Wild twistor $\mathscr{D}$-modules

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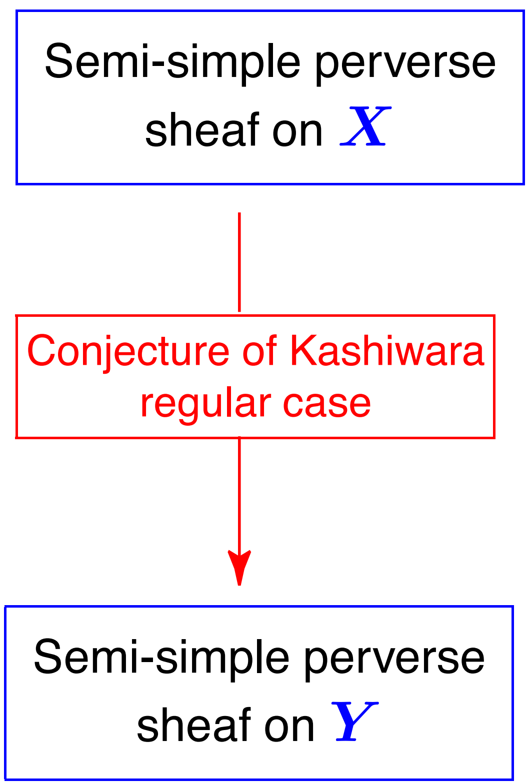

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↓ decomposition theorem (C.S.)

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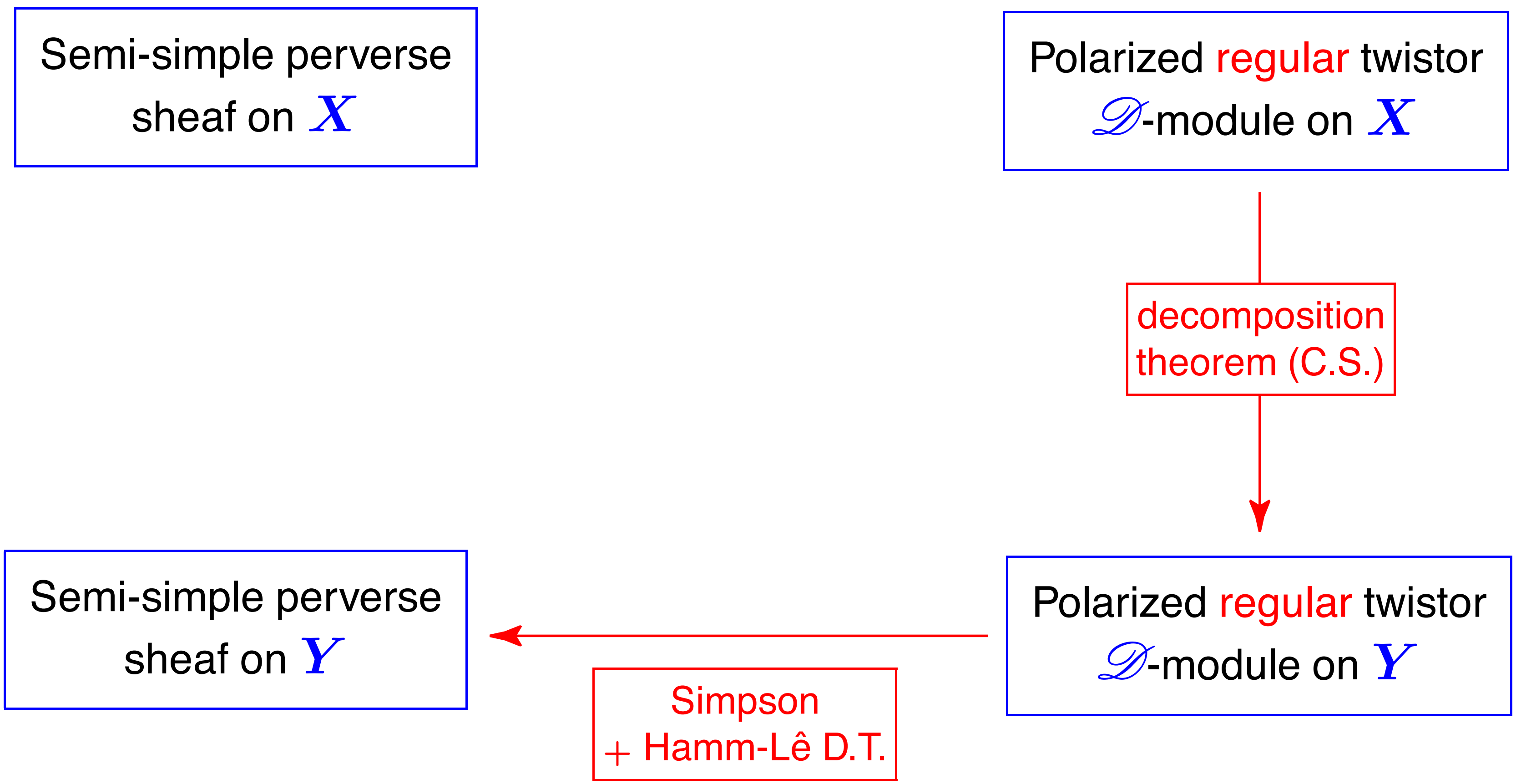

