Wild Hodge Theory

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Introduction

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Question:

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Question: What kind of Hodge theory can one develop in presence of ***irregular singularities*** ?

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  - Expect a Hodge filtration indexed by ***real*** numbers.
  - Lamentation: Do ***not*** expect a usual Hodge decomposition for this filtration.

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$$H^1\big(U, F^{\bullet}_{\mathrm{Del}}\,\mathrm{DR}(V, \nabla + df)\big) \Longrightarrow H^1_{\mathrm{DR}}\big(U, (V, \nabla + df)\big).$$

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    previously known (Deligne) if $\mathscr{V}$ underlies a PVHS.
  - The proof uses the existence of a ***harmonic metric*** on the associated flat bundle (V, ∇).
  - Equivalently: (V, ∇) ***underlies a variation of polarized twistor structure of weight*** 0.

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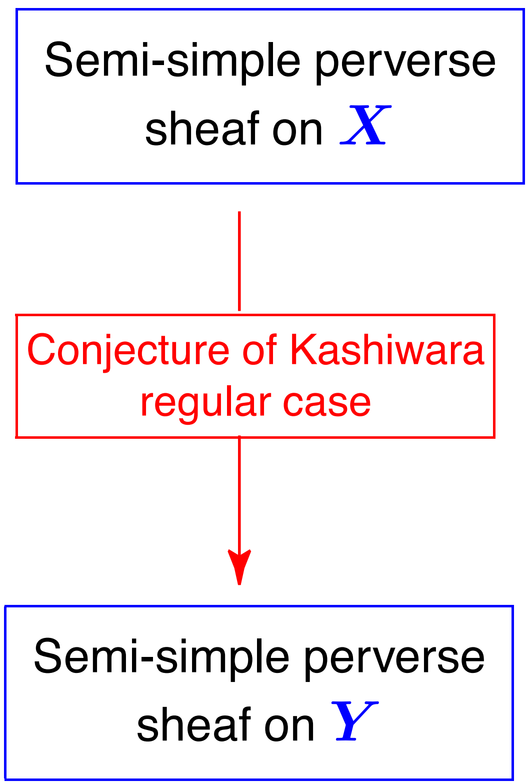

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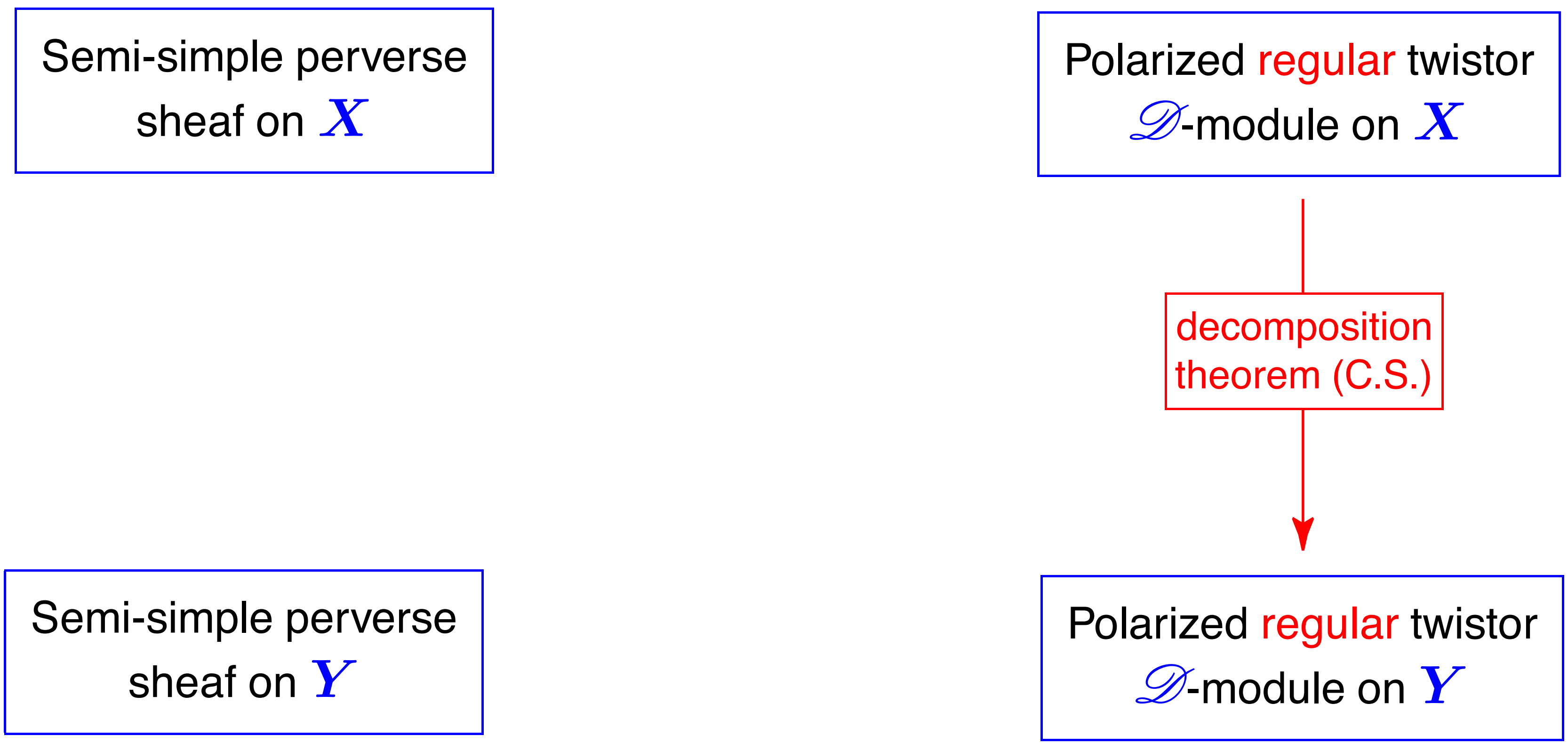

