irr}} H$ is obtained from $F^\bullet(\widetilde{\mathcal{M}} \otimes \mathcal{E}^t)$ by the standard procedure: the latter filtration can be used to filter the de Rham complex

$$F^\lambda_{\mathrm{Del}} {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}} \otimes \mathcal{E}^t) = \Big\{0 \longrightarrow F^{\lambda+1}_{\mathrm{Del}}(\widetilde{\mathcal{M}} \otimes \mathcal{E}^t) \xrightarrow{\ \nabla\ } \Omega^1_{\mathbb{P}^1} \otimes F^\lambda_{\mathrm{Del}}(\widetilde{\mathcal{M}} \otimes \mathcal{E}^t) \longrightarrow 0\Big\},$$

and by taking the filtration

$$\begin{aligned} &F^\lambda_{\mathrm{Del}} \boldsymbol{H}^0(\mathbb{P}^1, {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}} \otimes \mathcal{E}^t)) \\ &\qquad := \mathrm{image}\big[\boldsymbol{H}^0(\mathbb{P}^1, F^\lambda_{\mathrm{Del}} {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}} \otimes \mathcal{E}^t)) \to \boldsymbol{H}^0(\mathbb{P}^1, {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}} \otimes \mathcal{E}^t))\big]. \end{aligned}$$

Deligne [**Del07**] has considered this question when trying to define an irregular Hodge theory. The problem of extending the filtration is a local problem at infinity, so we will first work in the analytic setting of a disc X with coordinate x and a free $\mathscr{O}_X[1/x]$-module $\widetilde{\mathcal{M}}$ of finite rank with a connection having a regular singularity at $x = 0$ and no other singularity.

As a preliminary, let us consider the localization problem for coherent filtrations. Assume that $\widetilde{\mathcal{M}} = \mathscr{O}_X[1/x] \otimes_{\mathscr{O}_X} \mathcal{N}$ for some regular holonomic $\mathscr{D}_X$-module $\mathcal{N}$ and that $\mathcal{N}$ is endowed with a coherent filtration $F^\bullet\mathcal{N}$. How to obtain a natural coherent filtration on $\widetilde{\mathcal{M}}$ which "extends" $F^\bullet\mathcal{N}$. The point is that localizing each term of the filtration does not lead to a coherent filtration. The following construction is used in Hodge module theory. We consider the Deligne filtration $V^\bullet\mathcal{N}$. The natural morphism

$\mathcal{N} \to \widetilde{\mathcal{M}}$ induces an isomorphism $V^\beta\mathcal{N} \simeq V^\beta\widetilde{\mathcal{M}}$ for $\beta > -1$ and $V^{-1}\widetilde{\mathcal{M}} = x^{-1}V^0\mathcal{N}$. Moreover, $\widetilde{\mathcal{M}} = \mathscr{D}_X \cdot V^{-1}\mathcal{M}$. Then one sets

$$F^pV^\beta\widetilde{\mathcal{M}} = \begin{cases} F^p\mathcal{N} \cap V^\beta\widetilde{\mathcal{M}} & \text{if } \beta > -1, \\ x^{-1}F^pV^0\mathcal{N} & \text{if } \beta = 0. \end{cases}$$

Then we extend this as $F^p\widetilde{\mathcal{M}} := \sum_{j\geqslant 0} \partial_x^j F^{p+j}V^{-1}\widetilde{\mathcal{M}}$. This usually gives the desired coherent filtration (with some assumptions).

The construction of the Deligne filtration has some similarity. We identify $\widetilde{\mathcal{M}}\otimes\mathcal{E}^{1/x}$ with $\widetilde{\mathcal{M}}$ as an $\mathscr{O}_X$-module, but with the twisted $\mathscr{D}_X$-structure, i.e., ∂_x acts as $\partial_x - x^{-2}$. For each $\lambda \in \mathbb{R}$ we set

$$F^\lambda_{\mathrm{Del}}\widetilde{\mathcal{M}} := \sum_{k\geqslant 0} (\partial_x - x^{-2})^k x^{-1} F^{\lceil\lambda\rceil+k} V^{\lambda-\lceil\lambda\rceil}\widetilde{\mathcal{M}}.$$

We note that $\lambda - \lceil\lambda\rceil \in (-1, 0]$.