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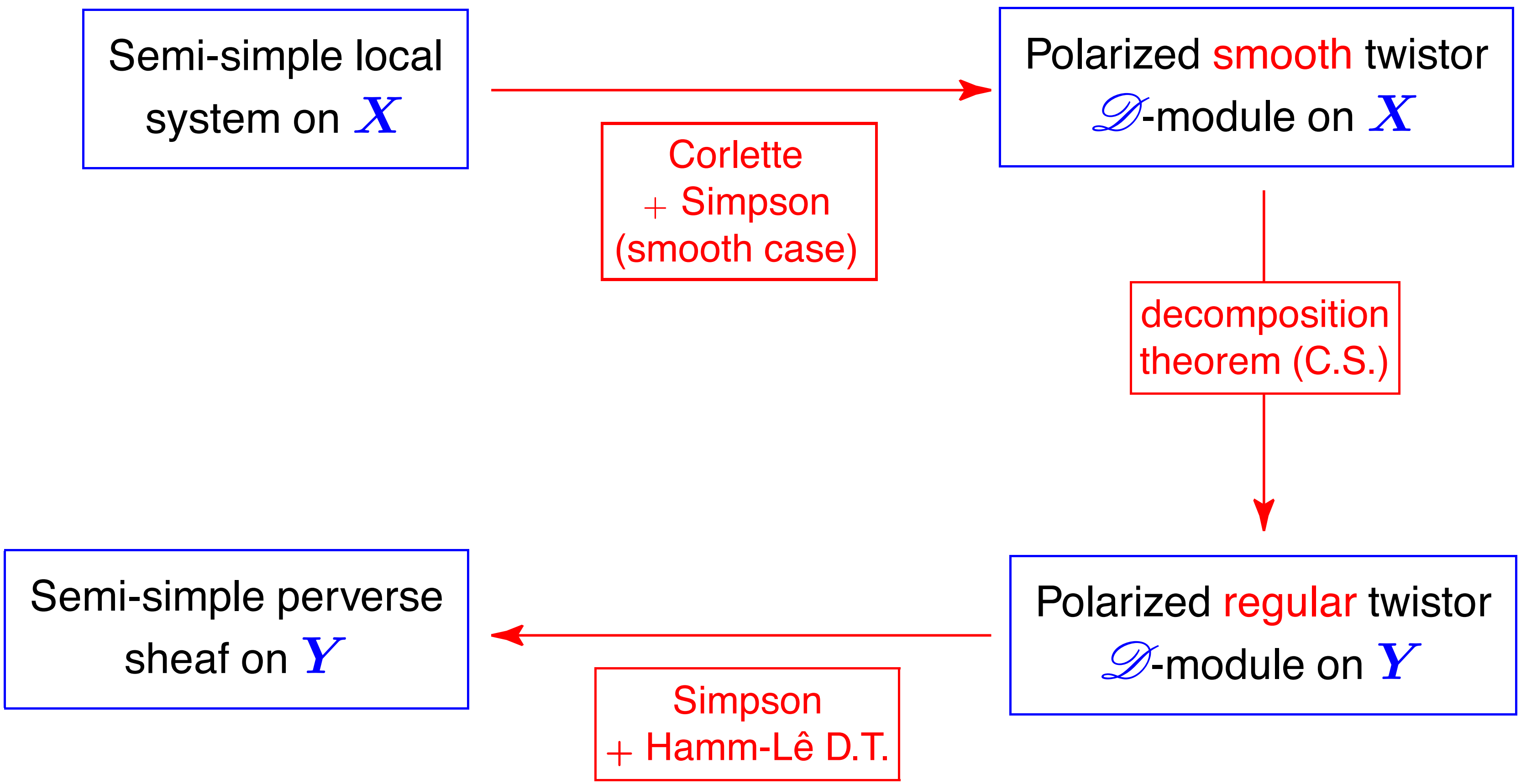

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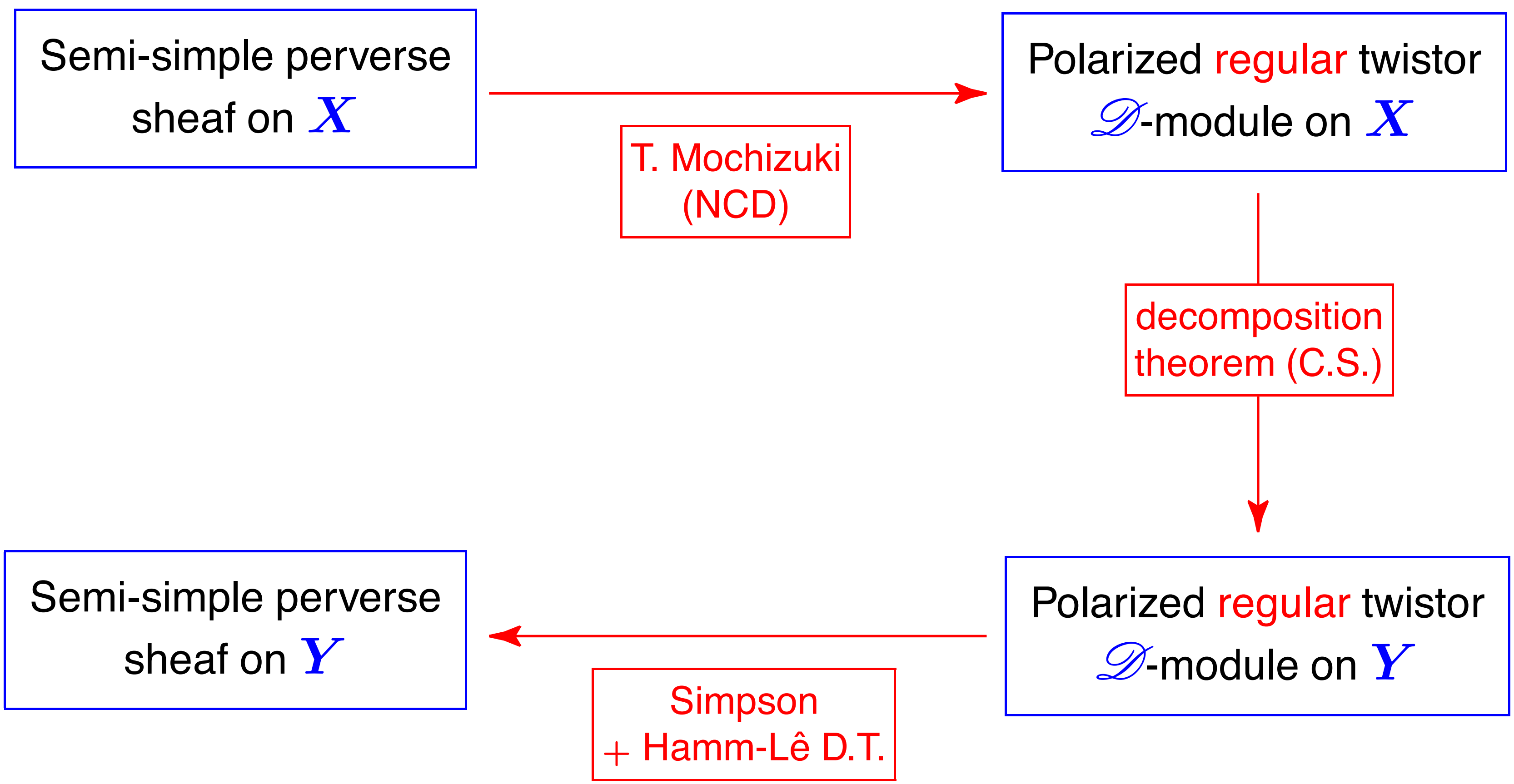