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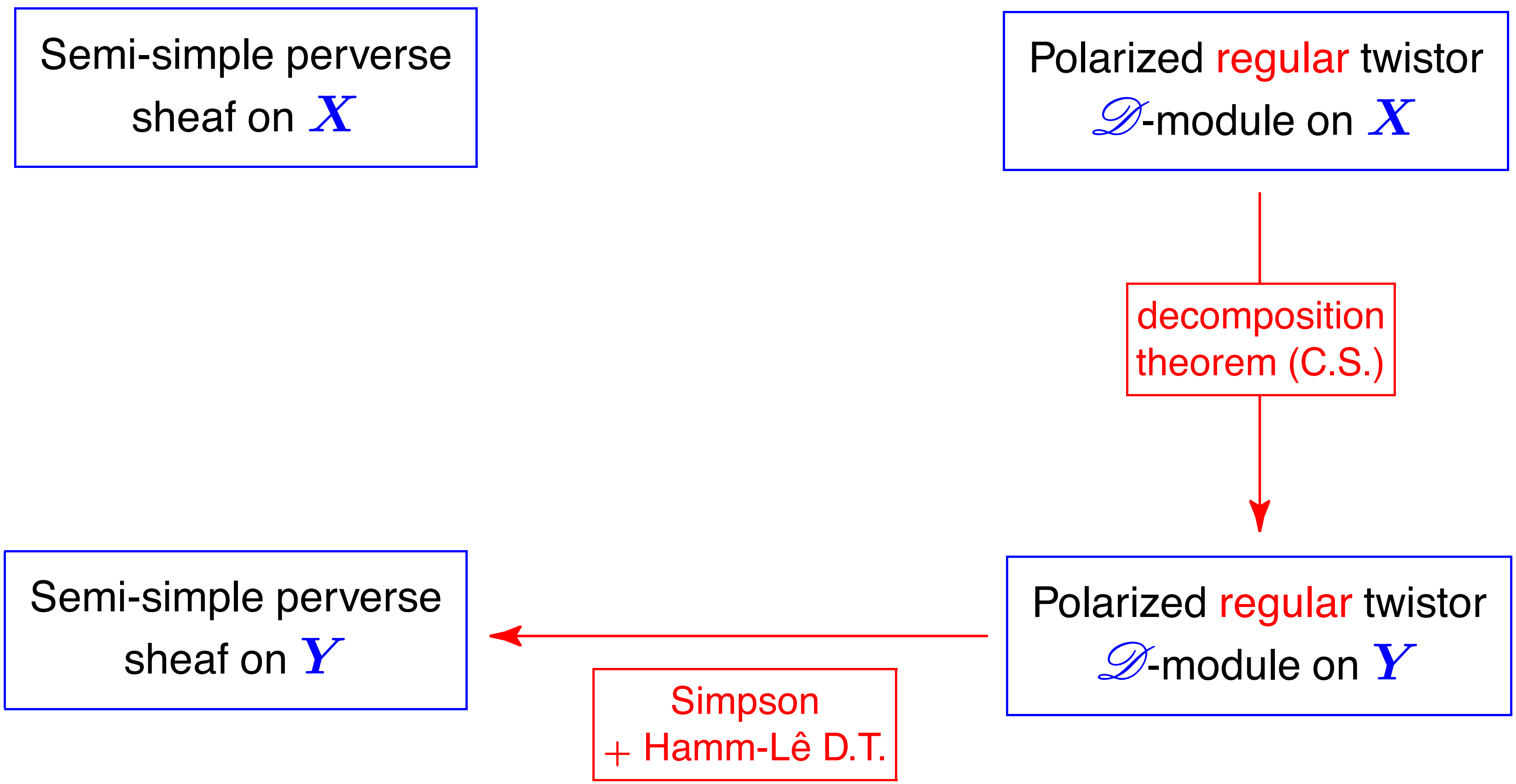

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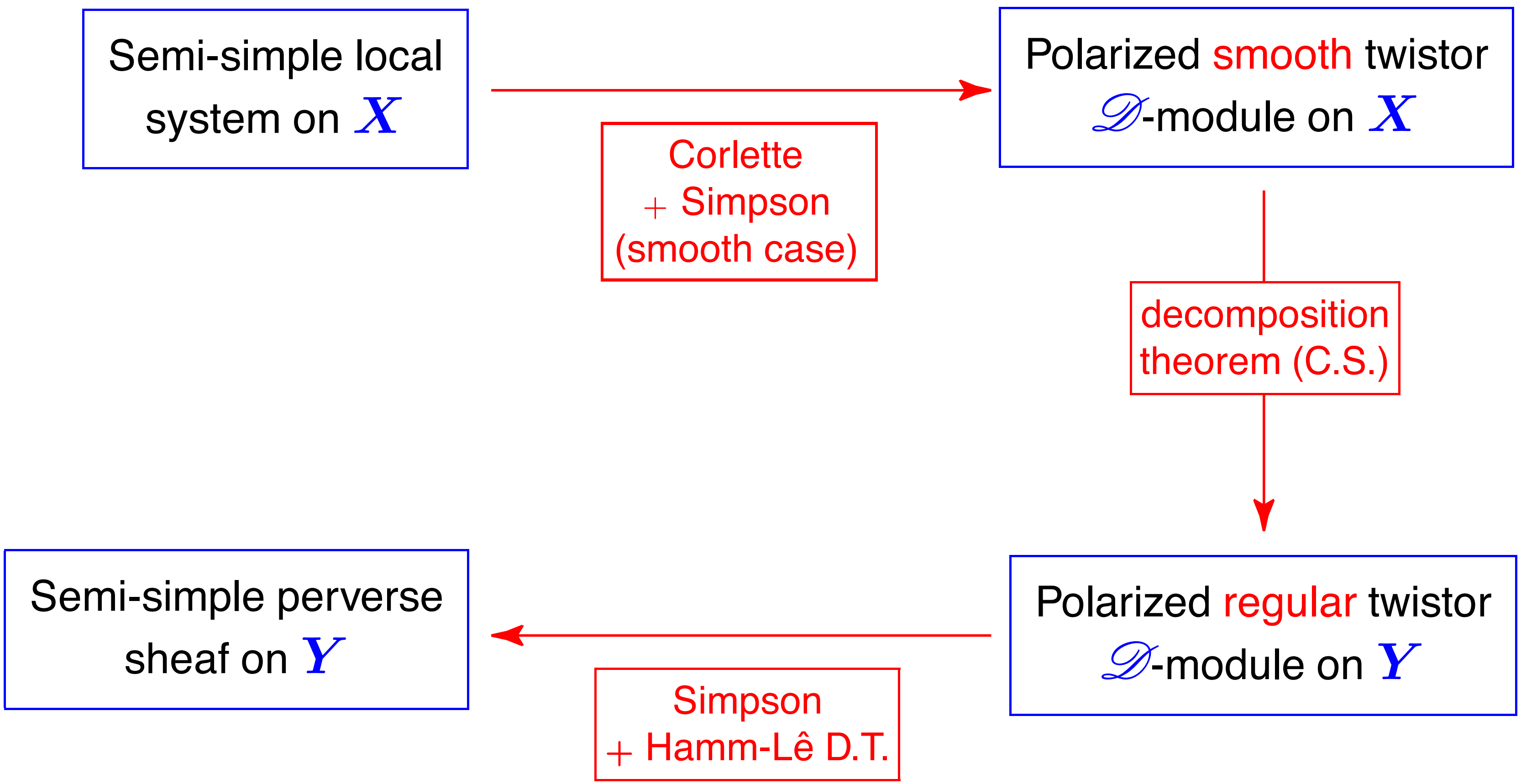

