STOKES SHELLS AND STOKES-FILTERED LOCAL SYSTEMS

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by

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1. Orders

We fix a finite set of polar parts $\widetilde{\mathfrak{I}} \subset z^{-1}\mathbb{C}[z^{-1}]$.

The standard order. We equip $\widetilde{\mathfrak{I}}$ with the usual family of orders $\leqslant_\theta$, with $\theta = \arg z \in S^1$ (be careful that Mochizuki takes the opposite convention; this does not affect the reasoning):

$$\widetilde{\mathfrak{b}} \leqslant_\theta \widetilde{\mathfrak{a}} \iff \exp(\widetilde{\mathfrak{b}} - \widetilde{\mathfrak{a}}) \text{ has moderate growth near } \theta$$

and

$$\begin{aligned}\widetilde{\mathfrak{b}} <_\theta \widetilde{\mathfrak{a}} &\iff \exp(\widetilde{\mathfrak{b}} - \widetilde{\mathfrak{a}}) \text{ has rapid decay near } \theta\\ &\iff \widetilde{\mathfrak{b}} \leqslant_\theta \widetilde{\mathfrak{a}} \quad\text{and}\quad \widetilde{\mathfrak{b}} \neq \widetilde{\mathfrak{a}}\end{aligned}$$

This (partial) order relation is open with respect to θ, so that, for any $\widetilde{\mathfrak{b}},\widetilde{\mathfrak{a}}\in\widetilde{\mathfrak{I}}$, the subsets $\{\theta\mid\widetilde{\mathfrak{b}}\leqslant_\theta\widetilde{\mathfrak{a}}\}$ and $\{\theta\mid\widetilde{\mathfrak{b}}<_\theta\widetilde{\mathfrak{a}}\}$ are *open*. In general, we define $<_\theta$ as ($\leqslant_\theta$ and $\neq$). Equivalently, $\leqslant_\theta$ is ($<_\theta$ or $=$).

Reduction to the maximal level. Let n be the maximal order of the pole of the elements of $\widetilde{\mathfrak{I}}$. The 'reduction to the maximal level' will be obtained by means of the projection

$$\pi_n:\widetilde{\mathfrak{I}}\longrightarrow\mathfrak{I}=(\widetilde{\mathfrak{I}}\bmod z^{-n+1})\subset z^{-n}\mathbb{C}.$$

We assume that $\mathfrak{I}$ contains zero and a nonzero element (to avoid trivialities).

Lemma 1. *For any $\theta\in S^1$ and any $\widetilde{\mathfrak{a}},\widetilde{\mathfrak{b}}\in\widetilde{\mathfrak{I}}$, we have*

$$\mathfrak{b}<_\theta\mathfrak{a}\implies\widetilde{\mathfrak{b}}<_\theta\widetilde{\mathfrak{a}}\quad\textit{and}\quad(\widetilde{\mathfrak{b}}<_\theta\widetilde{\mathfrak{a}}\ \textit{and}\ \mathfrak{b}\neq\mathfrak{a})\implies\mathfrak{b}<_\theta\mathfrak{a}.$$

It follows that $\widetilde{\mathfrak{b}}\leqslant_\theta\widetilde{\mathfrak{a}}\implies\mathfrak{b}\leqslant\mathfrak{a}$ but the converse does not hold. This lemma is easily checked.

Action of $\mathbb{R}_+^*$**.** The multiplicative group $\mathbb{R}_+^*$ acts on $z^{-n}\mathbb{C}$ by multiplication. Set $[\mathfrak{I}]=\operatorname{image}[\mathfrak{I}\to(z^{-n}\mathbb{C}/\mathbb{R}_+^*)]$ and let $\pi:\widetilde{\mathfrak{I}}\to[\mathfrak{I}]$ be the projection. For each $\alpha\in[\mathfrak{I}]$, set $\widetilde{\mathfrak{I}}(\alpha)=\{\widetilde{\mathfrak{a}}\in\widetilde{\mathfrak{I}}\mid[\mathfrak{a}]=\alpha\}$.

Action of $\mathbb{R}^*$**.** We denote $\overline{[\mathfrak{I}]}=[\mathfrak{I}]/\{\pm1\}=\mathfrak{I}/\mathbb{R}^*$ and, for $\alpha\in[\mathfrak{I}]$, we denote by $\overline{\alpha}$ the class of α in $\overline{[\mathfrak{I}]}$.

2. Stokes-filtered local system

For a sheaf $\mathscr{F}$ on S^1 and an open set U of S^1, $\mathscr{F}_U$ denotes the sheaf on S^1 obtained by restricting $\mathscr{F}$ to U and then extending it by zero to S^1.

2.a. Unramified $\widetilde{\mathfrak{I}}$-Stokes-filtered local system

$\widetilde{\mathfrak{I}}$-graded $\widetilde{\mathfrak{I}}$-Stokes-filtered local system. A $\widetilde{\mathfrak{I}}$-graded $\widetilde{\mathfrak{I}}$-Stokes-filtered local system $\mathscr{L}$ on some open set U of S^1 is an $\widetilde{\mathfrak{I}}$-graded local system $\bigoplus_{\tilde{\mathfrak{a}}\in\widetilde{\mathfrak{I}}}\mathscr{L}_{\tilde{\mathfrak{a}}}$, endowed with the Stokes filtration indexed by $(\widetilde{\mathfrak{I}},\leqslant)$:

$$\mathscr{F}_{<\tilde{\mathfrak{a}}}(\mathscr{L})=\bigoplus_{\tilde{\mathfrak{b}}\in\widetilde{\mathfrak{I}}}(\mathscr{L}_{\tilde{\mathfrak{b}}})_{\{\tilde{\mathfrak{b}}<\tilde{\mathfrak{a}}\}},\quad\mathscr{F}_{\leqslant\tilde{\mathfrak{a}}}(\mathscr{L})=\bigoplus_{\tilde{\mathfrak{b}}\in\widetilde{\mathfrak{I}}}(\mathscr{L}_{\tilde{\mathfrak{b}}})_{\{\tilde{\mathfrak{b}}\leqslant\tilde{\mathfrak{a}}\}}.$$

This is a concise way to write e.g. that, for any $\theta\in S^1$,

$$\mathscr{F}_{\leqslant\tilde{\mathfrak{a}}}(\mathscr{L})_\theta=\bigoplus_{\tilde{\mathfrak{b}}\leqslant_\theta\tilde{\mathfrak{a}}}\operatorname{gr}^{\mathscr{F}}_{\tilde{\mathfrak{b}}}(\mathscr{L})_\theta.$$

$\widetilde{\mathfrak{I}}$-Stokes-filtered local system. An unramified pre-Stokes structure indexed by $(\widetilde{\mathfrak{I}},\leqslant)$ on a local system $\mathscr{L}$ defined on some open set U of S^1 consists of a family of subsheaves $\mathscr{F}_{\leqslant\tilde{\mathfrak{a}}}(\mathscr{L})\subset\mathscr{L}$ such that, for any $\theta\in S^1$, $\mathscr{F}_{\leqslant\tilde{\mathfrak{a}}}(\mathscr{L})_\theta$ defines an exhaustive increasing filtration of $\mathscr{L}_\theta$. We then define $\mathscr{F}_{<\tilde{\mathfrak{a}}}(\mathscr{L})$ as the subsheaf of $\mathscr{F}_{\leqslant\tilde{\mathfrak{a}}}(\mathscr{L})$ which satisfies, for any θ:

$$\mathscr{F}_{<\tilde{\mathfrak{a}}}(\mathscr{L})_\theta = \sum_{\tilde{\mathfrak{b}}<_\theta\tilde{\mathfrak{a}}} \mathscr{F}_{\leqslant\tilde{\mathfrak{b}}}(\mathscr{L})_\theta.$$

(It is an exercise to show that it is a subsheaf.)

