HODGE STRUCTURES AND RIGID LOCAL SYSTEMS

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Abstract. Rigid irreducible local systems on the punctured Riemann sphere are known to underlie variations of complex Hodge structures if their monodromies satisfy some unitarity property. The first part of the talk will focus on the methods of computation of the corresponding Hodge numbers, that can be considered as hidden invariants. In the second part, the analogous theory with irregular singularities will be developed and some explicit computations will be explained for the confluent hypergeometric differential equations.

1. Introduction: Rigid bundles with connection

Settings.

- $j : U \subset \mathbb{P}^1$: open embedding of a Zariski nonempty open subset,
- $D := \mathbb{P}^1 \smallsetminus U \neq \varnothing$ finite set of points, so that U is affine.
- (V, ∇): alg. vector bundle on U with connection $\nabla : V \to \Omega^1_U \otimes V$. Then, giving V is equivalent to giving $\Gamma(U, V)$, which is a free $\mathscr{O}(U)$-module, i.e., $V \simeq \mathscr{O}^r_U$. By choosing a basis, one gets the matrix of the connection in this basis. Assume for example that $\infty \in D$ so that $U \subset \mathbb{A}^1_t$, then the matrix can be written as $A(t)\mathrm{d}t$ with $A(t) \in \mathrm{Mat}_{r\times r}(\mathscr{O}(U))$.
- We also consider j_*V as a free $\mathscr{O}_{\mathbb{P}^1}(*D)$-module with connection

$$\nabla : j_*V \longrightarrow \Omega^1_{\mathbb{P}^1} \otimes j_*V.$$

- $(j_*V)^{\mathrm{an}}$ on $\mathbb{P}^{1\mathrm{an}}$ and $\Gamma(U, V) = \Gamma(\mathbb{P}^{1\mathrm{an}}, (j_*V)^{\mathrm{an}})$.

Definition.

(1) (V, ∇) is *irreducible* (or simple) if it has non nontrivial sub-object.
(2) (V, ∇) irreducible is *rigid* if $\forall (V', \nabla')$ irred.,

$$\boxed{(V', \nabla')_{\widehat{x}} \simeq (V, \nabla)_{\widehat{x}} \quad \forall x \in \mathbb{P}^1 \smallsetminus U \quad \Longrightarrow \quad (V,', \nabla') \simeq (V, \nabla)}$$

Hukuhara-Levelt-Turrittin decomposition. *For all $x \in D$, $(V,\nabla)_{\widehat{x}}$ is the direct sum of elementary formal bundles with connection.*

***Definition* (Elementary formal bundles with connection).** Set $\mathscr{O}_{\mathbb{P}^1,\widehat{x}} = \mathbf{C}[\![t]\!]$ and $\mathscr{O}_{\mathbb{P}^1}(*D)_{\widehat{x}} = \mathbf{C}(\!(t)\!)$.

(1) $(V,\nabla)_{\widehat{x}}$ *Elementary regular*: $\exists$ a $\mathbf{C}(\!(t)\!)$-basis of $V_{\widehat{x}}$ in which $\mathrm{mat}(\nabla) = A_0\,\mathrm{d}t/t$ with A_0 cst.

Eigenvalues of formal monodromy: $\exp(-2\pi\mathrm{i}\alpha)$ with α eigenvalue of A_0.

(2) $(V,\nabla)_{\widehat{x}}$ *Elementary exponential*: $\exists p,q \geqslant 1$,

$$(V,\nabla)_{\widehat{x}} = \rho_*(\mathbf{C}(\!(u)\!), \mathrm{d}+\mathrm{d}\varphi), \quad \rho : u \longmapsto t = u^p,\ \varphi = c_o u^{-q}(1+\cdots).$$

Slope$= q/p$.

(3) $(V,\nabla)_{\widehat{x}}$ *Elementary*: direct sum of tensor products of an elementary regular formal bundles with connection with an elementary exponential one.

$$(V,\nabla)_{\widehat{x}} = \bigoplus_{\varphi} (\mathrm{El}(\rho,\varphi)\otimes R_\varphi).$$

Tame examples.

- Non-res. hypergeometric differential equations:

$$\mathcal{H}_{n,n}(\boldsymbol{\alpha},\boldsymbol{\beta},\gamma) = \prod_{i=1}^n (t\partial_t - \alpha_i) - \gamma t \prod_{j=1}^n (t\partial_t - \beta_j),$$

$\alpha_i - \beta_j \notin \mathbf{Z}$ (non-resonance condition), $\gamma \in \mathbf{C}^*$, $U = \mathbb{P}^1 \smallsetminus \{0,\infty,1/\gamma\}$.

$$H_{n,n}(\boldsymbol{\alpha},\boldsymbol{\beta},\gamma) = \mathbf{C}[t,t^{-1},(1-\gamma t)^{-1}]\langle t\partial_t\rangle / (\mathcal{H}_{n,n}(\boldsymbol{\alpha},\boldsymbol{\beta},\gamma)).$$

- Many examples obtained by applying the Katz algorithm (cf. Dettweiler-Reiter).

