UNIVERSAL UNFOLDINGS OF LAURENT POLYNOMIALS AND TT* STRUCTURES

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Introduction

Let $f:(\mathbb{C}^*)^n\to\mathbb{C}$ be a Laurent polynomial, that I assume to be convenient and non-degenerate, *e.g.* $f(u_1,\dots,u_n)=u_1+\cdots+u_n+1/u_1^{w_1}\cdots u_n^{w_n}$. The base space M of the "universal unfolding" of f is the analytic germ at the origin of the vector space $\mathbb{C}[u,u^{-1}]/J(f)$, where $J(f)=(u_1\partial_{u_1}f,\dots,u_n\partial_{u_n}f)$. It is known that this space carries a canonical Frobenius manifold structure. In this talk, I will explain how to equip it with a canonical Hermitian metric which is harmonic and endows it with a tt* structure, so that we get a CDV structure in the sense of Hertling.

1. Integrable variations of polarized twistor structures

1.1. Harmonic bundles. Let M be a complex manifold and let E be a holomorphic bundle on M, equipped with a Hermitian metric h and a holomorphic Higgs field θ (*i.e.* $\theta\wedge\theta=0$). Let $D=D'+d''$ be the Chern connection of h and let $\theta^\dagger$ be the h-adjoint of θ. We say that (E,h,θ) is *harmonic* if $D+\theta+\theta^\dagger$ is a flat connection.

1.2. Variation of twistor structures. Let me recall the terminology introduced by Carlos Simpson (1997). It is convenient to express this relation by adding a parameter z and to express it with the notion of a z-connection. A variation of twistor structure consists of the data of a $C^{\infty,\mathrm{an}}_{M\times\mathbb{P}^1}$ vector bundle $\mathcal{H}$ on $M\times\mathbb{P}^1$ with an integrable relative connection $\nabla:\mathcal{H}\to\Omega^1_{M\times\mathbb{P}^1/\mathbb{P}^1}(0,\infty)\otimes\mathcal{H}$. It is pure of weight 0 if each bundle $\mathcal{H}_{|\{x\}\times\mathbb{P}^1}$ is trivial.

Let $\sigma:\mathbb{P}^1\to\overline{\mathbb{P}}^1$ be the anti-linear map defined by $z\mapsto -1/\overline{z}$. The conjugate $\overline{\mathcal{H}}$ of $\mathcal{H}$ is defined as $\sigma^*\overline{\mathcal{H}}$. This is also a $C^{\infty,\mathrm{an}}$ vector bundle, equipped with a conjugate connection $\overline{\nabla}$. A sesquilinear form on the variation of twistor structure is a non-degenerate pairing $\mathcal{H} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \overline{\mathcal{H}} \to \mathcal{C}^{\infty,\mathrm{an}}_{M\times\mathbb{P}^1}$ which is compatible with the connections. The definition of "Hermitian" is then clear.

If the variation is pure of weight 0, then we say that a Hermitian non-degenerate pairing is a polarization if the pairing induced on $\mathbb{P}^1$-global sections is a Hermitian metric on the bundle H of $\mathbb{P}^1$-global sections of $\mathcal{H}$.

C. Simpson: *Equivalence between variations of polarized pure twistor structures of weight* 0 *and harmonic bundles* by taking $\mathbb{P}^1$-global sections.

1.3. Integrability. A variation of twistor structure is *integrable* if the relative connection ∇ comes from a absolute connection also denoted by ∇, which has Poincaré rank one. If there is a sesquilinear pairing, we also ask for the compatibility of the absolute connection with the pairing.

This notion is not new if M is compact Kähler, as is amounts to the notion of a variation of Hodge structure. The same holds if M is a punctured compact Riemann surface and the behaviour at the punctures is *tame*, as noticed by Christian Sevenheck. Probably such a result holds in the higher dimensional case, always with the assumption of tameness.

So one should think of non-tame, *i.e.* wild, possible degenerations of such variations.

C. Hertling: *Equivalence between integrable variations of polarized pure twistor structures of weight* 0 *and harmonic bundles* (E, h, θ) + *a holomorphic endomorphism* $\mathcal{U}$ *and a* C^∞ *endomorphism* $\mathcal{Q}$ *satisfying*

$$\begin{cases} [\theta, \mathcal{U}] = 0 \\ D'(\mathcal{U}) - [\theta, \mathcal{Q}] + \theta = 0 \\ D'(\mathcal{Q}) + [\theta, \mathcal{U}^\dagger] = 0. \end{cases} \tag{1}$$

On the other hand, giving an integrable variation of twistor structure with a non-degenerate Hermitian pairing is equivalent to giving

(1) A holomorphic vector bundle $\mathcal{H}'$ on $M \times \mathbb{C}$ with a meromorphic connection ∇ having Poincaré rank one along $M\times\{0\}$ (*i.e.* $z\nabla$ has at most a logarithmic pole along $M \times \{0\}$),

(2) A non-degenerate pairing $\mathcal{K} : \operatorname{Ker} \nabla_{|M\times\mathbf{S}} \otimes \sigma^*\overline{\operatorname{Ker} \nabla_{|M\times\mathbf{S}}} \to \mathbb{C}_{M\times\mathbf{S}}$, where $\mathbf{S} = \{|z|=1\}$.

