ASPECTS OF THE IRREGULAR HODGE FILTRATION

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Abstract. Given a regular function f on a smooth quasi-projective variety U, the de Rham complex of U relative to the twisted differential $\mathrm{d} + \mathrm{d}f$ can be equipped canonically with a filtration (the irregular Hodge filtration) for which the associated hypercohomology spectral sequence degenerates at E_1. A logarithmic version of this de Rham complex (relative to a suitable compactification of U) has been introduced by M. Kontsevich, who showed the independence of the dimension of the corresponding cohomologies with respect to the differential $ud + vdf$, for u, v arbitrary complex numbers. This leads to bundles on the projective line of the $(u : v)$ variable, on which we construct a natural connection for which the Harder-Narasimhan filtration satisfies the Griffiths transversality property and standard limiting properties at $v = 0$. This is a joint work with Hélène Esnault (Berlin) and Jeng-Daw Yu (Taipei).

1. Introduction et motivations

Since it is not usual to work with irregular singularities in Algebraic geometry, I will first insist on motivations for that.

1.a. Meromorphic connections with regular singularities and Hodge theory (curve case)

Let S be a smooth complex algebraic curve (possibly not projective) and let V be an algebraic vector bundle on S equipped with a connection $\nabla : V \to \Omega^1_S \otimes V$. The connection may have regular or irregular singularities at ∞.

Examples 1.1.

(1) Let $\omega \in H^0(S, \Omega^1_S)$ be an algebraic one-form. Then $\mathrm{d} + \omega$ defines a connection on the trivial bundle $\mathscr{O}_S$. If ω has poles of order $\geqslant 2$ in some (or any) smooth compactification $\overline{S}$ of S, the connection has irregular singularities.

(2) Set $j : S \hookrightarrow \overline{S}$, $\infty = \overline{S} \smallsetminus S$ and extend V as locally free $j_*\mathscr{O}_X$-module j_*V of finite rank, and for each $s \in \infty$, we get a free $(j_*\mathscr{O}_S)_s$-module $(j_*V)_s$. The connection extends in a natural way. It has regular singularities at s if for some $(j_*\mathscr{O}_S)_s$-basis of $(j_*V)_s$, the matrix of the connection has a pole of order at most one. Otherwise, it has regular singularities.

(3) Let $f : X \to S$ be a smooth proper morphism between smooth complex algebraic varieties. A celebrated theorem asserts that the Gauss-Manin connection on the vector bundle $R^k f_*\mathscr{O}_X$ has regular singularities at infinity.

(4) Assume that $(V^{\mathrm{an}}, \nabla^{\mathrm{an}})$ is a holomorphic vector bundle with connection on S^{an} which underlies a polarized variation of Hodge structure, with Hodge filtration $F^\bullet V^{\mathrm{an}}$. Then the subsheaf on $\overline{S}^{\mathrm{an}}$ consisting of sections of V^{an} whose norm has moderate growth near $\infty = \overline{S} \smallsetminus S$ is a locally free $\mathscr{O}_{\overline{S}^{\mathrm{an}}}(*\infty)$-module, and the connection ∇^{an} extends as a meromorphic connection $\overline{\nabla}^{\mathrm{an}}$. Moreover, this connection has regular singularities at infinity (theorem of Griffiths-Schmid, [**Sch73**, Th. 4.13(a)]). By using a GAGA-type result on the projective curve $\overline{S}$, one gets a bundle (V, ∇) with regular singularities at infinity.

1.b. Tameness vs non-tameness for harmonic bundles on noncompact curves. *À tout seigneur tout honneur*, let me start by quoting Carlos Simpson (Harmonic bundles on noncompact curves, JAMS 1990). In his paper "Harmonic bundles on noncompact curves", Simpson extends a great part of Griffiths-Schmid-Zucker's theory to harmonic bundles. However, the tameness (regularity) assumption is not automatic, and has to be added as an extra assumption.

> "The following are some of the topics which merit further study. First of all, the restriction of tameness is a fairly strong one, if one is interested in classifying all harmonic bundles. A question is whether one could set up a correspondence in which some nontame harmonic bundles correspond to systems of differential equations with irregular singularities..."

This led to various developments, firstly on curves (C.S., Biquard & Boalch), leading to a precise understanding of moduli spaces of meromorphic connections on "irregular curves" (Boalch), and in any dimension, leading to a far reaching generalization of mixed Hodge theory, called "mixed twistor theory" (T. Mochizuki).

