Universal unfoldings of Laurent polynomials and tt* structures

Claude Sabbah

Centre de Mathématiques Laurent Schwartz

UMR 7640 du CNRS

École polytechnique, Palaiseau, France

Integrable variations of twistor structures

Integrable variations of twistor structures

- **C. Simpson**: Equivalence between ***variations of polarized pure twistor structures of weight*** $\mathbf{0}$ and ***harmonic bundles*** by taking $\mathbb{P}^{\mathbf{1}}$-global sections.

Integrable variations of twistor structures

- **C. Simpson**: Equivalence between ***variations of polarized pure twistor structures of weight*** $\mathbf{0}$ and ***harmonic bundles*** by taking $\mathbb{P}^{1}$-global sections.
- **C. Hertling**: Equivalence between ***integrable*** variations of polarized pure twistor structures of weight $\mathbf{0}$ and harmonic bundles $(E, h, \theta) +$ a ***holomorphic endomorphism*** $\mathscr{U}$ and a C^{∞} ***endomorphism*** $\mathscr{Q}$ satisfying

$$\begin{cases} [\theta, \mathscr{U}] = 0 \\ D'(\mathscr{U}) - [\theta, \mathscr{Q}] + \theta = 0 \\ D'(\mathscr{Q}) + [\theta, \mathscr{U}^{\dagger}] = 0 \end{cases}$$

Corollary (C. Hertling).

- M *is* 1*-connected,* $x^o \in M$.

Corollary (C. Hertling).

- M *is* 1*-connected,* $x^o \in M$.
- $(\mathscr{H}', \nabla)$ *a holomorphic bundle on* $M \times \mathbb{C}$ *with a meromorphic connection* ∇ *having* ***Poincaré rank one*** *along* $M \times \{0\}$

Corollary (C. Hertling).

- M *is* 1*-connected,* $x^o \in M$.
- $(\mathscr{H}', \nabla)$ *a holomorphic bundle on* $M \times \mathbb{C}$ *with a meromorphic connection* ∇ *having* ***Poincaré rank one*** *along* $M \times \{0\}$
- $\mathscr{L} = \ker \nabla_{|M \times S}, \quad \mathscr{L}^o = \mathscr{L}_{|\{x^o\} \times S}$

Corollary (C. Hertling).

- M *is* 1*-connected,* $x^o \in M$.
- $(\mathscr{H}', \nabla)$ *a holomorphic bundle on* $M \times \mathbb{C}$ *with a meromorphic connection* ∇ *having* ***Poincaré rank one*** *along* $M \times \{0\}$
- $\mathscr{L} = \ker \nabla_{|M\times S}$, $\mathscr{L}^o = \mathscr{L}_{|\{x^o\}\times S}$
- *a non-degenerate Hermitian pairing* $\mathscr{K}^o : \mathscr{L}^o \otimes \iota^{-1}\overline{\mathscr{L}^o} \longrightarrow \mathbb{C}_S$.

Corollary (C. Hertling).

- M *is* 1*-connected,* $x^o \in M$.
- $(\mathscr{H}', \nabla)$ *a holomorphic bundle on* $M \times \mathbb{C}$ *with a meromorphic connection* ∇ *having* ***Poincaré rank one*** *along* $M \times \{0\}$
- $\mathscr{L} = \ker \nabla_{|M \times S}$, $\mathscr{L}^o = \mathscr{L}_{|\{x^o\} \times S}$
- *a non-degenerate Hermitian pairing* $\mathscr{K}^o : \mathscr{L}^o \otimes \iota^{-1}\overline{\mathscr{L}^o} \longrightarrow \mathbb{C}_S$.
- $\mathscr{K} : \mathscr{L} \otimes \iota^{-1}\overline{\mathscr{L}} \longrightarrow \mathbb{C}_{M \times S}$ *the unique non-degenerate Hermitian pairing extending* $\mathscr{K}^o$.

If the twistor structure at x^o corresponding to these data is ***pure of weight 0 and polarized***

If the twistor structure at x^o corresponding to these data is ***pure of weight 0 and polarized***

then there exists a (possibly empty) real analytic subvariety $\Theta \not\ni x^o$ of M such that, on the connected component of $M \smallsetminus \Theta$ containing x^o, the variation of twistor structure $(\mathscr{H}', \nabla, \mathscr{K})$ is pure of weight 0 and polarized.

Hodge-Simpson Theorem. *Given a variation of polarized twistor structure of weight* $\mathbf{0}$ *on a compact Kähler manifold* X*, its de Rham cohomology carries a polarized twistor structure (of some weight).*

Theorem. *The space of* L^2 *harmonic sections of* $E \otimes \mathscr{A}^1_{\mathbb{A}^1 \smallsetminus P}$, *with respect to the metric* h *and a metric on* $\mathbb{A}^1 \smallsetminus P$ *equivalent to the Poincaré metric near* $P \cup \{\infty\}$, *and with respect to the Laplace operator of* $d'' + \theta - dt + z(D' + \theta^\dagger - d\bar{t})$ *is finite dimensional and independent of* z.

Sketch of the proof

Sketch of the proof

- Starting from $f : U \longrightarrow \mathbb{A}^1$, we consider the Gauss-Manin system M (a mixed Hodge module on $\mathbb{A}^1$, after M. Saito).

Sketch of the proof

- Starting from $f : U \longrightarrow \mathbb{A}^1$, we consider the Gauss-Manin system M (a mixed Hodge module on $\mathbb{A}^1$, after M. Saito).
- We extract from it a polarized pure Hodge module $M_{!*}$.

Sketch of the proof

- Starting from $f : U \longrightarrow \mathbb{A}^1$, we consider the Gauss-Manin system M (a mixed Hodge module on $\mathbb{A}^1$, after M. Saito).
- We extract from it a polarized pure Hodge module $M_{!*}$.
- It defines a tame harmonic bundle on $\mathbb{A}^1 \smallsetminus P$, where P are the critical values of f, which corresponds to an integrable variation of polarized pure twistor structure.

Sketch of the proof

- Starting from $f : U \longrightarrow \mathbb{A}^1$, we consider the Gauss-Manin system M (a mixed Hodge module on $\mathbb{A}^1$, after M. Saito).
- We extract from it a polarized pure Hodge module $M_{!*}$.
- It defines a tame harmonic bundle on $\mathbb{A}^1 \smallsetminus P$, where P are the critical values of f, which corresponds to an integrable variation of polarized pure twistor structure.
- We twist this variation by $e^{-t/z}$ and take its de Rham cohomology: we get an integrable twistor structure.

Sketch of the proof

- Harmonic sections considered in the theorem above give a global frame of this twistor structure, hence purity.

Sketch of the proof

- Harmonic sections considered in the theorem above give a global frame of this twistor structure, hence purity.
- Positivity of the natural L^2 Hermitian form on the harmonic sections gives a polarization.

Sketch of the proof

- Harmonic sections considered in the theorem above give a global frame of this twistor structure, hence purity.
- Positivity of the natural L^2 Hermitian form on the harmonic sections gives a polarization.
- This polarization coincides with $\mathscr{K}^o$.