Theorem. *Assume that $(M, F^\bullet M)$ underlies a mixed Hodge module. Then*

$$F^\lambda_{\mathrm{Del}} H^0_{\mathrm{dR}}(M \otimes E^t) = F^\lambda_{\mathrm{irr}} H^0_{\mathrm{dR}}(M \otimes E^t)$$

and we have the E_1-degeneration property: the map

$$\boldsymbol{H}^0(\mathbb{P}^1, F^\lambda_{\mathrm{Del}}{}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}} \otimes \mathcal{E}^t)) \longrightarrow \boldsymbol{H}^0(\mathbb{P}^1, {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}} \otimes \mathcal{E}^t))$$

is injective.

Idea of proof. Let us fix $\beta \in (-1, 0]$ and consider the filtration $F^{p+\beta}_{\mathrm{Del}}$ indexed by $\mathbb{Z}$. It is enough to check the statement of the theorem for every fixed β. It is then usual ton consider the Rees $\mathbb{C}[z]$-module

$$R_{F^\beta_{\mathrm{Del}}}(\widetilde{\mathcal{M}} \otimes \mathcal{E}^t) := \bigoplus_p F^{p+\beta}_{\mathrm{Del}}(\widetilde{\mathcal{M}} \otimes \mathcal{E}^t) \cdot z^{-p}$$

and its de Rham complex. The E_1-degeneration property is equivalent to the property that the complex $\boldsymbol{H}^0(\mathbb{P}^1, {}^{\mathrm{p}}\mathrm{DR}(R_{F^\beta_{\mathrm{Del}}}(\widetilde{\mathcal{M}} \otimes \mathcal{E}^t)))$ is strict, i.e., $\mathbb{C}[z]$-flat, and $\boldsymbol{H}^k = 0$ for $k \neq 0$.

For that purpose, we introduce the Rees module R_FM and its Laplace transform ${}^F(R_FM)$ that we describe by an integral formula as in §1.2(3) starting from a module that we denote ${}^{\mathscr{F}}\!\mathscr{M}$ and integrating along t. The latter object has a V-filtration along $\tau = 0$ in a suitable sense, and the main idea is to express $R_{F^\beta_{\mathrm{Del}}}(\widetilde{\mathcal{M}} \otimes \mathcal{E}^t)$ in terms of the V-filtration $V^{\beta+\bullet}\,{}^{\mathscr{F}}\!\mathscr{M}$. In particular we find

$$R_{F^\beta_{\mathrm{Del}}}(\widetilde{\mathcal{M}} \otimes \mathcal{E}^t) \simeq V^\beta\,{}^{\mathscr{F}}\!\mathscr{M}/(\tau - z)V^\beta\,{}^{\mathscr{F}}\!\mathscr{M}.$$

The Hodge condition on $(M, F^\bullet M)$ is used by means of the property that $\widetilde{\mathcal{M}} \otimes \mathcal{E}^t$ underlies then a mixed twistor $\mathscr{D}$-module, hence ${}^{\mathscr{F}}\!\mathscr{M}$ also, and its V-filtration along $\tau = 0$ behaves strictly with respect to the projection to the τ-line. $\square$

LECTURE 2

RIGID IRREDUCIBLE HOLONOMIC D-MODULES

Summary. In the second lecture, we focus on rigid irreducible vector bundles with connection on a Zariski open set U of $\mathbb{P}^1$. We review results of Katz and Bloch-Esnault, and give a sketch of proof of the Katz-Arinkin-Deligne algorithm which reduces their study to rank-one objects by applying a sequence of simple functors. We then consider their Hodge properties, firstly in the regular case (Deligne, Simpson, Dettweiler-CS) and then in the possibly irregular case by means of the irregular Hodge filtration considered in the first lecture.

Notation. Now we consider U as an open subset of $\mathbb{P}^1$ and $k : U \hookrightarrow \mathbb{P}^1$ denotes the inclusion. The sheaf $\mathscr{D}_{\mathbb{P}^1}$ (with respect to the Zariski topology) is defined by considering differential operators on affine charts and gluing as usual. The notion of holonomic $\mathscr{D}_{\mathbb{P}^1}$-module is defined as in the previous lecture, in each affine chart.

Let $\mathcal{M}$ be an holonomic $\mathscr{D}_{\mathbb{P}^1}$-module. Then there exists a nonempty finite subset $\Sigma \subset \mathbb{P}^1$ such that $\mathcal{V} = \mathcal{M}_{|U}$ is a free $\mathscr{O}_U$-module with connection. Moreover, $k_*\mathcal{V}$ is also holonomic.