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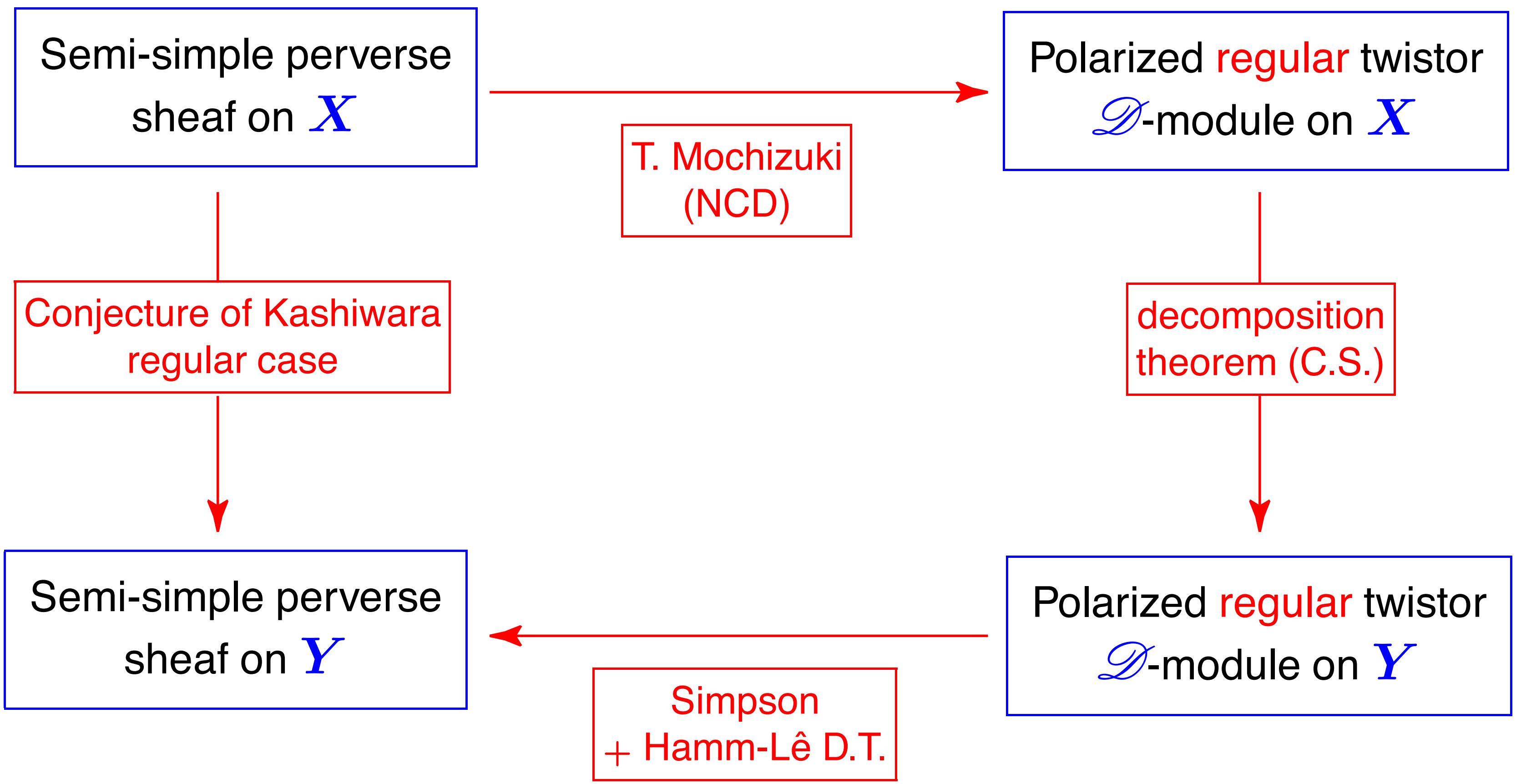

Twistor structures

(C. Simpson)

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Positivity	Positivity on $\Gamma(\mathbb{P}^1, \mathscr{H})$

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Positivity	Positivity on $\Gamma(\mathbb{P}^1, \mathscr{H})$
Tate twist (k), $k \in \mathbb{Z}$	$\otimes \mathscr{O}_{\mathbb{P}^1}(-2k) \quad (k \in \frac{1}{2}\mathbb{Z})$

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- Holomorphic vector bundle on $\mathbb{P}^1$

Twistor structures

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- $\mathscr{H}', \mathscr{H}''$ holomorphic on $\mathbb{A}^1$, "gluing":
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- Tate twist $(\mathscr{H}', \mathscr{H}'', C)(k) = (\mathscr{H}', \mathscr{H}'', (iz)^{-2k} C)$.
- $\mathscr{H}' = \mathscr{H}''$ and existence of a global frame ε of $\mathscr{H}'$
  such that, on S, $C(\varepsilon_i, \overline{\varepsilon_j}) = \delta_{ij}$.

Variation of twistor structures

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- ***Flat*** holomorphic relative connections ∇', ∇'':
  $\mathscr{H}'^{(\prime\prime)} \longrightarrow \frac{1}{z}\Omega^1_{\mathscr{X}/\mathbb{A}^1} \otimes \mathscr{H}'^{(\prime\prime)}$, compatibility with C:
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$\overset{z=0}{\longleftrightarrow}$ holom. vector bundle on X with a Higgs field θ, and Hermitian metric h which is ***harmonic***.