The second part of the data is of purely topological nature, as $\operatorname{Ker} \nabla_{|M\times\mathbb{C}^*}$ is a locally constant sheaf. Moreover, if ι denotes the involution $z \mapsto -z$, then ι coincides with σ on $\mathbf{S}$.

***Corollary* (C. Hertling).** *Let us assume that M is 1-connected and let $x^o \in M$. Let $(\mathcal{H}', \nabla)$ be a holomorphic bundle on $M \times \mathbb{C}$ with a meromorphic connection ∇ having Poincaré rank one along $M \times \{0\}$, let $\mathcal{L} = \operatorname{Ker} \nabla_{|M\times\mathbf{S}}$, $\mathcal{L}^o = \mathcal{L}_{|\{x^o\}\times\mathbf{S}}$ and let $\mathcal{K}^o : \mathcal{L}^o \otimes \iota^{-1}\overline{\mathcal{L}^o} \to \mathbb{C}_{\mathbf{S}}$ be a non-degenerate Hermitian pairing. Let $\mathcal{K} : \mathcal{L} \otimes \iota^{-1}\overline{\mathcal{L}} \to \mathbb{C}_{M\times\mathbf{S}}$ be the unique non-degenerate Hermitian pairing extending $\mathcal{K}^o$.*

Let us moreover assume that the twistor structure at x^o corresponding to these data is pure of weight 0 and polarized. Then there exists a (possibly empty) real analytic subvariety $\Theta \not\ni x^o$ of M such that, on the connected component of $M \smallsetminus \Theta$ containing x^o, the variation of twistor structure $(\mathcal{H}', \nabla, \mathcal{K})$ is pure of weight 0 and polarized.

2. Variation of twistor structure attached to a Laurent polynomial

To the convenient non-degenerate Laurent polynomial f is associated in a canonical way a Frobenius manifold structure on the germ $(M, 0)$ of the vector space $\mathbb{C}[u, u^{-1}]/J(f)$. I will explain this later. As a consequence, the tangent bundle TM comes equipped with a integrable holomorphic connection $\triangledown$, and, according to a construction of Dubrovin, the bundle $\mathcal{H}' \overset{\text{def}}{=} \pi^*TM$ has an associated integrable meromorphic connection ∇ with Poincaré rank one along $M \times \{0\}$, given by the formula

$$\begin{aligned} \nabla_\xi \eta &= \triangledown_\xi \eta - \frac{\xi \star \eta}{z} \\ \nabla_{\partial_z} \eta &= \mathfrak{E} \star \eta \cdot \frac{1}{z^2} - \triangledown_\eta \mathfrak{E} \cdot \frac{1}{z}. \end{aligned} \tag{2}$$

Let me recall that the construction of the Frobenius manifold structure uses many ingredients: the notion of primitive forms of K. Saito, the notion of "good basis" of M. Saito, and Hodge theory at $f = \infty$. Moreover, the question of how canonical such a structure is has been solved by A. Douai, using results of Hertling and Manin, when one assumes that the Newton polyhedron associated to f contains in its interior a basis of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. For the example given in the introduction, it is enough to use an argument due to B. Malgrange.

With this construction, the bundle $\mathcal{H}'^o = \mathbb{C}[u, u^{-1}]/J(f) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}}$ is identified to the anlytization of the Brieskorn lattice

$$\Omega^n(U)[z]/(zd - df\wedge)\Omega^{n-1}(U)[z]$$

equipped with the connection ∇^o defined by

$$z^2 \nabla^o_{\partial_z} \big[\textstyle\sum_k \omega_k z^k \big] = \big[\textstyle\sum_k k\omega_k z^{k+1} + \sum_k f\omega_k z^k \big].$$

On the other hand, it is classical (after the work of F. Pham) that the local system $\mathcal{L}^o$ is identified to the locally constant sheaf

$$H^n_{\Phi_z}(U,\mathbb{C}),$$

where Φ_z denotes the family of closed sets in U on which $\mathrm{Re}(f(u_1,\dots,u_n)/z) \leqslant c < 0$.

There is a natural intersection pairing (Poincaré duality pairing made sesquilinear)

$$\widehat{P}_z : H^n_{\Phi_z}(U,\mathbb{C}) \otimes \overline{H^n_{\Phi_{-z}}(U,\mathbb{C})} \longrightarrow \mathbb{C}.$$

***Definition* (C. Hertling).** The variation of twistor structure canonically attached to f on $(M,0)$ is the variation defined, according to the corollary above, by the data $(\mathcal{H}',\nabla)$ as in (2) and the pairing $\mathcal{K}^o = \dfrac{(-1)^{(n-1)n/2}}{(2i\pi)^n}\widehat{P}_z$.

***Theorem*.** *The twistor structure $(\mathcal{H}'^o,\nabla^o,\mathcal{K}^o)$ is pure of weight 0 and polarized.*

3. Proof of the theorem

3.1. Reduction of the problem to dimension one. How can one prove such a positivity statement? One should start with a variation of polarized twistor structure of weight 0 and get our twistor structure by a natural operation from the previous one.

