Isomonodromic deformations and degenerations of irregular singularities

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Introduction

Setting. Consider $\mathbb{C}^n$ with coordinates $t_1, \dots, t_n$ and, for a given $t^\circ \in \mathbb{C}^n$ with $t_i^\circ \neq t_j^\circ$ if $i \neq j$, consider the connection ∇° on the trivial bundle on the affine line (with coordinate z) having matrix

$$\Big(\frac{1}{z}\Lambda(t^\circ) + A^\circ\Big)\frac{\mathrm{d}z}{z}, \quad \Lambda(t^\circ) := \mathrm{diag}(t_i^\circ)_{i=1,\dots,n}, \quad A^\circ \in \mathrm{M}_n(\mathbb{C}).$$

This talk deals with a theorem that concerns the behaviour of an isomonodromic deformation of ∇° with parameters t when t tends to a value where $t_i = t_j$ for some $i \neq j$.

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This theorem was developed by ***Giordano Cotti***, ***Boris Dubrovin*** and ***Davide Guzzetti*** in various papers, where they have emphasized some properties of connections with irregular singularities which appear when studying Frobenius manifolds. These questions can be considered from a slightly more general perspective, and shade new light on the ***isomonodromic deformation theory*** of connections with ***irregular singularities***. These works are a source of inspiration for what follows, and I would encourage you to read them. I will not take exactly the same point of view, but the questions I address are similar.

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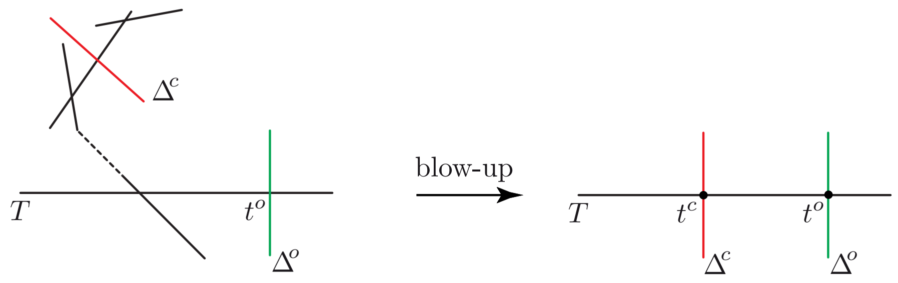
What is a turning point?

∇: an integrable conn. on $G = \mathscr{O}_{\Delta\times T}(*\,(0\times T))^d$, e.g. $\dim T = 1$.

$\exists$ a Zariski open set $T_0 \subset T$ s.t. the ***Hukuhara-Levelt-Turrittin theorem*** (dim. one with parameters) applies to ∇ in the nbd of each point of T_0.

Coalescing eigenvalues $\implies$ a ***turning point***.

The general situation at a turning point may be very complicated, however controlled by the theorem of ***Kedlaya-Mochizuki***:



After enough complex blowing-ups of $\Delta \times V$, $\nexists$ turning point for the pullback connection.

The first part of the theorem of C-D-G asserts that the turning point that is created at a coalescing value t^c is very simple.

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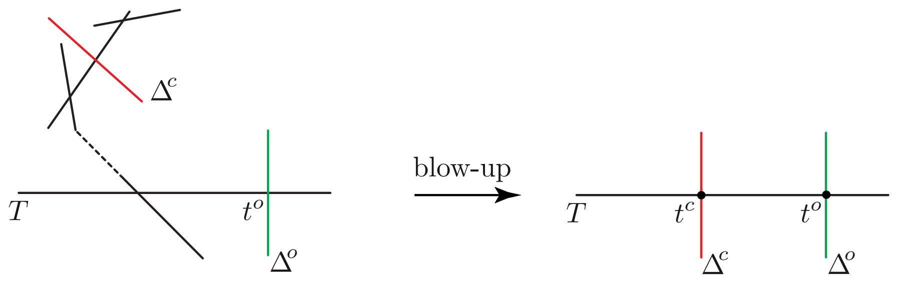

