Quadratic relations for periods of connections

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Quadratic relations for periods: the classical case

- X connected smooth complex projective mfld of dim. n.
- $(\gamma_i)_i$: basis of $H_m(X^{\mathrm{an}}, \mathbb{Q})$, $(\omega_j)_j$: basis of $H^m_{\mathrm{dR}}(X)$.
- Period matrix $\mathsf{P}_m = (\mathsf{P}_{m;ij})$, with

$$\mathsf{P}_{m;ij} := \int_{\gamma_i} \omega_j.$$

- De Rham duality pairing

$$\mathsf{Q}_m : H^m_{\mathrm{dR}}(X) \otimes H^{2n-m}_{\mathrm{dR}}(X) \longrightarrow H^{2n}_{\mathrm{dR}}(X) \xrightarrow[\sim]{\int_X} \mathbb{C}.$$

- Betti intersection pairing

$$\mathsf{B}_m : H_m(X^{\mathrm{an}}, \mathbb{Q}) \otimes H_{2n-m}(X^{\mathrm{an}}, \mathbb{Q}) \longrightarrow H_0(X^{\mathrm{an}}, \mathbb{Q}) \simeq \mathbb{Q}.$$

- Quadratic relations for the associated matrices, e.g. $m = n$:

$$(-1)^n \, \mathsf{B}_n = \mathsf{P}_n \cdot (\mathsf{Q}_n)^{-1} \cdot {}^t\mathsf{P}_n$$

- Set $\mathsf{S}_m := (2\pi \mathrm{i})^{-n} \mathsf{Q}_m$, $\forall m$.

$$\boxed{(-2\pi \mathrm{i})^n \, \mathsf{B}_n = \mathsf{P}_n \cdot (\mathsf{S}_n)^{-1} \cdot {}^t\mathsf{P}_n}$$

Example: the case of a curve.

- X: curve of genus $g \geqslant 1$.
- Pairing S:
  - $\omega_1, \dots \omega_g$ basis of $H^0(X, \Omega^1_X)$,
  - $f_1, \dots, f_g \in H^1(X, \mathscr{O}_X)$,
  - $\check{H}^1(X, \mathscr{O}_X) \otimes \check{H}^0(X, \Omega^1_X) \to \check{H}^1(X, \Omega^1_X) \overset{\mathrm{Res}}{\simeq} \mathbb{C}$.
  - $\mathsf{S}(f_i, \omega_j) = -\mathsf{S}(\omega_j, f_i) = [f_i \omega_j] \in H^1(X, \Omega^1_X) = \mathbb{C}$.
  - Can also represent $H^1(X, \mathscr{O}_X)$ by $(0,1)$ forms η_i and

$$\mathsf{S}(\eta_i, \omega_j) = -\mathsf{S}(\omega_j, \eta_i) = \frac{1}{2\pi \mathrm{i}} \int_X \eta_i \wedge \omega_j.$$

- Pairing B:
  - $(\alpha_i, \alpha_{g+i})_{i=1,\dots,g}$ symplectic basis of $H_1(X, \mathbb{Z})$.
  - $\rightsquigarrow$ $\mathsf{B}_{i,g+i} = -\mathsf{B}_{g+i,i} = 1$ and $\mathsf{B}_{k,\ell} = 0$ otherwise.

- $\rightsquigarrow$ Bilinear relations.

Sketch of proof.

$$\begin{tikzcd}
H^n_{\mathrm{dR}}(X^{\mathrm{an}}) \otimes H^n_{\mathrm{dR}}(X^{\mathrm{an}}) \arrow[r, "\mathsf{Q}_n"] \arrow[d, "\mathcal{P}_n"', "\wr"] \arrow[d, phantom] & \mathbb{C} \arrow[d, equal] \\
H_n(X^{\mathrm{an}}) \otimes H^n_{\mathrm{dR}}(X^{\mathrm{an}}) \arrow[r, "\mathsf{P}_n"] \arrow[d, "\wr"', "\mathcal{P}_n"] & \mathbb{C} \arrow[d, equal] \\
H_n(X^{\mathrm{an}}) \otimes H_n(X^{\mathrm{an}}) \arrow[r, "\mathsf{B}_n"] & \mathbb{C}
\end{tikzcd}$$

