THE BIRKHOFF PROBLEM AND FROBENIUS MANIFOLDS

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Introduction

A Frobenius structure on a manifold[(1)] M consists of the data of two objects on the tangent bundle TM: on the one hand a symmetric nondegenerate bilinear form g (we will call g a metric for short) which is *flat*, and a commutative and associative product $\star$ with unit on the other hand. These two objects are subject to natural compatibility relations.

As a consequence, there exist two kinds of local coordinate systems on such a manifold: on the one hand, flat coordinates with respect to the metric and on the other hand coordinates (x_i) (called canonical) in which the products of basic vector fields $\partial_{x_i} \star \partial_{x_j}$ are as simple as possible (*e.g.*, . $\partial_{x_i} \star \partial_{x_j} = \delta_{ij}\partial_{x_i}$, where δ_{ij} is the Kronecker symbol).

One of the many interesting features of Frobenius manifolds is that they produce various transcendantal functions by considering local coordinate changes going from a system of the first kind to a system of the second kind.

Two main families of examples are known:

• In the first one, canonical coordinates are naturally given, the flat structure is hidden and has to be revealed. The methods developed in this talk apply essentially to this kind of examples. The manifold is then the parameter space of a universal unfolding or a moduli space, which hence carries an affine structure. We owe it to K. Saito to have developed general tools (infinitesimal period mapping and primitive forms) to show the existence of such a structure in the base space of the miniversal unfolding of a holomorphic function with an isolated singularity. M. Saito has given complete arguments, using Hodge theory.

(1)In this talk, the manifolds are complex analytic and the mappings are holomorphic

• If on the other hand the flat structure is trivialized, the data of the associative and commutative product $\star$ is locally equivalent to the data of a function, called a potential, satisfying a system of nonlinear differential equations, also called WDVV. This approach comes from B. Dubrovin, who analysed in detail such structures, making the link with the existence of solutions to WDVV equations. This point of view sheds new light on the examples of the first kind. It is also particularly well suited to another family of examples, namely quantum cohomology of some algebraic manifolds, where it is deeply related to enumerative problems like counting the number of rational curves of certain kind on such manifolds. More recently, Yu. Manin brought into evidence the analogy between the relations on the coefficients of the Taylor expansion of a potential satisfying WDVV and the combinatorics which describes the cohomology of the moduli spaces $\overline{\mathcal{M}}_{0,n}$ of stable rational curves with n marked points.

1. The Birkhoff problem

Given a holomorphic vector bundle E on a complex analytic manifold X, consider on the inverse image $F = p^*E$ by $p : D \times X \to X$, where D is a disc, a *flat meromorphic connection* $\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_{D\times X}[*(0 \times X)]$ with poles along $0 \times X$.

Is it possible to extend ∇ on $\widetilde{F} \overset{\text{def}}{=} \widetilde{p}^*E$, with $\widetilde{p} : \mathbf{P}^1 \times X \to X$ as a flat meromorphic connection with at most logarithmic poles along $\infty \times X$ (and no poles in $\mathbf{C}^* \times X$)?

The following result is due to B. Malgrange:

Theorem 1.1. *Let X be a 1-connected analytic manifold, x^o a point of X, and let (F, ∇) be a rank d vector bundle on $D \times X$ equipped with a flat meromorphic connection with poles along $\{0\} \times X$. Assume that there exists a solution $(\widetilde{F}^o, \widetilde{\nabla}^o)$ to the Birkhoff problem for the restriction (F^o, ∇^o) of (F, ∇) at $D \times \{x^o\}$.*

Then there exists an analytic hypersurface Θ of X such that the Birkhoff problem has a solution for (F, ∇) on $X - \Theta$ extending the given one at x^o.

What do the solutions to the Birkhoff problem look like? I assume in the following that the type of the pole at $0 \times X$ is 1, *i.e.*, . in local coordinates $(z, \boldsymbol{x})$ and in any local basis of F, the matrix of $\nabla_{\partial/\partial z}$ as a pole of order 2 along $z = 0$ and the matrices of $\nabla_{\partial/\partial x_i}$ have a pole of order 1.

