FOURIER TRANSFORM OF INDICES AND INTERVALS
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by

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1. Introduction

We consider the formal Fourier transformation

$$F^{\infty,\infty} : \mathsf{Mod}(\mathbb{C}(\!(z)\!), \partial_z) \longrightarrow \mathsf{Mod}(\mathbb{C}(\!(u)\!), \partial_u)$$

with kernel $E^{1/zu}$. The aim of the workshop is to understand the description by Mochizuki of the extension of this functor to Stokes-filtered local system around ∞, so that the transformation of graded Stokes-filtered local system corresponds to $F^{\infty,\infty}$ via the Deligne-Malgrange RH correspondence. In this lecture, we focus on the behavior of the exponential factors $\widetilde{\mathfrak{I}}$ and the corresponding set of intervals $T(\overline{[\mathfrak{I}]})$ by this transformation.

Let $n > p \geqslant 1$ be positive integers. We consider a set $\widetilde{\mathfrak{I}}$ of p-ramified exponential factors with maximal order of the pole equal to n. We assume that $\mathfrak{I}$ contains at least two elements, one of them being 0, so that n is also the maximal level of $\widetilde{\mathfrak{I}}$, that is, the maximal order of the poles of the differences of elements of $\widetilde{\mathfrak{I}}$.

We fix a p-th root $\zeta = z_p$ of z, so that we regard $\widetilde{\mathfrak{I}}$ as a subset of $\zeta^{-1}\mathbb{C}[\zeta^{-1}]$. By the local stationary phase formula recalled below, we have to consider $(n-p)$-ramified exponential factors in the Fourier variable u, so we fix an $(n-p)$-th root $v = u_{n-p}$ of u.

2. Reminder of the formal Fourier transformation $F^{\infty,\infty}$

Let $\widetilde{\mathfrak{U}}(n,p)^* \subset \zeta^{-1}\mathbb{C}[\zeta^{-1}]\smallsetminus\{0\}$ be the subset of polar parts having a pole of order equal to n, and set $\widetilde{\mathfrak{U}}(n,p)=\widetilde{\mathfrak{U}}(n,p)^*\cup\{0\}$. We call a *ramification of order* p any $\rho\in\zeta^p\mathbb{C}(\!(\zeta)\!)\smallsetminus\zeta^{p+1}\mathbb{C}(\!(\zeta)\!)$.

Let $\widetilde{\mathfrak{a}}\in\widetilde{\mathfrak{U}}(n,p)^*$ and let ρ be a ramification of order p. The $(\pm)$ local formal Fourier transformation $F_{\pm}^{\infty,\infty}$ from ∞ to ∞ transforms an elementary connection $\mathrm{El}(\widetilde{\mathfrak{a}},\rho,R):=\rho_*(E^{\tilde{\mathfrak{a}}}\otimes R)$ (where R is finite-dimensional $\mathbb{C}(\!(\zeta)\!)$-vector space with a regular connection) to an elementary connection $\mathrm{El}(\widehat{\mathfrak{a}},\widehat{\rho},R(n/2))$, with $\widehat{\mathfrak{a}}\in\widetilde{\mathfrak{U}}(n,n-p)$, $\widehat{\rho}$ is a ramification of order $n-p$ and $(n/2)$ is the twist by the rank-one local system with monodromy $(-1)^n$, defined by the formulas, in the ζ variable (with $\rho(\zeta)=z^p$):

$$u=\widehat{\rho}(\zeta)=\pm\frac{\rho'(\zeta)}{\widetilde{\mathfrak{a}}'(\zeta)\rho(\zeta)^2},\quad \widehat{\mathfrak{a}}(\zeta)=\widetilde{\mathfrak{a}}(\zeta)+\frac{\rho(\zeta)}{\rho'(\zeta)}\,\widetilde{\mathfrak{a}}'(\zeta)=\widetilde{\mathfrak{a}}(\zeta)\pm\frac{1}{\rho(\zeta)\widehat{\rho}(\zeta)}.$$

