Some cohomological properties of semisimple representations of π_1

The Hard Lefschetz Theorem

- X: a ***compact Kähler manifold*** of dimension n, with Kähler form ω.
- $\mathcal{L}$: a ***local system of coefficients*** of rank d on X

$\Leftrightarrow$ a ***linear representation*** $\pi_1(X) \longrightarrow \mathrm{GL}_d(\mathbb{C})$

$\Leftrightarrow$ a ***locally constant sheaf*** of rank d $\mathbb{C}$-vector spaces

$\Leftrightarrow$ a ***holomorphic vector bundle*** V of rank d on X

with a ***flat holomorphic connection*** ∇.

$$\mathcal{L} = \mathrm{Ker}\, \nabla : V \longrightarrow V.$$

Hard Lefschetz Theorem

$$\omega^k \wedge : H^{n-k}(X, \mathcal{L}) \xrightarrow{\sim} H^{n+k}(X, \mathcal{L}) \quad \forall\, k = 1, \ldots, n.$$

The Hard Lefschetz Theorem is known to be true if

- $\mathcal{L}$ is the constant system of coefficients (Harmonic theory, Hodge).
- $\mathcal{L}$ is a ***unitary*** representation of $\pi_1(X)$ (same proof).
- $\mathcal{L}$ underlies a ***variation of polarized Hodge structures*** (Deligne).
- $\mathcal{L}$ is a ***semisimple*** local system on X (Simpson)

Variations of polarized Hodge structures

(V, ∇) a flat holomorphic vector bundle.

$$H = \mathcal{C}_X^\infty \bigotimes_{\mathcal{O}_X} V$$

$$D_V = D'_V + D''_V$$

the flat connection on the C^∞ bundle H obtained from ∇, so that

$$(V, \nabla) = (\operatorname{Ker} D''_V, D'_V).$$

One says that this is a ***variation of polarized complex Hodge structures of weight*** w if

H is a vector bundle equipped with a C^∞-decomposition

$$H = \bigoplus_{p\in\mathbb{Z}} H^{p,w-p} \qquad (w \in \mathbb{Z}),$$

and with a nondegenerate Hermitian form k such that

- the decomposition is k-***orthogonal***,
- $h \overset{\text{def}}{=} (-1)^p i^{-w} k$ on $H^{p,w-p}$ is ***positive definite***,
- and (Griffiths'***transversality relations***)

$$D'_V(H^{p,q}) \subset \left(H^{p,q} \oplus H^{p-1,q+1}\right) \otimes \mathcal{E}_X^{(1,0)}$$
$$D''_V(H^{p,q}) \subset \left(H^{p,q} \oplus H^{p+1,q-1}\right) \otimes \mathcal{E}_X^{(0,1)}$$

Perverse sheaves and the Decomposition Theorem

- Z: an irreducible projective variety.
- Z^o: a smooth Zariski open set in Z.
- $\mathcal{L}$: an irreducible local system on Z^o.

Irreducible perverse sheaf on Z

$$\Longleftrightarrow$$

Intersection complex $\mathrm{IC}_Z(\mathcal{L})$ (Goreski-MacPherson).

Example

If $Z = S$ is a compact Riemann surface, $S^o \xhookrightarrow{j} S$,

$$\mathrm{IC}_S(\mathcal{L}) = j_*\mathcal{L}$$

***Theorem* (Cattani-Kaplan-Schmid, Kashiwara-Kawai, M. Saito)**

The Hard Lefschetz Theorem holds for $\mathbf{IH}^*(Z, \mathcal{L})$ *if* $\mathcal{L}$ *underlies a* ***variation of polarized Hodge structures***.

***Decomposition Theorem* (Cattani-Kaplan-Schmid, Kashiwara-Kawai, M. Saito)**

Let $f : Z \longrightarrow Y$ be a morphism between projective varieties, then the direct image complex

$$Rf_* \mathrm{IC}_Z(\mathcal{L})$$

*decomposes as a finite direct sum of **irreducible perverse sheaves***

$$\bigoplus_i \mathrm{IC}_{Y_i}(\mathcal{L}_i)[n_i]$$

*if $\mathcal{L}$ underlies a **variation of polarized Hodge structures**.*

(complex analogue of the arithmetic Decomposition Theorem of Beilinson-Bernstein-Deligne-Gabber).

2 steps in the proof:

(1) To construct a category of ***polarizable Hodge*** $\mathcal{D}$***-modules*** (pure objects), such that the decomposition theorem holds in this category.

(2) To show that, for an irreducible perverse sheaf $\mathrm{IC}_Z(\mathcal{L})$ on an irreducible projective variety Z, such that $\mathcal{L}$ underlies a variation of Hodge structures, the perverse sheaf can be lifted to a polarizable Hodge $\mathcal{D}$-module.

A conjecture by M. Kashiwara

Conjecture

The properties of polarizable Hodge $\mathcal{D}$-modules which do not explicitly involve the Hodge filtration remain valid when one replaces ***"polarizable Hodge $\mathcal{D}$-module"*** *with* ***"semisimple holonomic $\mathcal{D}$-module"*** .

