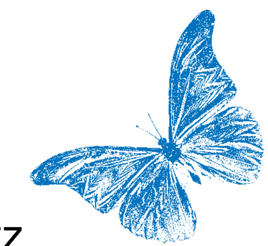
Hodge aspects of exponential motives

Claude Sabbah



Centre de Mathématiques Laurent Schwartz
École polytechnique, CNRS, Université Paris-Saclay
Palaiseau, France

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Pairs (X, f)

- X: quasi-proj. variety / field k.
- $f : X \to \mathbb{A}^1$: regular fn on X.

One finds these pairs e.g. in the following settings:

- $k = \mathbb{C}$, exponential periods

$$\int_\gamma e^f \omega, \qquad \omega\text{: alg. } n\text{-form, } \gamma\text{: semi-alg. } n\text{-chain.}$$

- $\mathrm{Char}\, k = p$, exponential sums.
- Mirror symmetry for Fano mflds: Landau-Ginzburg models.

Exp. motives (Fresán-Jossen)

- Data $[X, Y, f, n, i]$:
  - X quasi-proj. var. over $k \subset \mathbb{C}$,
  - $Y \subset X$ closed subvar.,
  - $n =$ degree of the cohom.,
  - i: Tate twist.

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  - $n =$ degree of the cohom.,
  - i: Tate twist.
- Choice of morphisms $\rightsquigarrow$ quiver $\mathbf{Q}^{\exp}(k)$.
- $\rho : \mathbf{Q}^{\exp}(k) \mapsto \mathbf{Vect}_{\mathbb{Q}}$,
  $\rho([X, Y, f, n, i]) := H^n_{\mathrm{rd}}(X, Y, f)(i)$.
- $\mathbf{Mot}^{\exp}(k)$: finite-dim. vect. spaces with an $\mathrm{End}(\rho)$-action.
- $(f \boxplus f')(x, x') := f(x) + f'(x') \rightsquigarrow$ tensor structure.

Exp. motives (Fresán-Jossen)

- TH. (Fresán-Jossen): $\mathbf{Mot}^{\exp}(k)$ *neutral Tannakian category.*
- $[X, Y, 0, i] \longleftrightarrow$ ***classical*** Nori motives.
- $\rightsquigarrow \mathbf{Mot}^{\mathrm{cl}}(k) \to \mathbf{Mot}^{\exp}(k)$ fully faithful.

Betti real. of exp. motives

- $H^n_{\mathrm{rd}}(X, Y, f) := H^n(X, Y \cup \{\mathrm{Re}(f) \gg 0\}; \mathbb{Q})$
- If $\gamma \in H_n(X, Y \cup \{\mathrm{Re}(f) \gg 0\}; \mathbb{Q})$ (dual vect. sp.), convergent period integral

$$\int_\gamma e^{-f}\omega, \qquad \omega \in \Gamma(X, \Omega^n_X).$$

- Poincaré duality, e.g. X smooth and $Y = \varnothing$:

$$H^n_{\mathrm{rd,c}}(X, f) \otimes H^{2d_X - n}_{\mathrm{rd}}(X, -f) \longrightarrow H^{2d_X}_{\mathrm{c}}(X) \simeq \mathbb{Q}.$$

De Rham real. of exp. motives

- Setting: X ***smooth***, $Y = \varnothing$.
- ***Twisted de Rham cohomology*** $H^n_{\mathrm{dR}}(X, \mathrm{d} + \mathrm{d}f)$: Hypercohomology of the alg. de Rham cplx.

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{\mathrm{d} + \mathrm{d}f} \Omega^1_X \longrightarrow \cdots \xrightarrow{\mathrm{d} + \mathrm{d}f} \Omega^d_X \longrightarrow 0$$

E.g. X affine:

$$0 \longrightarrow \mathscr{O}(X) \xrightarrow{\mathrm{d} + \mathrm{d}f} \cdots \xrightarrow{\mathrm{d} + \mathrm{d}f} \Omega^d(X) \longrightarrow 0$$

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E.g. $X = \mathbb{C}^n$:

$$0 \to \mathbb{C}[x] \to \bigoplus_i \mathbb{C}[x]\mathrm{d}x_i \to \cdots \to \bigoplus_i \mathbb{C}[x]\mathrm{d}\widehat{x_i} \to \mathbb{C}[x]\mathrm{d}x \to 0$$

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$$g(x) \longmapsto \sum_i (g'_{x_i} + g f'_{x_i})\mathrm{d}x_i$$

$$\sum_i h_i \mathrm{d}\widehat{x_i} \longmapsto \Big[\sum_i (-1)^{i-1}((h_i)'_{x_i} + h_i f'_{x_i})\Big]\mathrm{d}x$$

Logarithmic computation

- Choose $f : \overline{X} \to \mathbb{P}^1$ s.t.
  - $\overline{X}$ smooth projective,
  - $D := \overline{X} \smallsetminus X =$ ncd, $\quad D = |P| \cup H$,
  - $P = f^*(\infty)$ pole divisor with supp. $|P| \subset D$.