Remark. These formulas make clear the $\pm$-involutiveness of $F_{\pm}^{\infty,\infty}$, namely $F_{-}^{\infty,\infty}=(F_{+}^{\infty,\infty})^{-1}$. Indeed, we can write $\widetilde{\mathfrak{a}}=\widehat{\mathfrak{a}}-1/\widehat{\rho}\rho$, and if we write the definition of $\widehat{\rho}$ as $\widetilde{\mathfrak{a}}'\widehat{\rho}=-(1/\rho)'$, then we find the converse definition for ρ:

$$\widehat{\mathfrak{a}}'\rho=\rho\Big[\widetilde{\mathfrak{a}}'+\frac{1}{\widehat{\rho}}\Big(\frac{1}{\rho}\Big)'+\frac{1}{\rho}\Big(\frac{1}{\widehat{\rho}}\Big)'\Big]=\Big(\frac{1}{\widehat{\rho}}\Big)'.$$

In the following, we focus on $F_{+}^{\infty,\infty}$, simply denoted by $F^{\infty,\infty}/$

Expression in the variable v. Fixing the variable ζ as above allows for an explicit expression of the action of $\mathbb{Z}/p\mathbb{Z}$ on the connection $\rho^*\mathrm{El}(\widetilde{\mathfrak{a}},\rho,R)$, which is simple to compute if $\rho(\zeta)$ has the simple form $\rho(\zeta)=\zeta^p=z$:

$$\rho^*\mathrm{El}(\widetilde{\mathfrak{a}},\rho,R)\simeq\bigoplus_{k=1}^{p}(E^{\sigma_k^*\tilde{\mathfrak{a}}}\otimes R),$$

with $(\sigma_k^*\widetilde{\mathfrak{a}})(\zeta)=\widetilde{\mathfrak{a}}(\exp(2\pi ik/p)\zeta)$, and the action of $\ell\in\mathbb{Z}/p\mathbb{Z}$ is given by the natural isomorphisms

$$\tau_\ell:\sigma_\ell^*(E^{\sigma_k^*\tilde{\mathfrak{a}}}\otimes R)\simeq(E^{\sigma_{k+\ell}^*\tilde{\mathfrak{a}}}\otimes R),$$

due to the natural isomorphism $\sigma_\ell^*R\simeq R$. We set

$$\omega=n/p>1\quad\text{and}\quad\widehat{\omega}=\omega/(\omega-1)=n/(n-p),\quad\text{so}\ \frac{1}{\omega}+\frac{1}{\widehat{\omega}}=1,$$

and we write $\widetilde{\mathfrak{a}}(\zeta)=a_n\zeta^{-n}(1+\zeta O(\zeta))$. We have the following estimate:

$$(\sigma_k^*\widetilde{\mathfrak{a}})(\zeta)\equiv e^{-2\pi ik\omega}\widetilde{\mathfrak{a}}(\zeta)\mod\zeta^{-n+1}\mathbb{C}[\![\zeta]\!].$$

Similarly, the choice of the variable v is done in order to make explicit the action of $\mathbb{Z}/(n-p)\mathbb{Z}$ on $\widehat{\rho}^*\mathrm{El}(\widehat{\mathfrak{a}},\widehat{\rho},R(n/2))$: we have

$$\widehat{\rho}^*\mathrm{El}(\widehat{\mathfrak{a}},\widehat{\rho},R(n/2))\simeq\bigoplus_{j=1}^{n-p}E^{\widehat{\sigma}_j^*\hat{\mathfrak{a}}}\otimes R(n/2),$$

so that we first have to compute the change of variable $\zeta = \varphi(v)$ so that $\widehat{\rho}(\zeta) = v^{n-p}$. We have

$$\mathrm{El}(\widehat{\mathfrak{a}}, \widehat{\rho}, R(n/2)) \simeq \mathrm{El}(\widehat{\mathfrak{a}} \circ \varphi, \widehat{\rho} \circ \varphi, R(n/2)),$$

and

$$\sigma_j^*(\widehat{\mathfrak{a}} \circ \varphi)(v) = (\widehat{\mathfrak{a}} \circ \varphi)(e^{2\pi ji/(n-p)}v) \quad \text{for } j \in \mathbb{Z}/(n-p)\mathbb{Z}.$$

We set $A = -\omega a_n$, and we choose an $(n-p)$-th root $A^{1/n-p}$ of A. We have

$$\widehat{\rho}(\zeta) = \frac{p\zeta^{p-1}}{[-na_n\zeta^{-n-1}(1+\zeta O(\zeta)) \cdot \zeta^{2p}]} = A^{-1}\zeta^{n-p}(1+\zeta O(\zeta)).$$