2.1. Local and global numerical data attached to an holonomic $\mathscr{D}_{\mathbb{P}^1}$-module

Our first local numerical invariant attached to $\mathcal{M}$ is $\operatorname{rk}\mathcal{V}$. Let now $c \in \Sigma$ and set $t_c = t - c$. The theorem of Levelt-Turrittin gives the structure of $\widehat{\mathcal{M}}_c := \mathbb{C}[\![t_c]\!] \otimes_{\mathbb{C}[t_c]} \mathcal{M}$. There exists a unique decomposition (called the *slope decomposition*)

$$\widehat{\mathcal{M}}_c = \bigoplus_{\lambda \in \mathbb{Q}+} \widehat{\mathcal{M}}_c^{(\lambda)}$$

such that there exists $n_c \geqslant 1$ such that, setting $t'_c = t_c^{1/n_c}$ (the ramified variable) the $\mathbb{C}(\!(t'_c)\!)$-vector space $\mathbb{C}(\!(t'_c)\!) \otimes_{\mathbb{C}[\![t_c]\!]} \widehat{\mathcal{M}}_c^{(\lambda)}$ has a basis in which the matrix of the connection is block-diagonal, each block takes the form with $\mathrm{d}\varphi \operatorname{Id} + C/t'_c$, where $\varphi \in \mathbb{C}[1/t'_c]$ with pole order equal to $n_c\lambda$, and C is a constant matrix. If $\lambda \neq 0$, we have $\widehat{\mathcal{M}}_c^{(\lambda)} = \mathbb{C}(\!(t_c)\!) \otimes_{\mathbb{C}[\![t_c]\!]} \widehat{\mathcal{M}}_c^{(\lambda)}$.

- The *irregularity number* of $\mathcal{M}$ at c is defined by

$$\operatorname{irr}_c(\mathcal{M}) := \sum_{\lambda \in \mathbb{Q}_+^*} \lambda \dim_{\mathbb{C}(\!(t_c)\!)} \widehat{\mathcal{M}}_c^{(\lambda)}.$$

It only depends on $\mathcal{V}$.

- The *vanishing cycle number* of $\mathcal{M}$ at c is defined by

$\mu_c(\mathcal{M}) := \mathrm{irr}_c(\mathcal{M}) + \mathrm{rk}\,\mathcal{M} + \dim\mathrm{coker}[\partial_t : \widehat{\mathcal{M}}_c^{(0)} \to \widehat{\mathcal{M}}_c^{(0)}] - \dim\ker[\partial_t : \widehat{\mathcal{M}}_c^{(0)} \to \widehat{\mathcal{M}}_c^{(0)}].$

If $\mathcal{M} = k_*\mathcal{V}$, we have $\mu_c(\mathcal{M}) = \mathrm{irr}_c(\mathcal{M}) + \mathrm{rk}\,\mathcal{M}$, while if $\mathcal{M} = k_{!*}\mathcal{V}$, we have

$$\mu_c(\mathcal{M}) = \mathrm{irr}_c(\mathcal{M}) + \mathrm{rk}\,\mathcal{M} - \dim\ker[\partial_t : \widehat{\mathcal{M}}_c^{(0)} \to \widehat{\mathcal{M}}_c^{(0)}].$$

In the following, we will denote $\mu_c(\mathcal{V}) := \mu_c(k_{!*}\mathcal{V})$.

For a $\mathscr{D}_{\mathbb{P}^1}$-module $\mathcal{M}$, the (algebraic) de Rham complex ${}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$ is the complex

$$0 \longrightarrow \mathcal{M} \xrightarrow{\ \nabla\ } \underset{\bullet}{\Omega^1_{\mathbb{P}^1} \otimes \mathcal{M}} \longrightarrow 0.$$

The *rigidity index* of $\mathcal{V}$ is defined by

$$\mathrm{rig}\,\mathcal{V} := -\chi\big(\mathbb{P}^1, {}^{\mathrm{p}}\mathrm{DR}(k_{!*}\,\mathcal{E}nd(\mathcal{V}))\big).$$

It is not changed by tensoring $\mathcal{V}$ by a free rank-one $\mathscr{O}_U$-module with connection.