Twistor $\mathscr{D}$-modules

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Twistor $\mathscr{D}$-modules

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- Nevertheless, the first results for twistor $\mathscr{D}$-modules are obtained in the regular case.

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Semi-simple perverse sheaf on X

Polarized regular twistor $\mathscr{D}$-module on X

↓ decomposition theorem (C.S.)

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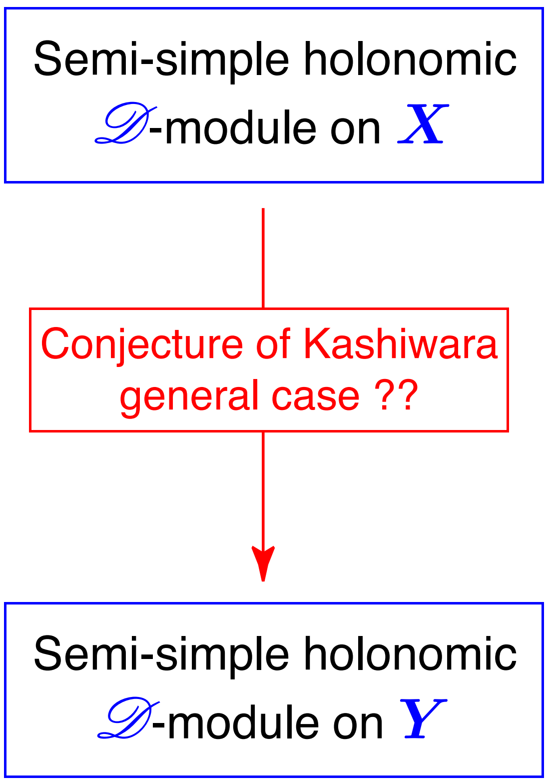

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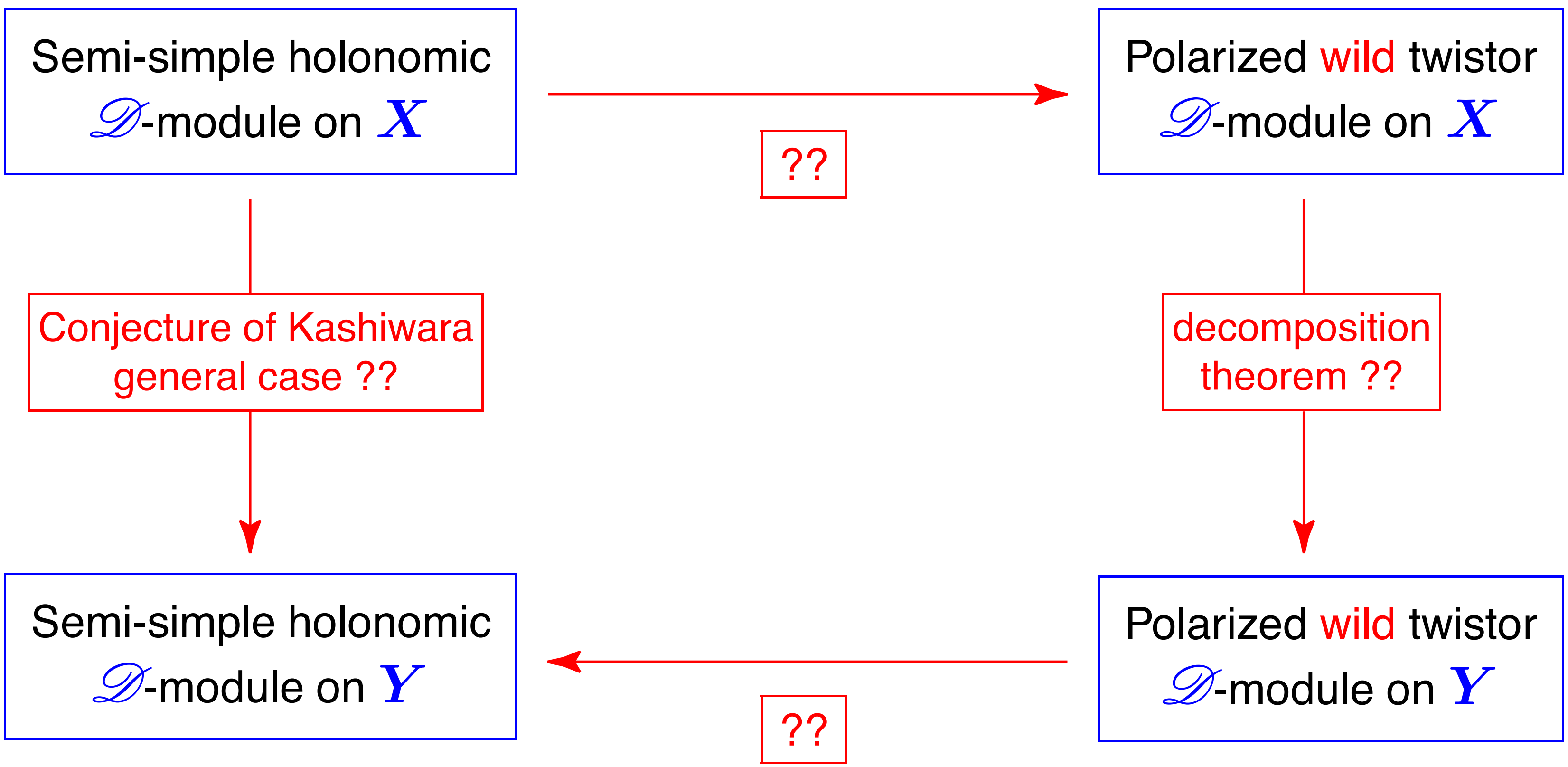

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- $M \longleftrightarrow \widehat{M} +$ Stokes structure.

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REMARK: This is analogous to part of the Nilpotent Orbit Theorem (Schmid).