1.c. Arithmetic analogies (Deligne). The motivations of Deligne in 1984 came from arithmetic.

"The analogy between vector bundles with integrable connection having irregular singularities on a complex algebraic variety X and ℓ-adic sheaves with wild ramification on an algebraic variety of characteristic p, leads one to ask how such a vector bundle with integrable connection can be part of a system of realizations analogous to what furnishes a family of motives parametrized by X...

In the "motivic" case, any de Rham cohomology group has a natural Hodge filtration. Can we hope for one on $H^i_{\mathrm{dR}}(U, V)$ for some classes of (V, ∇) with irregular singularities?"

Deligne hopes (1984) that such is the case when $(V, \nabla^{\mathrm{reg}})$ underlies a variation of polarized Hodge structure and $(V, \nabla) = (V, \nabla^{\mathrm{reg}} + \mathrm{d}f \otimes \mathrm{Id}_V)$.

1.d. Irregular Hodge theory and L-functions (Adolphson-Sperber)

Let f be a Laurent polynomial in n variables with coefficients in $\mathbb{Z}$. Assume that f is convenient and nondegenerate. For p a prime number away from a finite set, one considers the exponential sums

$$S_j(f) = \sum_{x \in \mathbb{G}_m(\mathbb{F}_{p^j})^n} \exp\Big(\frac{2\pi i}{p} \operatorname{Tr}_{\mathbb{F}_{p^j}/\mathbb{F}_p}(f(x))\Big)$$

and the corresponding L function $L(f,t) = \exp\big(\sum_j S_j(f) t^j / j\big)$. It is known that, with these assumptions on f, $L(f,t)^{(-1)^{n-1}}$ is a polynomial, with coefficients in $\mathbb{Q}(\zeta_p)$. A polygon is attached to such a polynomial by considering the p-adic valuation of the coefficients. On the other hand, a polygon is attached to f by means of the Newton polyhedron defined by f, so is not difficult to compute. Adolphson and Sperber (1989) prove that the L-polygon lies above the Newton polygon. This is motivated by an analogous result of Mazur for the zeta function of a smooth algebraic variety (Mazur-Katz theorem). In the latter situation, the Newton polygon is replaced by a Hodge polygon, so it is natural to ask for an interpretation of the Newton polygon attached to f in terms of "Hodge data" attached to the connection $\mathrm{d} + \mathrm{d}f$. This was also a motivation of Deligne.

1.e. Extended motivic exponential $\mathscr{D}$-modules (Kontsevich)

In [**Kon09**], M. Kontsevich defines the category of extended motivic-exponential $\mathscr{D}$-modules on smooth algebraic varieties over a field k of characteristic zero as the minimal class which is closed under extensions, sub-quotients, push-forwards and pull-backs, and contain all $\mathscr{D}_X$-modules of type $(\mathscr{O}_X, \mathrm{d} + \mathrm{d}f)$ for $f \in \mathscr{O}(X)$. Recall that a $\mathscr{D}_X$-module is a $\mathscr{O}_X$-module equipped with an integrable connection.

The natural question which arises, in the case the field k is equal to $\mathbb{C}$, is to endow the objects of this category of a "Hodge filtration".

1.f. Mirror symmetry (Kontsevich). For some Fano manifolds (or orbifolds), one looks for the mirror object as regular function $f : X \to \mathbb{A}^1$. Kontsevich conjectures that the Hodge numbers of the Fano manifolds can be read on a "Hodge filtration" on the cohomology $H^k_{\mathrm{dR}}(X, \mathrm{d} + \mathrm{d}f)$. This filtration is nothing but the irregular Hodge filtration as suggested by Deligne and extended in arbitrary dimension by J.-D. Yu.

2. The Kontsevich bundles

I will give an answer to a question Kontsevich asked us in march 2013, at IHES, after a lecture by Deligne.