Proposition. *We have*

$$\mathrm{rig}\,\mathcal{V} = 2(\mathrm{rk}\,\mathcal{V})^2 - \sum_{c\in\Sigma}\mu_c(\mathcal{E}nd(\mathcal{V})).$$

Moreover, $\mathrm{rig}\,\mathcal{V}$ *is even and, if* $\mathcal{V}$ *is irreducible, then*

$$\mathrm{rig}\,\mathcal{V} = 2 - h^0\big(\mathbb{P}^1, {}^{\mathrm{p}}\mathrm{DR}(k_{!*}\,\mathcal{E}nd(\mathcal{V}))\big) \leqslant 2.$$

Sketch of proof. The first statement is obtained by index theorems, due to the fact that $\boldsymbol{H}^i\big(\mathbb{P}^1, {}^{\mathrm{p}}\mathrm{DR}(k_{!*}\,\mathcal{E}nd(\mathcal{V}))\big) = \boldsymbol{H}^i\big(\mathbb{P}^1, {}^{\mathrm{p}}\mathrm{DR}^{\mathrm{an}}(k_{!*}\,\mathcal{E}nd(\mathcal{V}))\big)$, and Kashiwara's formula expressing $\mu_c(\mathcal{M})$ as the dimension of the vanishing cycle space at c of the constructible complex ${}^{\mathrm{p}}\mathrm{DR}^{\mathrm{an}}(k_{!*}\,\mathcal{E}nd(\mathcal{V}))$.

For the second statement, we note that there is a natural nondegenerate symmetric pairing $\mathcal{E}nd(\mathcal{V}) \otimes \mathcal{E}nd(\mathcal{V}) \to \mathscr{O}_U$ which is compatible with the connections: it is given by $(\varphi,\psi) \mapsto \mathrm{tr}(\varphi\psi)$. As a consequence, $\mathcal{E}nd(\mathcal{V}) \simeq \mathcal{E}nd(\mathcal{V})^\vee$, and thus $k_{!*}\,\mathcal{E}nd(\mathcal{V}) \simeq (k_{!*}\,\mathcal{E}nd(\mathcal{V}))^\vee$. Then, by Poincaré-Verdier duality, $\boldsymbol{H}^{-1}\big(\mathbb{P}^1, {}^{\mathrm{p}}\mathrm{DR}(k_{!*}\,\mathcal{E}nd(\mathcal{V}))\big) \simeq \boldsymbol{H}^{1}\big(\mathbb{P}^1, {}^{\mathrm{p}}\mathrm{DR}(k_{!*}\,\mathcal{E}nd(\mathcal{V}))\big)$ and $\boldsymbol{H}^{0}\big(\mathbb{P}^1, {}^{\mathrm{p}}\mathrm{DR}(k_{!*}\,\mathcal{E}nd(\mathcal{V}))\big)$ is equipped with a nondegenerate skew-symmetric pairing.

Lastly, we have an exact sequence

$$0 \longrightarrow k_{!*}\,\mathcal{E}nd(\mathcal{V}) \longrightarrow k_*\,\mathcal{E}nd(\mathcal{V}) \longrightarrow \mathcal{C} \longrightarrow 0$$

with $\mathcal{C}$ supported on Σ. Since $\boldsymbol{H}^{-1}\big(\mathbb{P}^1, {}^{\mathrm{p}}\mathrm{DR}(\mathcal{C})\big) = 0$, we obtain

$$\begin{aligned}\boldsymbol{H}^{-1}\big(\mathbb{P}^1, {}^{\mathrm{p}}\mathrm{DR}(k_{!*}\,\mathcal{E}nd(\mathcal{V}))\big) &\simeq \boldsymbol{H}^{-1}\big(\mathbb{P}^1, {}^{\mathrm{p}}\mathrm{DR}(k_*\,\mathcal{E}nd(\mathcal{V}))\big)\\ &= H^0(U, \mathrm{DR}(\mathcal{E}nd(\mathcal{V}))) = \mathrm{End}^\nabla(\mathcal{V}).\end{aligned}$$

If $\mathcal{V}$ is irreducible, the later space is one-dimensional. □

Definition. We say that an irreducible $\mathcal{V}$ is *rigid* if, for any $\mathcal{V}'$ such that $\widehat{k_*\mathcal{V}'_c} \simeq \widehat{k_*\mathcal{V}_c}$ for all $c \in \Sigma$, we have $\mathcal{V}' \simeq \mathcal{V}$.

***Theorem* (Katz [Kat96], Bloch-Esnault [BE04]).** *An irreducible* $\mathcal{V}$ *is rigid iff* $\mathrm{rig}(\mathcal{V}) = 2$, *i.e., iff* $\boldsymbol{H}^0\big(\mathbb{P}^1, {}^{\mathrm{p}}\mathrm{DR}(k_{!*}\,\mathcal{E}nd(\mathcal{V}))\big) = 0$. □

The hypergeometric systems as in the first lecture are irreducible and rigid ([**BH89, Kat96**]). We will indicate the proof of the following theorem.