- $\mathcal{P}_n$:= Poincaré isomorphism.
- Compatibility proved by de Rham by realizing $H_n(X^{\mathrm{an}})$ as currents.
- In term of matrices (e.g. $\mathsf{Q}_n(\omega, \omega') = {}^t\omega \cdot \mathsf{Q}_n \cdot \omega'$):

$$ {}^t\mathcal{P}_n \cdot \mathsf{P}_n = \mathsf{Q}_n, \qquad \mathsf{B}_n \cdot \mathcal{P}_n = \mathsf{P}_n. $$

$\Longrightarrow$

$$ \mathsf{B}_n = \mathsf{P}_n \cdot (\mathcal{P}_n)^{-1} = \mathsf{P}_n \cdot ({}^t\mathsf{Q}_n)^{-1} \cdot {}^t\mathsf{P}_n. $$

- Use $\quad {}^t\mathsf{Q}_n = (-1)^n \mathsf{Q}_n.$ $\square$

Quadratic relations for periods of vector bundles with log connection

Vector bundles with log connection.

- X connected smooth quasi-projective, (V, ∇): alg. vect. bdle on X with ***flat*** connection having ***reg. sing. at ∞ on*** X.
- $H^k_{\mathrm{dR}}(X, (V, \nabla))$, $H^k_{\mathrm{dR,c}}(X, (V, \nabla))$:
  - Choose $(\overline{X}, D)$ smooth proj. $D = $ ncd, $X = \overline{X} \smallsetminus D$.
  - Deligne's canonical extension (V_0, ∇):
    - $*$ V_0: vect. bdle on $\overline{X}$ extending V:
    - $*$ $\nabla : V_0 \to \Omega^1_{\overline{X}}(\log D) \otimes V_0$ extending ∇
    - $*$ eigenvalues of $\mathrm{res}_{D_i} \nabla$ have real part in $[0, 1)$.
  - $H^k_{\mathrm{dR}}(X, (V, \nabla)) \simeq \boldsymbol{H}^k(\overline{X}, (\Omega^\bullet_{\overline{X}}(\log D) \otimes V_0, \nabla))$,
  - $H^k_{\mathrm{dR,c}}(X, (V, \nabla)) \simeq \boldsymbol{H}^k(\overline{X}, (\Omega^\bullet_{\overline{X}}(\log D) \otimes V_0(-D), \nabla))$.
- Assume given pairing $\langle \bullet, \bullet \rangle : V \otimes V \to \mathcal{O}_X$ s.t.
  - nondegener. i.e., induces $V \xrightarrow{\sim} V^\vee$,
  - $\pm$-symmetric, i.e., $\langle w, v \rangle = \pm \langle v, w \rangle$,
  - compatible with ∇, i.e., $\mathrm{d}\langle v, w \rangle = \langle \nabla v, w \rangle + \langle v, \nabla w \rangle$.
- $\rightsquigarrow$

$$ \mathsf{S}_m : H^m_{\mathrm{dR,c}}(X, (V, \nabla)) \otimes H^{2n-m}_{\mathrm{dR}}(X, (V, \nabla)) \longrightarrow H^{2n}_{\mathrm{dR,c}}(X, (\mathcal{O}_X, \mathrm{d})) \overset{\mathrm{Tr}}{\simeq} \mathbb{C} $$

Intersection pairings between flat sections.

- $\mathscr{V} = V^{\mathrm{an},\nabla}$ loc. cst. sheaf of horiz. sections.
- $\rightsquigarrow$ $\pm$-sym. nondeg. pairing $\langle\bullet,\bullet\rangle : \mathscr{V}\otimes\mathscr{V}\to\mathbb{C}_X$.
- Assume defined over $\mathbb{Q}$:
  - $\mathscr{V} = \mathbb{C}\otimes_{\mathbb{Q}}\mathscr{V}_{\mathbb{Q}}$,
  - $\langle\bullet,\bullet\rangle : \mathscr{V}_{\mathbb{Q}}\otimes\mathscr{V}_{\mathbb{Q}}\to\mathbb{Q}_X$.
- $\rightsquigarrow$ $H_m(X^{\mathrm{an}},\mathscr{V}_{\mathbb{Q}})$, $H_m^{\mathrm{BM}}(X^{\mathrm{an}},\mathscr{V}_{\mathbb{Q}})$,
- $\rightsquigarrow$ $\mathsf{B}_m : H_m(X^{\mathrm{an}},\mathscr{V}_{\mathbb{Q}})\otimes H_{2n-m}^{\mathrm{BM}}(X^{\mathrm{an}},\mathscr{V}_{\mathbb{Q}})\to\mathbb{Q}$.