A pre-Stokes structure is a Stokes structure indexed by $(\widetilde{\mathfrak{I}},\leqslant)$ if, locally on S^1, $(\mathscr{L},\mathscr{F}_\bullet)$ is isomorphic to an $\widetilde{\mathfrak{I}}$-graded $\widetilde{\mathfrak{I}}$-Stokes-filtered local system. We also say that $(\mathscr{L},\mathscr{F}_\bullet)$ is an $\widetilde{\mathfrak{I}}$-Stokes-filtered local system.

Set $\mathrm{gr}^{\mathscr{F}}(\mathscr{L}) = \bigoplus_{\tilde{\mathfrak{a}}\in\widetilde{\mathfrak{I}}} \mathrm{gr}^{\mathscr{F}}_{\leqslant\tilde{\mathfrak{a}}}(\mathscr{L})$ with $\mathrm{gr}^{\mathscr{F}}_{\leqslant\tilde{\mathfrak{a}}}(\mathscr{L}) = \mathscr{F}_{\leqslant\tilde{\mathfrak{a}}}(\mathscr{L})/\mathscr{F}_{<\tilde{\mathfrak{a}}}(\mathscr{L})$, equipped with the induced filtration $\mathscr{F}_{\leqslant\tilde{\mathfrak{b}}}(\mathrm{gr}^{\mathscr{F}}(\mathscr{L})) = \bigoplus_{\tilde{\mathfrak{a}}\in\widetilde{\mathfrak{I}}} \mathscr{F}_{\leqslant\tilde{\mathfrak{b}}}(\mathrm{gr}^{\mathscr{F}}_{\leqslant\tilde{\mathfrak{a}}}(\mathscr{L}))$ with

$$\mathscr{F}_{\leqslant\tilde{\mathfrak{b}}}\big(\mathrm{gr}^{\mathscr{F}}_{\leqslant\tilde{\mathfrak{a}}}(\mathscr{L})\big) = \frac{\mathscr{F}_{\leqslant\tilde{\mathfrak{b}}}(\mathscr{L})\cap\mathscr{F}_{\leqslant\tilde{\mathfrak{a}}}(\mathscr{L})}{\mathscr{F}_{\leqslant\tilde{\mathfrak{b}}}(\mathscr{L})\cap\mathscr{F}_{<\tilde{\mathfrak{a}}}(\mathscr{L})}.$$

Lemma 2. *If $(\mathscr{L},\mathscr{F}_\bullet)$ is an $\widetilde{\mathfrak{I}}$-Stokes-filtered local system on S^1, then $(\mathrm{gr}^{\mathscr{F}}(\mathscr{L}),\mathscr{F}_\bullet)$ is an $\widetilde{\mathfrak{I}}$-graded $\widetilde{\mathfrak{I}}$-Stokes-filtered local system and locally on S^1, there exists a splitting of the Stokes filtration of $(\mathscr{L},\mathscr{F}_\bullet)$, that is, an isomorphism $(\mathscr{L},\mathscr{F}_\bullet)\simeq(\mathrm{gr}^{\mathscr{F}}(\mathscr{L}),\mathscr{F}_\bullet)$ whose associated graded morphism is the identity.*

Proof. Since the question is local, we can assume that $(\mathscr{L},\mathscr{F}_\bullet)$ is an $\widetilde{\mathfrak{I}}$-graded $\widetilde{\mathfrak{I}}$-Stokes-filtered local system, and then the claim is straightforward. □

Extending the index set. Let $\widetilde{\mathfrak{I}}'\supset\widetilde{\mathfrak{I}}$ be another finite set of polar parts. We extend the filtration $\mathscr{F}_\bullet(\mathscr{L})$ by setting, for any $\widetilde{\mathfrak{a}}'\in\widetilde{\mathfrak{I}}$ and any $\theta\in S^1$:

$$\mathscr{F}'_{\leqslant\tilde{\mathfrak{a}}'}(\mathscr{L})_\theta = \sum_{\substack{\tilde{\mathfrak{a}}\in\widetilde{\mathfrak{I}}\\ \tilde{\mathfrak{a}}\leqslant_\theta\tilde{\mathfrak{a}}'}} \mathscr{F}_{\leqslant\tilde{\mathfrak{a}}}(\mathscr{L})_\theta.$$

Lemma 3. *$(\mathscr{L},\mathscr{F}'_\bullet)$ is a Stokes filtration indexed by $\widetilde{\mathfrak{I}}'$ such that $\mathrm{gr}^{\mathscr{F}'}_{\tilde{\mathfrak{a}}'}(\mathscr{L})=0$ if $\widetilde{\mathfrak{a}}'\notin\widetilde{\mathfrak{I}}$ and is equal to $\mathrm{gr}^{\mathscr{F}}_{\tilde{\mathfrak{a}}'}(\mathscr{L})$ if $\widetilde{\mathfrak{a}}'\in\widetilde{\mathfrak{I}}$.*

Proof. The main point to check is the local triviality of $\mathscr{F}'$, so that we can assume that $\mathscr{F}$ is trivialized. We then have

$$\mathscr{F}'_{\leqslant\tilde{\mathfrak{a}}'}(\mathscr{L})_\theta = \sum_{\substack{\tilde{\mathfrak{a}}\in\widetilde{\mathfrak{I}}\\ \tilde{\mathfrak{a}}\leqslant_\theta\tilde{\mathfrak{a}}'}} \Big(\bigoplus_{\tilde{\mathfrak{b}}\leqslant_\theta\tilde{\mathfrak{a}}} \mathrm{gr}^{\mathscr{F}}_{\tilde{\mathfrak{b}}}(\mathscr{L})_\theta\Big).$$

Since, given $\widetilde{\mathfrak{a}}'\in\widetilde{\mathfrak{I}}'$, we have the equality of the two sets

$$\{\widetilde{\mathfrak{b}}\in\widetilde{\mathfrak{I}}\mid\exists\,\widetilde{\mathfrak{a}}\in\widetilde{\mathfrak{I}}\text{ s.t. }\widetilde{\mathfrak{b}}\leqslant_\theta\widetilde{\mathfrak{a}}\leqslant_\theta\widetilde{\mathfrak{a}}'\}=\{\widetilde{\mathfrak{b}}\in\widetilde{\mathfrak{I}}\mid\widetilde{\mathfrak{b}}\leqslant_\theta\widetilde{\mathfrak{a}}'\},$$

we deduce that

$$\mathscr{F}'_{\leqslant\tilde{\mathfrak{a}}'}(\mathscr{L})_\theta = \bigoplus_{\tilde{\mathfrak{b}}\leqslant_\theta\tilde{\mathfrak{a}}'} \mathrm{gr}^{\mathscr{F}}_{\tilde{\mathfrak{b}}}(\mathscr{L})_\theta,$$