***Theorem* (Katz [Kat96], Deligne [Del06], Arinkin [Ari10]).** *Let U be a Zariski open set in $\mathbb{P}^1$ and let $\mathcal{V}$ be an irreducible $\mathscr{O}_U$-module with connection. Assume that $\operatorname{rk}\mathcal{V} \geqslant 2$ and $\operatorname{rig}(\mathcal{V}) = 2$. Then, after tensoring $\mathcal{V}$ by a suitable rank-one $\mathscr{O}_U$-module with connection and choosing charts so that $\mathbb{P}^1 = \mathbb{A}^1_t \cup \{\infty\}$ in a suitable way, one of both possibilities occurs, and the first one always if $\mathcal{V}$ has regular singularities at Σ:*

(1) *There exists $\chi \in \mathbb{C}^*$ such that $\operatorname{rk}\mathrm{MC}_\chi(\mathcal{V}) < \operatorname{rk}\mathcal{V}$,*
(2) $1 \leqslant \operatorname{rk}{}^F(j_{!*}\mathcal{V}) < \operatorname{rk}\mathcal{V}$.

2.2. Convolutions

We fix charts $\mathbb{P}^1 = \mathbb{A}^1_t \cup \{\infty\}$ and consider Fourier transform with respect to it. Let P denote the sub-category of holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-modules N such that ${}^FN$ and ${}^FN^\vee$ are $\mathbb{C}[\tau]$-flat. In an equivalent way, N is in P if and only if ${}^FN$ has neither submodule nor quotient module which has $\mathbb{C}[\tau]$-torsion, that is, ${}^FN = j_{!*}({}^FN_{|U'})$ if U' is any Zariski open subset of $\mathbb{A}^1_\tau$ not containing any singularity of ${}^FN$. In particular, any irreducible N not equal to $(\mathbb{C}[t], \mathrm{d})$ is in P.

Definition. Given holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-modules M, N with N in P, their convolutions $\star_*$, $\star_!$, $\star_{\mathrm{mid}}$ are defined by means of the Laplace transformation as follows.

$$\begin{aligned}
{}^F(M \star_* N) &= {}^FM \otimes_{\mathbb{C}[\tau]} {}^FN,\\
{}^F(M \star_! N) &= (({}^FM)^\vee \otimes_{\mathbb{C}[\tau]} ({}^FN)^\vee)^\vee,\\
M \star_{\mathrm{mid}} N &= \operatorname{image}\big[M \star_! N \longrightarrow M \star_* N\big].
\end{aligned}$$

Remark. Let us explain the morphism $M \star_! N \to M \star_* N$. It amounts to understanding (in general) the morphism $({}^FM \otimes^L {}^FN)^\vee \to ({}^FM)^\vee \otimes^L ({}^FN)^\vee$. Recall that the (derived) tensor product is obtained from the external product ${}^FM \boxtimes {}^FN$ by applying the functor δ^+, where $\delta : \mathbb{A}^1 \hookrightarrow \mathbb{A}^1 \times \mathbb{A}^1$ is the diagonal inclusion. We are thus reduced to showing, for holonomic modules M, N on $\mathbb{A}^1$ (we apply this to ${}^FM, {}^FN$) and $\boldsymbol{M}$ on $\mathbb{A}^1 \times \mathbb{A}^1$:

- $(M \boxtimes N)^\vee \simeq M^\vee \boxtimes N^\vee$. This is straightforward.
- There is a natural morphism $\delta^+(\boldsymbol{M}^\vee) \to (\delta^+\boldsymbol{M})^\vee$. Let Δ be the diagonal. Then $\delta^+(\boldsymbol{M}^\vee)$ is the complex

$$\boldsymbol{M}^\vee \longrightarrow \underset{\bullet}{\boldsymbol{M}^\vee(*\Delta)}.$$

On the other hand, $(\delta^+\boldsymbol{M})^\vee$ is the complex

$$\underset{\bullet}{\boldsymbol{M}^\vee(!\Delta)} \longrightarrow \boldsymbol{M}^\vee,$$

with $\boldsymbol{M}^\vee(!\Delta) := (\boldsymbol{M}(*\Delta))^\vee$. The natural morphism $\boldsymbol{M}^\vee(!\Delta) \to \boldsymbol{M}^\vee(*\Delta)$ induces the desired morphism of complexes.