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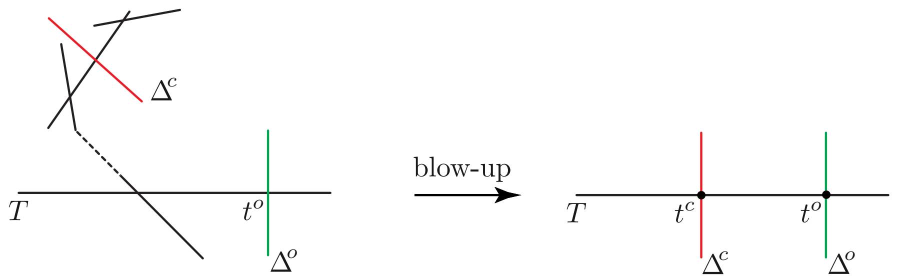

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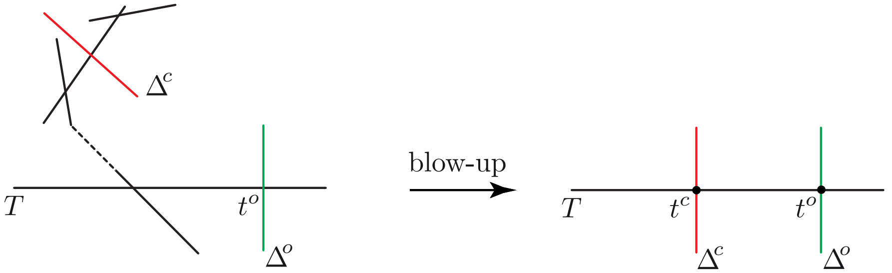

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$\Longrightarrow$ formula for the Stokes matrices of G° at $z=0$ in terms of monodromy data of M°.

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***Theorem* (Malgrange, DHMS).** *$\exists$ a pair of Stokes matrices (S°_+,S°_-) for G° at $z=0$, decomposed into blocks (i,j) $(i,j=1,\dots,n)$ s.t. the non-diagonal blocks (i,j) and (j,i) respectively read*

- $\mathrm{c}_j\circ\mathrm{v}_i$ *and* 0 *for* S°_+,
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A formula of Malgrange for Stokes matrices

$j : \mathbb{C}^*_\lambda(t^\circ) := \mathbb{C}_\lambda \smallsetminus \{\lambda = t^\circ_i \mid i = 1, \dots, n\} \hookrightarrow \mathbb{C}_\lambda$ (punctured affine line), with $t^\circ_i \neq t^\circ_{i'}$ if $i \neq i'$.

L°: a loc. const. sheaf of rank d on $\mathbb{C}^*_\lambda(t^\circ)$.

(V°, ∇°) free $\mathcal{O}(\mathbb{C}^*_\lambda(t^\circ))$-mod. with connection s.t. $L^\circ = (V^{\circ \mathrm{an}})^{\nabla^\circ}$ and ∇° has reg. sing. included at infinity.

$\rightsquigarrow$ $j_*(V^\circ, \nabla^\circ)$ is left module on the Weyl algebra $\mathbb{C}[\lambda]\langle\partial_\lambda\rangle$, and $\mathrm{DR}^{\mathrm{an}}\, j_*(V^\circ, \nabla^\circ) \simeq \boldsymbol{R}j_*L^\circ$: a perverse sheaf (up to a shift) on $\mathbb{C}_\lambda$.

More generally, can consider M°: a ***reg. holon.*** $\mathbb{C}[\lambda]\langle\partial_\lambda\rangle$-mod. s.t. $\mathcal{O}(\mathbb{C}^*_\lambda(t^\circ)) \otimes M^\circ = (V^\circ, \nabla)$.

Fourier transform ${}^{\mathrm{F}}M^\circ$: the same $\mathbb{C}$-vector space with an action of $\mathbb{C}[\zeta]\langle\partial_\zeta\rangle$ such that ζ acts as ∂_λ and ∂_ζ acts as $-\lambda$.

Setting $z = \zeta^{-1}$, the ***localization*** $G^\circ := \mathbb{C}[\zeta, \zeta^{-1}] \otimes_{\mathbb{C}[\zeta]} {}^{\mathrm{F}}M^\circ$ is a free $\mathbb{C}[z, z^{-1}]$-module with conn. having an ***irregular singularity of Poincaré rank one*** (exponential type) at $z = 0$.