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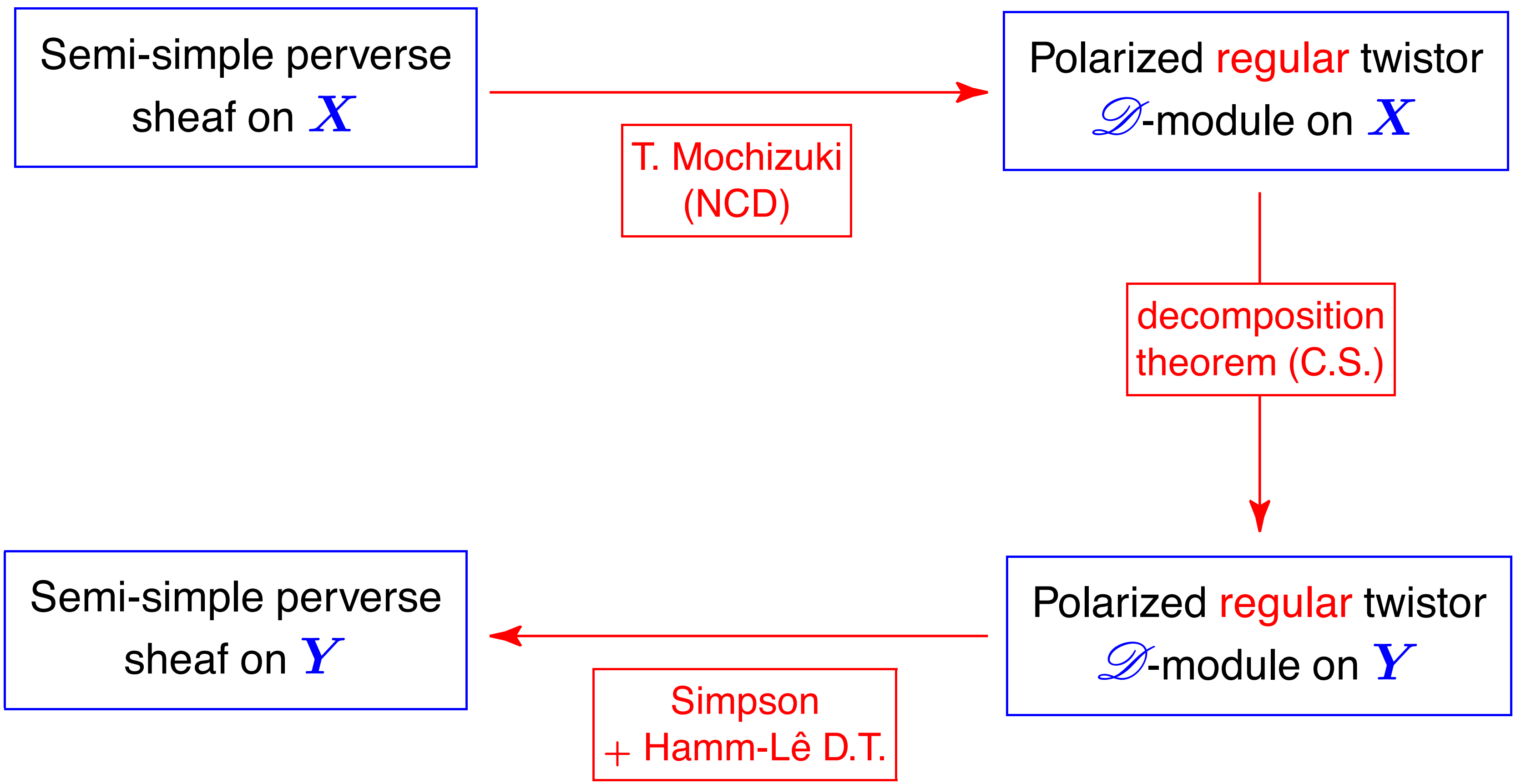

