ON THE FOURIER-LAPLACE TRANSFORM OF A VARIATION OF HODGE STRUCTURE

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Abstract. Generalizing the notion of (variations of) Hodge structure is needed by various recent mathematical developments (e.g. Mirror symmetry and tt* geometry). Harmonic Higgs bundles with supplementary data are good candidates. In the talk, I will explain how this new structure is stable by the Fourier-Laplace transform, a result related to previous work of S. Szabo on the Nahm transform.

1. Variation of polarized Hodge structure

Let $P = \{p_1, \dots, p_r, p_{r+1} = \infty\}$ be a non empty finite set of points on the Riemann sphere $\mathbb{P}^1$. We will denote by t the coordinate on the affine line $\mathbb{A}^1 = \mathbb{P}^1 \smallsetminus \{\infty\}$. The main object of interest in this talk will be a *(complex) variation of polarized Hodge structure* on $\mathbb{P}^1 \smallsetminus P$.

Examples.

(1) Let $\mathscr{V}$ be a *unitary* local system on $\mathbb{P}^1 \smallsetminus P$. This defines a complex variation of Hodge structure of type $(0, 0)$. Giving a local system is equivalent to giving a set of matrices (monodromy matrices) $T_1, \dots, T_r \in \mathrm{GL}(\mathbb{C}^{\operatorname{rk} V})$, up to conjugation by the same invertible matrix. The local system is unitary if one can find $T_1, \dots, T_r$ in the unitary group.

(2) Let $f : X \to \mathbb{P}^1$ be a projective morphism from a smooth complex projective variety X to the projective line. Away from the critical values P of f, each cohomology sheaf $H^k(f^{-1}(t), \mathbb{C})_{t \in \mathbb{P}^1 \smallsetminus P}$ forms a local system and underlies a variation of polarized Hodge structure.

(3) Let U be a smooth complex quasiprojective variety of dimension n (*e.g.* $\mathbb{C}^n$ or $(\mathbb{C}^*)^n$). Let $f : U \to \mathbb{C}$ be a regular function on U (*e.g.* a polynomial or a Laurent polynomial). Let us assume that f is *tame*, that is, f has only isolated critical points on U (and we denote by P the set of critical values including ∞) and has no "critical point at infinity with finite critical value". The local system $H^n(U, f^{-1}(t), \mathbb{C})$ underlies a variation of *mixed* Hodge structure, which contains, as a subquotient, a variation of polarized pure Hodge structure, all other

subquotients having no singular points at $p_1, \dots, p_r$. For the purpose of this talk, this will be as good as the pure case.

Definition. A *polarized variation of Hodge structure of weight* w on $\mathbb{P}^1 \smallsetminus P$ consists of a C^∞ vector bundle H on $\mathbb{P}^1 \smallsetminus P$ equipped with a flat connection D, a decomposition $H = \oplus_{p\in\mathbb{Z}} H^p$ (H^p is usually written as $H^{p,w-p}$) and a Hermitian metric h on H, satisfying the following properties:

- the decomposition is orthogonal with respect to h and the nondegenerate $(-1)^w$-Hermitian form $k = \oplus_p i^w(-1)^p h_{|H^p}$ is D-flat,
- (Griffiths' transversality)

$$
\begin{aligned}
D'(H^p) &\subset (H^p \oplus H^{p-1}) \otimes_{\mathscr{O}_{\mathbb{P}^1\smallsetminus P}} \Omega^1_{\mathbb{P}^1\smallsetminus P} \\
D''(H^p) &\subset (H^p \oplus H^{p+1}) \otimes_{\mathscr{O}_{\overline{\mathbb{P}^1\smallsetminus P}}} \Omega^1_{\overline{\mathbb{P}^1\smallsetminus P}}.
\end{aligned}
\tag{1}
$$

The Hodge filtration $F^\bullet H$ is

$$F^p H = \oplus_{q\geqslant p} H^q,$$

so that $D' F^p H \subset F^{p-1} H \otimes_{\mathscr{O}_{\mathbb{P}^1\smallsetminus P}} \Omega^1_{\mathbb{P}^1\smallsetminus P}$.

Griffiths' transversality (1) gives a decomposition $D = D^+ + \theta$, where D^+ is unitary with respect to h and θ is self-adjoint (Higgs field). Considering types and grading, the Higgs condition $(D^+)''(\theta') = 0$ is satisfied. Then $(H, (D^+)'', \theta')$ is a holomorphic Higgs bundle.