Main Theorem

- X: *a smooth projective manifold,*
- S: *a compact Riemann surface,*
- $\mathcal{L}$: *a* ***semisimple*** *locally constant sheaf on* X,
- $f : X \longrightarrow S$ *a holomorphic map.*

Then, the direct image complex $Rf_*\mathcal{L}$ *decomposes, in the derived category, in a direct sum of irreducible perverse sheaves (with shifts) on* S.

Harmonic metrics and Higgs bundles

Theorem **(K. Corlette 1988)**
Let (V, ∇) *be a holomorphic vector bundle equipped with a flat connection on a compact Kähler manifold* X. *Then,* (V, ∇) *has a* ***harmonic metric*** h *if and only if the locally constant sheaf* $\mathcal{L}$ *of its horizontal sections is* ***semisimple***.

Let h be any metric on H.
One may then find a unique connection

$$D_E = D'_E + D''_E \quad \text{on} \quad H$$

which is a metric connection for h such that, if

$$\theta'_E = D'_E - D'_V, \qquad \theta''_E = D''_E - D''_V,$$

the $(0,1)$-form θ''_E with values in $\mathrm{End}(H)$ is the h-***adjoint*** of the $(1,0)$-form θ'_E.

Definition

The metric $\boldsymbol{h}$ *is* ***harmonic*** *relatively to the flat holomorphic bundle* $(\boldsymbol{V}, \boldsymbol{\nabla})$ *if*

$$(D''_E + \theta'_E)^2 = 0$$

that is,

$$D''^2_E = 0, \qquad D''_E(\theta'_E) = 0, \qquad \theta'_E \wedge \theta'_E = 0.$$

$$E = \mathrm{Ker}\, D''_E : H \to H$$

E is a holomorphic bundle equipped with a 1-form θ'_E with values in $\mathrm{End}(E)$, which satisfies

$$\theta'_E \wedge \theta'_E = 0$$

θ'_E is a ***Higgs field*** for E.

The flatness of D_V also imposes relations as

$$D'^2_E = 0, \qquad D'_E(\theta'_E) = 0, \qquad D''_E(\theta''_E) = 0.$$

Consequently, for all $z_o \in \mathbb{C}$, the operator

$$D''_E + z_o \theta''_E$$

is a complex structure on H.

The associated holomorphic bundle

$$V_{z_o} = \mathrm{Ker}(D''_E + z_o \theta''_E)$$

is equipped, if $z_o \neq 0$, with a ***flat holomorphic connection***

$$\nabla_{z_o} = D'_E + \frac{1}{z_o}\theta'_E.$$

For $z_o = 1$ one recovers (V, ∇).

If h is harmonic, the identities of Kähler geometry apply to the mixed operators

$$\mathcal{D}_\infty = D'_E + \theta''_E, \quad \mathcal{D}_0 = D''_E + \theta'_E$$

$$D_V = \mathcal{D}_\infty + \mathcal{D}_0$$

$$\Delta_{D_V} = 2\Delta_{\mathcal{D}_\infty} = 2\Delta_{\mathcal{D}_0}.$$

Simpson deduces from them the ***Hard Lefschetz Theorem***.

Example

The case of variations of Hodge structures.

Family of flat bundles (V_{z_o}, ∇_{z_o}) for $z_o \neq 0$.

One also has operators which satisfy the identities of Kähler geometry:

$$\mathcal{D}_{z_o} = (z_o D'_E + \theta'_E) + (D''_E + z_o \theta''_E) = z_o \mathcal{D}_\infty + \mathcal{D}_0$$
$$\Delta_{z_o} = (1 + |z_o|^2) \Delta_{D_V}.$$

$\Longrightarrow$ all locally constant sheaves $\mathcal{L}_{z_o}$, $z_o \neq 0$, have the same cohomology.

Variations of polarized twistor structures

C. Simpson presents this notion by stating the

***Meta theorem* (C. Simpson)**

If the words ***"Hodge structure"*** *are replaced with* ***"twistor structure"*** *in the assumptions and conclusions of any theorem in Hodge theory, one still gets a true statement, the proof of which is analogous to that of its model.*

The notion of a twistor structure is a

"deshomogeneization"

of that of a Hodge structure, which has a notion of ***weight***.

The Hodge graduation on a bundle on $\boldsymbol{X}$ is replaced with the extension of this bundle as a bundle on $\boldsymbol{X} \times \mathbb{P}^1$.

The conjugation $\boldsymbol{H}^{q,p} = \overline{\boldsymbol{H}^{p,q}}$ is replaced with a geometric conjugation.

Polarizable twistor $\mathcal{D}_X$-modules

One may define the notion of a ***polarizable twistor*** $\mathcal{D}_X$***-module***.

The category of ***polarizable twistor*** $\mathcal{D}_X$***-modules*** on a smooth projective manifold X is ***abelian*** and ***semisimple***.