hence the local triviality property, and the second point of the statement. □

Recovering $(\mathscr{L},\mathscr{F}_\bullet)$ ***from*** $\bigoplus_{\widetilde{\mathfrak{a}}\in\widetilde{\mathfrak{I}}}(\mathrm{gr}^{\mathscr{F}}_{\widetilde{\mathfrak{a}}}(\mathscr{L}),\mathscr{F}_\bullet)$. One can recover $(\mathscr{L},\mathscr{F}_\bullet)$ from the graded object $(\mathrm{gr}^{\mathscr{F}}\mathscr{L},\mathscr{F})$ by specifying Stokes data (linear algebra). Let (U_i) be a finite covering of S^1 by open intervals on which there exists a splitting

$$\mathscr{L}|_{U_i}\xrightarrow{\sim}\bigoplus_{\widetilde{\mathfrak{b}}\in\widetilde{\mathfrak{I}}}\mathrm{gr}^{\mathscr{F}}_{\widetilde{\mathfrak{b}}}(\mathscr{L})|_{U_i}$$

so that $\mathscr{L}$ is recovered by means of gluing isomorphisms

$$G^i_j:\bigoplus_{\widetilde{\mathfrak{b}}\in\widetilde{\mathfrak{I}}}\mathrm{gr}^{\mathscr{F}}_{\widetilde{\mathfrak{b}}}(\mathscr{L})|_{U_{ij}}\xrightarrow{\sim}\bigoplus_{\widetilde{\mathfrak{b}}\in\widetilde{\mathfrak{I}}}\mathrm{gr}^{\mathscr{F}}_{\widetilde{\mathfrak{b}}}(\mathscr{L})|_{U_{ij}}\quad(U_{ij}:=U_i\cap U_j),$$

with two constraints:

(1) they satisfy the cocycle condition and $G^i_i=\mathrm{Id}$;

(2) they are compatible with the $\widetilde{\mathfrak{I}}$-Stokes filtrations, and the graded isomorphisms are the identity.

For the second condition, let $G^{i,\widetilde{\mathfrak{a}}}_{j,\widetilde{\mathfrak{b}}}$ be the components of G^i_j from $\mathrm{gr}^{\mathscr{F}}_{\widetilde{\mathfrak{a}}}(\mathscr{L})|_{U_{ij}}$ to $\mathrm{gr}^{\mathscr{F}}_{\widetilde{\mathfrak{b}}}(\mathscr{L})|_{U_{ij}}$. It is a morphism of local systems, hence is constant (assume that each U_{ij} is an interval), and it is compatible with the Stokes filtration. In particular, by considering $\mathscr{F}_{\leqslant\widetilde{\mathfrak{a}}}$, it sends $\mathrm{gr}^{\mathscr{F}}_{\widetilde{\mathfrak{a}}}(\mathscr{L})|_{U_{ij}}$ to $(\mathrm{gr}^{\mathscr{F}}_{\widetilde{\mathfrak{b}}}(\mathscr{L})_{\{\widetilde{\mathfrak{b}}\leqslant\widetilde{\mathfrak{a}}\}}|_{U_{ij}})$. Therefore, $G^{i,\widetilde{\mathfrak{a}}}_{j,\widetilde{\mathfrak{b}}}$ must be zero unless $\widetilde{\mathfrak{b}}\leqslant_\theta\widetilde{\mathfrak{a}}$ for each $\theta\in U_{ij}$, that we denote $\widetilde{\mathfrak{b}}\leqslant_{U_{ij}}\widetilde{\mathfrak{a}}$. It follows that G^i_j is block-triangular and the diagonal blocks $G^{i,\widetilde{\mathfrak{a}}}_{j,\widetilde{\mathfrak{a}}}$ are equal to the identity (because G^i_j is obtained by means of local splittings).

2.b. The p-ramified case. For $p\geqslant 1$, we consider the ramification $\rho:z_p\mapsto z=z_p^p$, together with the covering $\rho:S^1_p\to S^1$ induced by $\theta_p\mapsto\theta=p\theta_p$, and the action of $\mathbb{Z}/p\mathbb{Z}$ on $\mathbb{C}$ induced by $\sigma:z_p\mapsto e^{2\pi i/p}z_p$, together with the induced action on S^1_p. We start with a local system $\mathscr{L}$ on S^1 and we consider its pullback $\mathscr{L}_p$ by the ramification. We set $\omega=n/p$. A p-ramified Stokes-filtered local system is a Stokes-filtered local system $(\mathscr{L}_p,\mathscr{F}_\bullet)$ on S^1_p equipped with descent data for the map $\rho:z_p\mapsto z_p^p$. For the descent data to be defined, we assume that the set $\widetilde{\mathfrak{I}}$ is stable under the action of $\mathbb{Z}/p\mathbb{Z}$, i.e.,

$$\widetilde{\mathfrak{a}}(z_p)\in\widetilde{\mathfrak{I}}\implies(\sigma^*\widetilde{\mathfrak{a}})(z_p)=\widetilde{\mathfrak{a}}(\sigma(z_p))\in\widetilde{\mathfrak{I}}\quad\forall\,k\in\mathbb{Z}/p\mathbb{Z},$$

hence so are $\mathfrak{I}$ and $[\mathfrak{I}]$. The descent data on an $\widetilde{\mathfrak{I}}$-Stokes-filtered local system $(\mathscr{L}_p,\mathscr{F}_\bullet)$ consist then of an isomorphism

$$\tau:\sigma^{-1}\mathscr{L}_p\xrightarrow{\sim}\mathscr{L}_p,\quad\text{with }\tau^{\circ p}=\mathrm{Id},$$

such that τ sends isomorphically $\sigma^{-1}\mathscr{F}_{\leqslant\widetilde{\mathfrak{a}}}(\mathscr{L}_p)$ to $\mathscr{F}_{\leqslant\sigma^*\widetilde{\mathfrak{a}}}(\mathscr{L}_p)$ for any $\widetilde{\mathfrak{a}}\in\widetilde{\mathfrak{I}}$.

We call $\mathscr{F}_\bullet(\mathscr{L}_p)$ a *p-ramified $\widetilde{\mathfrak{I}}$-Stokes structure on $\mathscr{L}_p$*.

2.c. Reduction to the maximal level

We first start with the unramified case.

$\mathcal{I}$-graded $\widetilde{\mathcal{I}}$-Stokes-filtered local system. An $\mathcal{I}$-graded $\widetilde{\mathcal{I}}$-Stokes-filtered local system on some open set U of S^1 is an $\mathcal{I}$-graded Stokes-filtered local system $\bigoplus_{\mathfrak{a}\in\mathcal{I}}(\mathscr{L}_\mathfrak{a},\mathscr{F}^\mathfrak{a}_\bullet)$, where each $\mathscr{F}^\mathfrak{a}(\mathscr{L}_\mathfrak{a})$ is a Stokes filtration indexed by $\widetilde{\mathcal{I}}(\mathfrak{a}):=\pi_n^{-1}(\mathfrak{a})$. By regarding each $\mathscr{F}^\mathfrak{a}(\mathscr{L}_\mathfrak{a})$ as a Stokes filtration indexed by $\widetilde{\mathcal{I}}$, we can indeed regard $\bigoplus_{\mathfrak{a}\in\mathcal{I}}(\mathscr{L}_\mathfrak{a},\mathscr{F}^\mathfrak{a}_\bullet)$ as an $\widetilde{\mathcal{I}}$-Stokes-filtered local system.