Wild examples. Many examples with $U = \mathbb{P}^1 \smallsetminus \{0,\infty\}$, tame at 0, wild at ∞.

- Non-res. Confluent hypergeom. differential equations $\mathcal{H}_{n,m}(\boldsymbol{\alpha},\boldsymbol{\beta},\gamma)$, $n>m$.

Explanation of the word "confluent": For $\varepsilon > 0$, consider $H_{n,n}(\boldsymbol{\alpha},\boldsymbol{\beta}^\varepsilon,\varepsilon\gamma)$ with $\beta_i^\varepsilon = \beta_i$ for $i<n$ and $\beta_n^\varepsilon = \beta_n/\varepsilon$. Then,

$$\lim_{\varepsilon\to 0} H_{n,n}(\boldsymbol{\alpha},\boldsymbol{\beta}^\varepsilon,\varepsilon\gamma) = H_{n,n-1}(\boldsymbol{\alpha},\boldsymbol{\beta}',-\beta_n\gamma), \quad \boldsymbol{\beta}' = (\beta_1,\dots,\beta_{n-1}).$$

The regular singular points $1/\varepsilon\gamma$ and ∞ have merged to a single irregular singular point ∞.

- Examples of Frenkel-Gross: In these examples, the matrix of the connection

$$\mathrm{mat}(\nabla) = \mathrm{N}\frac{\mathrm{d}t}{t} + \mathrm{E}\,\mathrm{d}t, \quad \mathrm{N},\mathrm{E} \text{ some nilpotent endomorphisms}$$

- Examples with differential Galois group G_2 by K. Jakob (classification of all bundles with connection having an irregular singularity at ∞ with pure slope of the form $1/p$. Then p must be $2,3,6$ and there is only one other singular point, at the origin say, which is regular).

2. Polarizable variations of Hodge structures

***Definition* (Polarizable variation of complex Hodge structure)**

(V,∇) underlies a pVHS of weight w if $\mathcal{H} := \mathcal{C}^\infty_U \otimes V^{\mathrm{an}}$ endowed with the flat connection $D = \nabla + \overline{\partial}$ satisfies the following properties

- $\mathcal{H} \simeq \bigoplus_p \mathcal{H}^{p,w-p}$ as C^∞ bdles,
- Griffiths transversality:

$$\begin{cases} D' : \mathcal{H}^{p,w-p} \to \Omega^1_{U^{\mathrm{an}}} \otimes (\mathcal{H}^{p-1,w-p+1} \oplus \mathcal{H}^{p,w-p}) \\ D'' : \mathcal{H}^{p,w-p} \to \overline{\Omega}^1_{U^{\mathrm{an}}} \otimes (\mathcal{H}^{p+1,w-p-1} \oplus \mathcal{H}^{p,w-p}) \end{cases}$$

$\Longrightarrow F^pV^{\mathrm{an}} := \bigoplus_{p'\geqslant p} \mathcal{H}^{p,w-p}$ holom. subbundle of V^{an} such that $\nabla F^pV^{\mathrm{an}} \subset F^{p-1}V^{\mathrm{an}}$.

Hodge ranks: $\dim \mathrm{gr}^p_F V^{\mathrm{an}}$.

- *Polarization*: D-flat $\mathrm{S} : \mathcal{H} \otimes \overline{\mathcal{H}} \to \mathcal{C}^\infty$ such that
  (1) the Hodge decomposition is S-orthogonal,
  (2) the associated pairing h such that $\mathrm{h}_{|\mathcal{H}^{p,w-p}} = (-1)^p \mathrm{S}_{|\mathcal{H}^{p,w-p}}$ is a positive definite Hermitian form on each fibre of $\mathcal{H}$.

The results below hold in a more general situation, but we restrict to the setting of the talk. Let us recall the classical theorems of Griffiths and Schmid.

***Theorem* (Schmid).** *Let $(\mathcal{H}, F', F'', D, S)$ be a pVHS on U^{an}, and let V^{an} be the underlying holomorphic bundle $\ker D''$. Let $(j_*V^{\mathrm{an}})^{\mathrm{mod}}$ be the subsheaf of j_*V^{an} consisting of sections whose* h*-norm has moderate growth near D. Then $(j_*V^{\mathrm{an}})^{\mathrm{mod}}$ is a free $\mathscr{O}_{\mathbb{P}^{1\mathrm{an}}}$-module of finite rank.*

As a consequence, $(\mathcal{H}, F', F'', D, S)$ induces a natural algebraic structure on V^{an}. On the other hand, we say that (V,∇) underlies a pVHS if $(V,\nabla)^{\mathrm{an}}$ underlies a pVHS on U^{an} and if V is defined by the moderate growth condition with respect to h (say that V is *adapted* to h).

