ISOMONODROMIC DEFORMATIONS AND DEGENERATIONS OF IRREGULAR SINGULARITIES

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Abstract. Isomonodromic (that is, integrable) deformations of connections with irregular singularities in dimension one are well understood away from turning points of the parameter space. In general, at the turning points, the theorem of Kedlaya-Mochizuki is needed to understand the local behaviour of the Stokes structure, but it breaks the notion of deformation. Motivated by understanding boundaries of Frobenius manifolds, Cotti, Dubrovin and Guzzetti have analyzed some simple turning points and shown vanishing of certain entries of the Stokes matrices at the neighbourhood of these turning points. The talk will give a different point of view on these results.

1. Introduction

The papers [**CDG19**] (as well as the companion papers [**CG18, CG17, CDG20**]) and the recent preprint [**Guz21**] have emphasized some properties of connections with irregular singularities which appear when studying Frobenius manifolds. These questions can be considered from a slightly more general perspective, and shade new light on the isomonodromic deformation theory of connections with irregular singularities. These works are a source of inspiration for what follows, and I would encourage you to read them. I will not take exactly the same point of view, but the questions I address are similar. I will consider isomonodromic (that is, integrable) deformations of some irregular singularities and their degenerations. Details are developed in [**Sab21b**] (and also [**Sab21a**]). I also refer to [**Tey18**] for further developments.

2. A theorem of Cotti-Dubrovin-Guzzetti

This theorem concerns the behaviour of an isomonodromic family of irregular singularities near a point in the space of parameters where eigenvalues of the most polar part of the connection matrix coalesce.

Let me make precise the setting. We consider $\mathbb{C}^n$ with coordinates $t_1, \dots, t_n$ and, for a given $t \in \mathbb{C}^n$, we consider the connection ∇^t on the trivial bundle on the affine line (with coordinate z) having matrix

$$\Big(\frac{1}{z}\,\Lambda(t) + A^\circ\Big)\frac{\mathrm{d}z}{z}, \quad \Lambda(t) := \operatorname{diag}(t_i)_{i=1,\dots,n}, \quad A^\circ \in \mathrm{M}_n(\mathbb{C}).$$

If $t = t^\circ$ is such that $t_i^\circ \neq t_j^\circ$ for any pair $i \neq j$, then a famous theorem of Jimbo-Miwa-Ueno and Malgrange show the existence, in the neighbourhood of t°, of a universal integrable deformation of ∇°. Since the eigenvalues of $\Lambda(t^\circ)$ are pairwise distinct, we can write

$$A^\circ = D^\circ + [\Lambda(t^\circ), R^\circ]$$

for some matrix R°, whose diagonal can be chosen to be zero, and D° is the diagonal part of A°. In this specific case where we deal with a Birkhoff normal form, the deformation has an explicit form, which is much useful in the theory of Frobenius manifolds.

***Theorem* (Jimbo-Miwa-Ueno and Malgrange).** *There exists a neighbourhood $U(t^\circ)$ and a holomorphic matrix $R(t)$ $(t \in U(t^\circ))$ such that $R(t^\circ) = R^\circ$ and the connection ∇ on the trivial bundle $\mathscr{O}_{\mathbb{C}\times U}[z^{-1}]^n$ with matrix*

$$-\mathrm{d}(\Lambda(t)/z) + \big([\Lambda(t), R(t)] + D^\circ\big)\frac{\mathrm{d}z}{z} - [\mathrm{d}\Lambda(t), R(t)] \tag{A}$$

is a universal integrable deformation of ∇°. Furthermore, there exists a base change, formal with respect to z and holomorphic with respect to $t \in U(t^\circ)$, such that, after such a base change, the matrix of the connection reduces to

$$-\mathrm{d}(\Lambda(t)/z) + D^\circ\,\frac{\mathrm{d}z}{z}.$$

Let us now consider a partition $\{1, \dots, n\} = \bigsqcup_{a=1}^r I_a$ and let t^{c} be a "coalescing point" in $\mathbb{C}^n$ on the stratum defined by this partition, that is,

$$t_i^c = t_j^c \iff i \text{ and } j \in I_a \text{ for some } a.$$

Let us choose a simply connected neighbourhood $V(t^c)$ of the form $\prod_a V(t_a^c)$, and let $t^\circ \in V(t^c)$ be a generic point. We consider the possible extension of the JMU-M deformation defined on $\prod_a U(t_a^\circ)$ to $\prod_a V(t_a^c)$. The result I would like to discuss is due to Cotti-Dubrovin-Guzzetti.