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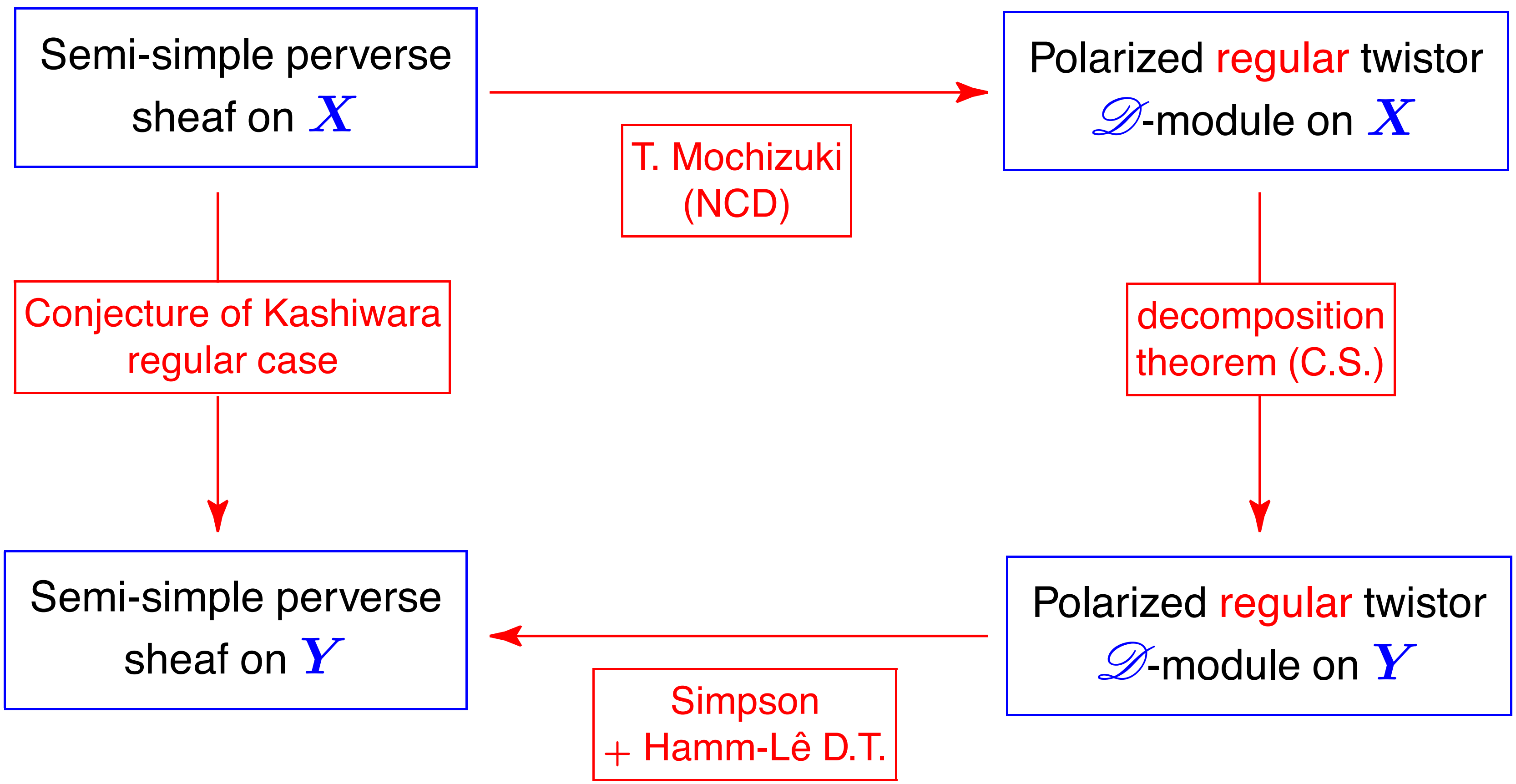

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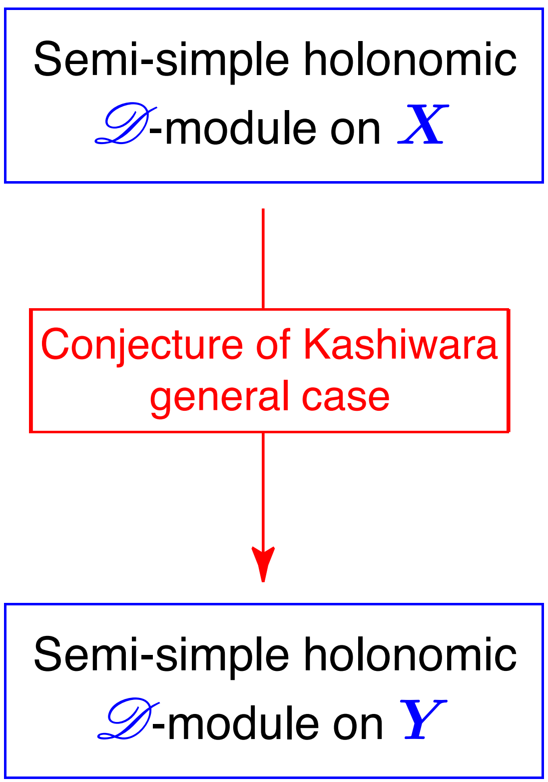

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C.S. (using ideals of Deligne, letter to Malgrange dec. 1983)

