HODGE PROPERTIES OF SOME DIFFERENTIAL EQUATIONS WITH IRREGULAR SINGULARITIES

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by

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Abstract. Some standard differential equations with irregular singularities, like Airy or Kloosterman and their symmetric products, behave in a way similar to Gauss-Manin differential equations, and their de Rham cohomology underlie a mixed Hodge structure, possibly with finite monodromy, enabling the use of tame arithmetic methods to handle the associated exponential sums. The talk will mainly focus on the Airy case, after a joint work with Jeng-Daw Yu.

1. The Kloosterman and Airy differential equations, and their symmetric powers

1.a. The Kloosterman connection and its symmetric powers. Let $n \geqslant 1$ be an integer. Consider the function

$$f : \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}^n \longrightarrow \mathbb{A}^1, \quad f(z, x_1, \ldots, x_n) = x_1 + \cdots + x_n + \frac{z}{x_1 \cdots x_n}$$

and the diagram (with $\pi(z, x) = z$)

$$\begin{array}{ccc} & \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}^n & \\ {}^{\pi}\swarrow & & \searrow^{f} \\ \mathbb{G}_{\mathrm{m}} & & \mathbb{A}^1. \end{array}$$

The Kloosterman connection has a definition in terms of D-modules:

$$\mathrm{Kl}_{n+1} = \pi_+ E^f = \pi_+(\mathscr{O}_{\mathbb{G}_{\mathrm{m}}^{n+1}}, \mathrm{d} + \mathrm{d}f) = \mathrm{H}^0 \pi_+(\mathscr{O}_{\mathbb{G}_{\mathrm{m}}^{n+1}}, \mathrm{d} + \mathrm{d}f).$$

It is isomorphic to the free $\mathscr{O}_{\mathbb{G}_{\mathrm{m}}}$-module of rank $n+1$ with connection ∇ such that $\nabla_{z\partial_z}$ has the matrix

$$A_{n+1}(z) = \begin{pmatrix} 0 & 0 & \cdots & 0 & z \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

One checks that ∇ has a regular singularity at $z=0$ with monodromy being unipotent with only one Jordan block. The singularity at $z=\infty$ is irregular with pure slope $1/(n+1)$. We are interested in $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ which has rank $\binom{n+k}{k}$. It is known to be irreducible as a bundle with connection on $\mathbb{G}_{\mathrm{m}}$ (a differential Galois group argument).

1.b. The Airy connection and its symmetric powers. For $n \geqslant 2$, we consider the function

$$f : \mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1, \quad f(z,x) = \tfrac{1}{n+1}\, x^{n+1} - zx,$$

and the diagram (with $\pi(z,x) = z$)

$$\begin{array}{ccc} & \mathbb{A}^1_z \times \mathbb{A}^1_x & \\ {}^{\pi}\swarrow & & \searrow^{f} \\ \mathbb{A}^1_z & & \mathbb{A}^1 \end{array}$$

The Airy differential equation is defined in terms of D-modules:

$$\mathrm{Ai}_n = \mathrm{H}^0 \pi_+ E^f \simeq \mathrm{Coker}\Big[\mathbb{C}[x,z] \xrightarrow{\ \partial_x + (x^n - z)\ } \mathbb{C}[x,z]\Big]$$

Then Ai_n is a free $\mathscr{O}_{\mathbb{A}^1}$-module of rank n with connection ∇ such that ∇_{∂_z} has matrix $A_n(z)$.

One checks that ∇ has an irregular singularity at infinity and no other singularity. It has pure slope $n/(n+1)$ at infinity. We are interested in $\mathrm{Sym}^k \mathrm{Ai}_n$ which has rank $\binom{n-1+k}{k}$ and is irreducible as a bundle with connection on $\mathbb{A}^1$.

1.c. De Rham cohomology. We are mostly interested by the middle de Rham cohomologies

$$\mathrm{H}^1_{\mathrm{dR,mid}} := \mathrm{image}\big[\mathrm{H}^1_{\mathrm{dR,c}} \to \mathrm{H}^1_{\mathrm{dR}}\big].$$

Although $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ and $\mathrm{Sym}^k \mathrm{Ai}_n$ have an irregular singularity at infinity, we have the following Hodge result.