***Theorem* (Cotti-Dubrovin-Guzzetti).** *Assume that the matrix $R(t)$, which is holomorphic on $U(t^\circ)$, extends holomorphically to $V(t^c)$. Then the connection with matrix* (A)*, which is defined on $V(t^c)$, is integrable on $V(t^c)$, and*

(1) *there exists a base change, formal with respect to z and holomorphic with respect to $t \in V(t^c)$, such that, after such a base change, the matrix of the connection reduces to*

$$-\mathrm{d}(\Lambda(t)/z) + D^\circ\,\frac{\mathrm{d}z}{z};$$

(2) *there exists a pair of Stokes matrices* $(S^{\circ}_{+}, S^{\circ}_{-})$ *attached to* ∇° *such that each entry* (i,j) *is zero if* $i \neq j$ *and* i,j *in the same subset* I_a.

What is a turning point? Let ∇ be an integrable connection on $M = \mathscr{O}_{\Delta\times T}(*(0\times T))^d$ with $\dim T = 1$ for example. There exists a Zariski open set $T_0 \subset T$ such that the Hukuhara-Levelt-Turrittin theorem (in dimension one with parameters) applies to ∇ in the neighbourhood of each point of T_0. The case of coalescing eigenvalues as above is typically a case where one expects a turning point. The general situation at a turning point may be very complicated. It is however controlled by the theorem of Kedlaya-Mochizuki: After enough complex blowing-ups of $\Delta \times V$, there remains no turning point. Comparing the connection ∇^{c} for the special value t^{c} of the parameter and ∇^t for the value t is obtained by considering the strict transforms of the slices $\Delta \times \{t^{\mathrm{c}}\}$ and $\Delta \times \{t\}$ in the blow-up space. They cut the exceptional divisor in distinct points and one can connect these points by a path that goes through various intersection loci of this exceptional divisor. Understanding what happens at these points is instrumental for understanding the structure of the original connection at the turning point. The first part of the theorem asserts that the turning point that is created at a coalescing value t^{c} is very simple.

Remark.

(1) The first part of the theorem can be seen as a degeneration statement, namely, the formal decomposition which exists for $t \in U(t^{\circ})$ extends to a neighbourhood of t^{c}. On the other hand, the second part is a deformation statement, that is, the property of Stokes matrices which holds when restricting to the slice $t = t^{\mathrm{c}}$ also holds for generic slices in the neighbourhood of t^{c}.

(2) The proof of the first part of the theorem is not too difficult: it is done by solving iteratively the equations to find the formal gauge transformation. This is why I will focus on the second part. Once the first part is proved, the remarkable fact is that the vanishing of Stokes entries is due to the constancy of the matrix D° with respect to t, which also follows from the 1-connectedness of $V(t^{\mathrm{c}})$. More precisely, the point is to relate constancy of D° on the open subset $V(t^{\mathrm{c}})^{\circ}$ where the coordinates are pairwise distinct to a theorem of Malgrange computing Stokes matrices by means of Fourier transformation. This theorem is a precursor of the much more general theorem proved by Mochizuki recently [**Moc18**], and does not need such a sophisticated technology.

(3) Once the first part is proved, one notices that the vanishing asserted in the second part holds if we replace t° with t^{c}. In general, the behaviour of the Stokes matrices near a turning point is far from trivial. We aim at showing that, under the assumption of the existence of a formal base change as in the theorem, part of this behaviour can be controlled (the vanishing property).