The meromorphic connection ∇ on $\widetilde{F}$ gives rise to two kinds of objects on E:

(1) A *flat* holomorphic connection $\triangledown$ and a horizontal endomorphism R_∞: this comes from the logarithmic singularity at infinity.

(2) If one chooses a coordinate z on the chart centered at 0 of $\mathbf{P}^1$, one constructs a linear endomorphism Φ on E with values in Ω^1_X, which satisfies $\Phi \wedge \Phi = 0$, and an endomorphism R_0 which commutes with Φ, *i.e.*, . for any local vector field ξ, the endomorphisms R_0 and $\Phi(\xi)$ of E commute.

In fact on can write

$$\nabla = p^* \triangledown + \left(\frac{R_0}{z} - R_\infty\right) \frac{dz}{z} + \frac{\Phi}{z}$$

and the integrability of ∇ is equivalent to the following relations:

$$\triangledown^2 = 0, \quad \triangledown(R_\infty) = 0, \quad \Phi \wedge \Phi = 0, \quad [R_0, \Phi] = 0$$

$$\triangledown(\Phi) = 0, \quad \triangledown(R_0) + \Phi = [\Phi, R_\infty].$$

2. Saito structures and Frobenius structures on a complex analytic manifold

Let M be a complex analytic manifold of dimension d, let TM denote its tangent bundle, Θ_M the sheaf of holomorphic vector fields and Ω^1_M the sheaf of holomorphic 1-forms.

A *Saito structure* on M (without metric) consists of

(1) a *torsionless flat connection* $\triangledown$ on the tangent bundle TM,

(2) a 1-form Φ with values in $\mathrm{End}(TM)$, namely a section of the sheaf $\mathrm{End}(\Theta_M) \otimes_{\mathcal{O}_M} \Omega^1_M$, which is *symmetric* when considered as a bilinear map $\Theta_M \otimes_{\mathcal{O}_M} \Theta_M \to \Theta_M$;

(3) two global sections (vector fields) e and $\mathfrak{E}$ of Θ_M.

These data are subject to the following conditions:

(1) the meromorphic connection ∇ on the vector bundle $\widetilde{p}^* TM$ on $\mathbf{P}^1 \times M$ defined by

$$\nabla = \widetilde{p}^* \triangledown - \left(\frac{\Phi(\mathfrak{E})}{z} + \triangledown \mathfrak{E}\right) \frac{dz}{z} + \frac{\pi^* \Phi}{z}$$

is *integrable* (in other words, the previous relations are satisfied by $\triangledown$, Φ, $R_0 \overset{\text{def}}{=} -\Phi(\mathfrak{E})$ and $R_\infty \overset{\text{def}}{=} \triangledown \mathfrak{E}$);

(2) the vector field e (*identity vector field*) is $\triangledown$-horizontal, *i.e.*, . $\triangledown(e) = 0$, and satisfies $\Phi(e) = -\,\mathrm{Id}$.

A "metric" adapted to the Saito structure is a symmetric nondegenerrate bilinear form g on TM such that $\triangledown(g) = 0$ (hence $\triangledown$ is the Levi-Civita connection of g) and the g-adjoint of Φ is Φ itself.

Consequences. • One defines an $\mathcal{O}_M$-bilinear product $\star : \Theta_M \otimes \Theta_M \to \Theta_M$ by the formula

$$\xi \star \eta = -\Phi(\xi)(\eta).$$

The symmetry of Φ means that this product is *commutative*. Moreover, e *is the identity*. The property $\Phi \wedge \Phi = 0$ is then equivalent to the fact that the product is *associative*: in local coordinates $(x_1, \dots, x_d)$, if one puts $\Phi(\partial_{x_i}) = \Phi_i$, the property means that the Φ_i are endomorphisms of TM which pairwise commute; using commutativity one gets,

$$\begin{aligned} \partial_{x_i} \star (\partial_{x_j} \star \partial_{x_k}) &= \Phi_i \circ \Phi_k(\partial_{x_j}) \\ (\partial_{x_i} \star \partial_{x_j}) \star \partial_{x_k} &= \Phi_k \circ \Phi_i(\partial_{x_j}). \end{aligned}$$