***Theorem A* (FSY, resp. SY).**

- *The middle de Rham cohomology* $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1})$ *underlies a natural pure Hodge structure of weight* $kn+1$.
- *The middle de Rham cohomology* $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{A}^1, \mathrm{Sym}^k \mathrm{Ai}_n)$ *underlies a natural pure* $\widehat{\mu}$*-Hodge structure of weight* $k+1$.

More precisely, we identify these middle de Rham cohomologies with the pure part of a suitable isotypic component of the de Rham cohomology of a variety with a group action, resp. and a finite automorphism.

1.d. $\mathsf{MHS}^{\widehat{\mu}}$**.** Objects of MHS consist of triples $((V_{\mathrm{dR}}, F^\bullet V_{\mathrm{dR}}), (V_{\mathrm{B}}, W_\bullet V_{\mathrm{B}}), \mathrm{comp})$:

- V_{dR} is a $\mathbb{C}$-vector space with a decreasing filtration $F^\bullet V_{\mathrm{dR}}$ indexed by $\mathbb{Z}$ (both possibly defined over a subfield K of $\mathbb{C}$),
- V_{B} is a finite dimensional $\mathbb{Q}$-vector space and $W_\bullet V_{\mathrm{B}}$ is a finite increasing filtration of it indexed by $\mathbb{Z}$,
- a *comparison isomorphism* $\mathrm{comp} : \mathbb{C} \otimes_{\mathbb{Q}} V_{\mathrm{B}} \xrightarrow{\sim} V_{\mathrm{dR}}$,

all subject to various conditions.

We now consider the category $\mathsf{MHS}^{\widehat{\mu}}$ of mixed Hodge structures with an automorphism of finite order. It is known (Scherk-Steenbrink, 1985) that $\mathsf{MHS}^{\widehat{\mu}}$ is endowed with a tensor structure, which is however not the natural one with respect to filtrations. If T is an automorphism of finite order m of V^{H}, we decompose its components with respect to eigenvalues:

- $(V_{\mathrm{B}}, W_\bullet)$ as $(V_{\mathrm{B},1}, W_\bullet) \oplus (V_{\mathrm{B},\neq 1}, W_\bullet)$,
- $(V_{\mathrm{dR}}, F^\bullet)$ as $\bigoplus_{\zeta^m=1} (V_{\mathrm{dR},\zeta}, F^\bullet)$ (over $\mathrm{K}(\zeta)$).

Define

$$W_\ell^{\widehat{\mu}} V_{\mathrm{B}} = W_\ell V_{\mathrm{B},1} \oplus W_{\ell-1} V_{\mathrm{B},\neq 1},$$
$$F_{\widehat{\mu}}^{p-a} V_{\mathrm{dR},\zeta} = F^p V_{\mathrm{dR},\zeta} \quad (\zeta = \exp(-2\pi \mathrm{i} a),\ a \in (-1, 0]).$$

Scherk-Steenbrink show that there exists a tensor structure $\star$ on $\mathsf{MHS}^{\widehat{\mu}}$ such that $W_\bullet^{\widehat{\mu}}, F_{\widehat{\mu}}^\bullet$ behave in the expected way.

***Theorem B* (FSY, resp. SY).**

- *The nonzero Hodge numbers* $\dim \mathrm{gr}_F^p \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ *are all equal to* 1*, and this occurs for*

$$p = 2, 4, \dots, k-1, \quad \textit{if } k \textit{ is odd},$$

and modified formulas for k even (one drops one or two values in the middle).

- *The nonzero $\widehat{\mu}$-Hodge numbers* $\dim \mathrm{gr}_F^{\mathsf{p}} \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{A}^1, \mathrm{Sym}^k \mathrm{Ai})$ *are all equal to* 1*, and this occurs for*

$$\mathsf{p} = \tfrac{1}{3}(k + 2i), \quad 1 \leqslant i \leqslant (k+1)/2 \quad \textit{if } k \textit{ is odd},$$

and similarly modified formulas for k even.