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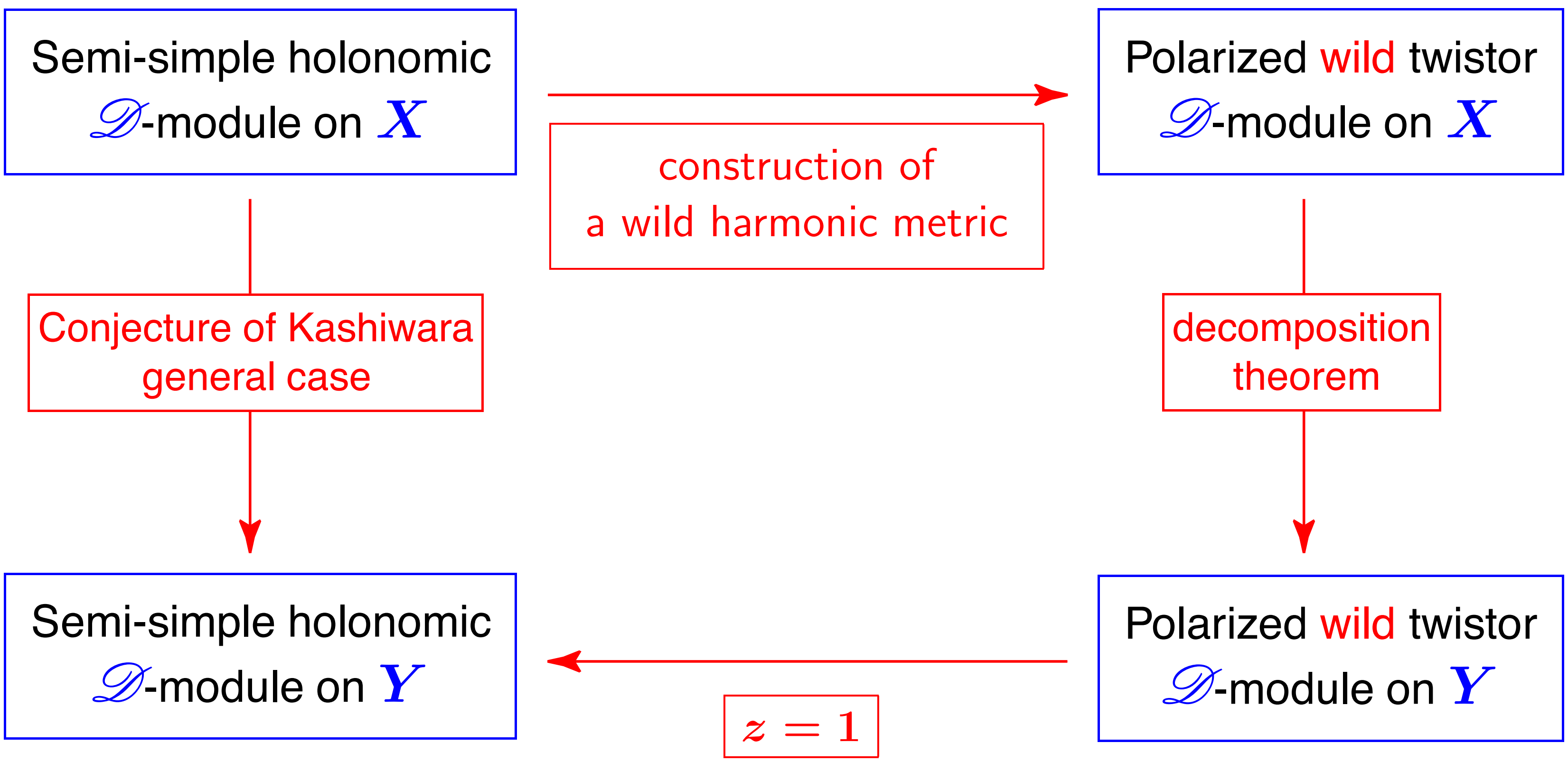
Sketch of T. Mochizuki's proof :



Twistor structures

(C. Simpson)

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Positivity of h	Positivity of $\Gamma(\mathbb{P}^1, \mathscr{S})$
Tate twist (k), $k \in \mathbb{Z}$	$\otimes \mathscr{O}_{\mathbb{P}^1}(-2k)$ $(k \in \frac{1}{2}\mathbb{Z})$

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Flatness: $\mathfrak{D}^2 = (\mathfrak{D}' + \mathfrak{D}'')^2 = 0.$

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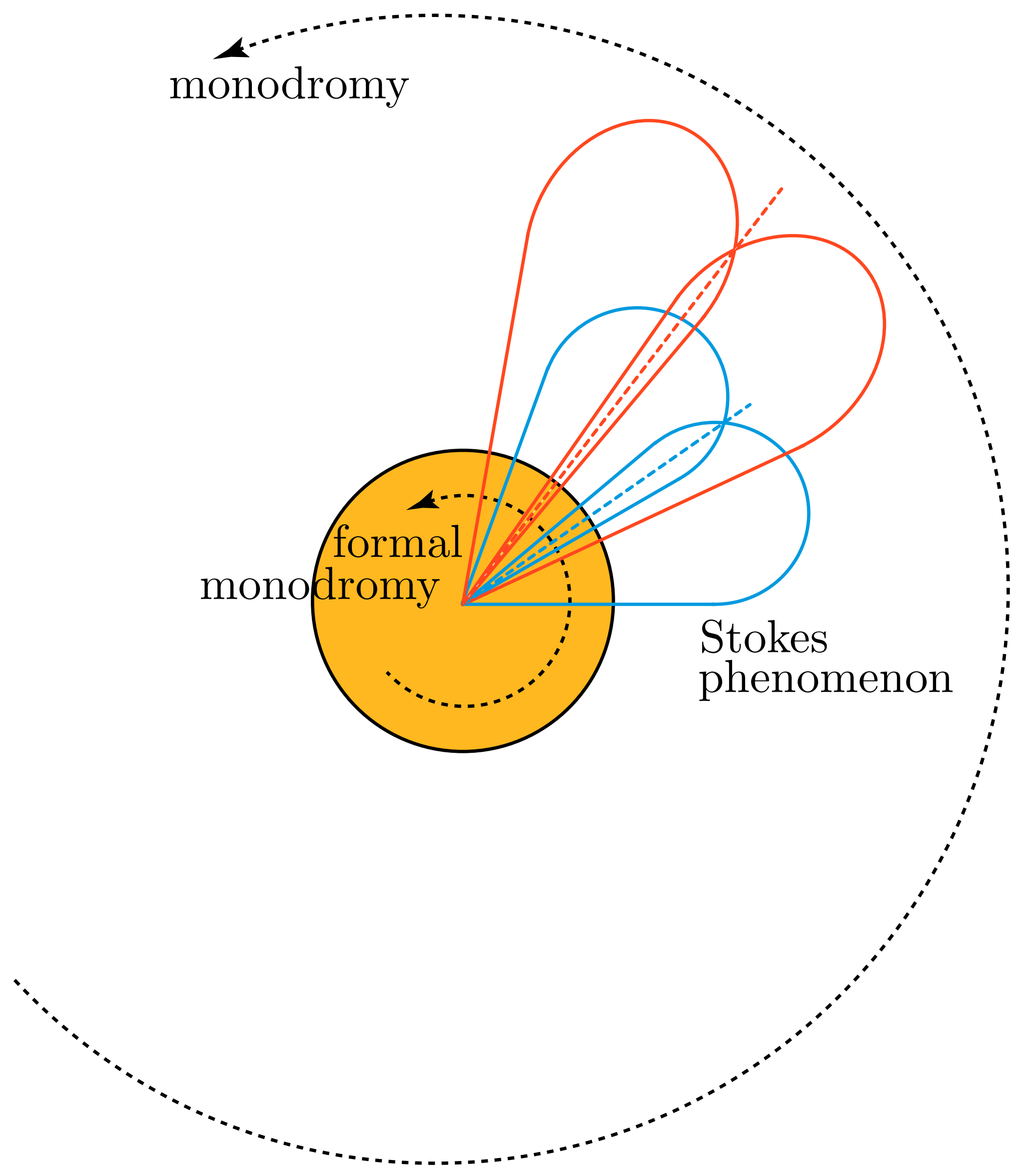