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***Example* (Middle extension).** Case $\mathrm{DR}^{\mathrm{an}}\, M^\circ \simeq j_*L^\circ$:

$\rightsquigarrow$ monodromy data are $(\Psi^\circ, \Phi^\circ_{i=1,\dots,n}, \mathrm{c}_i, \mathrm{v}_i)$ with $\Phi^\circ_i = \mathrm{im}(\mathrm{Id} - \mathrm{T}_i)$ and $\mathrm{v}_i = \text{inclusion} : \Phi^\circ_i \hookrightarrow \Psi^\circ$, $\quad \mathrm{c}_i = (\mathrm{Id} - \mathrm{T}_i) : \Psi^\circ \longrightarrow \Phi^\circ_i$.

Th. $\Longrightarrow$ for $i \neq j \in \{1, \dots, n\}$, (S°_+, S°_-) has vanishing blocks (i, j) and (j, i) iff

(Van) $\quad \boxed{(\mathrm{Id} - \mathrm{T}_j)_{|\,\mathrm{im}(\mathrm{Id} - \mathrm{T}_i)} = 0 \quad \text{and} \quad (\mathrm{Id} - \mathrm{T}_i)_{|\,\mathrm{im}(\mathrm{Id} - \mathrm{T}_j)} = 0.}$

$$(\Longleftrightarrow \quad \mathrm{T}_j\mathrm{T}_i = \mathrm{T}_j\mathrm{T}_i = \mathrm{T}_i + \mathrm{T}_j - \mathrm{Id}\,.)$$

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Dynamical version of Malgrange's theorem

Case of a ***coalescing point*** $t^{\mathrm{c}} \in \mathbb{C}^n$ with nbd $V(t^{\mathrm{c}}) = \prod_a V(t_a^{\mathrm{c}})$.

- $V(t^{\mathrm{c}})^\circ = \{t \in V(t^{\mathrm{c}}) \mid t_i \neq t_j \ \forall i \neq j\}$
- In $\mathbb{C}_\lambda \times V(t^{\mathrm{c}})^\circ$, hypersurface $H = \{\prod_i (\lambda - t_i) = 0\}$.
  $\rightsquigarrow$ disjoint union of the hyperplanes $H_i = \{\lambda - t_i = 0\}$.
- L: a locally const. sheaf of rk d on $(\mathbb{C}_\lambda \times V(t^{\mathrm{c}})^\circ) \smallsetminus H$.
- $j : (\mathbb{C}_\lambda \times V(t^{\mathrm{c}})^\circ) \smallsetminus H \hookrightarrow \mathbb{C}_\lambda \times V(t^{\mathrm{c}})^\circ$: the inclusion.
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Proposition. *For a given* $a = 1, \dots, r$*, Condition* (Van) *holds for any pair* $i \neq j \in I_a$ *iff* $\phi_{\lambda - t_i}(j_* L)$ *is* ***constant*** *for every* $i \in I_a$.

Sketch of proof. Represent the loc. constant sheaf $\phi_{\lambda - t_i}(j_* L)$ by the vector space $\mathrm{im}(\mathrm{Id} - \mathrm{T}_i)$ with autom. T_j for $j \neq i \in I_a$.

Constancy $\Longleftrightarrow$ $\mathrm{T}_{j|\,\mathrm{im}(\mathrm{Id} - \mathrm{T}_i)} = \mathrm{Id}$ for any $j \in I_a$. $\square$

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Consider:

- M: the reg. holonomic $\mathscr{D}$-module on $\mathbb{C}_\lambda \times V(t^c)^\circ$ whose de Rham complex is $j_* L$.
- ${}^{\mathrm{F}}M$: its partial Fourier transform relative to λ.
- $\widehat{G}$ be the formalization of ${}^{\mathrm{F}}M$ along $\{\zeta = \infty\} \times V(t^c)^\circ$.

The ***formal stationary phase formula with parameter t*** (Douai-CS 2003) $\Longrightarrow$

- $\widehat{G}$ has a decomposition
$$\widehat{G} \simeq \bigoplus_i (R_i[z^{-1}], \nabla_i + \mathrm{d}(t_i/z))$$
  with (R_i, ∇_i): log. connection with pole along $z = 0$.
- and L_i: sheaf of horiz. sections of the residual conn. $(R_i/zR_i, \nabla_{\mathrm{res}})$ on $V(t^c)^\circ$ ***isomorphic to*** $\phi_{\lambda - t_i}(j_* L)$.

