

Generalized Hodge Theory

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Hard Lefschetz Theorem

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$$n = \dim_{\mathbb{C}} X$$

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$$D'' F^p H \subset \overline{\Omega_X^1} \otimes F^p H, \quad D'(F^p H) \subset \Omega_X^1 \otimes F^{p-1} H$$

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Application to the Leray spectral sequence of
 $Y \xrightarrow{f} X \longrightarrow \text{pt.}$

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- The proof of M. Saito applies to \mathcal{V} on X_0 **underlying a PVHS**.

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- The proof uses the existence of a **harmonic metric** on the associated flat bundle (H, D) .
- Equivalently: \mathcal{V} **underlies a polarized variation of twistor structure of weight 0**.

Twistor structures

(C. Simpson)

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Hodge structures

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Filtered vect. sp. $(F^\bullet H, \bar{F}^\bullet H)$	Holom. vect. bdle \mathcal{H} on \mathbb{P}^1

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Pure Hodge structure wght w	$\sigma : z \mapsto -1/\bar{z}$ ($\bar{z} = -1/z$) $\mathcal{H} \simeq \mathcal{O}_{\mathbb{P}^1}(w)^d$

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Vector space H ($w = 0$)	$\mathcal{H} \simeq \mathcal{O}_{\mathbb{P}^1}(w)^d$ $\Gamma(\mathbb{P}^1, \mathcal{H})$

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$S : H \simeq H^*$	$\Gamma(\mathbb{P}^1, \mathcal{H})$
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$h : H \simeq H^*$	$\mathcal{S} : \mathcal{H} \simeq \mathcal{H}^* := \overline{\mathcal{H}}^\vee$
	$\Gamma(\mathbb{P}^1, \mathcal{S}) :$
	$\Gamma(\mathbb{P}^1, \mathcal{H}) \simeq \Gamma(\mathbb{P}^1, \mathcal{H})^*$

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Vector space H ($w = 0$)	$\mathcal{H} \simeq \mathcal{O}_{\mathbb{P}^1}(w)^d$
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Positivity of h	$\Gamma(\mathbb{P}^1, \mathcal{S}) :$ $\Gamma(\mathbb{P}^1, \mathcal{H}) \simeq \Gamma(\mathbb{P}^1, \mathcal{H})^*$ Positivity of $\Gamma(\mathbb{P}^1, \mathcal{S})$

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$(-1)^p$	Positivity of $\Gamma(\mathbb{P}^1, \mathcal{S})$

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Positivity of h	$\Gamma(\mathbb{P}^1, \mathcal{S}) :$
Tate twist (k) , $k \in \mathbb{Z}$	$\Gamma(\mathbb{P}^1, \mathcal{H}) \simeq \Gamma(\mathbb{P}^1, \mathcal{H})^*$
	Positivity of $\Gamma(\mathbb{P}^1, \mathcal{S})$
	$\otimes \mathcal{O}_{\mathbb{P}^1}(-2k) \quad (k \in \frac{1}{2}\mathbb{Z})$

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Flatness:

$$\mathcal{D}^2 = (\mathcal{D}' + \mathcal{D}'')^2 = 0.$$

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C. Simpson: Variations of pol. twistor struct. of weight 0

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Variation of twistor structures

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- The analytic proof uses **tame harmonic metrics** on X_0 (cf. works of Simpson, Biquard, Jost-Zuo) and **regular twistor \mathcal{D} -modules**.

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- Question: What kind of Hodge theory can one develop in presence of ***irregular singularities*** ?

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- The proof uses **wild harmonic metrics** on X_0 and **wild twistor \mathcal{D} -modules**.

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- \rightarrow The pol. var. of twistor str. attached to a PVHS satisfies the **Integrability property**.