3. A theorem of Malgrange for computing Stokes matrices

Let L° be a locally constant sheaf of rank d on $\mathbb{C}_\lambda \smallsetminus \{\lambda = t_i^\circ \mid i = 1, \dots, n\}$ (punctured affine line), with $t_i^\circ \neq t_i^\circ$ if $i \neq j$ and let j denote the inclusion in $\mathbb{C}_\lambda$. There exists a meromorphic bundle with connection (V°, ∇°) on $\mathbb{C}_\lambda$ with poles at t_i° $(i = 1, \dots, n)$ such that $L^\circ = V^{o\nabla^\circ}$ and such that ∇° has regular singularities included at infinity. We regard it as a left module on the Weyl algebra $\mathbb{C}[\lambda]\langle\partial_\lambda\rangle$, in which case we have $\mathrm{DR}^{\mathrm{an}}(V^\circ, \nabla^\circ) \simeq \boldsymbol{R}j_*L^\circ$, which is a perverse sheaf (up to a shift) on $\mathbb{C}_\lambda$.

More generally, let M° be a regular holonomic $\mathbb{C}[\lambda]\langle\partial_\lambda\rangle$-module whose restriction to $\mathbb{C}_\lambda \smallsetminus \{\lambda = t_i^\circ \mid i = 1, \dots, n\}$ is (V°, ∇). Its Fourier transform ${}^{\mathrm{F}}M^\circ$ is the same $\mathbb{C}$-vector space with an action of $\mathbb{C}[\zeta]\langle\partial_\zeta\rangle$ such that ζ acts as ∂_λ and ∂_ζ acts as $-\lambda$. Setting $z = \zeta^{-1}$, the localization $G^\circ := \mathbb{C}[\zeta, \zeta^{-1}] \otimes_{\mathbb{C}[\zeta]} {}^{\mathrm{F}}M^\circ$ is a free $\mathbb{C}[z, z^{-1}]$-module with connection having an irregular singularity of Poincaré rank one (exponential type) at $z = 0$. The theorem of Malgrange [**Mal91**, Chap. XII] (proved in a topological way in [**DHMS20**]) gives a formula for the Stokes matrices of G° at $z = 0$ in terms of monodromy data of M°.

One can represent the perverse sheaf $\mathrm{DR}\, M^\circ$ by a linear representation of a quiver:

- Vector spaces Ψ° (of rank d) and Φ_i° $(i = 1, \dots, n)$,
- linear morphisms $\mathrm{c}_i : \Psi^\circ \to \Phi_i^\circ$ and $\mathrm{v}_i : \Phi_i^\circ \to \Psi^\circ$,

subject to the relations that $\mathrm{Id} + \mathrm{c}_i \circ \mathrm{v}_i$ and $\mathrm{T}_i := \mathrm{Id} + \mathrm{v}_i \circ \mathrm{c}_i$ are invertible for each i.

***Theorem* (Malgrange, DHMS).** *There exists a pair of Stokes matrices (S_+°, S_-°) for G° at $z = 0$ which are decomposed into blocks (i, j) $(i, j = 1, \dots, n)$ such that the non-diagonal blocks (i, j) and (j, i) respectively read*

- $\mathrm{c}_j \circ \mathrm{v}_i$ *and* 0 *for* S_+°,
- 0 *and* $-\mathrm{c}_i \circ \mathrm{v}_j$ *for* S_-°.

***Example* (Intermediate extension).** If $\mathrm{DR}\, M^\circ \simeq j_*L^\circ$, then the monodromy data attached to the perverse sheaf j_*L° are $(\Psi^\circ, \Phi^\circ_{i=1,\dots,n}, \mathrm{c}_i, \mathrm{v}_i)$ with

$$\Phi_i^\circ = \mathrm{im}(\mathrm{Id} - \mathrm{T}_i), \quad \mathrm{v}_i = \text{inclusion} : \Phi_i^\circ \hookrightarrow \Psi^\circ, \quad \mathrm{c}_i = (\mathrm{Id} - \mathrm{T}_i) : \Psi^\circ \longrightarrow \Phi_i^\circ.$$