1.e. Motivations. For the Kloosterman connection (with $n = 1$): Conjectures of Broadhurst and Roberts on the L-function attached to the k-moments of Kloosterman sums. Their resolution (complete for k odd, with some indeterminacy if k is even, F-S-Y) make use of a theorem of Patrikis and Taylor, that needs the nonzero Hodge numbers are equal to 1 (Hodge-Tate weights equal to 1).

For Airy, we hope that an analogue of the technique used by F-S-Y can be extended to the case with an automorphism of finite order.

Remark. The theorem does not extend to $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ or $\mathrm{Sym}^k \mathrm{Ai}_n$ in general. For example, Yichen Qin (Polytechnique) has shown that it holds for $\mathrm{Sym}^k \mathrm{Kl}_3$ (with some other values for p) for $k \leqslant 9$, but breaks down if $k \geqslant 10$. There are also some examples for $\mathrm{Sym}^k \mathrm{Kl}_4$. For these values, the arithmetic consequences mentioned in the introduction also extend.

1.f. Remark on Hodge filtration with rational exponents. Such filtration occur in various settings:

- $\mathsf{MHS}^{\widehat{\mu}}$, as we have seen,
- the notion of arithmetic Hodge filtration, as introduced by Anderson (1986) as the Hodge realization of his theory of ulterior motives,
- the notion of irregular Hodge filtration, as introduced by Deligne in 1984 in this same room.

We will see that, in the present context, the first two filtrations are the same, and are a particular case of the third one.

More precisely, although the meromorphic bundles with connection $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ and $\mathrm{Sym}^k \mathrm{Ai}_n$ underlie a “variation of pure irregular Hodge structure”, their cohomology on $\mathbb{G}_{\mathrm{m}}$ or $\mathbb{A}^1$ underlie a usual mixed Hodge structure (or mixed Hodge structure with finite automorphism). This point of view appears to be effective for the computation of Hodge numbers.

2. ($\widehat{\mu}$-)Exponential mixed Hodge structures

2.a. Short preliminaries on mixed Hodge modules (M. Saito)

X: smooth quasi-proj. var. of dimension n over $\mathbb{C}$.

Category $\mathsf{MHM}(X)$: Objects are $M^{\mathrm{H}} := ((M, F^\bullet M), (\mathscr{F}_{\mathbb{Q}}, W_\bullet \mathscr{F}_{\mathbb{Q}}), \mathrm{comp})$, where

- M is a hol. $\mathscr{D}_X$-module (i.e., an $\mathscr{O}_X$-module with flat connection ∇, subject to suitable coherence and dimension properties, e.g. $\mathscr{O}_X$-locally free),
- $F^\bullet M$ is a (possibly infinite) filtration by coherent $\mathscr{O}_X$-modules such that $\nabla F^p M \subset \Omega^1_X \otimes F^{p-1} M$,
- $\mathscr{F}_{\mathbb{Q}}$ is a $\mathbb{Q}$-perverse sheaf on X^{an},
- $W_\bullet \mathscr{F}_{\mathbb{Q}}$ is a finite filtration by perverse subsheaves,
- $\mathrm{comp} : \mathbb{C} \otimes_{\mathbb{Q}} \mathscr{F}_{\mathbb{Q}} \xrightarrow{\sim} {}^{\mathrm{p}}\mathrm{DR}^{\mathrm{an}} M$ (comparison isomorphism),

subject to various compatibility conditions.

Example.

- ${}^{\mathrm{p}}\mathbb{Q}^{\mathrm{H}}_X = ((\mathscr{O}_X, \mathrm{d}), \text{triv. } F\text{-filtr.}, \mathbb{Q}_X[n], \mathrm{can})$ is a pure Hodge module of weight n.
- Admissible variations of MHS on $X \iff$ smooth objects of $\mathsf{MHM}(X)$.

Six operations and duality on $\mathsf{D}^{\mathrm{b}}(\mathsf{MHM}(X))$, lifting the six operations and duality in $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{Q}_X)$.