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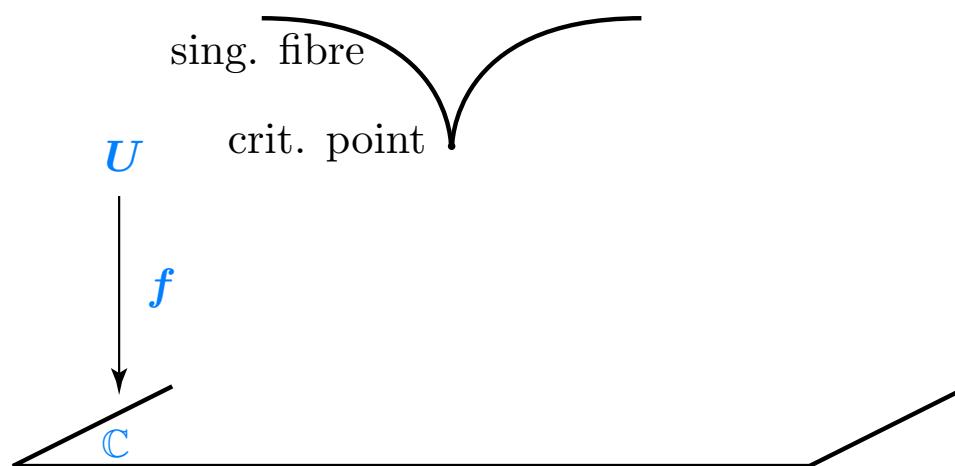
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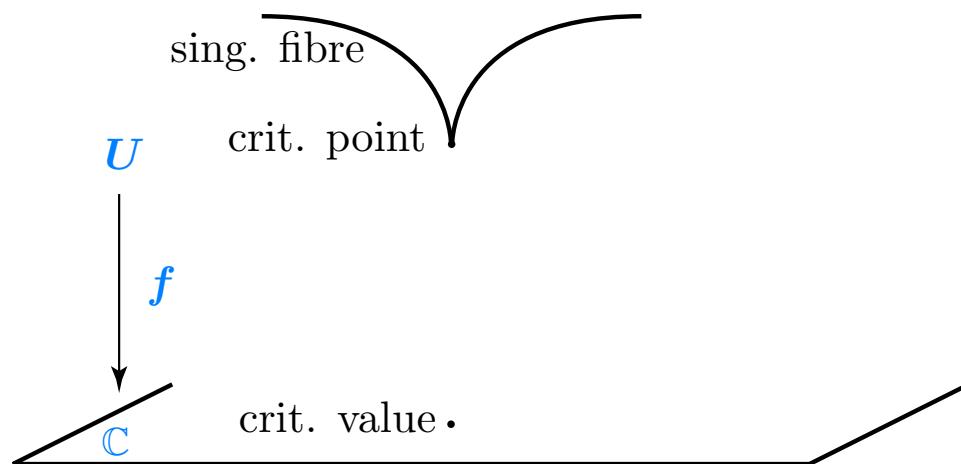
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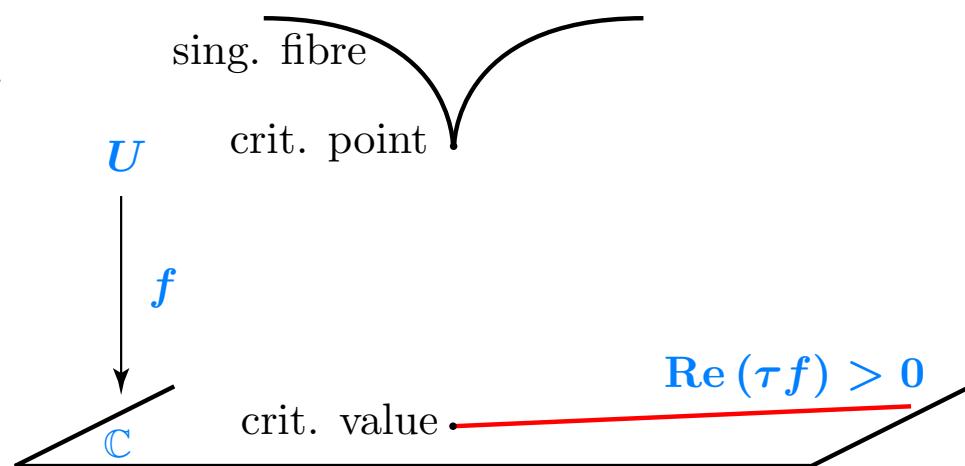
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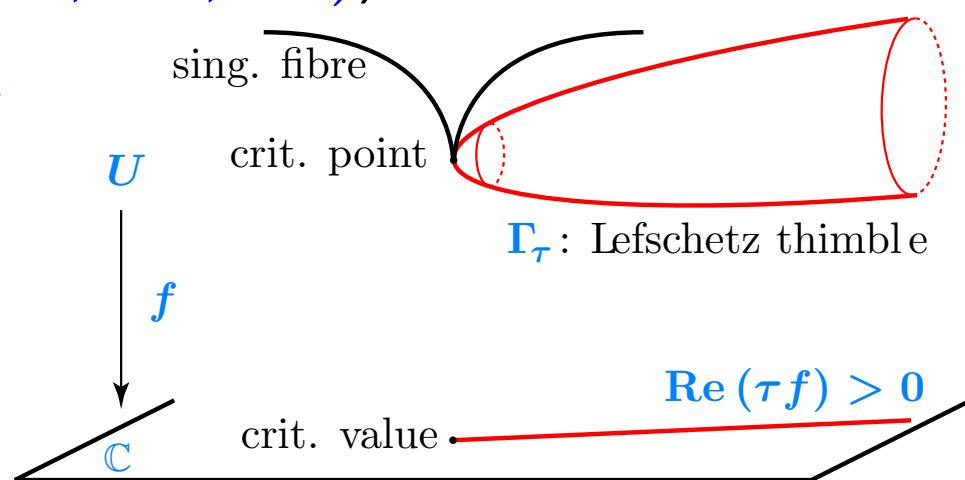
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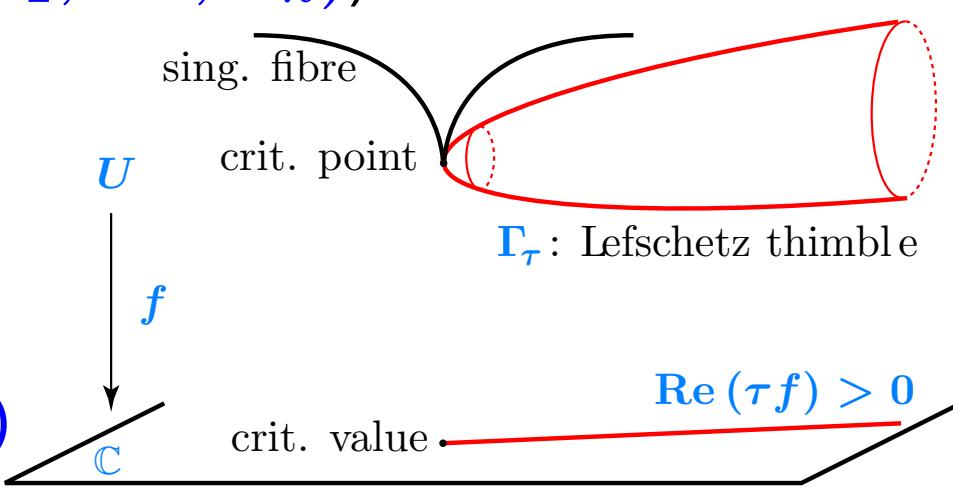
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