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- Kontsevich complex $(\Omega^\bullet_{\overline{X}}(\log D, f), \mathrm{d} + \mathrm{d}f)$:
- $\Omega^k_{\overline{X}}(\log D, f)$ defined as

$$\ker\Big[\mathrm{d}f : \Omega^k_{\overline{X}}(\log D) \longrightarrow \Omega^{k+1}_{\overline{X}}(\log D)(P)/\Omega^{k+1}_{\overline{X}}(\log D)\Big]$$

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- TH.: $\forall n$,

$$\boxed{H^n(\overline{X}, (\Omega^\bullet_{\overline{X}}(\log D, f), \mathrm{d} + \mathrm{d}f)) \xrightarrow{\sim} H^n_{\mathrm{dR}}(X, \mathrm{d} + \mathrm{d}f)}$$

Logarithmic computation

- $\rightsquigarrow$ dR cohom. with cpct supp. (recall $D = |P| \cup H$):

$$H^n_{\mathrm{dR,c}}(X, \mathrm{d}{+}\mathrm{d}f) := H^n(\overline{X}, (\Omega^\bullet_{\overline{X}}(\log D, f)(-H), \mathrm{d}{+}\mathrm{d}f))$$

- TH. (Yu): *Perfect pairing*

$$H^n_{\mathrm{dR,c}}(X, \mathrm{d}{+}\mathrm{d}f) \otimes H^{2d_X - n}_{\mathrm{dR}}(X, \mathrm{d}{-}\mathrm{d}f) \longrightarrow H^{d_X}(\overline{X}, \Omega^{d_X}_{\overline{X}}) \simeq \mathbb{C}$$

- Compatible with the Poincaré pairing via the comparison iso

$$H^k_{\mathrm{dR}}(X, \mathrm{d} + \mathrm{d}f) \simeq H^k_{\mathrm{rd}}(X, f; \mathbb{C})$$

and compact support analogue.

Rescaling parameter

- Hodge theory enters with the rescaling parameter

$$f \longmapsto f/u, \qquad u \in k^*.$$

- This defines an endofunctor of $\mathbf{Mot}^{\mathrm{exp}}$.
- Extension to $u = 0$:
  - ***Betti***: $\rightsquigarrow$ Stokes structure on the local system $H^n_{\mathrm{rd}}(X, f/u; \mathbb{Q})_{u \in \mathbb{C}^*}$.
  - ***De Rham***: Brieskorn $\mathbb{C}[u]$-modules

$$H^n_{\mathrm{dR}}(X, u\mathrm{d}+\mathrm{d}f) := H^n(\overline{X}, (\Omega^\bullet_{\overline{X}}[u](\log D, f), u\mathrm{d}+\mathrm{d}f))$$

- Th. (Barannikov-Kontsevich):

$$H^n_{\mathrm{dR}}(X, u\mathrm{d} + \mathrm{d}f) \quad \text{is } \mathbb{C}[u]\text{-free of finite rk.}$$

Rescaling parameter

- Yu: Perfect pairing of free $\mathbb{C}[u]$-modules

$$H^n_{\mathrm{dR,c}}(X, u\mathrm{d}+\mathrm{d}f)\otimes H^{2d_X-n}_{\mathrm{dR}}(X, u\mathrm{d}-\mathrm{d}f) \longrightarrow H^{d_X}(\overline{X}, \Omega^{d_X}_{\overline{X}})[u]$$

also known as K. Saito's ***residue pairing***.

- At $u = 0$: Serre duality for $(\Omega^{\bullet}_{\overline{X}}(\log D, f), \mathrm{d}f)$.