If one has a metric g adapted to the Saito structure, the condition $\Phi^* = \Phi$ is equivalent to $g(\xi_1 \star \xi_2, \xi_3) = g(\xi_1, \xi_2 \star \xi_3)$ for all vector fields ξ_1, ξ_2, ξ_3.

• The structure of sheaf of commutative and associative rings with identity, given by the product $\star$ on the sheaf Θ_M of vector fields on M (with coefficients in $\mathcal{O}_M$) allows one to define a surjective morphism of $\mathcal{O}_M$-algebras

$$\mathrm{Sym}_{\mathcal{O}_M} \Theta_M \longrightarrow \Theta_M$$

and, as $\mathrm{Sym}_{\mathcal{O}_M} \Theta_M$ is nothing other than the algebra $\mathcal{O}_M[TM]$ of functions on T^*M which are polynomial in the fibres of $T^*M \to M$, one identifies $\mathrm{Specan}\,\Theta_M$ with a closed analytic subspace L of T^*M. As Θ_M is a locally free of finite type $\mathcal{O}_M$-module, the morphism $L \to M$ is finite and surjective, one has $\dim L = \dim M$ and L is Cohen-Macaulay in T^*M. Moreover, the Euler vector field $\mathfrak{E}$ defines a global section of Θ_M, hence a function on L.

If the endomorphism R_0 is, generically on M, regular semi-simple, the manifold L is reduced and *Lagrangian* in T^*M: as L is Cohen-Macaulay, the fact that it is reduced is shown on an open dense set, on which one can assume that R_0 is regular semi-simple, and follows then from the previous remark; the fact that L is then Lagrangian is shown by M. Audin.

Assume M is simply connected and R_0 is regular semi-simple at all points. In this situation the algebra structure on T_xM is semi-simple for all $x \in M$. The eigenvalues of R_0 define d functions $x_1, \dots, x_d$ on M and the manifold L is nothing other than the disjoint union of the graphs of the dx_i.

The Potential. B. Dubrovin remarked that, if $t_1, \dots, t_n$ denotes a local system of flat coordinates on M, with corresponding vector fields $\partial_{t_1}, \dots, \partial_{t_d}$, there exists locally a holomorphic function $F(t_1, \dots, t_d)$ such that for all $i, j, k = 1, \dots, n$ we have

$$\frac{\partial^3 F}{\partial_{t_i}\partial_{t_j}\partial_{t_k}} = g(\partial_{t_i} \star \partial_{t_j}, \partial_{t_k}).$$

Moreover, such a function is homogeneous with respect to the action of the Euler vector field $\mathfrak{E}$.

The fact that the product is associative is then equivalent to the fact that F satisfies a system of third order (nonlinear) partial differential equations, called (WDVV).

The system (WDVV) can be written

$$\sum_{k,n} \frac{\partial^3 F}{\partial_{t_i}\partial_{t_j}\partial_{t_k}} g^{k,n} \frac{\partial^3 F}{\partial_{t_\ell}\partial_{t_m}\partial_{t_n}} = \sum_{k,n} \frac{\partial^3 F}{\partial_{t_\ell}\partial_{t_j}\partial_{t_k}} g^{k,n} \frac{\partial^3 F}{\partial_{t_i}\partial_{t_m}\partial_{t_n}}$$

for $i, j, \ell, m = 1, \dots, d$, where the matrix $(g^{k,n})$ is the inverse of the matrix $\left(g\left(\partial_{t_\alpha}, \partial_{t_\beta}\right)\right)_{\alpha,\beta}$.