2.a. Re-interpreting the mixed Hodge theory of an open smooth algebraic variety. Let X be a smooth complex projective variety, let D be a reduced divisor with normal crossings in X and set $U = X \smallsetminus D$. According to [**Del70**], we have

$$\dim \boldsymbol{H}^k\big(X, (\Omega_X^\bullet(\log D), \mathrm{d})\big) = \dim H^k(U, \mathbb{C})$$

and for each p, denoting by $\sigma^{\geqslant p}$ the stupid truncation of complexes, [**Del71**] tells us that

$$\boldsymbol{H}^k\big(X, \sigma^{\geqslant p}(\Omega_X^\bullet(\log D), \mathrm{d})\big) \longrightarrow \boldsymbol{H}^k\big(X, (\Omega_X^\bullet(\log D), \mathrm{d})\big)$$

is injective, its image defining the Hodge filtration of the latter space. The dimensions of the corresponding graded spaces are denoted here by $h^{p,k-p}(X,D) = \dim \boldsymbol{H}^{k-p}(X, \Omega_X^p(\log D))$. Let $\mathbb{P}^1$ denote the projective line covered by the charts $\mathbb{A}^1_v$ (coordinate v) and $\mathbb{A}^1_u$ (coordinate u) with $u = 1/v$ on the intersection $\mathbf{G}_{\mathrm{m}}$. We can thus define a filtered bundle

$$(\boldsymbol{H}^k(X,D), F^\bullet\boldsymbol{H}^k(X,D))$$

on $\mathbb{P}^1$ by gluing

$$\boldsymbol{H}^k\big(X, \sigma^{\geqslant p}(\Omega_X^\bullet(\log D)[v], \mathrm{d})\big) \quad \text{with} \quad \boldsymbol{H}^k\big(X, \sigma^{\geqslant p}(\Omega_X^\bullet(\log D)[u], u\mathrm{d})\big)$$

with the isomorphisms induced by

$$\Omega_X^j(\log D)[u,u^{-1}] \xrightarrow{\ v^j\ } \Omega_X^j(\log D)[v,v^{-1}],$$

which commute with the differentials. The pth graded bundle is easily seen to be isomorphic to $\mathscr{O}_{\mathbb{P}^1}(p)^{h^{p,k-p}}$, which proves both that $\boldsymbol{H}^k(X,D)$ decomposes as $\bigoplus_p \mathscr{O}_{\mathbb{P}^1}(p)^{h^{p,k-p}}$ and $F^\bullet\boldsymbol{H}^k(X,D)$ is its *Harder-Narasimhan filtration*. Moreover, this bundle comes equipped with a connection, induced by the action of ∂_v in the v-chart and by the action of $u\partial_u - k$ on $\Omega_X^k(\log D)[u]$ in the u-chart. This connection has thus a unique pole at $u=0$, and it is logarithmic. The Harder-Narasimhan filtration is stable under the connection.

2.b. The Kontsevich bundles attached to f. Let now $f: X \to \mathbb{P}^1$ be a morphism to the projective line $\mathbb{P}^1 = \mathbb{A}^1_t \cup \{\infty\}$, and assume that $U \subset f^{-1}(\mathbb{A}^1_t)$. M. Kontsevich has associated to these data and to $k \geqslant 0$ the subsheaf Ω_f^k of $\Omega_X^k(\log D)$ consisting of logarithmic k-forms ω such that $\mathrm{d}f \wedge \omega$ remains a logarithmic $(k+1)$-form, a condition which carries on the restriction of ω to a neighbourhood of the reduced pole divisor $P_{\mathrm{red}} = f^{-1}(\infty)$. For each $\alpha \in [0,1)$, let us denote by $[\alpha P]$ the divisor supported on P_{red} with multiplicity $[\alpha e_i]$ on the component P_i of $P := f^*(\infty)$ with multiplicity e_i. One can also define a subsheaf $\Omega_f^k(\alpha)$ of $\Omega_X^k(\log D)([\alpha P])$ by the condition that $\mathrm{d}f \wedge \omega$ is a section of $\Omega_X^{k+1}(\log D)([\alpha P])$, so that the case $\alpha = 0$

corresponds to the case considered by Kontsevich. Clearly, only those α such that $\alpha e_i \in \mathbb{Z}$ for some i are relevant.

For each pair $(u,v) \in \mathbb{C}^2$ and each $\alpha \in [0,1)$, one can form a complex $(\Omega_f^\bullet(\alpha), u\mathrm{d} + v\mathrm{d}f)$.