Irregular Hodge filtration

- TH.: $\dim H^n(\overline{X}, (\Omega^\bullet_{\overline{X}}(\log D, f), u_o \mathrm{d} + v_o \mathrm{d}f))$ *is independent of* $u_o, v_o \in \mathbb{C}$.
- Various proofs: Katzarkov-Kontevich-Pantev, Esnault-CS-Yu, M. Saito, T. Mochizuki.
- COR.: *Degeneration at* E_1 *of the spectr. seq. for the filtration of* $(\Omega^\bullet_{\overline{X}}(\log D, f), \mathrm{d} + \mathrm{d}f)$ *by the stupid truncation.*
- $\rightsquigarrow$ ***irregular Hodge filtration*** $F^p H^n_{\mathrm{dR}}(X, \mathrm{d} + \mathrm{d}f)$ s.t.

$$\boxed{\mathrm{gr}^p_F H^{p+q}_{\mathrm{dR}}(X, \mathrm{d} + \mathrm{d}f) \simeq H^q(\overline{X}, \Omega^p_{\overline{X}}(\log D, f))}$$

Irregular Hodge-Tate property

- $H^n_{\mathrm{rd}}(X,f) \simeq H^n(X,f^{-1}(t);\mathbb{Q})$ for $|t| \gg 0$.
- Monodromy T on $H^n(X,f^{-1}(t);\mathbb{Q})$.
  ($=$ Monodr. of the u-loc. syst. $H^n_{\mathrm{rd}}(X,f/u;\mathbb{Q})_{u\in\mathbb{C}^*}$)
- DEF.: $H^n(X,f)$ is ***irreg. Hodge-Tate*** if
  - T is unipotent, $\rightsquigarrow$ monodromy weight filtr. $W_\bullet H^n_{\mathrm{rd}}(X,f)$ centered at n,
  - $\forall p$, $\dim \mathrm{gr}^p_F H^n_{\mathrm{dR}}(X,f) = \dim \mathrm{gr}^W_{2p} H^n_{\mathrm{rd}}(X,f)$

Irregular Hodge-Tate property

- CONJ. (Katzarkov-Kontsevich-Pantev):
  If (X, f) arises as a tame Landau-Ginzburg model of a smooth projective Fano variety then $H^{d_X}(X, f)$ is irregular Hodge-Tate.
- Various partial results in small dim.

Irregular Hodge-Tate property

- CONJ. (Katzarkov-Kontsevich-Pantev): *If (X, f) arises as a tame Landau-Ginzburg model of a smooth projective Fano variety then $H^{d_X}(X, f)$ is irregular Hodge-Tate.*
- Various partial results in small dim.
- TH. (CS): *The conj. holds for Landau-Ginzburg models arising from **toric** Fano manifolds.*
- Uses Batyrev's classification of toric Fano mflds M by ***reflexive polytopes***. Then $X \simeq (\mathbb{C}^*)^d$ and $f = \sum x^m$ where m is a vertex of the polytope. Then $f \longleftrightarrow c_1(TM)$ and satisfies Hard Lefschetz on $H^*(M)$.

Sym. Kloosterman connections

- Kloosterman connection: $\mathbb{C}$ analogue of Kloosterman sheaf

$$\begin{array}{ccc} & \mathbb{G}_{\mathrm{m}}^2 & \\ {\scriptstyle \mathbf{prod}} \swarrow & & \searrow {\scriptstyle \mathbf{sum}} \\ \mathbb{G}_{\mathrm{m},z} & & \mathbb{A}^1_y \end{array}$$

$\mathrm{Kl}_2 := \mathrm{prod}_* \mathrm{sum}^*(\mathscr{O}_{\mathbb{A}^1}, \mathrm{d} + \mathrm{d}y)$
rk 2 vect. bdle on $\mathbb{G}_{\mathrm{m},z}$ with conn.
$\rightsquigarrow \mathrm{Sym}^k \mathrm{Kl}_2$:
rk $k+1$ vect. bdle with conn.

- PROP. (Fresán-CS-Yu): $H^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ *is the dR real. of an exp. motive* $H^{k+1}(\mathbb{G}_{\mathrm{m}}^{k+1}, f_k)^{\mathfrak{S}_k}$*.*

$$f_k(x_1, \ldots, x_k, z) = \sum_{i=1}^{k} x_i + z \sum_{i=1}^{k} 1/x_i.$$

Sym. Kloosterman connections

- Recall $\mathbf{Mot}^{\mathrm{cl}}(k) \to \mathbf{Mot}^{\exp}(k)$ fully faithful.
- PROP. (Fresán-CS-Yu):
  *The exp. motive $H^{k+1}(\mathbb{G}_{\mathrm{m}}^{k+1}, f_k)^{\mathfrak{S}_k}$ is **classical**.*
- Defined from $H_{\mathrm{c}}^{k-1}(\mathscr{K})$,

$$\mathbb{G}_{\mathrm{m}}^{k} \supset \mathscr{K} : \qquad \sum_{i=1}^{k} x_i + \sum_{i=1}^{k} 1/x_i = 0.$$

Sym. Kloosterman connections

- $\Longrightarrow H^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ underlies a MHS, with Hodge filtr. $=$ irreg. Hodge filtr.