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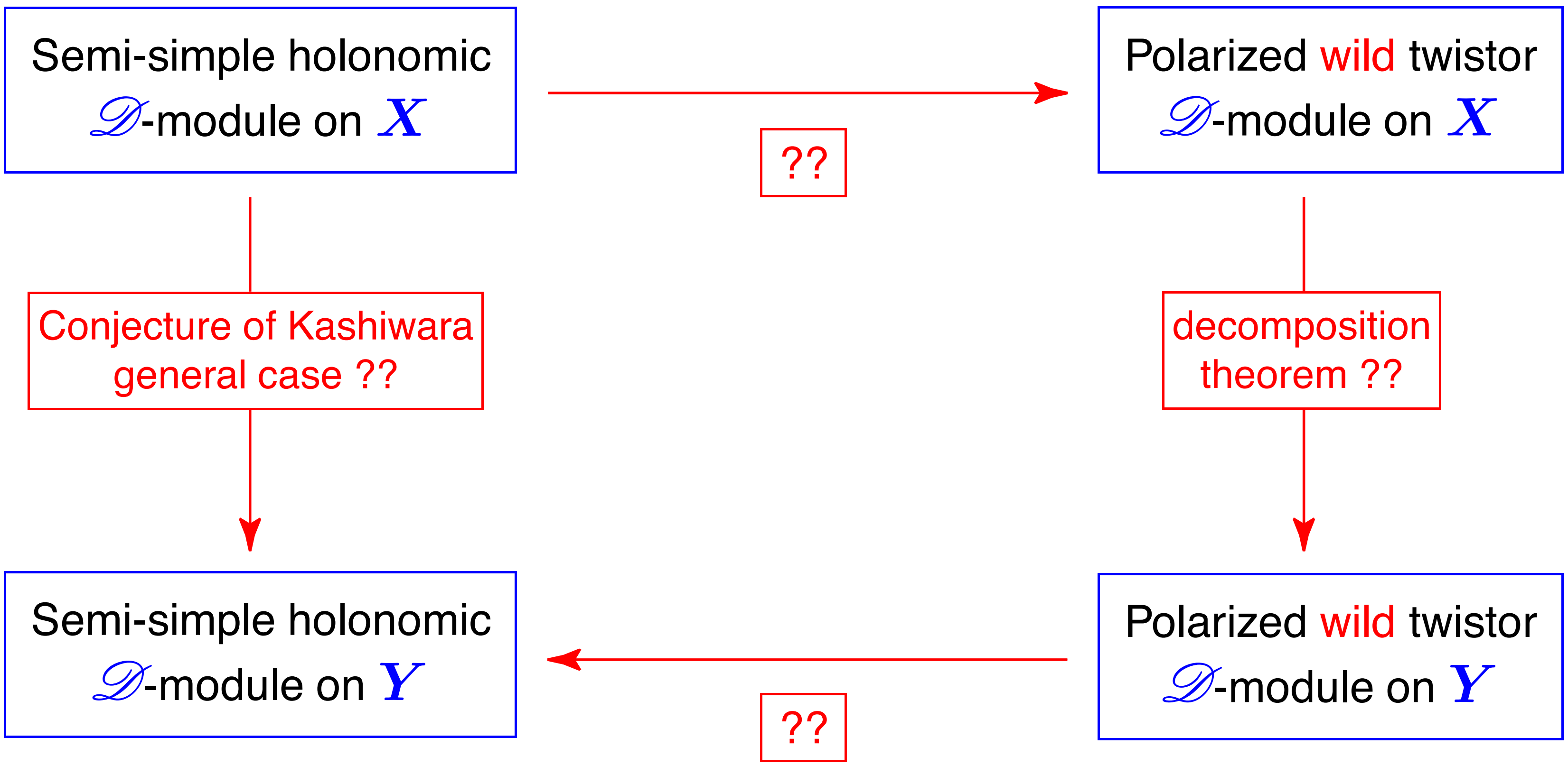
$\rightarrow \mathrm{MT}^{(\mathrm{s})}_{\leqslant d}(X, w)$ ('s' is for 'sauvage', *i.e.*, 'wild').

Wild twistor $\mathscr{D}$-modules



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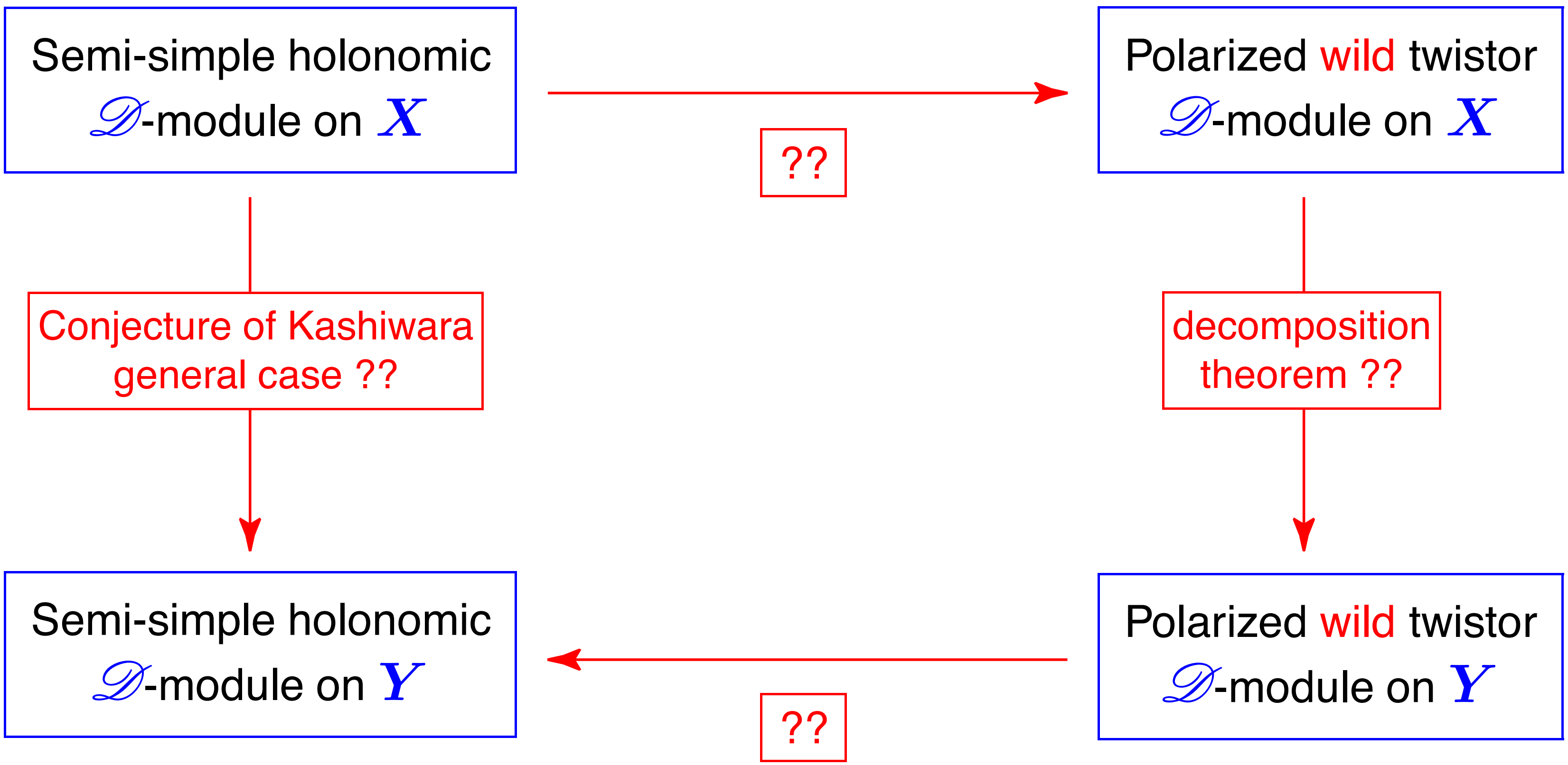