Some properties.

(1) For $N \in \mathsf{P}$, $\star_* N$ and $\star_! N$ are exact functors on $\mathrm{Mod}_{\mathrm{hol}}(\mathbb{C}[t]\langle\partial_t\rangle)$. On the other hand, $\star_{\mathrm{mid}} N$ preserves injective morphisms as well as surjective morphisms.

(2) Say that an irreducible N is *constant* if it is equal to $(\mathbb{C}[t], \mathrm{d} + a\mathrm{d}t)$ $(a \in \mathbb{C}^*)$. Then ${}^FN$ is supported on a point and $N \notin \mathsf{P}$.

(3) For $N_1, N_2 \in \mathsf{P}$ such that $N_1 \star_{\mathrm{mid}} N_2$ is also in P, we have associativity $M \star_{\mathrm{mid}} (N_1 \star_{\mathrm{mid}} N_2) = (M \star_{\mathrm{mid}} N_1) \star_{\mathrm{mid}} N_2$.

The Kummer module with eigenvalue $\chi \in \mathbb{C}^* \smallsetminus \{1\}$ is $K_\chi := (\mathbb{C}[t, t^{-1}], \mathrm{d} + \frac{1}{2\pi i} \log \chi \cdot \mathrm{d}t/t)$. It is irreducible and non-constant, hence belongs to P.

Definition. For $\chi \in \mathbb{C}^* \smallsetminus \{1\}$, the functor MC_χ is the middle convolution with K_χ.

Some properties. If $\chi\chi' \neq 1$, then $\mathrm{MC}_{\chi\chi'} = \mathrm{MC}_\chi \circ \mathrm{MC}_{\chi'}$. It satisfies $\mathrm{MC}_\chi \circ \mathrm{MC}_{1/\chi} = \mathrm{Id}$ on non-constant irreducible holonomic modules. In particular, it sends non-constant irreducible holonomic modules to non-constant irreducible holonomic modules.

2.3. Consequence of the theorem: the algorithm of Katz-Deligne-Arinkin

The theorem is completed by the following property. Let us fix charts as above and let V be a free $\mathscr{O}(U)$-module with connection. Set $M = j_{!*}V$.

***Theorem* (Katz [Kat96], Bloch-Esnault [BE04]).** *If M is irreducible and non-constant, we have $\operatorname{rig}{}^F M = \operatorname{rig} M$.*

As a consequence, one also obtains that, for M is irreducible and non-constant and for $\chi \in \mathbb{C}^* \smallsetminus \{1\}$, we have $\operatorname{rig} \mathrm{MC}_\chi(M) = \operatorname{rig}(M)$.

Corollary. *An irreducible bundle $\mathcal{V}$ with connection on $U = \mathbb{P}^1 \smallsetminus \Sigma$ is rigid if and only if it can be reduced to a rank-one bundle with connection on some Zariski open set of $\mathbb{P}^1$ by a sequence of the following operations:*

- *Tensoring with a rank-one bundle with connection,*
- *Pull-back by an homography of $\mathbb{P}^1$,*
- *Fourier transformation.*

If it has regular singularities only, then after an initial homography, the a sequence of the following operations (which preserve regularity) is enough:

- *Tensoring with a rank-one bundle with regular singular connection,*
- *middle convolution MC_χ ($\chi \in \mathbb{C}^* \smallsetminus \{1\}$).*

2.4. Sketch of proof of the theorem

The first step is to find the right rank-one bundle with connection to tensor $\mathcal{V}$. It is based on the following lemma.

Lemma. *Let $\widehat{\mathcal{V}}$ be a finite-dimensional $\mathbb{C}(\!(t)\!)$-vector space with connection, and let $(\widehat{\mathcal{V}}_a)_{a \in A}$ its irreducible components. Let $a_o \in A$ be such that the function*

$$a \longmapsto \frac{\mu(\mathcal{H}om(\widehat{\mathcal{V}}_a, \widehat{\mathcal{V}}))}{\operatorname{rk} \widehat{\mathcal{V}}_a}$$

achieves its minimum for $a = a_o$. Then, setting $\widehat{\mathcal{L}} := \widehat{\mathcal{V}}_{a_o}$, we have

$$\mu(\mathcal{E}nd(\widehat{\mathcal{V}})) \geqslant \frac{\operatorname{rk} \widehat{\mathcal{V}}}{\operatorname{rk} \widehat{\mathcal{L}}} \cdot \mu(\mathcal{H}om(\widehat{\mathcal{L}}, \widehat{\mathcal{V}})).$$