2.b. Exponential mixed Hodge modules (Kontsevich-Soibelman)

In this section, we set $X = \mathbb{A}^1$ with coordinate θ. We define EMHS as the full subcategory of $\mathsf{MHM}(\mathbb{A}^1)$ whose objects N^{H} satisfy $\mathbf{H}^k(\mathbb{A}^1, \mathscr{F}_{\mathbb{Q}}) = 0$ for all k (i.e., $k = 0, 1, 2$). The convolution $\star$ induces a tensor structure on EMHS, and there is a projector $\Pi : \mathsf{MHM}(\mathbb{A}^1) \to \mathsf{EMHS}$ defined by $N^{\mathrm{H}} \to N^{\mathrm{H}} \star ({}_{\mathrm{H}}j_! {}^{\mathrm{p}}\mathbb{Q}^{\mathrm{H}}_{\mathbb{G}_{\mathrm{m}}})$.

Then there is an embedding $\mathsf{MHS} \hookrightarrow \mathsf{EMHS}$, $V^{\mathrm{H}} \mapsto \Pi({}_{\mathrm{H}}i_* V^{\mathrm{H}})$. The essential image of MHS is $\mathsf{EMHS}^{\mathrm{cst}}$, consisting of objects of EMHS whose underlying perverse sheaf is constant on $\mathbb{G}_{\mathrm{m}}$. This is compatible with the tensor structure, if MHS is endowed with its natural tensor structure.

Similarly, there is an embedding $\mathsf{MHS}^{\widehat{\mu}} \hookrightarrow \mathsf{EMHS}$ whose essential image $\mathsf{EMHS}^{\widehat{\mu}}$ consists of objects whose underlying perverse sheaf is locally constant with finite monodromy on $\mathbb{G}_{\mathrm{m}}$.

Given an object of $\mathsf{EMHS}^{\widehat{\mu}}$, one recovers an object of $\mathsf{MHS}^{\widehat{\mu}}$ by taking its vanishing cycles at the origin.

Example. Let $f : \mathbb{A}^1 \to \mathbb{A}^1$ defined by $x \mapsto x^m$. The pushforward mixed Hodge module $f_* {}^{\mathrm{p}}\mathbb{Q}^{\mathrm{H}}_{\mathbb{A}^1}$ (pure of weight 1) gives rise to $\mathrm{H}^1(\mathbb{A}^1, x^m) \in \mathsf{EMHS}^{\widehat{\mu}}$ by applying Π. The associated $\widehat{\mu}$-mixed Hodge structure is $\phi_{x^m} {}^{\mathrm{p}}\mathbb{Q}^{\mathrm{H}}_{\mathbb{A}^1}$ (of dimension $m-1$) endowed with it finite monodromy.

***Definition* (De Rham fibre).** For $N^{\mathrm{H}} \in \mathsf{EMHS}$, the de Rham fibre is the $\mathbb{C}$-vector space

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1, N \otimes E^\theta) := \mathrm{Coker}[(\nabla + (\mathrm{d}\theta \otimes \mathrm{Id})) : N \to \Omega^1_{\mathbb{A}^1} \otimes N].$$

(Convolution induces $\otimes$ on $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1, N \otimes E^\theta)$, and $W_\bullet N$ induces $W_\bullet \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1, N \otimes E^\theta)$.)

***Theorem C* (S-Y & F-S-Y).** *To any object of* EMHS*, one can associate canonically a filtration* $F^\bullet_{\mathrm{irr}} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1, N \otimes E^\theta)$ *indexed by* $\mathbb{Q}$ *which is compatible with convolution and tensor product. Furthermore, for each* $V^{\mathrm{H}} \in \mathsf{MHS}^{\widehat{\mu}}$ *with image* $N^{\mathrm{H}} \in \mathsf{EMHS}^{\widehat{\mu}}$*, there exists an isomorphism of bi-filtered vector spaces, compatible with the tensor structures:*

$$(V_{\mathrm{dR}}, F^\bullet_{\widehat{\mu}} V_{\mathrm{dR}}, W^{\widehat{\mu}}_\bullet V_{\mathrm{dR}}) \simeq (\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1, N \otimes E^\theta), F^\bullet_{\mathrm{irr}}, W_\bullet).$$