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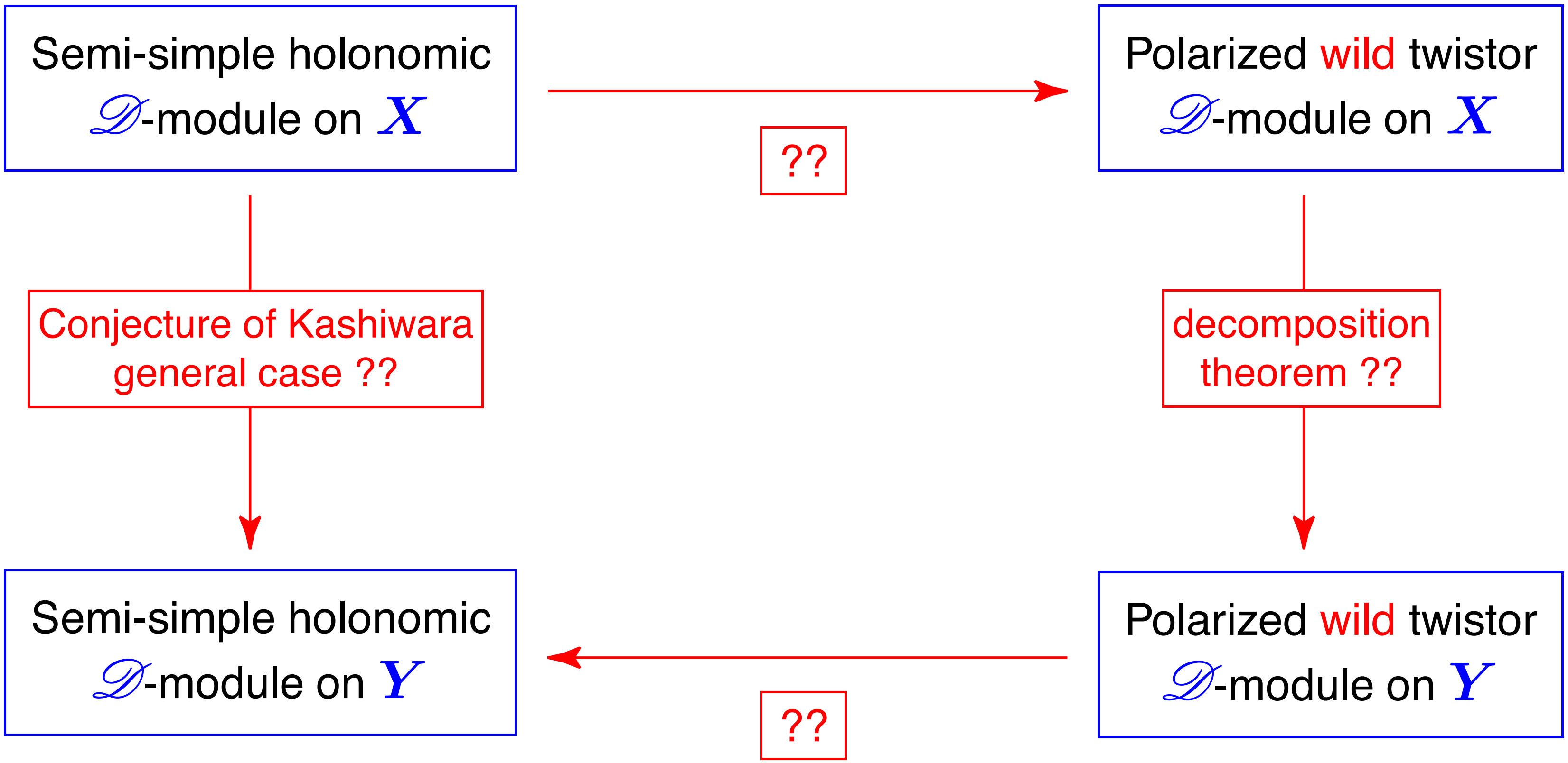

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- $f : X \longrightarrow \mathbb{A}^1, \quad (x, y) \longmapsto y$,
- $M = \mathscr{O}_{\mathbb{A}^2}[1/y]$, $\nabla = d + d(x/y) = d + \dfrac{dx}{y} - \dfrac{x\,dy}{y^2}$.

Then $\psi_f^{\mathrm{Del}} M = 0 \qquad (M = \mathscr{E}^{x/y})$.

Solution: Consider $\psi_g^{\mathrm{Del}} M$ for various g in order to recover information on $\widehat{M}$.