Period pairings.

- Two period pairings (by using $\langle\bullet,\bullet\rangle$):

$$\mathsf{P}_m : H_m(X^{\mathrm{an}},\mathscr{V}_{\mathbb{Q}})\otimes H_{\mathrm{dR}}^{2n-m}(X,(V,\nabla))\longrightarrow\mathbb{C}$$

$$\mathsf{P}_m^{\mathrm{BM}} : H_m^{\mathrm{BM}}(X^{\mathrm{an}},\mathscr{V}_{\mathbb{Q}})\otimes H_{\mathrm{dR,c}}^{2n-m}(X,(V,\nabla))\longrightarrow\mathbb{C}$$

***Theorem* (Matsumoto & al., 1994).**

- P_m *and* $\mathsf{P}_m^{\mathrm{BM}}$ *are nondeg.*
- *"Quadratic relations" e.g. for $m=n$:*

$$\pm(-2\pi\mathrm{i})^n\mathsf{B}_n = \mathsf{P}_n\cdot(\mathsf{S}_n)^{-1}\cdot{}^t\mathsf{P}_n^{\mathrm{BM}}.$$

Middle quadratic relations.

- $H_{\mathrm{dR,mid}}^m(X,(V,\nabla)) := \mathrm{im}\Big[H_{\mathrm{dR,c}}^m(X,(V,\nabla))\to H_{\mathrm{dR}}^m(X,(V,\nabla))\Big]$,
- $H_m^{\mathrm{mid}}(X^{\mathrm{an}},\mathscr{V}_{\mathbb{Q}}) := \mathrm{im}\Big[H_m(X^{\mathrm{an}},\mathscr{V}_{\mathbb{Q}})\to H_m^{\mathrm{BM}}(X^{\mathrm{an}},\mathscr{V}_{\mathbb{Q}})\Big]$
- $\rightsquigarrow$ Nondeg. $\pm$-sym. pairings, e.g. for $m=n$:

$$\mathsf{S}^{\mathrm{mid}} : H_{\mathrm{dR,mid}}^n(X,(V,\nabla))\otimes H_{\mathrm{dR,mid}}^n(X,(V,\nabla))\longrightarrow\mathbb{C},$$

$$\mathsf{B}^{\mathrm{mid}} : H_n^{\mathrm{mid}}(X^{\mathrm{an}},\mathscr{V}_{\mathbb{Q}})\otimes H_n^{\mathrm{mid}}(X^{\mathrm{an}},\mathscr{V}_{\mathbb{Q}})\longrightarrow\mathbb{Q},$$

$$\mathsf{P}^{\mathrm{mid}} : H_n^{\mathrm{mid}}(X^{\mathrm{an}},\mathscr{V}_{\mathbb{Q}})\otimes H_{\mathrm{dR,mid}}^n(X,(V,\nabla))\longrightarrow\mathbb{C}.$$

***Corollary* (Quadratic relations).**

$$\boxed{\pm(-2\pi\mathrm{i})^n\mathsf{B}^{\mathrm{mid}} = \mathsf{P}^{\mathrm{mid}}\cdot(\mathsf{S}^{\mathrm{mid}})^{-1}\cdot{}^t\mathsf{P}^{\mathrm{mid}}}$$

Example (Matsumoto, 1994).

- Quadratic relations for generalized hypergeometric functions (Appell, Lauricella...).

A conjecture of Broadhurst and Roberts

Bessel moments and Bernoulli matrices.

- ***Bessel moments***:
  - Special values of some Feynman integrals expressed as period of Laurent polynomials. E.g.

$$f(x,y,z) = (1+x+y+z)(1+x^{-1}+y^{-1}+z^{-1}).$$

  - These periods are also expressed as k-moments of the "modified Bessel functions" $I_0(t), K_0(t)$ (e.g. k = odd integer):

$$\mathrm{BM}_k(i,j) = \star \int_0^\infty I_0^i(t) K_0^{k-i}(t) \cdot t^{2j} \frac{\mathrm{d}t}{t}.$$