***Theorem* (Griffiths regularity theorem).** *If (V,∇) underlies a pVHS, then (V,∇) is tame.*

***Theorem* (Deligne).** *Let (V,∇) be irreducible tame on U. Assume that its monodromy eigenvalues have absolute value equal to one. Then $(V,\nabla)^{\mathrm{an}}$ underlies at most one (up to a shift of the Hodge filtration) polarizable variation of complex Hodge structure on U^{an}.*

***Theorem* (Simpson).** *Assume moreover (V,∇) is rigid. Then such a pVHS exists.*

Natural questions.

(1) What about the wild case?

(2) How to compute the Hodge numbers (i.e., ranks of the Hodge bundles) for a given rigid irreducible tame (V, ∇)?

3. Polarizable variation of integrable twistor structure

As a consequence, if (V, ∇) is wild, one cannot not expect it underlies an adapted pVHS. The idea of a twistor structure, introduced by Simpson in 1997, is to replace a vector space endowed with two opposite filtrations (i.e., a Hodge decomposition) with an object without filtrations, but keeping track of the opposedness condition.

Twistor structure.

- $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$, coord. z fixed on $\mathbb{A}^1$ (twistor variable, not that it has nothing to do with the $\mathbb{P}^1$ already used).
- Twistor structure $\mathcal{T}$: locally free $\mathscr{O}_{\mathbb{P}^1}$-module (i.e., vector bundle on $\mathbb{P}^1$).

E. g., $H = \bigoplus H^{p,w-p}$, $F'^p = \bigoplus_{p' \geqslant p} H^{p',w-p'}$, $F''^q = \bigoplus_{q' \geqslant q} H^{w-q',q'}$ corresponds to $(R_{F'}H, R_{F''}H, \text{gluing})$. and connection $\triangledown$ with log pole at $0, \infty$.

- Pure of weight $w \iff$ of pure slope w.
- Dual $\mathcal{T}^\vee$: dual vect. bdle. Hermitian dual $\mathcal{T}^* = \sigma^* \overline{\mathcal{T}}^\vee$, $\sigma : \mathbb{P}^1 \to \overline{\mathbb{P}}^1$.
- For a pure twistor structure $\mathcal{T}$ of weight 0, two naturally associated vector spaces of the same dimension: $\mathcal{T}_1 := \mathcal{T}_{\{z=1\}}$ and $\Gamma(\mathbb{P}^1, \mathcal{T})$.
- There is a notion of a polarization: isom. $\mathcal{T} \simeq \mathcal{T}^*$, such that the induced isom $\Gamma(\mathbb{P}^1, \mathcal{T}) \simeq \Gamma(\mathbb{P}^1, \mathcal{T})^*$ is positive definite.
- Notion of pVTS (more complicated to define) $\overset{\text{Simpson}}{\iff}$ Flat C^∞ vect. bdle with *harmonic metric*.

Integrability. Origin of the notion: tt*-structures of Cecotti-Vafa in mirror symmetry, interpretation by Hertling.
The twistor structure is *integrable* if it is endowed with $\triangledown : \mathcal{T} \to \Omega^1_{\mathbb{P}^1}(2 \cdot 0 + 2 \cdot \infty) \otimes \mathcal{T}$.

- Deligne merom. extension $(\mathcal{T}^{\mathrm{Del}}, \triangledown^{\mathrm{Del}})$ at ∞ of $(\mathcal{T}_{|\mathbf{C}_z}, \triangledown)$.
- Deligne log. extensions $\mathcal{T}^\alpha$ such that $\mathrm{Res}_{z=\infty} \triangledown_{\mathcal{T}^\alpha}$ has eigenvalues in $[\alpha, \alpha+1)$ $(\alpha \in \mathbf{R})$.
- $F^p \mathcal{T}^\alpha$: Harder-Narasimhan filtration of $\mathcal{T}^\alpha$.
- $F^{p+\alpha} \mathcal{T}_1 = F^p \mathcal{T}^\alpha_{z=1}$: *irregular Hodge filtration* on $\mathcal{T}_1$.
- Notion of pVITS .

Polarizable variation of integrable twistor structure

Origin: Simpson, CS, Biquard-Boalch, T. Mochizuki.

Theorem.