From the Stokes filtration $\mathscr{F}_\bullet(\mathscr{L})$ indexed by $\widetilde{\mathcal{I}}$ one constructs a filtration ${}^n\mathscr{F}_\bullet$ indexed by $\mathcal{I}$ by setting

$$ {}^n\mathscr{F}_{<\mathfrak{a}}(\mathscr{L})=\sum_{\tilde{\mathfrak{b}}\in\widetilde{\mathcal{I}}}(\mathscr{F}_{\leqslant\tilde{\mathfrak{b}}}(\mathscr{L}))_{\{\mathfrak{b}<\mathfrak{a}\}}\quad\text{and}\quad {}^n\mathscr{F}_{\leqslant\mathfrak{a}}(\mathscr{L})=\sum_{\tilde{\mathfrak{b}}\in\widetilde{\mathcal{I}}}(\mathscr{F}_{\leqslant\tilde{\mathfrak{b}}}(\mathscr{L}))_{\{\mathfrak{b}\leqslant\mathfrak{a}\}}. $$

Lemma 4. *This is an $\mathcal{I}$-Stokes filtration.*

Proof. The question is local, so we can assume that $(\mathscr{L},\mathscr{F}_\bullet)$ is $\widetilde{\mathcal{I}}$-graded and we write $\mathscr{L}=\bigoplus_{\tilde{\mathfrak{c}}\in\widetilde{\mathcal{I}}}\mathrm{gr}^{\mathscr{F}}_{\tilde{\mathfrak{c}}}(\mathscr{L})$ and $\mathscr{F}_{\leqslant\tilde{\mathfrak{b}}}(\mathscr{L})=\bigoplus_{\tilde{\mathfrak{c}}\in\widetilde{\mathcal{I}}}\mathrm{gr}^{\mathscr{F}}_{\tilde{\mathfrak{c}}}(\mathscr{L})_{\{\tilde{\mathfrak{c}}\leqslant\tilde{\mathfrak{b}}\}}$, so that

$$ {}^n\mathscr{F}_{\leqslant\mathfrak{a}}(\mathscr{L})=\sum_{\tilde{\mathfrak{b}}}\Big(\bigoplus_{\tilde{\mathfrak{c}}\in\widetilde{\mathcal{I}}}\mathrm{gr}^{\mathscr{F}}_{\tilde{\mathfrak{c}}}(\mathscr{L})_{\{\tilde{\mathfrak{c}}\leqslant\tilde{\mathfrak{b}}\}}\Big)_{\{\mathfrak{b}\leqslant\mathfrak{a}\}}=\bigoplus_{\tilde{\mathfrak{c}}\in\widetilde{\mathcal{I}}}\Big(\sum_{\tilde{\mathfrak{b}}}\big(\mathrm{gr}^{\mathscr{F}}_{\tilde{\mathfrak{c}}}(\mathscr{L})\big)_{\{\mathfrak{b}\leqslant\mathfrak{a}\}\cap\{\tilde{\mathfrak{c}}\leqslant\tilde{\mathfrak{b}}\}}\Big). $$

For $\tilde{\mathfrak{c}}$ and $\mathfrak{a}$ fixed, we set $U_{\mathfrak{a},\tilde{\mathfrak{c}}}=\bigcup_{\tilde{\mathfrak{b}}\in\widetilde{\mathcal{I}}}(\{\tilde{\mathfrak{c}}\leqslant\tilde{\mathfrak{b}}\}\cap\{\mathfrak{b}\leqslant\mathfrak{a}\})$. Then

$$ {}^n\mathscr{F}_{\leqslant\mathfrak{a}}\big(\mathrm{gr}^{\mathscr{F}}_{\tilde{\mathfrak{c}}}(\mathscr{L})\big)=\big(\mathrm{gr}^{\mathscr{F}}_{\tilde{\mathfrak{c}}}(\mathscr{L})\big)_{U_{\mathfrak{a},\tilde{\mathfrak{c}}}}. $$

By Lemma 1, we have $(\{\tilde{\mathfrak{c}}\leqslant\tilde{\mathfrak{b}}\}\cap\{\mathfrak{b}\leqslant\mathfrak{a}\})\subset\{\mathfrak{c}\leqslant\mathfrak{a}\}$, for each $\tilde{\mathfrak{b}}$, hence $U_{\mathfrak{a},\tilde{\mathfrak{c}}}\subset\{\mathfrak{c}\leqslant\mathfrak{a}\}$. On the other hand, $U_{\mathfrak{a},\tilde{\mathfrak{c}}}\supset\{\mathfrak{c}\leqslant\mathfrak{a}\}$, as seen by considering the component of $U_{\mathfrak{a},\tilde{\mathfrak{c}}}$ with $\tilde{\mathfrak{b}}=\tilde{\mathfrak{c}}$. Therefore, ${}^n\mathscr{F}_{\leqslant\mathfrak{a}}(\mathscr{L})=\bigoplus_{\tilde{\mathfrak{c}}}\mathrm{gr}^{\mathscr{F}}_{\tilde{\mathfrak{c}}}(\mathscr{L})_{\{\mathfrak{c}\leqslant\mathfrak{a}\}}$, which concludes the proof. □

Each summand $\mathrm{gr}^{{}^n\mathscr{F}}_{\mathfrak{a}}(\mathscr{L})$ ($\mathfrak{a}\in\mathcal{I}$) of the graded local system $\mathrm{gr}^{{}^n\mathscr{F}}(\mathscr{L})$ inherits from $\mathscr{F}_\bullet(\mathscr{L})$ a filtration $\mathscr{F}^\mathfrak{a}_\bullet(\mathrm{gr}^{{}^n\mathscr{F}}_{\mathfrak{a}}(\mathscr{L}))$, which is a Stokes filtration indexed by $\widetilde{\mathcal{I}}(\mathfrak{a})$, and that we can consider as indexed by $\widetilde{\mathcal{I}}$: we have

$$ \mathscr{F}^\mathfrak{a}_{\leqslant\tilde{\mathfrak{b}}}(\mathrm{gr}^{{}^n\mathscr{F}}_{\mathfrak{a}}(\mathscr{L}))=\frac{\mathscr{F}_{\leqslant\tilde{\mathfrak{b}}}(\mathscr{L})\cap{}^n\mathscr{F}_{\leqslant\mathfrak{a}}(\mathscr{L})}{\mathscr{F}_{<\tilde{\mathfrak{b}}}(\mathscr{L})\cap{}^n\mathscr{F}_{<\mathfrak{a}}(\mathscr{L})},\qquad \tilde{\mathfrak{b}}\in\widetilde{\mathcal{I}}(\mathfrak{a}). $$

We define $\mathscr{F}_\bullet(\mathrm{gr}^{{}^n\mathscr{F}}(\mathscr{L}))=\bigoplus_{\mathfrak{a}\in\mathcal{I}}\mathscr{F}^\mathfrak{a}_\bullet(\mathrm{gr}^{{}^n\mathscr{F}}_{\mathfrak{a}}(\mathscr{L}))$, that we consider as indexed by $\widetilde{\mathcal{I}}$.