There exists a solution for φ of the form

$$\varphi(v) = A^{1/(n-p)}v(1+vO(v)).$$

Equivalently, if we consider the two-variable function $G(\zeta, v) = \widetilde{\mathfrak{a}}(\zeta) + 1/v^{n-p}\zeta^p$, then $\varphi(v)$ is a solution of

$$(\partial_\zeta G)(\varphi(v), v) \equiv 0,$$

as follows from the definition of $\widehat{\rho}(\zeta) = (1/\widetilde{\mathfrak{a}}'(\zeta)) \cdot (-1/\rho(\zeta))'$.

We first notice that

$$\widehat{\mathfrak{a}}(\zeta) \equiv (1-\omega)a_n\zeta^{-n} \mod \zeta^{-n+1}\mathbb{C}[\![\zeta]\!],$$

and thus in the variable v (we now write $\widehat{\mathfrak{a}}(v)$ instead of $\widehat{\mathfrak{a}} \circ \varphi(v)$):

$$\begin{aligned}\widehat{\mathfrak{a}}(v) &\equiv (1-\omega)a_n\varphi(v)^{-n} \mod v^{-n+1}\mathbb{C}[\![v]\!]\\ &\equiv \frac{\omega-1}{\omega} A \cdot A^{-n/(n-p)}v^{-n} \mod v^{-n+1}\mathbb{C}[\![v]\!]\\ &\equiv \frac{1}{\widehat{\omega}} A^{-\widehat{\omega}/\omega}v^{-n} \mod v^{-n+1}\mathbb{C}[\![v]\!],\end{aligned}$$

so that we can write

$$\widehat{a}_n = \star(-a_n)^{-\widehat{\omega}/\omega},$$

where $\star$ is the positive constant $\omega^{-\widehat{\omega}}(\omega-1)$. On the other hand, the action of $\mathbb{Z}/(n-p)\mathbb{Z}$ has the estimate

$$(\widehat{\sigma}_j^*\widehat{\mathfrak{a}})(v) \equiv e^{-2\pi ij\widehat{\omega}}\widehat{\mathfrak{a}}(v) \mod v^{-n+1}\mathbb{C}[\![v]\!].$$

We can also regard $\widehat{\sigma}_j^*\widehat{\mathfrak{a}}$ as the effect of changing the choice of the $(n-p)$-th root of A.

3. Transformation of intervals

The aim of this section is to describe the correspondence $T(\widetilde{\mathfrak{a}}) \mapsto T(\widehat{\mathfrak{a}})$ induced by $F^{\infty,\infty}$, and the inverse correspondence.

3.a. The simplest example. We consider the simplest case (Gauss type) where $n=2$ and $p=n-p=1$, so that $\omega=2$. We have $\zeta=z$ and $\widetilde{\mathfrak{a}}(\zeta)=a_2\zeta^{-2}(1+\zeta O(\zeta))$ and we set $a_2=|a_2|e^{i\vartheta_2}$. Let $J_\zeta=I(\vartheta_0;\pi/4)$ be an interval centered at ϑ_0 and of radius $\pi/4$. We have

$$\widetilde{\mathfrak{a}}(\zeta)<_{J_\zeta}0 \text{ (resp. } 0<_{J_\zeta}\widetilde{\mathfrak{a}}(\zeta)) \iff \vartheta_2=2\vartheta_0+\pi \bmod 2\pi \text{ (resp. } \vartheta_2=2\vartheta_0 \bmod 2\pi).$$

In other words, $T(\widetilde{\mathfrak{a}})$ consists of two intervals $J_\zeta\ (\mathrm{mod}\,2\pi)$ of length $\pi/2$ on which $\widetilde{\mathfrak{a}}<_{J_\zeta}0$, that we denote $J_\zeta(\widetilde{\mathfrak{a}})_{<0}$, and two intervals $J_\zeta(\widetilde{\mathfrak{a}})_{>0}$ on which $0<_{J_\zeta}\widetilde{\mathfrak{a}}$. These intervals are respectively

$$I(\vartheta_2/2+\pi/2 \bmod \pi;\pi/4) \quad\text{and}\quad I(\vartheta_2/2 \bmod \pi;\pi/4).$$