Moreover, if $\operatorname{rk}\widehat{\mathcal{L}} \geqslant 2$*, we have* $\mu(\mathcal{H}om(\widehat{\mathcal{L}},\widehat{\mathcal{V}})) \geqslant \operatorname{rk}\widehat{\mathcal{L}}\cdot\operatorname{rk}\widehat{\mathcal{V}}$*, hence*

$$\mu(\mathcal{E}nd(\widehat{\mathcal{V}})) \geqslant (\operatorname{rk}\widehat{\mathcal{V}})^2.$$

Proof. In an exact sequence $0 \to \widehat{\mathcal{V}}' \to \widehat{\mathcal{V}} \to \widehat{\mathcal{V}}'' \to 0$ of finite-dimensional $\mathbb{C}(\!(t)\!)$-vector spaces with connection, rk and irr behave in an additive way, while the sequence of $\ker\partial_t$ can be non-exact on the right. As a consequence, $\mu(\widehat{\mathcal{V}}) \geqslant \mu(\widehat{\mathcal{V}}') + \mu(\widehat{\mathcal{V}}'')$. Then

$$\mu(\mathcal{E}nd(\widehat{\mathcal{V}})) \geqslant \sum_{a\in A}\mu(\mathcal{H}om(\widehat{\mathcal{V}}_a,\widehat{\mathcal{V}})) \geqslant \frac{\sum_a \operatorname{rk}\widehat{\mathcal{V}}_a}{\operatorname{rk}\widehat{\mathcal{L}}}\cdot\mu(\mathcal{H}om(\widehat{\mathcal{L}},\widehat{\mathcal{V}}))$$

by the choice of a_o.

For the second part of the proof it is enough to check, by upper additivity of μ, that $\mu(\mathcal{H}om(\widehat{\mathcal{L}},\widehat{\mathcal{V}}_a)) \geqslant \operatorname{rk}\widehat{\mathcal{L}}\cdot\operatorname{rk}\widehat{\mathcal{V}}_a$, and this can be done by an explicit computation. □

The component $\widehat{\mathcal{L}}$ has rank one if and only if it is non-ramified. This occurs automatically if we assume that $\widehat{\mathcal{V}}$ has a regular singularity, or if it is non-ramified. The rigidity assumption also helps in controlling the rank of $\widehat{\mathcal{L}}$ at each singular point in Σ.

Lemma. *Assume that* $\mathcal{V}$ *is irreducible and rigid. Then there is at most one* $c\in\Sigma$ *such that* $\operatorname{rk}\widehat{\mathcal{L}}_c > 1$*.*

Proof. We have $\operatorname{rig}\mathcal{V} = 2(\operatorname{rk}\mathcal{V})^2 - \sum_{c\in\Sigma}\mu_c(\mathcal{E}nd(\mathcal{V}))$. If there are two $c\in\Sigma$ with $\operatorname{rk}\widehat{\mathcal{L}}_c > 1$, then $\operatorname{rig}\mathcal{V} \leqslant 0$ by the second par of the above lemma. □

The proof of the theorem goes as follows.

- If for any singularity $c\in\Sigma$ the space $\widehat{\mathcal{L}}_c$ has rank one, then one chooses ∞ as point in U, so that $\Sigma\subset\mathbb{A}^1_t$. One checks that any rank-one bundle with connection $\mathcal{L}'$ on $\mathbb{A}^1\smallsetminus\Sigma$ such that $\widehat{\mathcal{L}}'_c \simeq \widehat{\mathcal{L}}_c$ for every $c\in\Sigma$ and regular at infinity has a non-trivial monodromy $\chi\in\mathbb{C}^*\smallsetminus\{1\}$ at infinity. Let us choose one such $\mathcal{L}'$. Then one checks that $\operatorname{rk}\mathrm{MC}_\chi(\mathcal{V}\otimes\mathcal{L}'^\vee) < \operatorname{rk}\mathcal{V}$.
- If there is $c\in\Sigma$ such that $\operatorname{rk}\widehat{\mathcal{L}}_c \geqslant 2$, then one applies an homography to assume that such a point is at infinity. One can find rank-one bundle with connection $\mathcal{L}'$ on U such that $\widehat{\mathcal{L}}'_c \simeq \widehat{\mathcal{L}}_c$ for every $c\in\Sigma\smallsetminus\{\infty\}$ and that the slope $\widehat{\mathcal{L}}'^\vee_\infty\otimes\widehat{\mathcal{L}}_\infty$ is not an integer. Then one shows that $\operatorname{rk}{}^F(j_{!*}(\mathcal{V}\otimes\mathcal{L}'^\vee)) < \operatorname{rk}\mathcal{V}$.