Theorem

If $f : X \to Y$ *is a morphism between smooth projective manifolds, the direct image of a* ***polarizable twistor*** $\mathcal{D}_X$***-module*** *decomposes in* ***direct sum*** *of its cohomology modules, which are* ***polarizable twistor*** $\mathcal{D}_Y$***-modules*** *(the weight is obtained in the usual way).*

Conjecture

If X is smooth projective, the restriction functor to $z = 1$ is an equivalence between the category of ***polarizable twistor $\mathcal{D}_X$-modules*** *of weight 0 and that of* ***semisimple*** *perverse sheaves on X.*

According to C. Simpson's work, this conjecture is true in the following cases:

- **Locally constant sheaves.**
- **X is a compact Riemann surface.**

This implies the ***Main Theorem***.

Geometric conjugation

Let $f(x)$ be a holomorphic function on an open set of X. Its conjugate

$$\overline{f}(x) \overset{\text{def}}{=} \overline{f(x)}$$

is a holomorphic function on the ***complex conjugate manifold*** $\overline{X}$.

Let $g(z)$ be a holomorphic function on an open set U of $\mathbb{P}^1$. Its ***"conjugate"***

$$\overline{g}(z) \overset{\text{def}}{=} g(-1/z)$$

is a holomorphic function on the ***"complex conjugate set"*** $\overline{U}$.

Mix these two notions to define $\overline{f}(x,z)$.

If $\mathcal{F}$ is a $\mathcal{O}_{X\times U}$-module, then $\overline{\mathcal{F}}$ is a $\mathcal{O}_{\overline{X}\times\overline{U}}$-module.

Variations of polarized twistor structures

$\mathbb{P}^1 = U_0 \cup U_\infty$, z is the coordinate on U_0,
$\mathbf{A}$ is an open annulus $\rho < |z| < 1/\rho$, with $0 < \rho < 1$.

- Two locally free $\mathcal{O}_{X\times U_0}$-modules $\mathcal{H}', \mathcal{H}''$ of rank d equipped with a ***flat connection***

$$\nabla : \mathcal{H}' \longrightarrow \mathcal{H}' \bigotimes_{\mathcal{O}_{X\times U_0}} \frac{1}{z}\,\Omega^1_{X\times U_0}$$

- and a nondegenerate $\mathcal{C}^\infty_X(\mathcal{O}(\mathbf{A}))$-bilinear pairing (***glueing***)

$$C : \pi_{\mathbf{A}*}\mathcal{H}' \bigotimes_{\mathcal{O}(\mathbf{A})} \overline{\pi_{\mathbf{A}*}\mathcal{H}''} \longrightarrow \mathcal{C}^\infty_X(\mathcal{O}(\mathbf{A}))$$

  compatible with the connection.

One gets a bundle $\widetilde{\mathcal{H}}$ on $\mathbb{P}^1$ by glueing $\mathcal{H}'^*$ and $\overline{\mathcal{H}''}$.

$\widetilde{\mathcal{H}}$ is C^∞ with respect to variables of X and holomorphic with respect to the variable of $\mathbb{P}^1$.

Twistor condition of weight w:
the restriction to any $\{x\} \times \mathbb{P}^1$ is $\simeq \mathcal{O}_{\mathbb{P}^1}(w)^d$.

Polarization:

an isomorphism ***compatible with the connections***

$$S : \mathcal{H}'' \xrightarrow{\sim} \mathcal{H}'$$

which satisfies a ***positivity condition***:

If H is the bundle $\pi_*\widetilde{\mathcal{H}}(-w)$, the polarization defines a ***hermitian metric*** h on H.

The bundle $\mathcal{H}'_{|z=1}$ is a holomorphic subbundle of H equipped with a ***flat holomorphic connection***.

The bundle $\mathcal{H}'_{|z=0}$ is a holomorphic subbundle of H equipped with a ***Higgs field***.

Weil operator: $\widetilde{\mathcal{T}} = (\mathcal{H}', \mathcal{H}'', (iz)^{-w}C)$ has weight $\mathbf{0}$.

Tate twist: $\mathcal{T}(k) = (\mathcal{H}', \mathcal{H}'', (iz)^{-2k}C)$.

Hermitian duality:

$$\mathcal{T}^* = (\mathcal{H}'', \mathcal{H}', C^*) \quad \text{with} \quad C^*(x, \overline{y}) = \overline{C(y, \overline{x})}.$$

$$w(\mathcal{T}^*) = -w(\mathcal{T}),$$

$$w(\mathcal{T}(k)) = w(\mathcal{T}) - 2k,$$

$$\mathcal{T}(k)^* = \mathcal{T}^*(-k).$$

***Theorem* (C. Simpson)**

*Such a metric is **harmonic**. Conversely, any harmonic metric on H is obtained in this way.*

Hodge-Simpson Theorem

If X is compact Kähler, L the Lefschetz operator, and if

$$\mathcal{T} = (\mathcal{H}', \mathcal{H}'', C)$$

is a variation of polarized twistor structure of weight w on X, then for any $k \geqslant 0$, the primitive part of the kth de Rham cohomology is a polarized twistor structure of weight $w + k$.