Lemma 5 **(analogous to Lemma 2).** *Let $(\mathscr{L},\mathscr{F}_\bullet)$ be an $\widetilde{\mathcal{I}}$-Stokes-filtered local system. Then $(\mathrm{gr}^{{}^n\mathscr{F}}(\mathscr{L}),\mathscr{F}_\bullet)$ is an $\mathcal{I}$-graded $\widetilde{\mathcal{I}}$-Stokes-filtered local system and, locally on S^1, there exists an $\mathcal{I}$-splitting of the Stokes filtration of $(\mathscr{L},\mathscr{F}_\bullet)$, that is, an isomorphism $(\mathscr{L},\mathscr{F}_\bullet)\simeq(\mathrm{gr}^{{}^n\mathscr{F}}(\mathscr{L}),\mathscr{F}_\bullet)$ whose associated ${}^n\mathscr{F}$-graded morphism is the identity.*

Proof. Since the assertion is local, we can assume that $(\mathscr{L},\mathscr{F}_\bullet)$ is $\widetilde{\mathcal{I}}$-trivial, and the assertion is then straightforward. □

The reason to consider the reduction to the maximal level is to obtain canonical splittings on intervals of length $(\pi/n)+\varepsilon$.

Theorem. *Let U be any open interval in S^1 of length $(\pi/n)+\varepsilon$ (with $\varepsilon>0$ small). Then there exists an $\mathfrak{I}$-splitting of the Stokes filtration $\mathscr{F}_\bullet(\mathscr{L})$ on any sub-interval of U, and this splitting is unique on any sub-interval of length $>\pi/n$.* □

For an interval U of length $(\pi/n)+\varepsilon$, we denote by $\operatorname{can}(U)$ the unique splitting

$$\operatorname{can}(U):(\mathscr{L},\mathscr{F}_\bullet)|_U \xrightarrow{\sim} \bigoplus_{\mathfrak{a}\in\mathfrak{I}}(\operatorname{gr}^{n\mathscr{F}}_{\mathfrak{a}}(\mathscr{L}),\mathscr{F}^{\mathfrak{a}}_\bullet)|_U.$$

The p-ramified case. We set $\omega=n/p$. We assume that $\widetilde{\mathfrak{I}}$ is stable by the $\mathbb{Z}/p\mathbb{Z}$-action and $(\mathscr{L}_p,\mathscr{F}_\bullet)$ is p-ramified. The set $\mathfrak{I}$ is also stable by the $\mathbb{Z}/p\mathbb{Z}$-action, the $\mathfrak{I}$-Stokes filtration ${}^n\mathscr{F}_\bullet(\mathscr{L}_p)$ is $\mathbb{Z}/p\mathbb{Z}$-equivariant, and the graded local system $\operatorname{gr}^{n\mathscr{F}}(\mathscr{L}_p)$ is the pullback by ρ of a well-defined local system that we denote by $\operatorname{gr}^{n\mathscr{F}}(\mathscr{L})$ (although, strictly speaking, ${}^n\mathscr{F}_\bullet(\mathscr{L})$ is not defined as a family of subsheaves of $\mathscr{L}$). Furthermore, the induced Stokes filtration is also p-ramified, that is, each $(\bigoplus_{k\in\mathbb{Z}/p\mathbb{Z}}(\operatorname{gr}^{n\mathscr{F}}_{\mathfrak{a}\circ\sigma^k}(\mathscr{L}_p),\mathscr{F}^{\mathfrak{a}\circ\sigma_k}_\bullet)$ is stable by τ.

Furthermore, the unique splittings in the theorem are compatible with τ (by uniqueness). Therefore, such a splitting exists and is unique on any union of intervals $U_p=\rho^{-1}(U)$, with U of length $\pi/\omega+\varepsilon$.

3. $\widetilde{\mathfrak{I}}$-Stokes shells

3.a. Unramified $\widetilde{\mathfrak{I}}$-Stokes shells

Choice of the covering. To any $\overline{\alpha}\neq 0$ in $\overline{[\mathfrak{I}]}$ is associated a family $T(\overline{\alpha})$ of $2n$ disjoint open intervals J of length π/n in S^1. The centers θ_J of such intervals J are the numbers $\bmod 2\pi$ such that $e^{2\pi in\theta_J}=\overline{\alpha}$. We set

$$T(\overline{[\mathfrak{I}]})=\{J\mid J\in T(\overline{\alpha})\text{ for some }\overline{\alpha}\in\overline{[\mathfrak{I}]}\}.$$

For any $J\in T(\overline{[\mathfrak{I}]})$, if $J\in T(\overline{\alpha})$ for some $\overline{\alpha}\in\overline{[\mathfrak{I}]}\smallsetminus\{0\}$, then there is no other $\overline{\beta}$ such that $J\in T(\overline{\beta})$. Furthermore, one of the representatives $\alpha,-\alpha$ of $\overline{\alpha}$ is $<_\theta 0$ for any $\theta\in J$ and the other one is ordered conversely, and the order switches on $J+\pi/n$, that is denoted by J_{next}. We denote the positive one by $\alpha_+(J)$ and the negative one by $\alpha_-(J)$.

Deformation data. A $[\mathfrak{I}]$-graded $(\widetilde{\mathfrak{I}},\leqslant)$-Stokes-filtered local system $(\mathscr{K}_\bullet,\mathscr{F}_\bullet)$ is a $[\mathfrak{I}]$-graded local system $\mathscr{K}_\bullet=\bigoplus_{\alpha\in[\mathfrak{I}]}\mathscr{K}_\alpha$ such that each $\mathscr{K}_\alpha$ is equipped with an $\widetilde{\mathfrak{I}}(\alpha)$-Stokes filtration $\mathscr{F}^\alpha_\bullet(\mathscr{K}_\alpha)$. We can regard each $(\mathscr{K}_\alpha,\mathscr{F}^\alpha_\bullet)$ as a Stokes-filtered local system indexed by $\widetilde{\mathfrak{I}}$, so that $(\mathscr{K}_\bullet,\mathscr{F}_\bullet)$ is indeed a Stokes-filtered local system indexed by $\widetilde{\mathfrak{I}}$.

By *deformation data* $\mathcal{R}$ on $(\mathscr{K}_\bullet,\mathscr{F}_\bullet)$ we mean a subfamily of the family of morphisms of $\widetilde{\mathfrak{I}}$-Stokes-filtered local system

$$\mathcal{R}^{i,\alpha}_{j,\beta}:(\mathscr{K}_\alpha,\mathscr{F}^\alpha_\bullet)|_{J_i\cap J_j}\longrightarrow(\mathscr{K}_\beta,\mathscr{F}^\beta_\bullet)|_{J_i\cap J_j}$$

for those pairs (J_i, J_j) in $T(\overline{[\mathfrak{I}]})$ such that $J_i \in T(\overline{\alpha})$ and $J_j \in T(\overline{\beta})$. If $\alpha \neq \beta$, we can extend the filtrations $\mathscr{F}_\bullet$ on $\mathscr{K}_\alpha$ and $\mathscr{K}_\beta$ so that they are indexed by $\widetilde{\mathfrak{I}}$ (cf. Lemma 3), so that the compatibility $\mathcal{R}^{i,\alpha}_{j,\beta}$ with the Stokes filtrations is meaningful.