Corollary. *If the sheaves L_i are **constant** on $V(t^c)^\circ$, then: $\forall t^\circ \in V(t^c)^\circ$, $\forall a = 1, \dots, r$ and $\forall i \neq j \in I_a$, the (i, j) entries of the Stokes matrices (S_+°, S_-°) are zero.*

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Conclusion: Proof of the theorem of C-D-G

Consider a partition $\{1, \dots, n\} = \bigsqcup_{a=1}^r I_a$ and let t^c be a "coalescing point" in $\mathbb{C}^n$ on the stratum defined by this partition, that is,

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$V(t^c)$: a 1-connected nbd of the form $\prod_a V(t_a^c)$

$t^\circ \in V(t^c)$: a generic point.

Assumption: $\exists R(t)$ holom. on $V(t^c) = \prod_a V(t_a^c)$ and integr. conn.

$$\text{(JMUM)} \quad -\mathrm{d}\Big(\frac{\Lambda(t)}{z}\Big) + \big([\Lambda(t), R(t)] + D^\circ\big)\frac{\mathrm{d}z}{z} - [\mathrm{d}\Lambda(t), R(t)]$$

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***Theorem* (Cotti-Dubrovin-Guzzetti).** *Furthermore,*

(1) *$\exists$ a base change, formal with respect to z and holom. w.r.t. $t \in V(t^c)$, s.t., after this base change, the matrix of $\widehat{\nabla}$ is*

$$-\mathrm{d}\Big(\frac{\Lambda(t)}{z}\Big) + D^\circ \frac{\mathrm{d}z}{z};$$

(2) *$\exists$ a pair of Stokes matrices (S_+°, S_-°) attached to ∇° s.t. each entry (i,j) is zero if $i \neq j$ and i,j in the same subset I_a.*

- Proof of (1) omitted (not much difficult).
- (1) $\Longrightarrow$ L_i ***constant of rk one*** on $V(t^c)^\circ$.
- Proof of (2): relate (JMUM) with the above corollary.

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Setting.

- $F^\circ := (\mathbb{C}[z]^n, {}^F\nabla^\circ)$ with matrix
$$\Big(\frac{\Lambda^\circ}{z}+A^\circ\Big)\frac{\mathrm{d}z}{z},\quad \Lambda^\circ:=\mathrm{diag}(t_1^\circ,\dots,t_n^\circ).$$
- $\widetilde{G}^\circ := \mathbb{C}[z,z^{-1}]\otimes_{\mathbb{C}[z]}F^\circ$ with merom. conn. ${}^F\nabla^\circ$.
- Can assume (add $c\,\mathrm{Id}_n\,\mathrm{d}z/z$ with suitable $c\in\mathbb{C}$):
  - integral eigenvalues of A° are $\geqslant 1$,
  - no diagonal entry of A° is an integer.
- Set $\lambda=z^2\partial_z$ and $E^\circ:=F^\circ$ regarded as a $\mathbb{C}[\lambda]$-mod.
- Action of z^{-1} ⇝ merom. connect. ∇° on E°.

Lemma. *E° is $\mathbb{C}[\lambda]$-free of rk n and ∇° is log. with matrix*

$$B^\circ=(A^\circ-\mathrm{Id}_n)(\lambda\,\mathrm{Id}_n-\Lambda^\circ)^{-1}\mathrm{d}\lambda=\sum_{i=1}^n\frac{B_i^\circ}{\lambda-t_i^\circ}.$$

- Each matrix B_i° has rank one and a unique nonzero eigenvalue: the ith diagonal entry of $A^\circ-\mathrm{Id}_n$, that is ***non integral***.
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- Set $(V^\circ, \nabla^\circ) = \big(\mathbb{C}\big[\lambda, (\prod_i(\lambda - t_i^\circ))^{-1}\big] \otimes E^\circ, \nabla^\circ\big)$.

Lemma. *The $\mathbb{C}[\lambda]\langle\partial_\lambda\rangle$-submodule of (V°, ∇°) generated by E° is the middle extension (M°, ∇°) of (V°, ∇°), whose localized Laplace transform $(G^\circ, {}^F\nabla^\circ)$ is equal to $(\widetilde{G}^\circ, {}^F\nabla^\circ)$.*

Proof.