2.5. Hodge filtration

Assume that $\mathcal{V}$ has rank one and regular singularities only. If moreover the monodromy $\lambda_c\in\mathbb{C}^*$ around $c\in\Sigma$ has absolute value equal to one, then the local system $\ker\nabla^{\mathrm{an}}$ is unitary. How much does such a property extend to a rigid irreducible $\mathcal{V}$ with only regular singularities at Σ?

***Theorem* (Deligne [Del87], Simpson [Sim90]).** *Assume that* $\mathcal{V}$ *is irreducible and rigid, and that the eigenvalues of the monodromy operators around any* $c\in\Sigma$ *have absolute value equal to one. Then there exists a variation of polarizable Hodge structure on* $\mathcal{V}^{\mathrm{an}}$*. This variation is unique up to a shift of the filtration.*

In other words, there exist a filtration of $\mathcal{V}$ by sub-bundles $F^p\mathcal{V}$ corresponding to a Hodge decomposition of the associated C^∞ bundle, and it admits a polarization. A natural question is then to compute the rank of the bundles $F^p\mathcal{V}$. The algorithm of Katz can serve to compute inductively these numbers, if one is able to compute the behaviour by the two basic operations: twist by a unitary rank-one bundle, and MC_χ for $\chi \in \mathbb{C}^* \smallsetminus \{1\}$ and $|\chi| = 1$.

There is a basic set of Hodge invariants (integers), among which the rank of each Hodge bundle $\mathrm{gr}_F^p\mathcal{V}$, whose behaviour can be computed along the Katz algorithm. This leads to examples of computation of the Hodge ranks for interesting examples of rigid irreducible vector bundles with connection. E.g., Examples with monodromy group dense in the exceptional Lie group G_2 [**DS13**] (also considered by group-theoretic considerations applied to Mumford-Tate groups in [**KP16**]), and hypergeometric systems with regular singularities [**Fed15**].

Local invariants. Let X be a disc with coordinate x centered at one singular point. At the end of the first lecture, we have already considered the bundles $F^p\mathcal{V}^\beta$ for $\beta \in (-1, 0]$. The theory of Schmid enables us to define invariants (nearby cycles)

$$\nu^p_{e^{-2\pi i\beta}} := \dim F^p\mathcal{V}^\beta / (F^{p+1}\mathcal{V}^\beta + F^p\mathcal{V}^{>\beta}),$$

whose sum over β is the rank of the Hodge bundles $h^p(\mathcal{V}) := \mathrm{rk}\,\mathrm{gr}_F^p\mathcal{V}$. These invariants can be refined by taking into account the action of monodromy on $\mathcal{V}^\beta/\mathcal{V}^{>\beta}$. If $\beta \neq 0$, we also denote them by $\mu^p_{e^{-2\pi i\beta}}$ (vanishing cycles), and for $\beta = 0$ we also define μ^p_1, as well as the refined variant taking into account the monodromy.

Global invariants. Let us now go back to the global setting. At each singular point, we can extend the Hodge bundle $F^p\mathcal{V}$ by gluing with the locally analytically defined bundle $F^p\mathcal{V}^0$. The degree of the quotient bundle $F^p\mathcal{V}^0/F^{p+1}\mathcal{V}^0$ is denoted by $\delta^p(\mathcal{V})$.

***Theorem* (Dettweiler-CS [DS13]).** *The set of local and global invariants can be controlled explicitly along the Katz algorithm.*

2.6. Irregular Hodge filtration

How does the previous result extends in the general case of rigid irreducible $\mathcal{V}$ with possibly irregular singularities. The notion of Deligne Hodge filtration, that we now call irregular Hodge filtration, can be extended to cases more general than that treated in the first lecture. One finds a generalization of the theorem of Deligne and Simpson for rigid irreducible $\mathcal{V}$'s on $U = \mathbb{P}^1 \smallsetminus \Sigma$.

Theorem. *Assume that $\mathcal{V}$ is irreducible and rigid, and that the eigenvalues of the* formal *monodromy operators around any $c \in \Sigma$ have absolute value equal to one. Then there exists a variation of polarizable irregular Hodge structure on $\mathcal{V}^{\mathrm{an}}$. This variation is unique up to a shift of the filtration.*

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