On the other hand, we have $v=(-1/2a_2)\zeta$ and $\widehat{a}_2=\star(-a_2)^{-1}$ with $\star=1/4$. Therefore,

$$\widehat{\vartheta}_2=-(\vartheta_2+\pi) \mod 2\pi$$

and the transformation of the arguments is

$$\theta_v=\theta_\zeta-(\vartheta_2+\pi) \mod 2\pi.$$

By this transformation, the interval

$$I(\vartheta_2/2+\pi/2 \bmod \pi;\pi/4),\quad \text{resp. } I(\vartheta_2/2 \bmod \pi;\pi/4),$$

is mapped to

$$I(\widehat{\vartheta}_2/2 \bmod \pi;\pi/4),\quad \text{resp. } I(\widehat{\vartheta}_2/2+\pi/2 \bmod \pi;\pi/4).$$

In other words, $T(\widetilde{\mathfrak{a}})_{<0}$, resp. $T(\widetilde{\mathfrak{a}})_{>0}$, is mapped to $T(\widehat{\mathfrak{a}})_{>0}$, resp. $T(\widehat{\mathfrak{a}})_{<0}$.

3.b. The p-ramified case. We consider a pair $(\widetilde{\mathfrak{a}}(\zeta),\rho)$, with $\rho(\zeta)=\zeta^p=z$ and we set $z=|z|e^{i\theta}$ and $\zeta=|\zeta|e^{i\theta_p}$. The intervals J are considered either in $\mathbb{R}_\theta$, in which case they are denoted by J_z, or in $\mathbb{R}_{\theta_p}$, in which case they are denoted by J_ζ. The bijection $\theta_p\mapsto\theta=p\theta_p$ induces a one-to-one correspondence $J_\zeta\mapsto J_z=pJ_\zeta$. As we do not consider intervals in S^1, this is indeed one-to-one.

***The sets* $T(\sigma_k^*\widetilde{\mathfrak{a}})$.** Assume that $J_\zeta\in T(\widetilde{\mathfrak{a}})_\zeta$ and that $\widetilde{\mathfrak{a}}<_{J_\zeta}0$. Then $\widetilde{\mathfrak{a}}<_{J_\zeta+m\pi/n}0$ for any even $m\in\mathbb{Z}$, and the converse inequality for any odd m. If $a_n=|a_n|e^{i\vartheta_n}$, then $T(\widetilde{\mathfrak{a}})_{\zeta,<0}$ consists of the intervals $I_\zeta((\vartheta_n+\pi)/n;\pi/2n)+2\ell\pi/n$ and $T(\widetilde{\mathfrak{a}})_{\zeta,>0}$ consists of the intervals $I_\zeta(\vartheta_n/n;\pi/2n)+2\ell\pi/n$ ($\ell\in\mathbb{Z}$).

On the other hand, for any $k\in\mathbb{Z}/p\mathbb{Z}$, let us consider the intervals for $\sigma_k^*\widetilde{\mathfrak{a}}$ with $(\sigma_k^*\widetilde{\mathfrak{a}})(\zeta)=\widetilde{\mathfrak{a}}(e^{2\pi ik/p}\zeta)$. We have $\sigma_k^*\widetilde{\mathfrak{a}}<0$ on $J_\zeta-2k\pi/p+m\pi/n$ for any even m, and the converse inequality for any odd m. Then $T(\sigma_k^*\widetilde{\mathfrak{a}})_{\zeta,<0}$ consists of the intervals $I_\zeta((\vartheta_n+\pi)/n;\pi/2n)+2\ell\pi/n-2k\pi/p$ and $T(\widetilde{\mathfrak{a}})_{\zeta,<0}$ consists of the intervals $I_\zeta(\vartheta_n/n;\pi/2n)+2\ell\pi/n-2k\pi/p$ ($\ell\in\mathbb{Z}$).