- Properties of eigenvalues of $B_i^\circ \implies$ first assertion.
- Set G°: localized Laplace transform of M°.
- $E^\circ \hookrightarrow M^\circ \implies F^\circ \hookrightarrow G^\circ$, hence $\widetilde{G}^\circ \subset G^\circ$.
- For equality, enough to show $\mathrm{rk}\, G^\circ = n$.
- Known: $\mathrm{rk}\, G^\circ = \sum_{i=1}^n \phi_{t_i^\circ} M^\circ$.
- $\implies$ enough to show that, for each local monodromy T_i of $L^\circ = (V^\circ)^{\nabla^\circ}$ around t_i°, we have $\boxed{\mathrm{rk}(\mathrm{Id}_n - \mathrm{T}_i) = 1}$.

By our assumption on B°, the local monodromy T_i is conjugate to $\exp -2\pi\mathrm{i}B_i^\circ$, hence $\mathrm{T}_i - \mathrm{Id}$ has rank one, as desired. □

Setting.

- $F^\circ := (\mathbb{C}[z]^n, {}^F\nabla^\circ)$ with matrix
$$\Big(\frac{\Lambda^\circ}{z} + A^\circ\Big)\frac{\mathrm{d}z}{z}, \quad \Lambda^\circ := \mathrm{diag}(t_1^\circ, \dots, t_n^\circ).$$
- $\widetilde{G}^\circ := \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}[z]} F^\circ$ with merom. conn. ${}^F\nabla^\circ$.
- Can assume (add $c\,\mathrm{Id}_n\,\mathrm{d}z/z$ with suitable $c \in \mathbb{C}$):
  - integral eigenvalues of A° are $\geqslant 1$,
  - no diagonal entry of A° is an integer.
- Set $\lambda = z^2\partial_z$ and $E^\circ := F^\circ$ regarded as a $\mathbb{C}[\lambda]$-mod.
- Action of $z^{-1} \rightsquigarrow$ merom. connect. ∇° on E°.

Lemma. *E° is $\mathbb{C}[\lambda]$-free of rk n and ∇° is log. with matrix*
$$B^\circ = (A^\circ - \mathrm{Id}_n)(\lambda\,\mathrm{Id}_n - \Lambda^\circ)^{-1}\mathrm{d}\lambda = \sum_{i=1}^n \frac{B_i^\circ}{\lambda - t_i^\circ}.$$

- Each matrix B_i° has rank one and a unique nonzero eigenvalue: the ith diagonal entry of $A^\circ - \mathrm{Id}_n$, that is ***non integral***.
- Set $(V^\circ, \nabla^\circ) = \big(\mathbb{C}\big[\lambda, (\prod_i(\lambda - t_i^\circ))^{-1}\big] \otimes E^\circ, \nabla^\circ\big)$.

Lemma. *The $\mathbb{C}[\lambda]\langle\partial_\lambda\rangle$-submodule of (V°, ∇°) generated by E° is the middle extension (M°, ∇°) of (V°, ∇°), whose localized Laplace transform $(G^\circ, {}^F\nabla^\circ)$ is equal to $(\widetilde{G}^\circ, {}^F\nabla^\circ)$.*

Proof.

- Properties of eigenvalues of $B_i^\circ \implies$ first assertion.
- Set G°: localized Laplace transform of M°.
- $E^\circ \hookrightarrow M^\circ \implies F^\circ \hookrightarrow G^\circ$, hence $\widetilde{G}^\circ \subset G^\circ$.
- For equality, enough to show $\mathrm{rk}\, G^\circ = n$.
- Known: $\mathrm{rk}\, G^\circ = \sum_{i=1}^n \phi_{t_i^\circ} M^\circ$.
- $\implies$ enough to show that, for each local monodromy T_i of $L^\circ = (V^\circ)^{\nabla^\circ}$ around t_i°, we have $\boxed{\mathrm{rk}(\mathrm{Id}_n - \mathrm{T}_i) = 1}$.

By our assumption on B°, the local monodromy T_i is conjugate to $\exp -2\pi\mathrm{i} B_i^\circ$, hence $\mathrm{T}_i - \mathrm{Id}$ has rank one, as desired. □

Conclusion. Up to adding $c\,\mathrm{Id}_n\,\mathrm{d}z/z$ to the matrix, the connection (JMUM) on $V(T^{\mathrm{c}})^\circ$ is the localized Fourier transform G of a middle extension M. Furthermore, the constancy condition of $\phi_{\lambda - t_i} M$ is satisfied because L_i is constant (of rank one).
Dynamical Malgrange theorem $\implies$ ***vanishing of Stokes entries***.

□