Let (V, ∇) be the holomorphic bundle with connection $(\operatorname{Ker} D'', D')$ and $F^p V = F^p H \cap V$. We have $\nabla F^p V \subset F^{p-1} V \otimes_{\mathscr{O}_{\mathbb{P}^1\smallsetminus P}} \Omega^1_{\mathbb{P}^1\smallsetminus P}$. We can identify $(H, (D^+)'', \theta')$ with $(\operatorname{gr}_F V, \operatorname{gr}_F^{-1} \nabla)$.

2. The Fourier-Laplace transform

2.1. The twisted L^2-complex. On H we consider the twisted connection $D - 2dt$. Using a metric on $\mathbb{P}^1 \smallsetminus P$ which is equivalent to the Poincaré metric on the punctured disc near each puncture $p_i \in P$ (hence a complete metric), and the Hermitian metric h on H, we make the L^2 de Rham complex $\mathscr{L}_{(2)}(\mathbb{P}^1 \smallsetminus P, H, D - 2dt, h)$.

***Theorem 1* (S. Szabo, CS).** *This complex has cohomology in degree* 1 *at most, and this cohomology is a finite dimensional vector space, equipped in a natural way with a Hermitian metric. Moreover, it is canonically identified with the cohomology of the* L^2 *complex* $\mathscr{L}_{(2)}(\mathbb{P}^1 \smallsetminus P, H, (D^+)'' + \theta' - dt, h)$.

In particular, as the metric on $\mathbb{P}^1 \smallsetminus P$ is complete, we can apply Hodge theory and compute this cohomology with L^2 harmonic forms.

2.2. Algebraic interpretation of the L^2 complex. The holomorphic bundle with flat connection (V, ∇) extends in a unique way as an algebraic bundle with flat algebraic connection and, according to results of Schmid, this extensions is obtained by considering holomorphic sections whose h-norm has moderate growth at P. Note that the twist of the de Rham complex is of no consequence at finite distance. The work of Zucker tells us that, near a puncture at finite distance, in order to compare the de Rham complex with the previous L^2 complex, we should replace the algebraic bundle with a D-module called the "intermediate" (or minimal) extension. Taking global sections on $\mathbb{A}^1$ of this D-module gives a $\mathbb{C}[t]\langle\partial_t\rangle$-module M which is holonomic and has regular singularities everywhere.

Theorem 2. *The twisted algebraic de Rham complex $M \xrightarrow{e^t\nabla e^{-t}} M \otimes dt$ has cohomology in degree 1 at most. This cohomology can be identified with that of the previous L^2 complex.*

One can give the following interpretation of the dimension μ of this cohomology: let $\mathscr{V} = \operatorname{Ker} \nabla$ be the local system of horizontal section of (V, ∇) (or equivalently, (H, D)); near each puncture $p_i \neq \infty$, define $\mu_i(\mathscr{V}) = \operatorname{rk} \mathscr{V} - \dim \Gamma(\mathrm{nb}(p_i)^*, \mathscr{V})$; then $\mu = \sum_i \mu_i(\mathscr{V})$.

Examples.

(1) Given unitary matrices $T_1, \dots, T_r$ of size $\operatorname{rk} V$, $\mu_i = \operatorname{rk} V - \dim(\operatorname{Ker} T_i - \mathrm{Id})$.

(2) Let $f : X \to \mathbb{P}^1$ projective, X smooth, p_i the critical values at finite distance. Then $\mu_i^{(k)} = \dim \mathbb{H}^k(f^{-1}(p_i), \phi_{f,p_i}(\mathbb{C}))$.

(3) Let $f : U \to \mathbb{C}$ be a tame regular function. For any critical value p_i of f, corresponding to critical points $x_i^{(1)}, \dots, x_i^{(k_i)}$, the corresponding number μ_i it the sum of the Milnor numbers of f at $x_i^{(j)}$, $j = 1, \dots, k_i$, and μ is the total sum of Milnor numbers of f at its critical points.