Irregular nearby cycles, after Deligne

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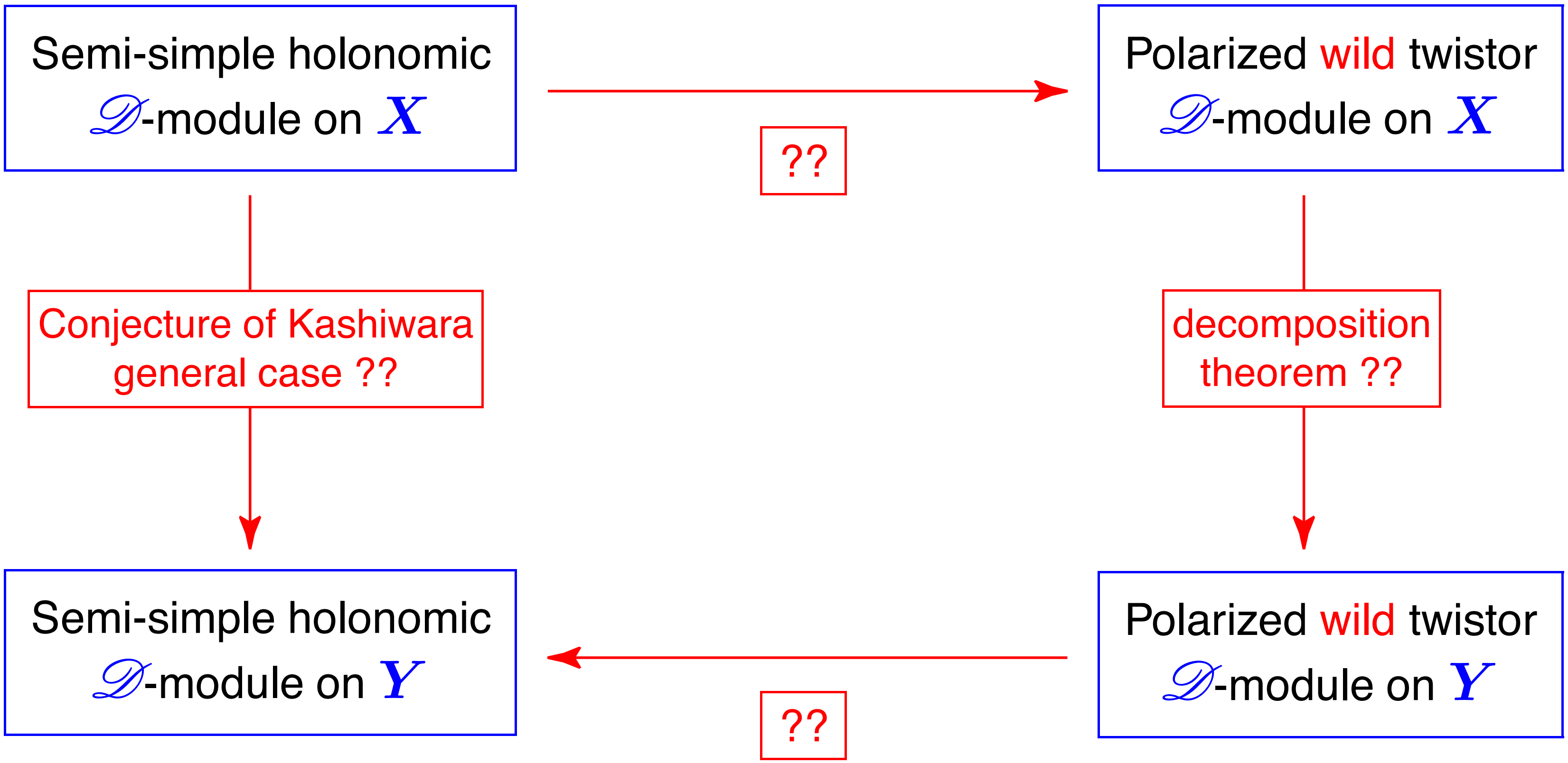
$\longrightarrow \mathrm{MT}^{(\mathrm{s})}_{\leqslant d}(X, w)$ (‘s’ is for ‘sauvage’, *i.e.*, ‘wild’).