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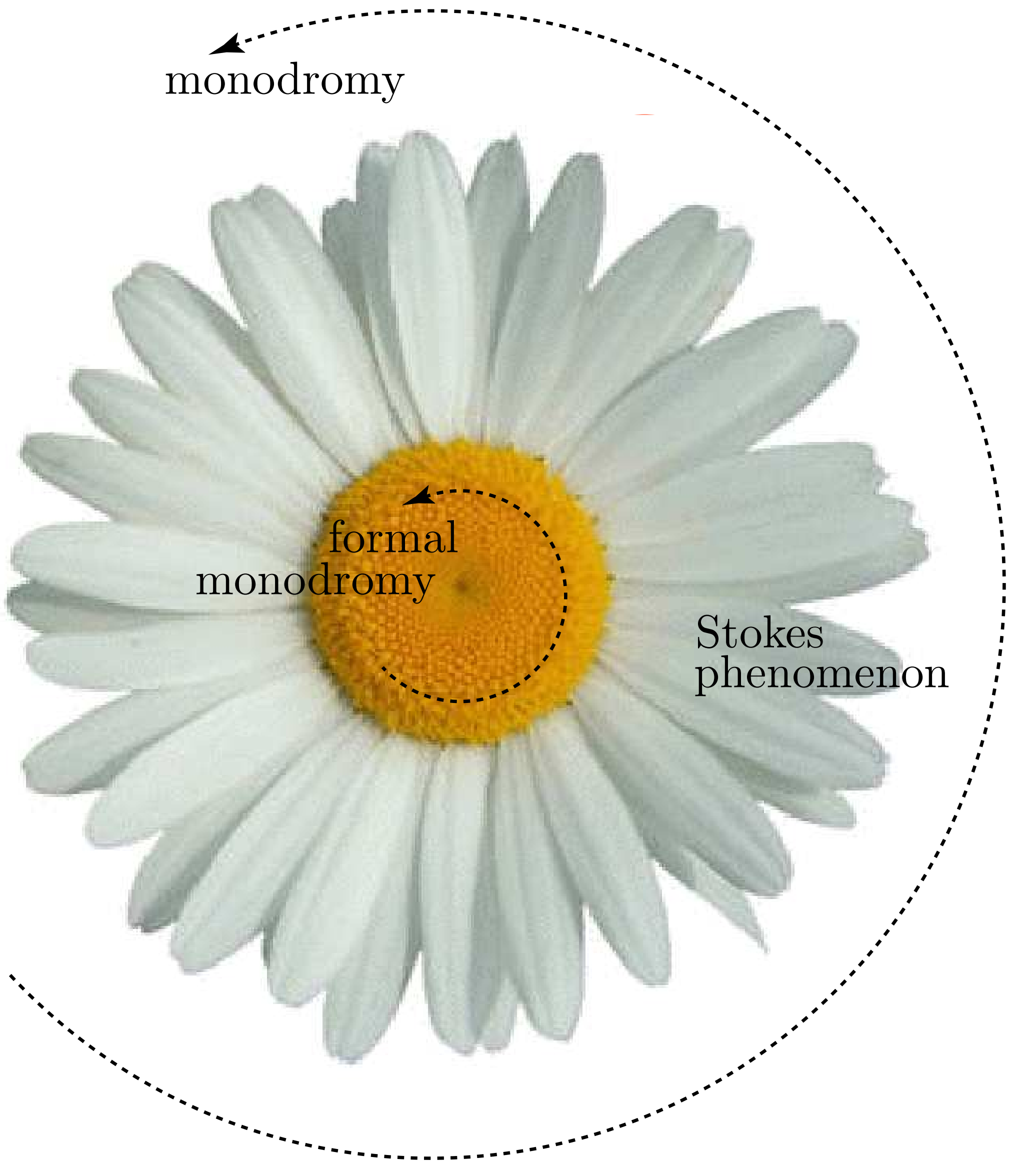

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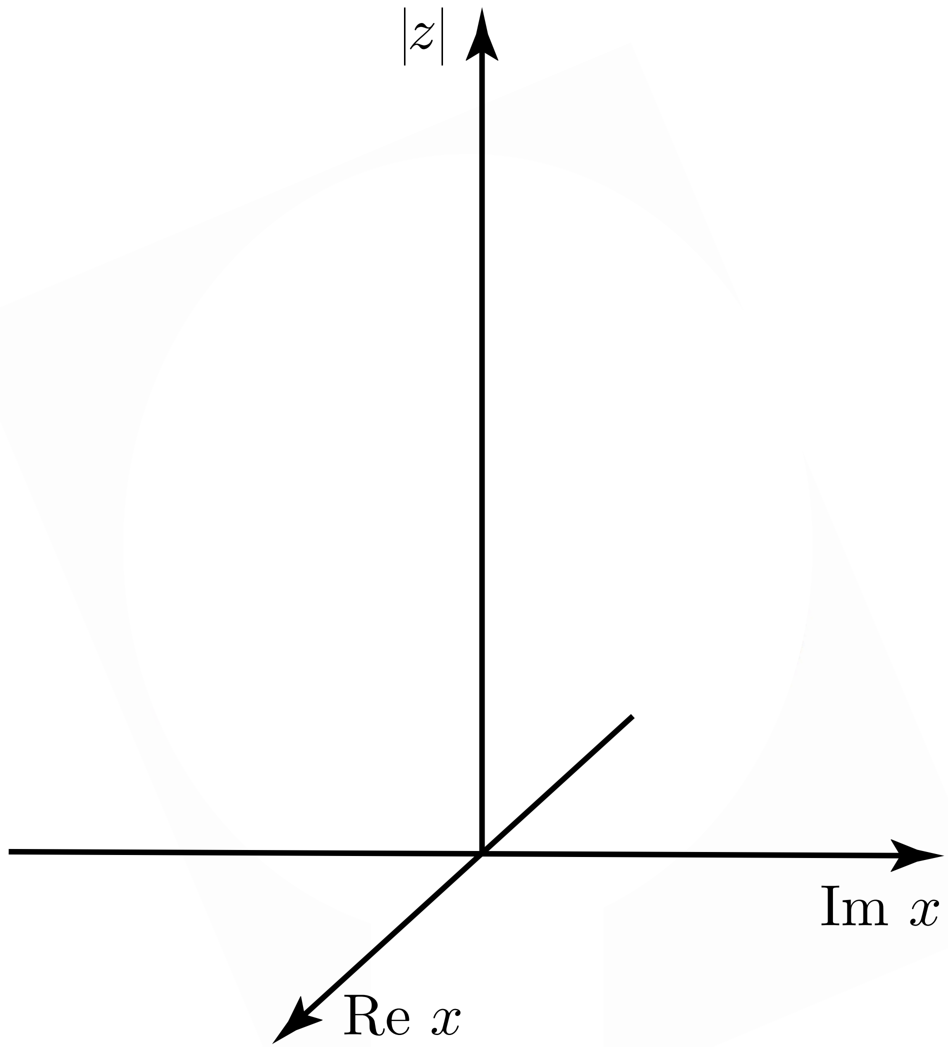