2.3. Rescaling parameter. We now rescale the variable t with a nonzero complex parameter τ. From the algebraic point of view, this consists in considering the twisted de Rham complex $\mathbb{C}[\tau, \tau^{-1}] \otimes_{\mathbb{C}} M \xrightarrow{\nabla_{\partial_t} - \tau} \mathbb{C}[\tau, \tau^{-1}] \otimes_{\mathbb{C}} M$. It has cohomology in degree 1 at most, and this cohomology $\widehat{V}$ is a free $\mathbb{C}[\tau, \tau^{-1}]$-module of rank μ. It comes equipped with a algebraic connection $\widehat{\nabla}$.

The twisted L^2 de Rham complex also defines a flat C^∞ bundle $(\widehat{H}, \widehat{D})$ on $\mathbb{C}^*$, equipped with a Hermitian metric $\widehat{h}$.

***Theorem 3* (S. Szabo, CS).** *The metric flat bundle $(\widehat{H}, \widehat{D}, \widehat{h})$ is harmonic.*

Remark. Up to now, the theory uses less than the variation of Hodge structure: it only uses the harmonicity property of the original Hermitian metric h.

Remark. On the other hand, one cannot expect, in general, that the new metric has a tame behaviour at $\tau=\infty$. In particular, this implies that it does not correspond to a usual variation of polarized Hodge structure.

Question. *What kind of a supplementary structure does the flat harmonic bundle* $(\widehat{H},\widehat{D},\widehat{h})$ *underlie?*

3. The supersymmetric index

The answer to this question (or a variant of it) has been given in 1991 by two physicists, Cecotti and Vafa. They give us two operators $\widehat{\mathscr{U}}$ and $\widehat{\mathscr{Q}}$ on $\widehat{H}$, where $\widehat{\mathscr{Q}}$ is selfadjoint with respect to $\widehat{h}$, $\widehat{\mathscr{U}}$ is $(\widehat{D}^+)''$-holomorphic, and which satisfy a series of differential equations:

$$\begin{aligned}\widehat{\mathscr{Q}}^\dagger &= \widehat{\mathscr{Q}},\\ [\widehat{\theta}',\widehat{\mathscr{U}}] &= 0,\\ (\widehat{D}^+)'(\widehat{\mathscr{U}})-[\widehat{\theta}',\widehat{\mathscr{Q}}]+\widehat{\theta}' &= 0,\\ (\widehat{D}^+)'(\widehat{\mathscr{Q}})+[\widehat{\theta}',\widehat{\mathscr{U}}^\dagger] &= 0.\end{aligned}$$

These differential equations are better interpreted as an integrability condition, by adding a new variable z.

For instance, the operators associated to the variation of Hodge structure are $\mathscr{U}=0$ and $\mathscr{Q}=-\bigoplus_p p\,\mathrm{Id}_{H^{p,w-p}}$. On the other hand, the eigenvalues of $\widehat{\mathscr{Q}}$ need not be constant.

3.1. The spectrum.

At each $p_i\in\mathbb{P}^1$ is associated the spectrum of the variation of Hodge structure. For $p_{r+1}=\infty$, we call it the spectrum at infinity. In Example (3), this spectrum coincides with the Varchenko-Steenbrink spectrum of the critical points of f. In any case, at $p_i\neq\infty$, the corresponding polynomial has degree μ_i.

Let me explain the definition of the spectrum at finite distance. I will set $SP_{p_i}(T)=\prod_\gamma(T-\gamma)^{\nu_\gamma^{(i)}}$. For any $\alpha\in(-1,0]$, let V^α be the holomorphic bundle on $\mathbb{A}^1$ with connection having a logarithmic pole at each p_i, extending (V,∇), and such that the residue of the connection on V^α has eigenvalues in $[\alpha,\alpha+1[$. If $\alpha\neq0$ and $p\in\mathbb{Z}$, I set $\nu_{\alpha+p}^{(i)}=\dim(F^p\cap V^\alpha)/(F^{p+1}\cap V^\alpha+F^p\cap V^{>\alpha})$. When $\alpha=0$, the definition has to be modified a little bit. At infinity, we have a similar definition, and there is also a small change to be done at $\alpha=0$, but different from that done at finite distance.

Theorem 4. $\lim_{\tau\to\infty} \chi(\widehat{\mathscr{Q}}(\tau))(T) = \prod_{i=1}^{r} \mathrm{SP}_{p_i}(T)$ *and* $\lim_{\tau\to 0} \chi(\widehat{\mathscr{Q}}(\tau))(T) = \mathrm{SP}_\infty(T)$.

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