Example of such a result:

***Hodge-Simpson Theorem*.** *Given a variation of polarized twistor structure of weight 0 on a compact Kähler manifold X, its de Rham cohomology carries a polarized twistor structure (of some weight).*

The main ingredient in the proof is the fact that, for any $z \in \mathbb{C}$, the Laplace operator Δ_z relative to the operator $D_z = d''+\theta+z(D'+\theta^\dagger)$ and the Kähler metric is essentially constant: $\Delta_z = (1+|z|^2)\Delta_0$ (this is the analogue of the classical Kähler identity $\Delta_d = 2\Delta_{d'} = 2\Delta_{d''}$). Hence, the space of harmonic sections does not depend on z. This will give the pure weight 0 property. The positivity is obtained by a standard argument, on primitive sections first.

One can obtain $(\mathcal{H}'^o,\nabla^o)$ in the theorem by

(1) considering the trivial variation of twistor structures $(\mathcal{O}_U[z], zd)$ (the Higgs field is equal to 0),

(2) twisting it by $e^{-f/z}$, that is, adding a new Higgs field $\theta = -df$,

(3) and taking the de Rham cohomology of this new variation.

The operator D_z is now $d''-df+z(d'-d\overline{f})$. We are faced with two problems: U is non-compact and f is not bounded on U (so that e^{-f} can have an exponential growth). The Hodge theory for the corresponding Laplacian can be difficult to develop (although it has been developped in some special cases).

Instead, we use *Horatio's method*: if we face numerous enemies, we fake escaping by running fast, then kill the enemy running faster when he reaches us, then kill the next one, etc. Here, we escape by falling down along the fibres of $f : U \to \mathbb{A}^1$.

Let t be the coordinate on $\mathbb{A}^1$. The Gauss-Manin connection of f gives a bundle with connection on $\mathbb{A}^1 \smallsetminus \{\text{critical values of } f\}$. The interesting bundle has fibre $H^{n-1}(f^{-1}(t), \mathbb{C})$. It underlies a variation of mixed Hodge structure (M. Saito). The assumption made on f (cohomological tameness) implies that this mixed Hodge structure is an extension of pure Hodge structures for which one sub-quotient is a variation of polarized Hodge structure and any other quotient is a trivial variation of Hodge structure on $\mathbb{A}^1$. The generic fibre is identified to the intersection cohomology of a suitable compactification of $f^{-1}(t)$.

The variation of polarized Hodge structure induces a variation of polarized twistor structure of the same weight.

Remark. To be precise, one needs to have a control at the critical values of f. This is done by considering the Gauss-Manin system of f, which underlies a mixed Hodge module, and to extract from it a sub-quotient which is a polarizable pure Hodge module, all other sub-quotients being isomorphic to a power of $(\mathbb{C}[t], d)$. M. Saito's theory also makes precise the polarization.

3.2. Exponentially twisted harmonic bundles in dimension one. Let E be a holomorphic bundle on $\mathbb{A}^1 \smallsetminus P$, equipped with a Hermitian metric h and a holomorphic Higgs field θ. We assume that (E, h, θ) is a *tame harmonic bundle* in the sense of Simpson (1990). Let us consider the exponentially twisted harmonic bundle $(E, h, \theta - dt)$. This remains a harmonic bundle which is tame at P, but *wild* at ∞.

Theorem. *The space of L^2 harmonic sections of $E \otimes \mathcal{A}^1_{\mathbb{A}^1 \smallsetminus P}$, with respect to the metric h and a metric on $\mathbb{A}^1 \smallsetminus P$ equivalent to the Poincaré metric near $P \cup \{\infty\}$, and with respect to the Laplace operator of $d'' + \theta - dt + z(D' + \theta^\dagger - d\bar{t})$ is finite dimensional and independent of z.*

Remark. The proof of this theorem that I gave first by using a degeneration argument $\tau \to 0$ was only valid for $(E, h, \theta - \tau dt)$ for $|\tau|$ small (depending on (E, h, θ)). On the other hand, S. Szabo gave a proof in a nearby context. One can arrange the proof I gave by using an argument similar to an argument used by Szabo, which says:

The harmonic sections have an exponential decay when $t \to \infty$.

3.3. End of the proof of the theorem. The proof proceeds as follows:

(1) Starting from $f : U \to \mathbb{A}^1$, we consider the Gauss-Manin system M (a mixed Hodge module on $\mathbb{A}^1$, after M. Saito).

(2) We extract from it a polarized pure Hodge module $M_{!*}$.

(3) It defines a tame harmonic bundle on $\mathbb{A}^1 \smallsetminus P$, where P are the critical values of f, which corresponds to an integrable variation of polarized pure twistor structure.

(4) We twist this variation by $e^{-t/z}$ and take its de Rham cohomology: we get an integrable twistor structure.

(5) Harmonic sections considered in the theorem above give a global frame of this twistor structure, hence purity.

(6) Positivity of the natural L^2 Hermitian form on the harmonic sections gives a polarization.

(7) This polarization coincides with $\mathcal{K}^o$.

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