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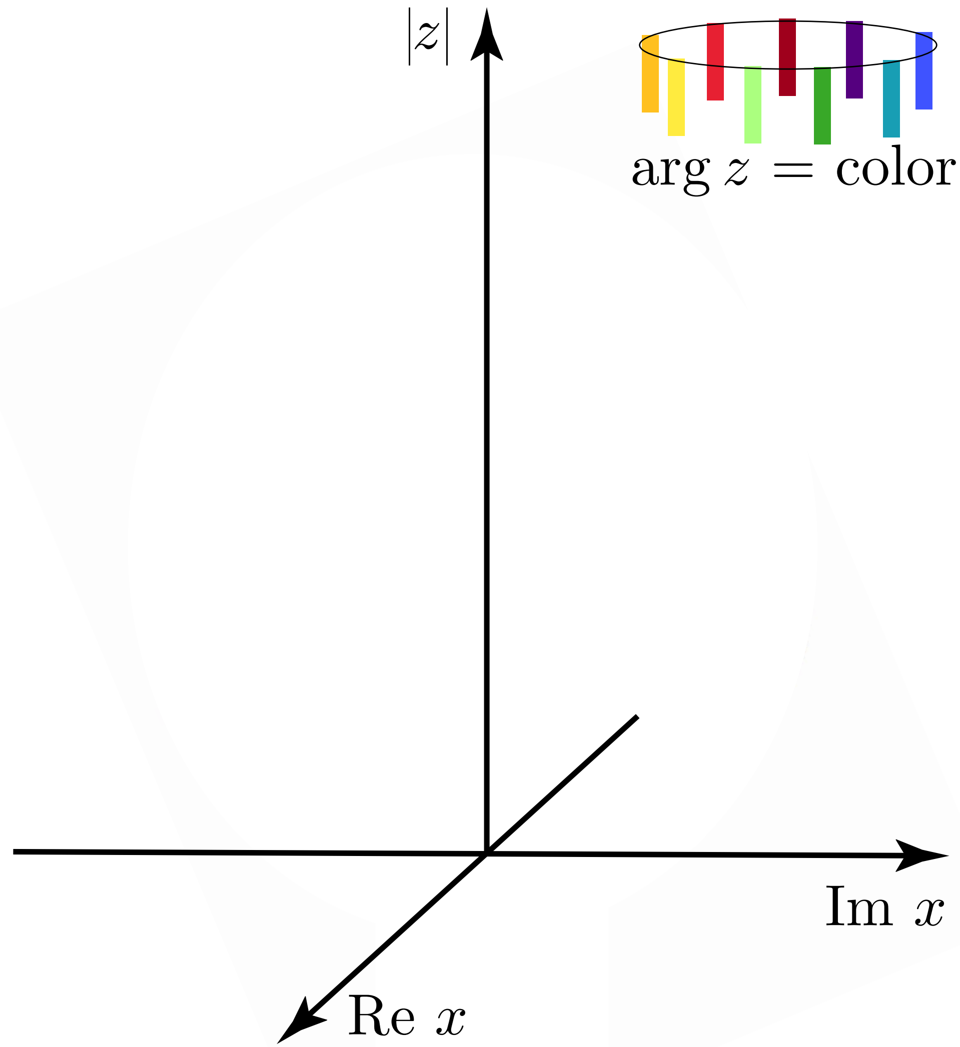