The subfamily $\mathcal{R}$ is the following (bigger to smaller):

(1) If $J_i = J_j =: J$, the only morphisms which occur are

$$\begin{aligned} R(J)^{\alpha_+(J)}_0 &: (\mathscr{K}_{\alpha_+(J)}, \mathscr{F}^{\alpha_+(J)}_\bullet)|_J \longrightarrow (\mathscr{K}_0, \mathscr{F}^0_\bullet)|_J \\ R(J)^{\alpha_+(J)}_{\alpha_-(J)} &: (\mathscr{K}_{\alpha_+(J)}, \mathscr{F}^{\alpha_+(J)}_\bullet)|_J \longrightarrow (\mathscr{K}_{\alpha_-(J)}, \mathscr{F}^{\alpha_-(J)}_\bullet)|_J \\ R(J)^0_{\alpha_-(J)} &: (\mathscr{K}_0, \mathscr{F}^0_\bullet)|_J \longrightarrow (\mathscr{K}_{\alpha_-(J)}, \mathscr{F}^{\alpha_-(J)}_\bullet)|_J, \end{aligned}$$

and it may be convenient to consider the block morphism

$$R(J) : \bigoplus_{\alpha\in\{\alpha_-(J),0,\alpha_+(J)\}} (\mathscr{K}_\alpha, \mathscr{F}^\alpha_\bullet)|_J \longrightarrow \bigoplus_{\alpha\in\{\alpha_-(J),0,\alpha_+(J)\}} (\mathscr{K}_\alpha, \mathscr{F}^\alpha_\bullet)|_J$$

by also considering the blocks $R(J)^\alpha_\alpha = \mathrm{Id}$, and the other blocks are zero. It is then clear that $R(J)$ is block-lower triangular and invertible.

(2) If $J_i \neq J_j$ and $J_i \cap J_j \neq \varnothing$, we consider the morphism

$$R^{i,\alpha_+(J_i)}_{j,\alpha_-(J_j)} : (\mathscr{K}_{\alpha_+(J_i)}, \mathscr{F}^{\alpha_+(J_i)}_\bullet)|_{J_i\cap J_j} \longrightarrow (\mathscr{K}_{\alpha_-(J_j)}, \mathscr{F}^{\alpha_-(J_j)}_\bullet)|_{J_i\cap J_j},$$

where both filtrations are considered as $\widetilde{\mathfrak{I}}$-filtrations.

Definition. An unramified $\widetilde{\mathfrak{I}}$-Stokes shell consists of a direct sum $\bigoplus_{\alpha\in[\mathfrak{I}]}(\mathscr{K}_\alpha, \mathscr{F}^\alpha_\bullet)$, where the α-summand is an $\widetilde{\mathfrak{I}}(\alpha)$-Stokes-filtered local system, together with a family of deformation data $\mathcal{R}$.

3.b. The p-ramified case. Notation as in Section 2.b. Then $\mathbb{Z}/p\mathbb{Z}$ acts on the set $T_p(\overline{[\mathfrak{I}]})$ of intervals of length π/n in S^1_p and $\sigma^*\big(\alpha_\pm(J)\big) = \alpha_\pm(\sigma^{-1}(J))$. Then there is a naturally defined notion of equivariance of an $\widetilde{\mathfrak{I}}$-Stokes shell, and we can define a p-ramified $\widetilde{\mathfrak{I}}$-Stokes shell as a $\mathbb{Z}/p\mathbb{Z}$-equivariant $\widetilde{\mathfrak{I}}$-Stokes shell.

In his paper, Takuro presents the intervals $T(\overline{[\mathfrak{I}]})$ and p-ramified $\widetilde{\mathfrak{I}}$-Stokes shells a little differently. We consider the commutative diagram, with $\theta = p\theta_p$:

$$\begin{array}{ccc} \theta_p & \longmapsto & \theta \\ \mathbb{R}_p & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow \\ S^1_p & \longrightarrow & S^1 \\ e^{i\theta_p} & \longmapsto & e^{i\theta} \end{array}$$

Then $T(\overline{[\mathfrak{I}]})$ is a set of intervals of length π/ω in $\mathbb{R}$. An interval J in $T(\overline{[\mathfrak{I}]})$ is obtained by taking a connected component of the pullback in $\mathbb{R}_p$ of an interval in $T_p(\overline{[\mathfrak{I}]})$ and its image by the dilation $\theta_p \mapsto p\theta_p$. Then $\mathscr{K}_\bullet = \bigoplus_{\alpha\in[\mathfrak{I}]}\mathscr{K}_\alpha$ is a (trivial) $[\mathfrak{I}]$-graded local system on $\mathbb{R}$ equipped with Stokes filtration $\mathscr{F}^\alpha_\bullet$ indexed by $\widetilde{\mathfrak{I}}(\alpha)$. This object is equipped with an action of $2\pi\mathbb{Z}$, but one has to take care that each $\mathscr{K}_\alpha$ is a priori not equivariant, as $\alpha(z)$ is sent to $\alpha(e^{2\pi i(\theta+k/p)}z)$ by the action of $2k\pi$. Only the sum over the orbit of α by this action is stable by the $2\pi\mathbb{Z}$-action.

In this presentation, the set $T(\overline{[\mathfrak{I}]})$ and of deformation data is infinite, but the relations between the deformation data due to the $2\pi\mathbb{Z}$-equivariance (called Ψ in the paper of Mochizuki) reduce them to a finite set. The two approaches are easily seen to be equivalent.

4. $\widetilde{\mathfrak{I}}$-Stokes shell $\mapsto$ $\widetilde{\mathfrak{I}}$-Stokes-filtered local system in the simplest case

I only consider, in this section and the next ones, the unramified case. The extension to the p-ramified case, when considering $\mathbb{Z}/p\mathbb{Z}$-equivariant objects, is straightforward, and by using the equivalence indicated in Section 3.b, one recovers the statement of the article of Mochizuki.

Assume that $\overline{[\mathfrak{I}]} = \{\overline{\alpha}, 0\}$ with $\overline{\alpha} \neq 0$ (i.e., $\widetilde{\mathfrak{I}}$ is aligned). From an $\widetilde{\mathfrak{I}}$-Stokes shell $\big(\bigoplus_{\alpha\in[\mathfrak{I}]}(\mathscr{K}_\alpha, \mathscr{F}^\alpha_\bullet), \mathcal{R}\big)$, we wish to recover an $\widetilde{\mathfrak{I}}$-Stokes-filtered local system $(\mathscr{L}(\overline{\alpha}, 0), \mathscr{F}_\bullet)$. We consider the open covering (U_i) of S^1 by the intervals $J_\pm$ with $J \in T(\overline{\alpha})$ and J_-, resp. J_+, is the interval J extended a little near the initial, resp. final, boundary point. Up to shortening the intervals $J_\pm$, we can assume that for any J, the intervals J_- and $J_{\mathrm{next}-}$ do not intersect. For any $\widetilde{\mathfrak{a}} \in \widetilde{\mathfrak{I}} \setminus \{0\}$, we have $[\mathfrak{a}] \in \{\pm\alpha\}$, and we wish to construct $(\mathscr{L}(\overline{\alpha}, 0), \mathscr{F}_\bullet)$ so that

$$\mathrm{gr}^{\mathscr{F}}_{\widetilde{\mathfrak{a}}} \mathscr{L}(\overline{\alpha}, 0) := \mathrm{gr}^{\mathscr{F}^{[\mathfrak{a}]}}_{\widetilde{\mathfrak{a}}} \mathscr{K}_{[\mathfrak{a}]} \quad \forall\, \widetilde{\mathfrak{a}} \in \widetilde{\mathfrak{I}}.$$

For that purpose, we define $(\mathscr{L}(\overline{\alpha}, 0), \mathscr{F}_\bullet)|_{U_i} = \bigoplus_{\alpha\in[\mathfrak{I}]}(\mathscr{K}_\alpha, \mathscr{F}^\alpha_\bullet)|_{U_i}$ and we need to construct a family of gluings G^i_j with respect to this covering. The components $G^{i,\alpha}_{j,\alpha}$ are the identity.