In this example, for $i \neq j \in \{1, \dots, n\}$, the representative (S_+°, S_-°) of Stokes matrices for G° has vanishing blocks (i, j) and (j, i) if and only if

$$\text{(B)} \qquad (\mathrm{Id} - \mathrm{T}_j)_{|\,\mathrm{im}(\mathrm{Id} - \mathrm{T}_i)} = 0 \quad \text{and} \quad (\mathrm{Id} - \mathrm{T}_i)_{|\,\mathrm{im}(\mathrm{Id} - \mathrm{T}_j)} = 0,$$

that is,

$$\mathrm{T}_j\mathrm{T}_i = \mathrm{T}_j\mathrm{T}_i = \mathrm{T}_i + \mathrm{T}_j - \mathrm{Id}\,.$$

4. Dynamical version of the theorem of Malgrange

Let us go back to the case of a coalescing point $t^c \in \mathbb{C}^n$ and a neighbourhood $V(t^c) = \prod_a V(t_a^c)$, and let $V(t^c)^\circ$ be the open subset consisting of points t with pairwise distinct entries. On the product space $\mathbb{C}_\lambda \times V(t^c)^\circ$ we consider the hypersurface H defined by $\prod_i(\lambda - t_i) = 0$. By assumption, it is the disjoint union of the hyperplanes H_i defined by the functions $\lambda - t_i$.

Let L be a locally constant sheaf of rank d on $(\mathbb{C}_\lambda \times V(t^c)^\circ) \smallsetminus H$ and let

$$j : (\mathbb{C}_\lambda \times V(t^c)^\circ) \smallsetminus H \hookrightarrow \mathbb{C}_\lambda \times V(t^c)^\circ$$

denote the inclusion. For each $i = 1, \dots, n$ consider the vanishing cycle sheaf $\phi_{\lambda - t_i}(j_*L)$ which is easily seen to be locally constant on H_i. It comes equipped with an automorphism T_i.

Let j_*L° be the restriction of j_*L to $\mathbb{C}_\lambda \times \{t^\circ\}$.

Proposition. *For a given $a = 1, \dots, r$, Condition* (B) *holds for any pair $i \neq j \in I_a$ if and only if the vanishing cycle sheaf $\phi_{\lambda - t_i}(j_*L)$ is constant for every $i \in I_a$.*

Sketch of proof. We represent the locally constant sheaf $(\phi_{\lambda - t_i}(j_*L), \mathrm{T}_i)$ by the vector space $(\mathrm{im}(\mathrm{Id} - \mathrm{T}_i), \mathrm{T}_i)$ with automorphisms T_j for $j \neq i \in I_a$. Its constancy is equivalent to $\mathrm{T}_{j|\,\mathrm{im}(\mathrm{Id} - \mathrm{T}_i)} = \mathrm{Id}$ for any $j \in I_a$. $\square$

Let M be the regular holonomic $\mathscr{D}$-module on $\mathbb{C}_\lambda \times V(t^c)^\circ$ whose de Rham complex is j_*L, let ${}^{\mathrm{F}}M$ be its partial Fourier transform relative to λ and let $\widehat{G}$ be the formalization of ${}^{\mathrm{F}}M$ along $\{\zeta = \infty\} \times V(t^c)^\circ$. The formal stationary phase formula with parameter t proved in [**DS03**] shows that $\widehat{G}$ has a decomposition

$$\widehat{G} \simeq \bigoplus_i (R_i[z^{-1}], \nabla_i + \mathrm{d}(t_i/z))$$

where (R_i, ∇_i) is a logarithmic connection with pole along $z = 0$, corresponding to a locally constant sheaf $\mathscr{L}_i$ on $\mathbb{C}^*_z \times V(t^c)^\circ$. Furthermore, the sheaf L_i of horizontal sections of the residual connection $(R_i/zR_i, \nabla_{\mathrm{res}})$ on $V(t^c)^\circ$ is isomorphic to $\phi_{\lambda - t_i}(j_*L)$.