2.c. Gauss-Manin exponential mixed Hodge modules. Let $f : X \to \mathbb{A}^1$ be a regular function X (smooth, quasi-proj., $\dim X = n$). The push-forwards $\mathscr{H}^r{}_{\mathrm{H}}f_!{}^{\mathrm{p}}\mathbb{Q}_X^{\mathrm{H}}$ and $\mathscr{H}^r{}_{\mathrm{H}}f_*{}^{\mathrm{p}}\mathbb{Q}_X^{\mathrm{H}}$ are objects of $\mathsf{MHM}(\mathbb{A}^1)$.

Definition **(Gauss-Manin exponential mixed Hodge modules)**
We associate with (X, f) the following exponential mixed Hodge structures:

$$\mathrm{H}^j_{\mathrm{c}}(X, f) = \Pi(\mathscr{H}^{j-n}{}_{\mathrm{H}}f_!{}^{\mathrm{p}}\mathbb{Q}_X^{\mathrm{H}}), \quad \mathrm{H}^j(X, f) = \Pi(\mathscr{H}^{j-n}{}_{\mathrm{H}}f_*{}^{\mathrm{p}}\mathbb{Q}_X^{\mathrm{H}}).$$

The de Rham fibres are $(? = \mathrm{c}, \varnothing)$:

$$\mathrm{H}^j_{\mathrm{dR},?}(X, f) = \mathbf{H}^j_?(X, (\Omega^\bullet_X, \mathrm{d} + \mathrm{d}f)).$$

Computation of the irregular Hodge filtration. Let $\overline{X}$ be smooth projective compactification of X such that $\overline{X} \smallsetminus X = D$ is a simple normal crossing divisor. We regard f as a rational function $f : X \dashrightarrow \mathbb{P}^1$. It has a pole divisor P with support $|P|$ and a zero divisor Z. The compactification $\overline{X}$ is called *non-degenerate w.r.t. f* (T. Mochizuki) if in some analytic neighbourhood U of $|P|$, $Z \cap U$ is reduced and non-singular, and $U \cap (Z \cup D)$ has normal crossings.

For $\mathsf{p} \in \mathbb{Q}$, consider the filtration

$$\begin{aligned} &F^{\mathsf{p}}_{\mathrm{Yu}}(\Omega^\bullet_{\overline{X}}(*D), \mathrm{d} + \mathrm{d}f) \\ &\quad = \Big\{0 \to \mathscr{O}_{\overline{X}}([-\mathsf{p}P]_+) \xrightarrow{\ \mathrm{d}+\mathrm{d}f\ } \Omega^1_{\overline{X}}(\log D)([(-\mathsf{p}+1)P]_+) \xrightarrow{\ \mathrm{d}+\mathrm{d}f\ } \cdots \\ &\qquad\qquad \xrightarrow{\ \mathrm{d}+\mathrm{d}f\ } \Omega^n_{\overline{X}}(\log D)([(-\mathsf{p}+n)P]_+) \to 0\Big\}. \end{aligned}$$

Theorem D **(E-S-Y, K-K-P, Yu, M. Saito, T. Mochizuki).** *The spectral sequence associated to*

$$\mathbf{H}^j_?\big(\overline{X}, F^{\mathsf{p}}_{\mathrm{Yu}}(\Omega^\bullet_{\overline{X}}(*D), \mathrm{d} + \mathrm{d}f)\big)$$

degenerates at E_1 and the induced filtration $F^\bullet_{\mathrm{Yu}}\mathrm{H}^j_{\mathrm{dR},?}(X, f)$ does not depend on the choice of the compactification of f as above. It is equal to the irregular Hodge filtration $F^\bullet_{\mathrm{irr}}\mathrm{H}^j_{\mathrm{dR},?}(X, f)$.