- ***Bernoulli matrix*** ($B_n := n$th Bernoulli nbr):

$$\mathcal{B}_k(i,j) = (-1)^{k-i} \frac{(k-i)!(k-j)!)}{k!} \cdot \frac{B_{k-i-j-1}}{(k-i-j-1)!}.$$

***Conjecture* (B-R, by computation, e.g. k odd).** *Set $k' = (k-1)/2$. Consider the $k' \times k'$ matrices*

$$\mathrm{BM}_k = (\mathrm{BM}_k(i,j))_{1\leqslant i,j\leqslant k'} \quad \textit{and} \quad \mathcal{B}_k = (\mathcal{B}_k(i,j))_{1\leqslant i,j\leqslant k'}.$$

There exists $\mathcal{D}_k \in \mathrm{GL}_{k'}(\mathbb{Q})$ defined by an explicit algorithm s.t.

$$(-2\pi \mathrm{i})^{k+1}\mathcal{B}_k = \mathrm{BM}_k \cdot \mathcal{D}_k \cdot {}^t\mathrm{BM}_k\,.$$

¿ Interpret the conj. in terms of quadratic relations for periods ?

Generalization of the quadratic relations (F-S-Y).

- Since I_0, K_0 are sols of a diff. eq. with ***irreg. sing.*** need to extend quadratic relations to this case.
- $\rightsquigarrow$ Consider (Kl_2, ∇) rk 2 vect. bdle on $\mathbb{G}_\mathrm{m}$ $\longleftrightarrow$ "modified Bessel diff. eq." and $(\mathrm{Sym}^k \mathrm{Kl}_2, \nabla)$.
- $\rightsquigarrow$ Nondegen. de Rham pairing

$$\mathsf{S}_k : H^1_{\mathrm{dR,c}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2) \otimes H^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2) \longrightarrow \mathbb{C}.$$

- $\rightsquigarrow$ ***Rapid decay*** and ***moderate*** twisted homology and

$$H_1^{\mathrm{mid}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2) := \mathrm{im}\Big[H_1^{\mathrm{rd}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2) \to H_1^{\mathrm{mod}}(\dots)\Big].$$

- $\rightsquigarrow$ Nondegen. Betti intersection pairing:

$$\mathsf{B}_k : H_1^{\mathrm{rd}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2) \otimes H_1^{\mathrm{mod}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2) \longrightarrow \mathbb{Q}.$$

- $\rightsquigarrow$ Nondegen. Period pairings

$$\mathsf{P}_k^{\mathrm{rd,mod}} : H_1^{\mathrm{rd}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2) \otimes H^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2) \longrightarrow \mathbb{C}$$

$$\mathsf{P}_k^{\mathrm{mod,rd}} : H_1^{\mathrm{mod}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2) \otimes H^1_{\mathrm{dR,c}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2) \longrightarrow \mathbb{C}.$$

- $\rightsquigarrow$ Middle quadratic relations:

$$\boxed{(-2\pi \mathrm{i})^{k+1}\mathsf{B}_k^{\mathrm{mid}} = \mathsf{P}_k^{\mathrm{mid}} \cdot (\mathsf{S}_k^{\mathrm{mid}})^{-1} \cdot {}^t\mathsf{P}_k^{\mathrm{mid}}}$$

Theorem **(Fresán-S-Yu, 2020).**

- *There exists an explicit basis of* $H_1^{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ *such that*
$$\mathsf{B}_k^{\mathrm{mid}} = \mathcal{B}_k.$$
- *There exists an explicit basis of* $H^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ *s.t.*
$$\mathsf{P}_k^{\mathrm{mid}} = \mathrm{BM}_k\,.$$
- *The de Rham matrix* $\mathsf{S}_k^{\mathrm{mid}} \in \mathrm{GL}_{k'}(\mathbb{Q})$ *has an algorithmic computation (the matrix* $(\mathsf{S}_k^{\mathrm{mid}})^{-1}$ *checked to agree with the matrix* $\mathcal{D}_k$ *suggested by Broadhurst-Roberts for* $k \leqslant 22$*).*
- $(\mathsf{S}_k^{\mathrm{mid}}, \mathsf{B}_k^{\mathrm{mid}}, \mathsf{P}_k^{\mathrm{mid}})$ *also enter in a quadratic relation for a motive (hence the Bessel moments are periods).*

In other words, the period structure
$$(H^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2), H_1^{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2), \mathsf{P}_k^{\mathrm{mid}})$$
coincides with the period structure of a ***Nori motive***.