- Assume that $U_i = J_+$ and $U_j = J_{\mathrm{next}-}$. Then the order any pair $\widetilde{\mathfrak{a}} \neq \widetilde{\mathfrak{a}}'$ in $\widetilde{\mathfrak{I}}$ changes at the boundary point. It follows that $G^{i,\alpha}_{j,\alpha'}$ must be zero in this case.
- Assume that $U_i = J_-$ and $U_j = J_+$, so that $U_{ij} = J$. Then the order any pair $\widetilde{\mathfrak{a}} \neq \widetilde{\mathfrak{a}}'$ in $\widetilde{\mathfrak{I}}$ is constant on J, and the automorphism $R(J)$ defines a gluing. As there is no triple intersection, no compatibility condition needs to be checked.

5. $\widetilde{\mathfrak{I}}$-Stokes shell $\mapsto$ $\widetilde{\mathfrak{I}}$-Stokes-filtered local system

For $\widetilde{\mathfrak{I}}$ general, i.e., not aligned, we can perform the above construction independently for each $\overline{\alpha} \neq 0$ in $\overline{[\mathfrak{I}]}$.

We would like to obtain $(\mathscr{L}, \mathscr{F}_\bullet)$ as a compound of the various $(\mathscr{L}(\overline{\alpha}, 0), \mathscr{F}_\bullet)$ previously constructed. The difficulty comes from the fact that the open coverings for constructing each of them are distinct and we do not want to use the open covering by all the intervals $J_\pm$ with $J \in T(\overline{[\mathfrak{I}]})$, as it is much redundant. Furthermore, each

$(\mathscr{L}(\overline{\alpha},0),\mathscr{F}_\bullet)$ involves the same $(\mathscr{K}_0,\mathscr{F}^0_\bullet)$ in its construction, and the final construction should not duplicate this object.

Instead, one uses a similar procedure inductively with respect to the order of the arguments in $S(\widetilde{\mathfrak{I}})=\bigcup_{\widetilde{\mathfrak{a}}\in\widetilde{\mathfrak{I}}\smallsetminus\{0\}}\mathrm{St}(\widetilde{\mathfrak{a}},0)$. We choose one such argument denoted by θ_0, and we denote the others successively by $\theta_1,\theta_2,\dots,\theta_m$, and $\theta_{m+1}=\theta_0$, in increasing order.

For any $k=0,\dots,m$, we define the $\widetilde{\mathfrak{I}}$-Stokes-filtered local system $(\mathscr{L},\mathscr{F}_\bullet)|_{(\theta_k,\theta_{k+2})}$ as

$$(\mathscr{L},\mathscr{F}_\bullet)|_{(\theta_k,\theta_{k+2})}:=\bigoplus_{\alpha\in[\mathfrak{I}]}(\mathscr{K}_\alpha,\mathscr{F}^\alpha_\bullet)|_{(\theta_k,\theta_{k+2})}.$$

Assume we have constructed $(\mathscr{L},\mathscr{F}_\bullet)$ on (θ_0,θ_k) for $k\geqslant 2$ with an $\widetilde{\mathfrak{I}}$-splitting as above on each (θ_j,θ_{j+2}) with $j=0,\dots,k-1)$. We define $(\mathscr{L},\mathscr{F}_\bullet)$ on (θ_0,θ_{k+1}) by gluing $(\mathscr{L},\mathscr{F}_\bullet)|_{(\theta_0,\theta_k)}$ and $(\mathscr{L},\mathscr{F}_\bullet)|_{(\theta_{k-1},\theta_{k+1})}$ on (θ_k,θ_{k+1}). We now define the gluing isomorphism $G(k)$ and we consider the subset $T_k(\overline{[\mathfrak{I}]})$ consisting of the J's which contain (θ_k,θ_{k+1}). Then $G(k)$ is represented by a block-morphism, with the following blocks:

- The diagonal blocks $\mathscr{K}_\alpha|_{(\theta_k,\theta_{k+1})}\to\mathscr{K}_\alpha|_{(\theta_k,\theta_{k+1})}$ $(\alpha\in[\mathfrak{I}])$ are the identity;
- the morphisms $R^{\alpha_+(J_1)}_{\alpha_-(J_2)}|_{(\theta_k,\theta_{k+1})}$ for any pair $J_1\neq J_2$ in $T_k(\overline{[\mathfrak{I}]})$;
- the blocks entering $R(J)$ restricted to (θ_k,θ_{k+1}) for any $J\in T_k(\overline{[\mathfrak{I}]})$.

Lemma 6. *The gluing morphism $G(k)$ is an isomorphism compatible with the Stokes filtration.*

Proof. If we order the blocks as $0,(\alpha_+(J))_{J\in T_k(\overline{[\mathfrak{I}]})},(\alpha_-(J))_{J\in T_k(\overline{[\mathfrak{I}]})})$, we see that the morphism $G(k)$ is block lower-triangular with the identity as diagonal blocks. Compatibility with the Stokes filtration follows from the block lower-triangular form of $G(k)$ together with the compatibility of each block with the Stokes filtration, as assumed in the definition of deformation data. ☐

6. $\widetilde{\mathfrak{I}}$-Stokes-filtered local system $\mapsto$ $\widetilde{\mathfrak{I}}$-Stokes shell

Let $(\mathscr{L},\mathscr{F}_\bullet)$ be an $\widetilde{\mathfrak{I}}$-Stokes-filtered local system.

6.a. Construction of $(\mathscr{K}_\alpha,\mathscr{F}^\alpha_\bullet)$. We first construct $(\mathscr{K}_\alpha,\mathscr{F}^\alpha_\bullet)$ indexed by $\widetilde{\mathfrak{I}}(\alpha)$ for any $\alpha\in[\mathfrak{I}]$.