Corollary. *With this notation, if the sheaves L_i are constant on $V(t^c)^\circ$, then for any $t^\circ \in V(t^c)^\circ$, any $a = 1, \dots, r$, and any pair $i \neq j \in I_a$, the (i,j) entries of the Stokes matrices obtained by Malgrange's construction are zero.*

5. Conclusion: Proof of the theorem of C-D-G

It remains to interpret the meromorphic connection of the theorem in the framework of the above corollary, since the constancy assumption on L_i has already been checked. We only need to show that the $\mathscr{O}_{V(t^c)^\circ}[z, z^{-1}]$-free module with connection given by (A) is the localization of the Fourier transform of a regular holonomic $\mathscr{D}$-module on $\mathbb{C}_\lambda \times V(t^c)^\circ$ whose de Rham complex takes the form j_*L. It is enough to check this when restricting to a special value $t^\circ \in V(t^c)^\circ$.

We consider the free $\mathbb{C}[z]$-module F° of rank n endowed with the connection ${}^F\nabla^\circ$ having matrix

$$\Big(\frac{\Lambda^\circ}{z} + A^\circ\Big)\frac{\mathrm{d}z}{z}, \quad \Lambda^\circ := \mathrm{diag}(t_1^\circ, \dots, t_n^\circ).$$

For our purpose, we can tensor F° with a rank one bundle with logarithmic connection having a regular singularity at the origin, that is, by adding $c\,\mathrm{Id}_n\,\mathrm{d}z/z$ to A° for some complex number c. We thus assume that the only possible integral eigenvalues of A° are $\geqslant 1$ and that no diagonal entry of A° is an integer. The inverse Fourier lattice (E°,∇°) is F° regarded as a $\mathbb{C}[\lambda]$-module, where λ acts as $z^2\partial_z$. It is free of rank n, with the same canonical basis as F° (see e.g. [**Sab02**, Prop. V.2.10]), and the matrix of ∇° is

$$B^\circ=(A^\circ-\mathrm{Id}_n)(\lambda\,\mathrm{Id}_n-\Lambda^\circ)^{-1}\mathrm{d}\lambda=\sum_{i=1}^n\frac{B_i^\circ}{\lambda-t_i}.$$

Furthermore, each matrix B_i° has rank one and a unique nonzero eigenvalue, which is the ith diagonal entry of $A^\circ-\mathrm{Id}_n$, that is non integral by the choice of c. Set $(V^\circ,\nabla^\circ)=(\mathbb{C}[\lambda,(\prod_i(\lambda-t_i^\circ))^{-1}\otimes E^\circ,\nabla^\circ)$.

Lemma. *The $\mathbb{C}[\lambda]\langle\partial_\lambda\rangle$-submodule of (V°,∇°) generated by E° is the middle extension (M°,∇°) of (V°,∇°), whose localized Laplace transform (G°,∇°) is equal to $\mathbb{C}[z,z^{-1}]\otimes_{\mathbb{C}[z]}F^\circ$ with connection having matrix A°.*

Proof. Let G° be the localized Laplace transform of M°. By definition, ${}^F\!M^\circ$ contains F°. Let us check that the localization map $F^\circ\to G^\circ$ is injective: the kernel of ${}^F\!M^\circ\to G^\circ$ is the ζ-torsion submodule of ${}^F\!M^\circ$; if $f\in F^\circ$ satisfies $\zeta^kf=0$ in ${}^F\!M^\circ$, then, since F° is a $\mathbb{C}[z]$-module, z^kf also satisfies $\zeta^k(z^kf)=0$, that is, $f=0$.

Then G° contains F°, hence $\mathbb{C}[z,z^{-1}]\otimes_{\mathbb{C}[z]}F^\circ$. It is enough to show that the rank of G° is equal to that of F°, that is, n. Since the rank of G° is equal to the sum of the dimensions of the vanishing cycles ϕ_i° of M° at $t_1^\circ,\dots,t_n^\circ$, it is enough to show that, for each local monodromy T_i of the local system $L^\circ=(V^\circ)^{\nabla^\circ}$ around t_i°, the rank of $\mathrm{Id}_n-\mathrm{T}_i$ is equal to one.

By our assumption on B°, the local monodromy T_i is conjugate to $\exp-2\pi\mathrm{i}B_i^\circ$, hence $\mathrm{T}_i-\mathrm{Id}$ has rank one, as desired. □

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