Theorem 2.1 **(Kontsevich, E-S-Y, M. Saito, cf. [ESY13]).** *For each k and $\alpha \in [0,1)$, $\dim \boldsymbol{H}^k\big(X, (\Omega_f^\bullet(\alpha), u\mathrm{d}+v\mathrm{d}f)\big)$ is independent of $(u,v)\in\mathbb{C}^2$ and α. It is equal to*

$$\dim \boldsymbol{H}^k\big(U, (\Omega_U^\bullet, \mathrm{d}+\mathrm{d}f)\big) = \sum_{p+q=k} \dim \boldsymbol{H}^q(X, \Omega_f^p(\alpha)).$$

The *irregular Hodge numbers* are then defined as

$$h_\alpha^{p,q}(f) = \dim \boldsymbol{H}^q(X, \Omega_f^p(\alpha)). \tag{2.2}$$

We have $h_\alpha^{p,q}(f) \neq 0$ only if $p,q \geqslant 0$ and $p+q \leqslant 2\dim X$.

Let us note that, if f is the constant map, we recover the results mentioned above. In a similar way we can therefore construct the *Kontsevich bundles* $\mathscr{K}_\alpha^k$ on $\mathbb{P}^1$. We set

$$\begin{aligned} \mathscr{K}^k_{\alpha\,|\mathbb{A}^1_v} &:= \boldsymbol{H}^k\big(X, (\Omega_f^\bullet(\alpha)[v], \mathrm{d}+v\mathrm{d}f)\big),\\ \mathscr{K}^k_{\alpha\,|\mathbb{A}^1_u} &:= \boldsymbol{H}^k\big(X, (\Omega_f^\bullet(\alpha)[u], u\mathrm{d}+\mathrm{d}f)\big). \end{aligned} \tag{2.3}$$

Using the isomorphism $\mathbb{C}[u,u^{-1}] \xrightarrow{\sim} \mathbb{C}[v,v^{-1}]$ given by $u\mapsto v^{-1}$, we have a natural quasi-isomorphism

$$u^\bullet : (\Omega_f^\bullet(\alpha)[u,u^{-1}], u\mathrm{d}+\mathrm{d}f) \xrightarrow{\sim} (\Omega_f^\bullet(\alpha)[v,v^{-1}], \mathrm{d}+v\mathrm{d}f) \tag{2.4}$$

induced by the multiplication by u^{-p} on the pth term of the first complex. Since we know by the above mentioned results that both modules $\mathscr{K}^k_{\alpha\,|\mathbb{A}^1_v}, \mathscr{K}^k_{\alpha\,|\mathbb{A}^1_u}$ are free over their respective ring $\mathbb{C}[v]$ or $\mathbb{C}[u]$, the identification

$$\boldsymbol{H}^k(u^\bullet) : \boldsymbol{H}^k\big(X, (\Omega_f^\bullet(\alpha)[u,u^{-1}], u\mathrm{d}+\mathrm{d}f)\big) \simeq \boldsymbol{H}^k\big(X, (\Omega_f^\bullet(\alpha)[v,v^{-1}], \mathrm{d}+v\mathrm{d}f)\big)$$

allows us to glue these modules as a bundle $\mathscr{K}_\alpha^k$ on $\mathbb{P}^1$. The E_1-degeneration property can be expressed by the injectivity

$$\begin{aligned} \boldsymbol{H}^k\big(X, \sigma^{\geqslant p}(\Omega_f^\bullet(\alpha)[v], \mathrm{d}+v\mathrm{d}f)\big) &\hookrightarrow \boldsymbol{H}^k\big(X, (\Omega_f^\bullet(\alpha)[v], \mathrm{d}+v\mathrm{d}f)\big),\\ \boldsymbol{H}^k\big(X, \sigma^{\geqslant p}(\Omega_f^\bullet(\alpha)[u], u\mathrm{d}+\mathrm{d}f)\big) &\hookrightarrow \boldsymbol{H}^k\big(X, (\Omega_f^\bullet(\alpha)[u], u\mathrm{d}+\mathrm{d}f)\big), \end{aligned}$$

where $\sigma^{\geqslant p}$ denotes the stupid truncation. Since this truncation is compatible with the gluing $u^\bullet$, this defines a filtration $\sigma^{\geqslant p}\mathscr{K}_\alpha^k$. The pth graded bundle is then isomorphic to $\mathscr{O}_{\mathbb{P}^1}(p)^{h_\alpha^{p,k-p}}$, so this filtration is the Harder-Narasimhan filtration $F^\bullet\mathscr{K}_\alpha^k$ and the Birkhoff-Grothendieck decomposition of $\mathscr{K}_\alpha^k$ reads

$$\mathscr{K}_\alpha^k \simeq \bigoplus_{p=0}^k \mathscr{O}_{\mathbb{P}^1}(p)^{h_\alpha^{p,k-p}}.$$

In particular, all components are nonnegative and we have

$$\deg \mathscr{K}_\alpha^k = \sum_{p=0}^{k} p\cdot h_\alpha^{p,k-p}.$$

However, defining a connection on this bundle is not straightforward, due to the term $\mathrm{d}f$. The main result of this article, which answers questions of M. Kontsevich, defines this connection and extends the behaviour seen for $\boldsymbol{H}^k(X,D)$ with respect to the Harder-Narasimhan filtration.