2.d. The case of a product $f = t^m g$. We assume that $X = \mathbb{A}^1_t \times V$, $V :=$ affine space and $f = t^m g$, $g : V \to \mathbb{A}^1$, $m \geqslant 1$, V_m^* is the cyclic covering of $V^* = V \smallsetminus g^{-1}(0)$.

Theorem E **(F-S-Y, S-Y).** *Under this assumption, $\mathrm{H}^j_?(X, t^m g) \in \mathsf{EMHS}^{\widehat{\mu}}$ for any j. Furthermore,*

$$\mathrm{H}^j_{\mathrm{c}}(X, t^m g)_1 \simeq \mathrm{H}^j_{\mathrm{c}}(\mathbb{A}^1 \times g^{-1}(0)) \simeq \mathrm{H}^{j-2}_{\mathrm{c}}(g^{-1}(0))(-1),$$

$$\mathrm{H}^j_{\mathrm{c}}(X, t^m g)_{\neq 1} \simeq \Big[\mathbb{E}^{\mathrm{H}}_m \otimes \mathrm{H}^{j-1}_{\mathrm{c}}(V_m^*)\Big]^{\mu_m}, \quad \mathbb{E}^{\mathrm{H}}_m := \mathrm{H}^1(\mathbb{A}^1, x^m).$$

3. Airy

3.a. The $\widehat{\mu}$-Hodge structure. We focus on the classical Airy differential equation ($n=2$). We identify

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1,\textstyle\bigotimes^k\mathrm{Ai})\simeq\mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{A}^{k+1},E^{f_k})=:\mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{A}^{k+1},f_k),\quad f_k:=\sum_{i=1}^k(\tfrac13 x_i^3-zx_i),$$

and

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1,\mathrm{Sym}^k\mathrm{Ai})\simeq\mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{A}^{k+1},f_k)^{\mathfrak{S}_k,\chi}.$$

This already gives an exponential mixed Hodge structure to $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1,\mathrm{Sym}^k\mathrm{Ai})$. In order to refine it, we notice that, when $z\in\mathbb{G}_{\mathrm{m}}$, setting $z=t^2$ and $y_i=x_i/t$,

$$f_k(z,x_1,\dots,x_k)=t^3g_k(y_1,\dots,y_k).$$

Then

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}},\mathrm{Sym}^k\mathrm{Ai})\simeq\mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}\times\mathbb{A}^k,t^3g_k)^{\mu_2\times\mathfrak{S}_k,\chi}$$

and this has the same pure part as $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1,\mathrm{Sym}^k\mathrm{Ai})$, so

$$\begin{aligned}\mathrm{H}^1_{\mathrm{mid}}(\mathbb{A}^1,\mathrm{Sym}^k\mathrm{Ai})_{\mathrm{cl}}&=\Big[\mathrm{gr}^W_{k-1}\mathrm{H}^{k-1}_{\mathrm{c}}(g_k^{-1}(0))^{\mu_2\times\mathfrak{S}_k,\chi}\Big](-1),\\ \mathrm{H}^1_{\mathrm{mid}}(\mathbb{A}^1,\mathrm{Sym}^k\mathrm{Ai})_{\neq1}&=\Big(\mathrm{H}^1(\mathbb{A}^1,t^3)\otimes\mathrm{gr}^W_k\mathrm{H}^k(U_3)^{\mu_2\times\mathfrak{S}_k,\chi}\Big)^{\mu_3},\end{aligned}$$

where U_3 is the cyclic covering of order 3 of $U=\mathbb{A}^k\smallsetminus g_k^{-1}(0)$.