Conversely, the datum of a solution F of this system, which is homogeneous with respect to $\mathfrak{E}$ can define an associative product on Θ_M.

A *local Frobenius structure* on M is the datum of F satisfying (WDVV) and homogeneous with respect to $\mathfrak{E}$.

3. Examples

3.1. Isomonodromic deformations. Let $\widetilde{F}^o$ be the trivial bundle on $\mathbf{P}^1$ equipped with the connexion ∇^o with matrix

$$\left(\frac{B_0^o}{z} + B_\infty\right)\frac{dz}{z}$$

where B_0^o and B_∞ are in $M_d(\mathbf{C})$. Assume furthermore that

- $B_0^o = \operatorname{diag}(x_1^o, \dots, x_d^o)$ with $x_i^o \neq x_j^o$ for $i \neq j$,
- $B_\infty - (m/2)\,\mathrm{Id}$ is skewsymmetric, for some $m \in \mathbf{Z}$,
- there exists an eigenvector ω^o of B_∞, all entries of which are nonzero.

Let X be the complement of the diagonals in $\mathbf{C}^d$, with base point x^o, and let $(\widetilde{X}, \widetilde{x}^o)$ be its universal covering.

***Theorem 3.1* (B. Dubrovin).** *There exists an analytic hypersurface Θ_{ω^o} in $\widetilde{X}$ not going through $\widetilde{x}^o$ and a unique structure of a semi-simple Frobenius manifold on $\widetilde{X} - \Theta_{\omega^o}$ with initial values B_0^o and $-B_\infty$ for R_0 and R_∞, such that $e = \sum_i \partial_{x_i}$ and $\mathfrak{E} = \sum_i x_i \partial_{x_i}$. Any simply connected semi-simple Frobenius manifold is isomorphic to a simply connected open set of a Frobenius manifold of this kind.*

3.2. Unfoldings of singularities. Let $f(u_1,\dots,u_n)$ be a germ of analytic function with isolated singularities, or let $f(u_1,\dots,u_n)$ be a polynomial on $\mathbf{C}^n$ with isolated singularities and "noncharacteristic at infinity" for some compactification of $\mathbf{C}^n$. Consider the vector space

$$\mathcal{Q}_f \overset{\text{def}}{=} \frac{\mathbf{C}\{u_1,\dots,u_n\}}{\left(\frac{\partial f}{\partial u_1},\dots,\frac{\partial f}{\partial u_n}\right)} \quad \text{or} \quad \frac{\mathbf{C}[u_1,\dots,u_n]}{\left(\frac{\partial f}{\partial u_1},\dots,\frac{\partial f}{\partial u_n}\right)}$$

which has dimension μ. The following is due to K. Saito and M. Saito in the case of germs, and is proved by C.S. for the so called convenient nondegenerate polynomials.

Theorem 3.2. *There exists a Frobenius structure on a neighbourhood V of 0 in $\mathcal{Q}_f$ for which the initial data on $T_0V = \mathcal{Q}_f$ are:*

- *the identity $e(0)$ is the class of 1, and $\mathfrak{E}(0)$ is the class of f,*
- *the metric g restricted at T_0V is given by the Grothendieck residue.*

The proof uses two results of M. Saito: the existence of a particularly nice solution of the Birkhoff problem for the micro-local (or for the Fourier transform of the) Gauss-Manin system of f, and the existence of a "primitive form". One then applies the theorem of Malgrange.

3.3. Quantum cohomology. For some kind of smooth projective varieties V (*e.g.*, . Fano manifolds) one can define a potential F on a (formal) neighbourhood of 0 in the even cohomology $H^{2*}(V,\mathbf{C})$, called the *Gromov-Witten potential*, the coefficients of which are related to enumerative properties of parametrized rational curves on V. It can be shown that it satisfies (WDVV) when the metric is the Poincaré duality. This gives a (maybe formal) perturbation of the cup product on $H^{2*}(V,\mathbf{C})$, called the quantum cup product.

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