Motivic interpretation.

- (Kl_2, ∇) is the Gauss-Manin conn. of $\big(\mathcal{O}_{\mathbb{G}_{\mathrm{m}}^2}, \mathrm{d} + \mathrm{d}(x + z/x)\big)$ by the proj. $\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}} \quad (x, z) \mapsto z$.
- $\big(\bigotimes^k \mathrm{Kl}_2, \nabla\big)$: G-M conn. of $\big(\mathcal{O}_{\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}^k}, \mathrm{d} + \mathrm{d}(f_k)\big)$
$$f_k(x_1, \ldots, x_k, z) = \textstyle\sum_i (x_i + z/x_i)$$
- Set $\widetilde{\mathrm{Kl}}_2 = [2]^* \mathrm{Kl}_2, \quad [2] : t \mapsto t^2$. Set $y_i = x_i/t$.
- Then $\big(\bigotimes^k \widetilde{\mathrm{Kl}}_2, \nabla\big)$: G-M conn. of $E^{t \cdot g_k} := \big(\mathcal{O}_{\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}^k}, \mathrm{d} + \mathrm{d}(t \cdot g_k)\big)$
$$g_k(y_1, \ldots, y_k) = \textstyle\sum_i (y_i + 1/y_i) : \mathbb{G}_{\mathrm{m}}^k \to \mathbb{A}^1.$$
- $$\begin{aligned} H^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) &\simeq H^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \textstyle\bigotimes^k \widetilde{\mathrm{Kl}}_2)^{\mathfrak{S}_k \times \mu_2} \\ &\simeq \boxed{H^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}^k, t \cdot g_k)^{\mathfrak{S}_k \times \mu_2}} \end{aligned}$$
- General fact (Fresán-Jossen, Yu, F-S-Y): U smooth quasi-proj., $g : U \to \mathbb{A}^1$ regular, $H^n_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}} \times U, t \cdot g)$ underlies a Nori motive, hence endowed with a canonical MHS.
- Analogue of Fourier inversion formula for $h : \mathbb{R} \to \mathbb{R}$:
$$h(0) = \star \int_{\mathbb{R}} \widehat{h}(t)\, \mathrm{d}t = \star \int_{\mathbb{R}^2} e^{2\pi i\, t \cdot h(x)} \mathrm{d}t\, \mathrm{d}x.$$
- Set $\mathcal{K} = g_k^{-1}(0) \subset \mathbb{G}_{\mathrm{m}}^k$. Variant of what we want:
$$H^{k+1}(\mathbb{A}^1 \times \mathbb{G}_{\mathrm{m}}^k, t \cdot g_k) \simeq H_{\mathrm{c}}^{k-1}(\mathcal{K})^\vee(-k).$$

Quadratic relations for irregular periods

Irreg. singularities.

- X smooth quasi-proj., (V,∇) on X with possibly irreg. sing. at ∞ on X
  - $\Longrightarrow$ $\nexists(V_0,\nabla)$ log. connection on $(\overline{X},D)$ extending (V,∇).
  - But (Kedlaya-Mochizuki, 2011): $\exists\,(\overline{X},D)$, $D=$ strict ncd and $\exists\,(V_0,\nabla)$ ***good Deligne-Malgrange lattice***:
    - $*\ \forall x\in D$, $\exists\Phi\subset\mathscr{O}_{\overline{X},x}(*D)$ finite,
    - $*\ \forall\varphi\in\Phi$, $\exists\,(R_\varphi,\nabla)$ with reg. sing. on $(\mathrm{nb}(x),D)$,
    - $*\ (\mathscr{O}_{\widehat{x}}\otimes V_0,\nabla)\simeq\bigoplus_{\varphi\in\Phi}\Big[(\mathscr{O}_{\widehat{x}},\mathrm{d}+\mathrm{d}\varphi)\otimes(R_{\varphi,0},\nabla)\Big]$.
  - $j:X\hookrightarrow\overline{X}$,
  - $\forall i\geqslant 1,\quad V_i:=V_{i-1}+\Theta_{\overline{X}}(-\log D)\cdot V_{i-1}\subset j_*V$.
  - $\rightsquigarrow$ $\nabla: V_{i-1}\to\Omega^1_{\overline{X}}(\log D)\otimes V_i$

De Rham cohomologies (T. Mochizuki, Esnault-S).

$$H^k_{\mathrm{dR}}(X,(V,\nabla))\simeq\boldsymbol{H}^k\big(\overline{X},(\Omega^\bullet_{\overline{X}}(\log D)\otimes V_\bullet,\nabla)\big)$$

$$H^k_{\mathrm{dR,c}}(X,(V,\nabla))\simeq\boldsymbol{H}^k\big(\overline{X},(\Omega^\bullet_{\overline{X}}(\log D)\otimes V_\bullet(-D),\nabla)\big)$$

Rapid decay and moderate homologies.