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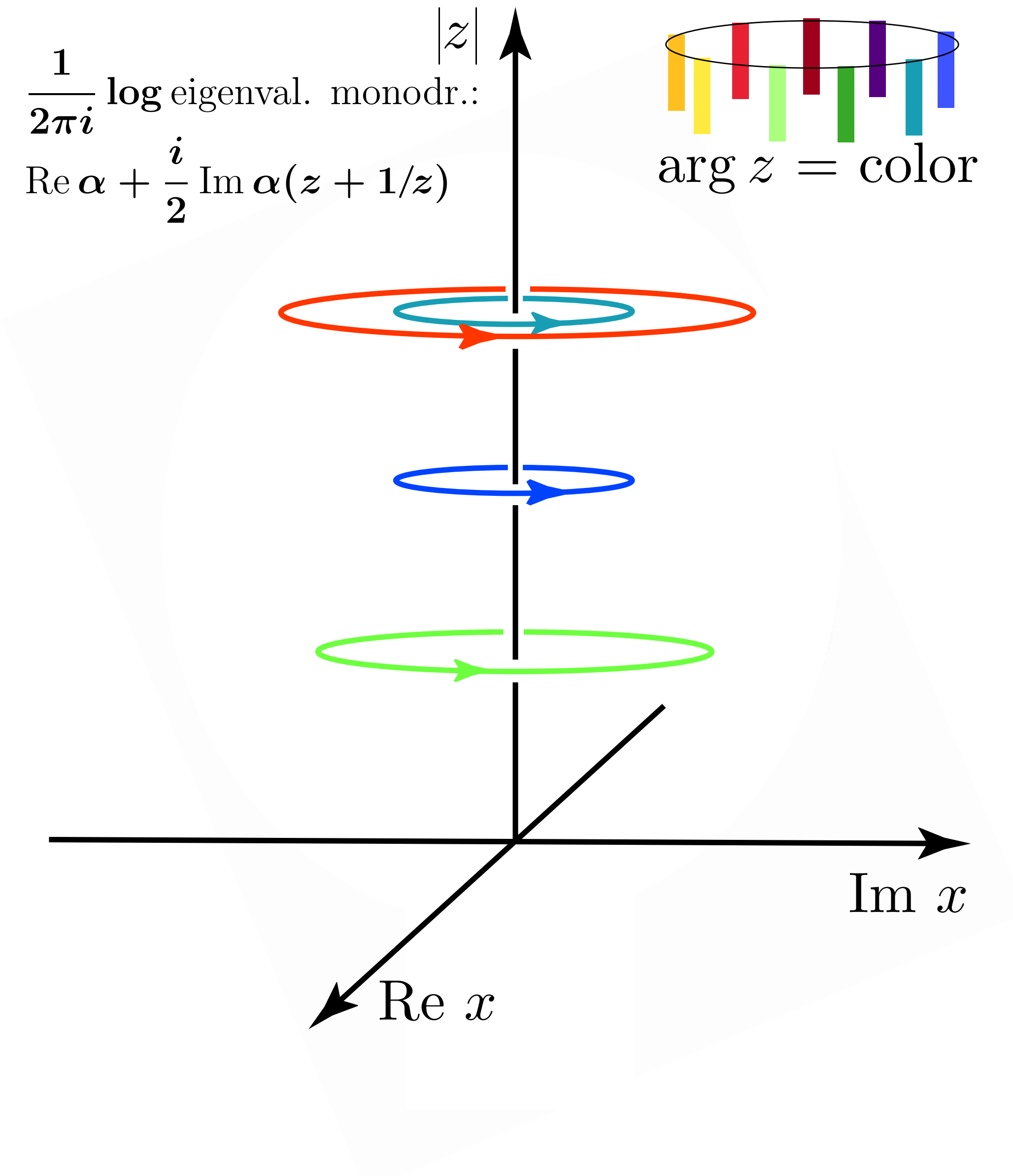

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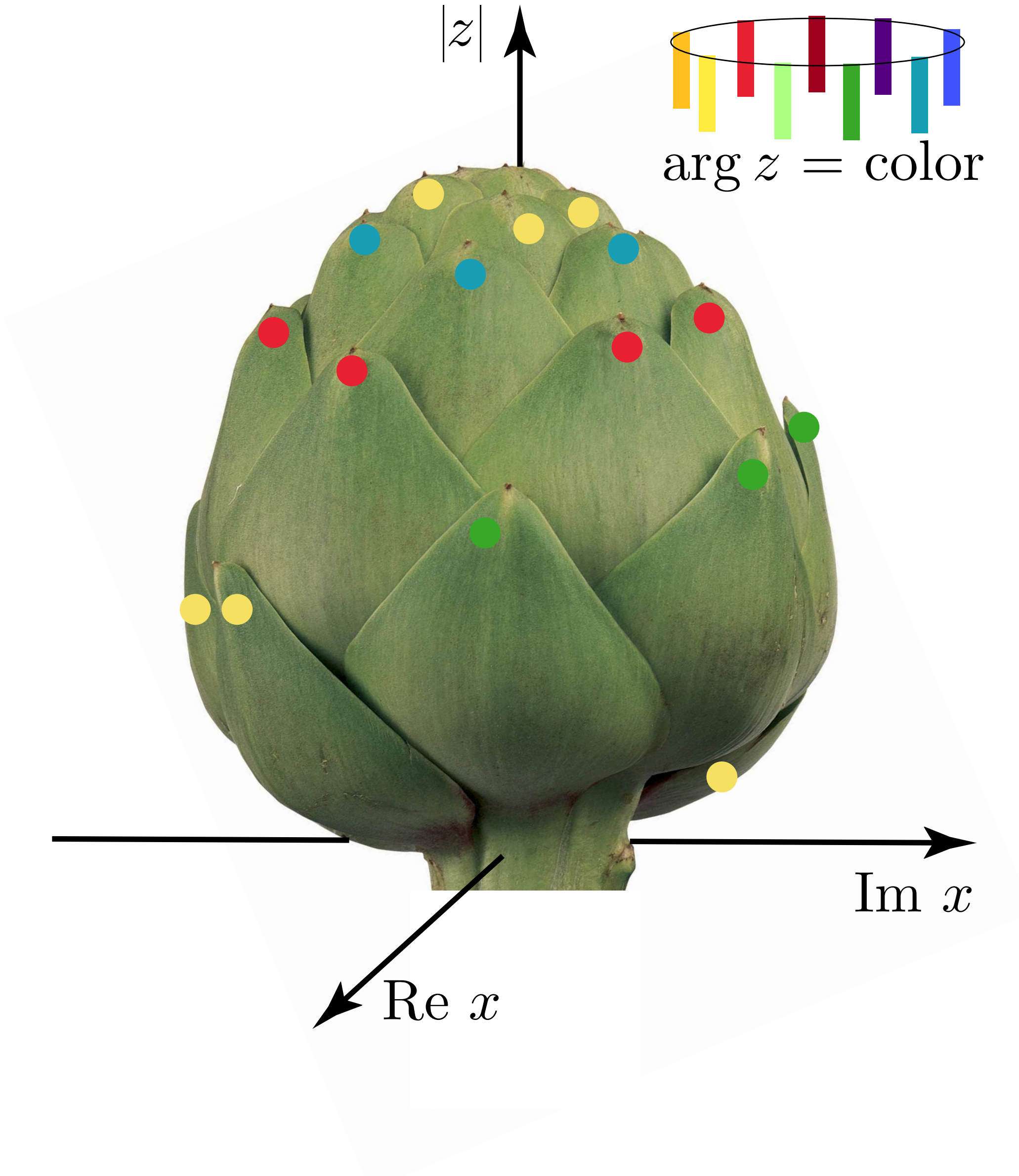

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