We write these properties in terms of intervals J_z. Then $T(\widetilde{\mathfrak{a}})_{z,>0}$, resp. $T(\widetilde{\mathfrak{a}})_{z,<0}$, consists of the intervals ($\ell \in \mathbb{Z}$)

$$J_z(\widetilde{\mathfrak{a}}, \ell, <0) = I_z((\vartheta_n + \pi)/\omega; \pi/2\omega) + 2\ell\pi/\omega,$$

resp.

$$J_z(\widetilde{\mathfrak{a}}, \ell, >0) = I_z(\vartheta_n/\omega; \pi/2\omega) + 2\ell\pi/\omega.$$

Similarly, $T(\sigma_k^*\widetilde{\mathfrak{a}})_{z,<0}$, resp. $T(\sigma_k^*\widetilde{\mathfrak{a}})_{z,>0}$, consists of the intervals ($\ell \in \mathbb{Z}$)

$$J_z(\sigma_k^*\widetilde{\mathfrak{a}}, \ell, <0) = I_z((\vartheta_n + \pi)/\omega; \pi/2\omega) + 2\ell\pi/\omega - 2k\pi,$$

resp.

$$J_z(\sigma_k^*\widetilde{\mathfrak{a}}, \ell, >0) = I_z(\vartheta_n/\omega; \pi/2\omega) + 2\ell\pi/\omega - 2k\pi.$$

The sets $T(\sigma_j^*\widehat{\mathfrak{a}})$**.** By the formula for $\widehat{a}_n$ in terms of a_n, we have

$$\widehat{\vartheta}_n = -\frac{\widehat{\omega}}{\omega}(\vartheta_n + \pi).$$

We have $v \equiv (-\omega a_n)^{-1/(n-p)}\zeta$, which leads to the correspondence of the arguments

$$\theta_v = \theta_\zeta - (\vartheta_n + \pi)/(n-p), \quad \textit{i.e.}, \ \theta_u = (\omega - 1)\theta_z - (\vartheta_n + \pi), \quad \textit{i.e.}, \ \widehat{\omega}\theta_u = \omega(\theta_z + \widehat{\vartheta}_n).$$

Similarly, $T(\sigma_j^*\widehat{\mathfrak{a}})_{u,<0}$, resp. $T(\sigma_j^*\widehat{\mathfrak{a}})_{u,>0}$, consists of the intervals ($\widehat{\ell} \in \mathbb{Z}$)

$$J_u(\sigma_j^*\widehat{\mathfrak{a}}, \widehat{\ell}, <0) = I_u((\widehat{\vartheta}_n \pm \pi)/\widehat{\omega}; \pi/2\widehat{\omega}) + 2\widehat{\ell}\pi/\widehat{\omega} - 2j\pi,$$

resp.

$$J_u(\sigma_j^*\widehat{\mathfrak{a}}, \widehat{\ell}, >0) = I_u(\widehat{\vartheta}_n/\widehat{\omega}; \pi/2\widehat{\omega}) + 2\widehat{\ell}\pi/\widehat{\omega} - 2j\pi.$$

Fourier transformation of intervals. Let us consider $J_z(\widetilde{\mathfrak{a}}, 0, <0)$. The notation of Mochizuki is then $\vartheta_o = (\vartheta_n + \pi)/\omega$. Therefore, $\widehat{\vartheta}_n/\widehat{\omega} = -\vartheta_o$. We have found

$$J_z(\widetilde{\mathfrak{a}}, 0, <0) = I_z(\vartheta_o, \pi/2\omega) \quad \text{and} \quad J_u(\sigma_j^*\widehat{\mathfrak{a}}, 0, >0) = I_z(-\vartheta_o, \pi/2\widehat{\omega}) - 2j\pi$$

The map $\theta_z \mapsto \theta_u = (\omega/\widehat{\omega})\theta_z - \vartheta_n$ induces a diffeomorphism between the interval $J_z(\widetilde{\mathfrak{a}}, 0, <0)$ and the interval $J_u(\widehat{\mathfrak{a}}, 0, >0)$. By affine transformations, we obtain a diffeomorphism from $J_z(\sigma_k^*\widetilde{\mathfrak{a}}, \ell, <0)$ to $J_u(\sigma_j^*\widehat{\mathfrak{a}}, \widehat{\ell}, >0)$.

On the other hand, by inverse Fourier transformation, we relate $J_u(\widehat{\mathfrak{a}}, 0, <0)$ with $J_z(\sigma_k^*\widetilde{\mathfrak{a}}, 0, >0)$, and then $J_u(\sigma_j^*\widehat{\mathfrak{a}}, \widehat{\ell}, <0)$ to $J_z(\sigma_k^*\widetilde{\mathfrak{a}}, \ell, >0)$.

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