Theorem 2.5.

(1) *There exists a natural connection ∇ on $\mathscr{K}_\alpha^k$, which has a logarithmic pole at $v=0$ and a pole of order at most two at $u=0$, and the eigenvalues of its residue $\mathrm{Res}_{v=0}\nabla$ belong to $(-\alpha-1,-\alpha]$.*

(2) *The Harder-Narasimhan filtration $F^\bullet\mathscr{K}_\alpha^k$ satisfies the Griffiths transversality property with respect to ∇, that is, $\nabla F^p\mathscr{K}_\alpha^k\subset F^{p-1}\mathscr{K}_\alpha^k$.*

(3) *On each generalized eigenspace of $\mathrm{Res}_{v=0}\nabla$ the nilpotent part of the residue strictly shifts by -1 the filtration naturally induced by the Harder-Narasimhan filtration.*

The fibre of $F^p\mathscr{K}_\alpha^k$ at a point $(v:1)$ with $v\neq 0$ is the pth term of the irregular Hodge filtration relative to the function vf. When restricted to $\mathbb{C}^*$, the filtered bundle $F^\bullet\mathscr{K}_\alpha^k$ together with its connection can therefore be regarded as a variation of the irregular Hodge filtration with respect to the rescaling parameter v, and 2.5(3) asserts a limiting behaviour when $v\to 0$ similar to the behaviour of a variation of Hodge structure. It is important to note that one cannot expect a similar behaviour when $v\to\infty$, since the singularity of the connection is irregular in general.

3. Idea for the proof of Theorem 2.5

Let us give a very brief sketch. It consists in constructing the connection as a Gauss-Manin connection for a proper morphism, but with a starting $\mathscr{D}$-module which has irregular singularities. The main problem will then be to ensure that, when dealing with Hodge type filtrations, the E_1-degeneration property still holds for the filtered Gauss-Manin connection.

(1) Introduce $\mathcal{E}^{vf}=\big(\mathscr{O}_{X\times\mathbb{A}^1_v}(*(P_{\mathrm{red}}\times\mathbb{A}^1_v)),\mathrm{d}+\mathrm{d}(vf)\big)$, together with its filtration

$$F_{\alpha+p}\mathcal{E}^{vf}=\Big(\sum_{k\leqslant p}F_k\mathscr{O}_X(*P_{\mathrm{red}})\big([(\alpha+p)P]\big)v^k\Big)[v]\cdot e^{vf}$$

and $\mathcal{E}:=\mathcal{E}^{vf}(*(H\times\mathbb{A}^1_v))$ with filtration

$$F_{\alpha+p}\mathcal{E}=\sum_{q+q'\leqslant p}F_x\mathscr{O}_X(*H)\cdot F_{\alpha+q'}\mathcal{E}^{vf}.$$

(2) On $\mathbb{A}^1_v \smallsetminus \{v=0\}$, the push-forward filtered $\mathscr{D}$-module $\mathscr{H}^k q_+(\mathcal{E}, F_{\alpha+\bullet}\mathcal{E})$ by the projection $q : X \times (\mathbb{A}^1_v \smallsetminus \{v=0\}) \to (\mathbb{A}^1_v \smallsetminus \{v=0\})$ satisfies the E_1-degeneration, according to Th. 2.1, and we recover $(\sigma^\bullet \mathscr{K}^k_\alpha)_{|\mathbb{A}^1_v \smallsetminus \{v=0\}}$.

(3) The limiting filtered $\mathscr{D}$-module when $v \to 0$ of $(\mathcal{E}, F_{\alpha+p}\mathcal{E})$ underlies a mixed Hodge module. Using results of M. Saito on filtered $\mathscr{D}$-modules, one obtains the desired behaviour when $v \to 0$.

(4) The behaviour at $v = \infty$ needs an adaptation of M. Saito's technique, since one does not have a Hodge limiting object. Here, one uses Brieskorn lattices.

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