- TH. (Fresán-CS-Yu):
  Hodge nbrs of $H^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ *are* 0 *or* 1 (explicit formula).

- Proof by easier comput. of irreg. Hodge filtr. (by applying the toric method of Adolphson & Sperber).

- $\rightsquigarrow$ Arithmetic consequences for symmetric power moments of Kloosterman sums: functional equation for the corresponding L-function.

Rigid irred. connections on $\mathbb{P}^1$

SETTING:

- U: Zar. open in $\mathbb{P}^1$, (V, ∇) bdle with connect. on U.
- Assumptions on (V, ∇):
  - irreducible,
  - ***rigid***, *i.e.*, $\forall (V', \nabla')$ irred.,

$$(V', \nabla')_{\widehat{x}} \simeq (V, \nabla)_{\widehat{x}} \; \forall x \in \mathbb{P}^1 \smallsetminus U \quad \Longrightarrow \quad (V,', \nabla') \simeq (V, \nabla)$$

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MANY INTERESTING EXAMPLES:

- Hypergeometrics,
- Frenkel-Gross examples,
- Examples with diff. Galois group G_2 (Dettweiler-Reiter-Katz, K. Jakob)
- But $\mathrm{Sym}^k \mathrm{Kl}_2$ rigid iff $k = 1, 2$.

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$$(V',\nabla')_{\widehat{x}} \simeq (V,\nabla)_{\widehat{x}}\ \forall x\in\mathbb{P}^1\smallsetminus U \quad\Longrightarrow\quad (V,',\nabla') \simeq (V,\nabla)$$

RESULTS (Katz-Arinkin-Deligne algorithm): Any rigid irred. (V,∇) on $\mathbb{P}^1$ can be obtained from the trivial bdle $(\mathscr{O}_U,\mathrm{d})$ by applying successively elementary operations:

- tensor product by a rk-one (L,∇),
- Middle convolution by a rk-one (L,∇),
- Fourier transform.

Tame rigid irred. connections on $\mathbb{P}^1$

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CONSEQUENCE (Katz): Assume eigenvalues of local monodromies at each $x \in \mathbb{P}^1 \smallsetminus U$ are roots of unity.

- $\rightsquigarrow$ Geometric expression of (V, ∇) as a subquotient of a Gauss-Manin connection.
- $\rightsquigarrow$ pVHS on (V, ∇) (ess. unique, cf. Deligne).
- $\rightsquigarrow$ Comput. of Hodge nbrs through the algorithm (Dettweiler-CS).

Wild rigid irred. connections on $\mathbb{P}^1$

RESULTS (Deligne-Arinkin algorithm): Any ***wild*** rigid irred. (V, ∇) on $\mathbb{P}^1$ can be obtained from the trivial bdle $(\mathscr{O}_U, \mathrm{d})$ by applying successively elementary operations:

- tensor product by a rk-one (L, ∇),
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Wild rigid irred. connections on $\mathbb{P}^1$

RESULTS (Deligne-Arinkin algorithm): Any ***wild*** rigid irred. (V, ∇) on $\mathbb{P}^1$ can be obtained from the trivial bdle $(\mathscr{O}_U, \mathrm{d})$ by applying successively elementary operations:

- tensor product by a rk-one (L, ∇),
- Fourier transform.

CONSEQUENCE: Assume eigenvalues of local ***formal*** monodromies at each $x \in \mathbb{P}^1 \smallsetminus U$ are roots of unity.

- $\rightsquigarrow$ Geometric expression of (V, ∇) as a subquotient of a Gauss-Manin connection ***twisted by some*** e^f.
- $\rightsquigarrow$ var. of ***irreg. HS*** on (V, ∇) (ess. unique).
- $\rightsquigarrow$ Examples of comput. of irreg. Hodge nbrs (Castaño Domínguez-Sevenheck, Yu-CS).