- $\varpi:\widetilde{X}\longrightarrow\overline{X}$ real oriented blow up of the components of D.
- $\widetilde{\jmath}:X^{\mathrm{an}}\hookrightarrow\widetilde{X}$
- $\mathscr{V}=\ker\nabla$ loc. cst. sheaf on X^{an}, e.g. defined over $\mathbb{Q}$.
- $\mathscr{V}^{\mathrm{rd}}\subset\mathscr{V}^{\mathrm{mod}}\subset\widetilde{\jmath}_*\mathscr{V}$: $\mathbb{R}$-constructible sheaves on $\widetilde{X}$.
- $H^{\mathrm{rd}}_m(X^{\mathrm{an}},\mathscr{V}):=H_m(\widetilde{X},\mathscr{V}^{\mathrm{rd}})$, $\quad H^{\mathrm{mod}}_m(X^{\mathrm{an}},\mathscr{V}):=H_m(\widetilde{X},\mathscr{V}^{\mathrm{mod}})$

Pairings. $\langle\bullet,\bullet\rangle$: nondeg. $\pm$-sym. pairing on (V,∇).

- $\rightsquigarrow$ Nondeg. pairings, e.g. for $m=n$:

$$\mathsf{S}:H^n_{\mathrm{dR,c}}(X,(V,\nabla))\otimes H^n_{\mathrm{dR}}(X,(V,\nabla))\longrightarrow\mathbb{C},$$
$$\mathsf{B}:H^{\mathrm{rd}}_n(X^{\mathrm{an}},\mathscr{V}_{\mathbb{Q}})\otimes H^{\mathrm{mod}}_n(X^{\mathrm{an}},\mathscr{V}_{\mathbb{Q}})\longrightarrow\mathbb{Q},$$
$$\mathsf{P}^{\mathrm{rd,mod}}:H^{\mathrm{rd}}_n(X^{\mathrm{an}},\mathscr{V}_{\mathbb{Q}})\otimes H^n_{\mathrm{dR}}(X,(V,\nabla))\longrightarrow\mathbb{C},$$
$$\mathsf{P}^{\mathrm{mod,rd}}:H^{\mathrm{mod}}_n(X^{\mathrm{an}},\mathscr{V}_{\mathbb{Q}})\otimes H^n_{\mathrm{dR,c}}(X,(V,\nabla))\longrightarrow\mathbb{C}.$$

- $\rightsquigarrow$ Nondeg. $\pm$-sym. pairings

$$\mathsf{S}^{\mathrm{mid}}:H^n_{\mathrm{dR,mid}}(X,(V,\nabla))\otimes H^n_{\mathrm{dR,mid}}(X,(V,\nabla))\longrightarrow\mathbb{C},$$
$$\mathsf{B}^{\mathrm{mid}}:H^{\mathrm{mid}}_n(X^{\mathrm{an}},\mathscr{V}_{\mathbb{Q}})\otimes H^{\mathrm{mid}}_n(X^{\mathrm{an}},\mathscr{V}_{\mathbb{Q}})\longrightarrow\mathbb{Q},$$
$$\mathsf{P}^{\mathrm{mid}}:H^{\mathrm{mid}}_n(X^{\mathrm{an}},\mathscr{V}_{\mathbb{Q}})\otimes H^n_{\mathrm{dR,mid}}(X,(V,\nabla))\longrightarrow\mathbb{C}.$$

***Corollary* (Middle quadratic relations).**

$$\boxed{\pm(-2\pi\mathrm{i})^n\mathsf{B}^{\mathrm{mid}}=\mathsf{P}^{\mathrm{mid}}\cdot(\mathsf{S}^{\mathrm{mid}})^{-1}\cdot{}^t\mathsf{P}^{\mathrm{mid}}}$$