Wild twistor $\mathscr{D}$-modules



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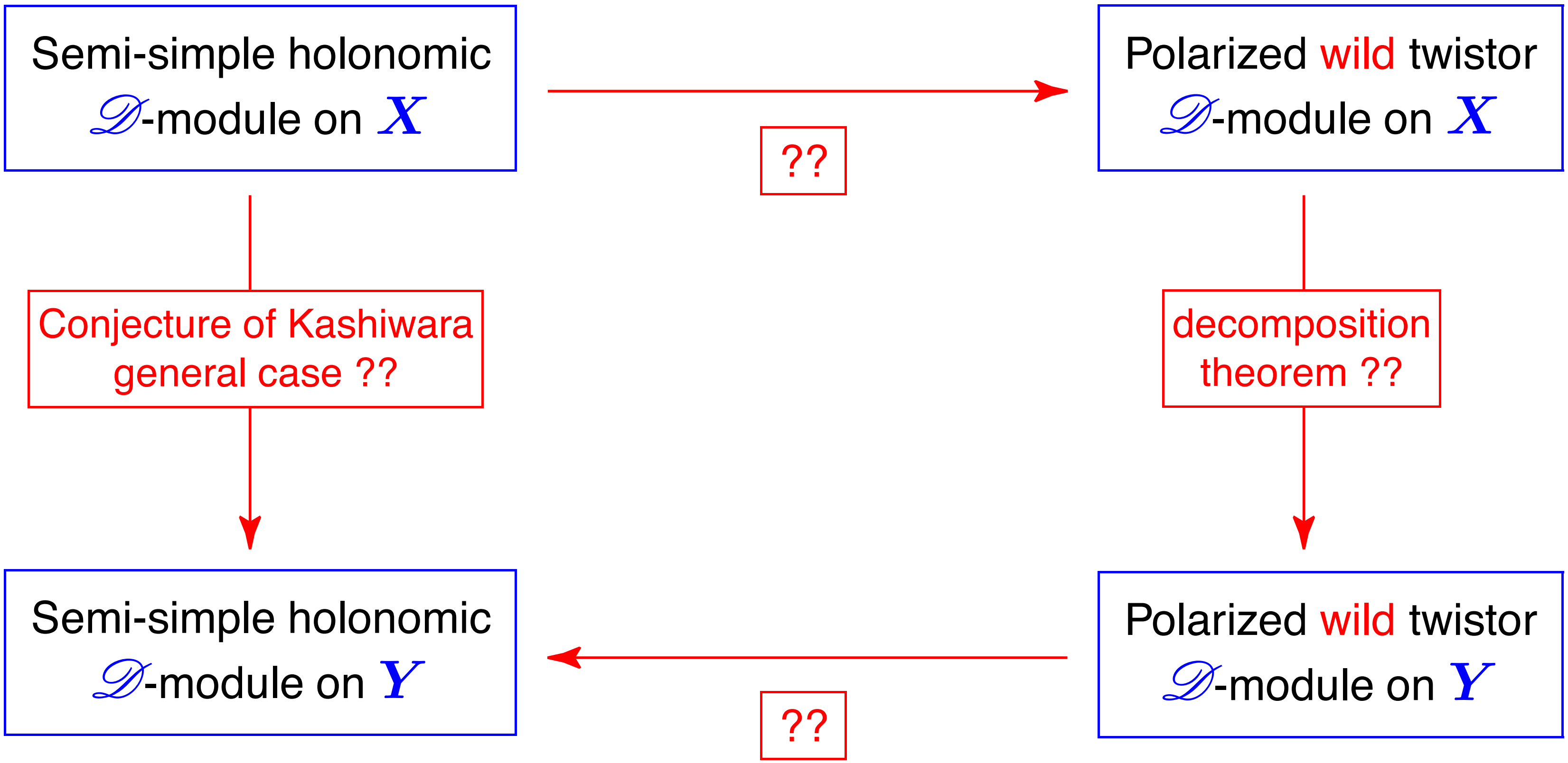
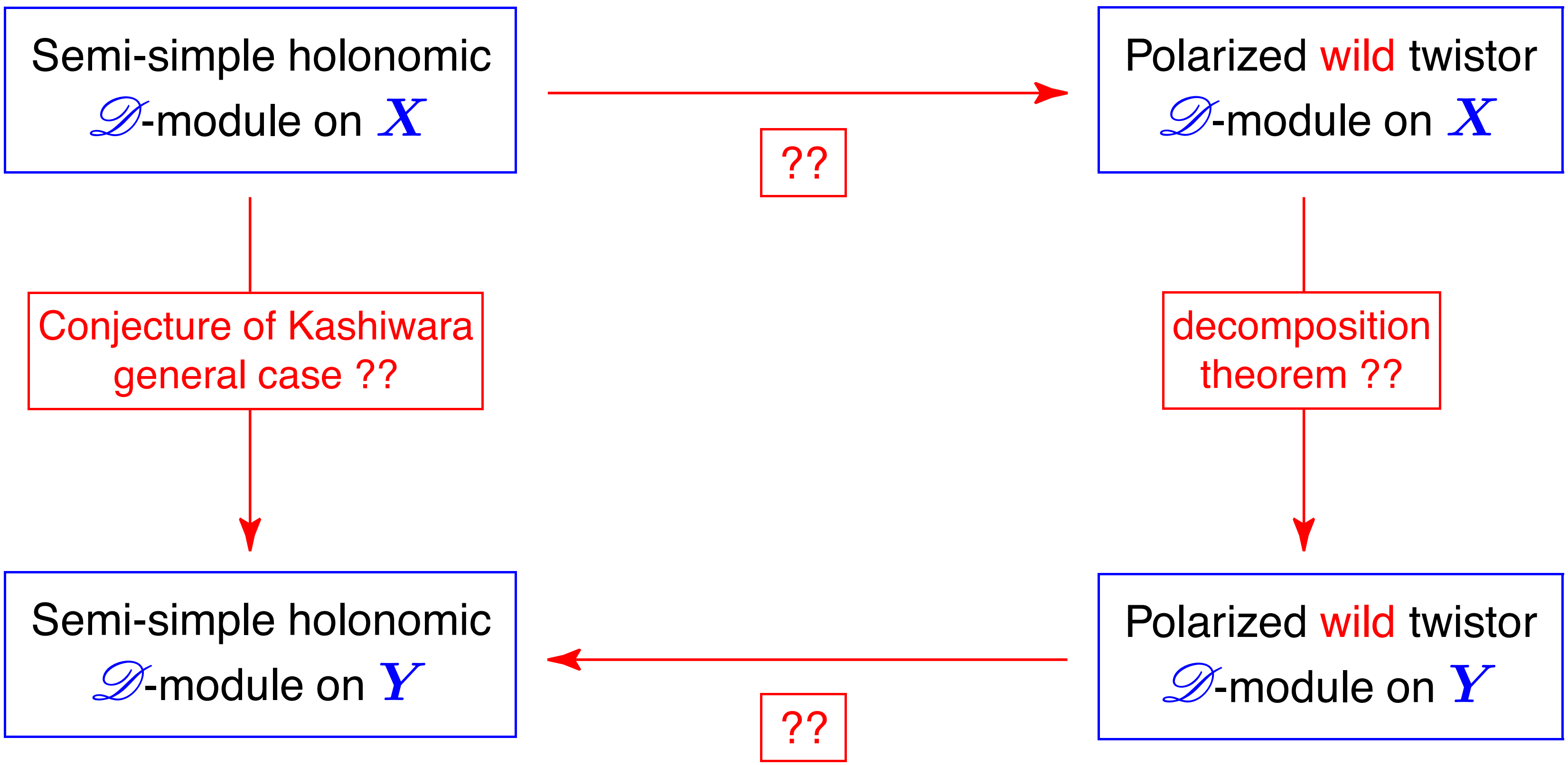


Work of O. Biquard and Ph. Boalch on curves.

Wild twistor $\mathscr{D}$-modules



Work of O. Biquard and Ph. Boalch on curves.

Recent work of T. Mochizuki.