Case when $\alpha\neq 0$. We consider the covering of S^1 by the open subset $J_\pm$ with $J\in T(\overline{\alpha})$. For any $J\in T(\overline{\alpha})$, there exist unique splittings of ${}^n\mathscr{F}_\bullet(\mathscr{L})$:

$$\mathrm{can}(J_\pm):(\mathscr{L},\mathscr{F}_\bullet)|_J\xrightarrow{\ \sim\ }\bigoplus_{\mathfrak{a}\in\mathfrak{I}}(\mathrm{gr}^{{}^n\mathscr{F}}_{\mathfrak{a}}(\mathscr{L}),\mathscr{F}^{\mathfrak{a}}_\bullet)|_J.$$

For $J \in T(\overline{\alpha})$ and $(i=-, j=+)$ or $(i=+, j=\text{next}-)$, we consider the gluing isomorphism

$$G^i_j(J) = \operatorname{can}(J_j)\circ\operatorname{can}(J_i)^{-1} : \bigoplus_{\mathfrak{a}\in\mathfrak{I}}(\operatorname{gr}^{n\mathscr{F}}_{\mathfrak{a}}(\mathscr{L}), \mathscr{F}^{\mathfrak{a}}_{\bullet})|_{J_i\cap J_j} \xrightarrow{\sim} \bigoplus_{\mathfrak{a}\in\mathfrak{I}}(\operatorname{gr}^{n\mathscr{F}}_{\mathfrak{a}}(\mathscr{L}), \mathscr{F}^{\mathfrak{a}}_{\bullet})|_{J_i\cap J_j}.$$

We note that the components $G^{i,\mathfrak{a}}_{j,\mathfrak{a}}$ are equal to Id because of the splitting condition. Furthermore, by the compatibility with the Stokes filtration ${}^n\mathscr{F}_{\bullet}$, the block $G^{i,\mathfrak{a}}_{j,\mathfrak{b}}$ is zero unless $\mathfrak{b}\leqslant_{J_i\cap J_j}\mathfrak{a}$.

We claim that the blocks $G^{i,\alpha}_{j,\alpha}$, consisting of sub-blocks $G^{i,\mathfrak{a}}_{j,\mathfrak{b}}$ for $[\mathfrak{a}]=[\mathfrak{b}]=\alpha$, is invertible. Assume first that $J_i=J_+$ and $J_j=J_{\text{next}-}$. Then $G^{i,\mathfrak{a}}_{j,\mathfrak{b}}=0$ if $\mathfrak{a}\neq\mathfrak{b}$ because the final boundary of J_+ is a Stokes direction for $(\mathfrak{a},\mathfrak{b})$. Since $G^{i,\mathfrak{a}}_{j,\mathfrak{a}}=\mathrm{Id}$, hence the claim is true in this case.

Assume now that $J_i=J_-$ and $J_j=J_+$. Then $G^{i,\mathfrak{a}}_{j,\mathfrak{b}}=0$ unless $\mathfrak{b}\leqslant_J\mathfrak{a}$ (i.e., $\mathfrak{b}\leqslant_\theta\mathfrak{a}$ for any $\theta\in J$), and this gives a block-triangular form to $G^{i,\alpha}_{j,\alpha}$ with identity blocks on the diagonal. Moreover, it is clear by this triangular form that $G^{i,\alpha}_{j,\alpha}$ is compatible with the filtration induced by the order on $\widetilde{\mathfrak{I}}(\alpha)$.

It follows that we can glue the various $\bigoplus_{\mathfrak{a}\in\mathfrak{I}|[\mathfrak{a}]=\alpha}(\operatorname{gr}^{n\mathscr{F}}_{\mathfrak{a}}(\mathscr{L}),\mathscr{F}^{\mathfrak{a}}_{\bullet})|_{J_\pm}$ (considered as an $\mathfrak{I}(\alpha)$-graded $\widetilde{\mathfrak{I}}(\alpha)$-Stokes-filtered local system by means of the gluing morphisms $G^{i,\alpha}_{j,\alpha}$, and we obtain the $\widetilde{\mathfrak{I}}(\alpha)$-Stokes-filtered local system $(\mathscr{K}_\alpha,\mathscr{F}^\alpha_\bullet)$.

Case when $\alpha=0$. We do not have $T(0)$ at our disposal, so we have to change the argument. To show that the blocks $G^{i,0}_{j,0}$ are invertible, we take the opportunity that 0 is the single element of $\mathfrak{I}$ mapping to 0 in $[\mathfrak{I}]$ to make the construction directly from the Stokes filtration ${}^n\mathscr{F}_\bullet(\mathcal{L})$. We set

$$(\mathscr{K}_0,\mathscr{F}^0_\bullet)=(\operatorname{gr}^{n\mathscr{F}}_0(\mathscr{L}),\mathscr{F}^0_\bullet).$$

6.b. Deformation data associated to an $\widetilde{\mathfrak{I}}$-Stokes-filtered local system

We consider the following pairs (J_{1-},J_{2+}) of intervals in associated to pairs (J_1,J_2) in $T(\overline{[\mathfrak{I}]})$ such that

- $J_1=J_2=J$,
- $J_i=(\theta_i,\theta_i+\pi/n)$ $(i=1,2)$ with

$$\theta_1<\theta_2<\theta_1+\pi/n.$$

On these intervals, we have the canonical $\mathfrak{I}$-splittings $\operatorname{can}(J_{1-})$ and $\operatorname{can}(J_{2+})$ of $(\mathscr{L},\mathscr{F}_\bullet)$, and the canonical $\mathfrak{I}$-splittings $\operatorname{can}^\alpha(J_{1-})$ and $\operatorname{can}^\alpha(J_{2+})$ of $(\mathscr{K}_\alpha,\mathscr{F}^\alpha_\bullet)$ $(\alpha\in[\mathfrak{I}]\smallsetminus\{0\})$. We also set

$$\operatorname{can}^0(J_{1-})=\mathrm{Id},\quad \operatorname{can}^0(J_{2+})=\mathrm{Id}$$

and

$$\operatorname{can}'(J_{1-})=\bigoplus_{\alpha\in[\mathfrak{I}]}\operatorname{can}^\alpha(J_{1-}),\quad \operatorname{can}'(J_{2+})=\bigoplus_{\alpha\in[\mathfrak{I}]}\operatorname{can}^\alpha(J_{2+}).$$

Lastly, we obtain canonical isomorphisms

$$\operatorname{can}''(J_{1-}) := \operatorname{can}'(J_{1-})^{-1} \circ \operatorname{can}(J_{1-}) : (\mathscr{L}, \mathscr{F}_\bullet)|_{J_{1-}} \xrightarrow{\sim} \bigoplus_{\alpha\in[\mathfrak{I}]} (\mathscr{K}_\alpha, \mathscr{F}_\bullet^\alpha)|_{J_{1-}},$$

$$\operatorname{can}''(J_{2+}) := \operatorname{can}'(J_{2+})^{-1} \circ \operatorname{can}(J_{2+}) : (\mathscr{L}, \mathscr{F}_\bullet)|_{J_{2+}} \xrightarrow{\sim} \bigoplus_{\alpha\in[\mathfrak{I}]} (\mathscr{K}_\alpha, \mathscr{F}_\bullet^\alpha)|_{J_{2+}},$$

from which we extract the gluing isomorphisms

$$\operatorname{can}''(J_{2+}) \circ \operatorname{can}''(J_{1-})^{-1} : \bigoplus_{\alpha\in[\mathfrak{I}]} (\mathscr{K}_\alpha, \mathscr{F}_\bullet^\alpha)|_{J_1\cap J_2} \xrightarrow{\sim} \bigoplus_{\alpha\in[\mathfrak{I}]} (\mathscr{K}_\alpha, \mathscr{F}_\bullet^\alpha)|_{J_1\cap J_2}.$$

Then, in these gluing morphisms,

- if $J_1 = J_2 = J$, the deformation data on J are the blocks $(\alpha_+(J), 0)$, $(0, \alpha_-(J))$, $(\alpha_+(J), \alpha_-(J))$;
- if $J_1 \neq J_2$, the deformation data on $J_1 \cap J_2$ are the blocks $(\alpha_+(J_{1-}), \alpha_-(J_{2+}))$.

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