3.b. Basis of the de Rham cohomology. The advantage of working with $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1,\mathrm{Sym}^k\mathrm{Ai})$ is that it is easy to guess a candidate Hodge filtration. Starting from a basis (v_0,v_1) of Ai for which the connection reads

$$\nabla_{\partial_z}(v_0,v_1)=(v_0,v_1)\cdot\begin{pmatrix}0&z\\1&0\end{pmatrix},$$

we obtain a basis $(u_a=v_0^{k-a}v_1^a)_{a=0,\dots,k}$ of $\mathrm{Sym}^k\mathrm{Ai}$, and one checks that

$$\omega_i=z^{i-1}v_0^k\mathrm{d}z,\quad i=1,\dots,\lfloor(k\pm1)/2\rfloor\quad(+\text{ for }k\text{ odd},-\text{ for }k\text{ even})$$

is a basis of $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1,\mathrm{Sym}^k\mathrm{Ai})$. Furthermore, if $4\nmid k$, $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{A}^1,\mathrm{Sym}^k\mathrm{Ai})\simeq\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1,\mathrm{Sym}^k\mathrm{Ai})$.

It is a pity not to see the occurrence of Airy functions. They come in when $4\mid k$. We define the (rational) number $\gamma_{k,i}$ as the coefficient of $(1/z)^i$ in the expansion at infinity of $2\pi\,\mathsf{Ai}(z)\,\mathsf{Bi}(z)$. Classical formulas show that $\gamma_{k,k/4}=1$, $\gamma_{k,i}>0$ for $i\in k/4+3\mathbb{N}$ as is zero otherwise. Then $(\omega_i-\gamma_{k,i}\omega_{k/4})$, for $i=1,\dots,\lfloor(k-1)/2\rfloor$ and $i\neq k/4$, is a basis of $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{A}^1,\mathrm{Sym}^k\mathrm{Ai})$ when $4\mid k$.

3.c. Computation of the $\widehat{\mu}$-Hodge numbers, case k odd. We assume k odd. In such a case, $g_k^{-1}(0)$ is smooth and a non-degenerate compactification $\overline{X}$ of $\mathbb{A}^1\times\mathbb{A}^k$ for t^3g_k is simply obtained by blowing up in $\mathbb{P}^1\times\mathbb{P}^k$ the divisor $\{0\}\times\mathbb{P}^{k-1}_\infty$. If P' denotes the strict transform of $\mathbb{P}^1\times\mathbb{P}^{k-1}_\infty$ in $\overline{X}$, the pole divisor of the rational function t^3g_k on $\overline{X}$ is $P=3((\{\infty\}\times\mathbb{P}^k)+P')$.

The image of ω_i in $\mathrm{H}^{k+1}(\mathbb{A}^{k+1},t^3g_k)$ is also that of $w_i:=z^{i-1}\mathrm{d}z\wedge\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_k$ via the morphism

$$\Gamma(\overline{X},\Omega^{k+1}_{\overline{X}}(\log D)(*P))\longrightarrow\mathrm{H}^{k+1}(\mathbb{A}^{k+1},t^3g_k).$$

In view of Theorem **??**, it is enough to compute the order of the pole of w_i along the components of P. One checks that $w_i\in\Gamma(\overline{X},\Omega^{k+1}_{\overline{X}}(\log D)(\frac{1}{3}(k+2i)P))$, hence has image in $F_{\mathrm{irr}}^{k+1-\frac{1}{3}(k+2i)}\mathrm{H}^{k+1}(\mathbb{A}^{k+1},t^3g_k)$.

Defining $G^{\mathsf{p}}\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{A}^1,\mathrm{Sym}^k\mathrm{Ai})$ as the subspace generated by $\{\omega_i\mid i\leqslant k+\frac{3}{2}(1-\mathsf{p})\}$. Then $G^{\mathsf{p}}\subset F^{\mathsf{p}}_{\mathrm{irr}}$ for all p. An argument Hodge symmetry shows equality, and this proves the theorem when k is odd.

3.d. Computation of the $\widehat{\mu}$-Hodge numbers, case k even. The compactification $\overline{X}$ constructed above is degenerated in this case, due to singularities (ordinary double points) on $g_k^{-1}(0)$, that also need to be resolved. The argument for k odd can be extended, but, due to these new divisors, one can only show the relation $w_i\in\Gamma(\overline{X},\Omega^{k+1}_{\overline{X}}(\log D)(\frac{1}{3}(k+2i)P))$ for half of the basis (ω_i). A completely different argument is used, by considering first the pullback of Ai by $t\mapsto z=t^2$.

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