(1) *Assume that (V, ∇) is irreducible. Then $(V, \nabla)^{\mathrm{an}}$ underlies a unique (up to obvious ambiguity) structure of a pVTS.*

(2) *If* (V,∇) *is moreover rigid, then this pVTS is a pVITS iff the local formal eigenvalues of the monodromies have absolute value equal to* 1.

(3) *In such a case, it can be equipped with a canonical irregular Hodge filtration, which is unique up to a shift.*

Remark. The first point is already interesting if (V,∇) is tame, since it does not assume that the eigenvalues of monodromies have absolute value equal to one.

4. The Katz-Arinkin-Deligne algorithm

***Theorem* (Katz, Deligne, Arinkin).** *Let* U *be a Zariski open set in* $\mathbb{P}^1$ *and let* (V,∇) *be a rigid irreducible* $\mathscr{O}_U$*-module with connection. Assume that* $\operatorname{rk} V \geqslant 2$. *Then, after tensoring* (V,∇) *by a suitable rank-one* $\mathscr{O}_U$*-module with connection and choosing charts so that* $\mathbb{P}^1 = \mathbb{A}^1_t \cup \{\infty\}$ *in a suitable way, one of both possibilities occurs, and the first one always if* (V,∇) *is tame:*

(1) *There exists* $\chi \in \mathbf{C}^*$ *such that* $\operatorname{rk}\operatorname{MC}_\chi(V,\nabla) < \operatorname{rk} V$,

(2) $1 \leqslant \operatorname{rk} {}^F(i_{!*}V) < \operatorname{rk} V$, $i : U \hookrightarrow \mathbb{A}^1$.

***Theorem* (Katz, Bloch-Esnault).** *Let* (V,∇) *be an irreducible* $\mathscr{O}_U$*-module with connection. Then* (V,∇) *is rigid iff* ${}^F(i_{!*}V)_{|U'}$ *is rigid, iff* $\operatorname{rk}\operatorname{MC}_\chi(V,\nabla)$ *is rigid for some (any)* $\chi \in \mathbf{C}^*$.

$\rightsquigarrow$ Algorithm starting from a rank-one object.

Computation of Hodge numbers. One defines a set of numerical Hodge data attached to a pVHS on U: rank and degrees of Hodge bundles, Hodge numbers of vanishing cycles at the singular points.

***Theorem* (Dettweiler-CS, N. Martin).** *Formulas for the behaviour of the numerical Hodge data by* MC_χ *and tensor product by a rank-one local system (endowed with its trivial pVHS of weight* 0*).*

$\rightsquigarrow$ Explicit computations via the Katz algorithm. Assume $\alpha_i, \beta_j \in [0,1)$ and that the sequences $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are increasing.

***Theorem* (Fedorov).** *Assume moreover that* $\boldsymbol{\alpha}, \boldsymbol{\beta}$ *have the same size* n *(tame case). Then the jumps of the Hodge filtration occur at the numbers* $p = \#\{j \mid \beta_j < \alpha_k\} - k$ *when* k *varies from* 1 *to* n, *and the size of the jump at such a* p *is*

$$\boxed{\operatorname{rk}\operatorname{gr}_F^p H(\boldsymbol{\alpha},\boldsymbol{\beta},\gamma) = \#\big\{k \mid \#\{j \mid \beta_j < \alpha_k\} - k = p\big\}}$$

Remark. Beukers-Heckman already have a condition in order that $H(\boldsymbol{\alpha},\boldsymbol{\beta},\gamma)$ is unitary, that is, the Hodge filtration has only one jump. This coincides with the previous statement for one jump, that is, the sequences are like $\alpha_1 < \beta_1 < \cdots < \alpha_n < \beta_n$.

On the other hand, if the sequences are not interlaced at all, e.g. $\alpha_n < \beta_1$, then the Hodge ranks are all equal to one.

We now relax the condition that $\boldsymbol{\alpha}, \boldsymbol{\beta}$ have the same size.

***Theorem* (Castaño Domínguez-Sevenheck, CS-Yu).** *Assume* $\alpha_i, \beta_j \in [0,1)$ $n > m$. *Set* $\mu = n - m$. *Then the jumps of the Hodge filtration occur at the* real *numbers* $p = \#\{j \mid \beta_j < \alpha_k\} - k + \mu\alpha_k$ *when* k *varies from* 1 *to* n. *Moreover, the size of the jump at such a* p *is*

$$\boxed{\operatorname{rk}\operatorname{gr}_F^p H(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma) = \#\bigl\{k \mid \#\{j \mid \beta_j < \alpha_k\} = p + k - \mu\alpha_k\bigr\}}$$

No explicit computation of the irregular Hodge numbers via the Arinkin-Deligne algorithm. Nevertheless, one can use an interpretation due to Katz of a confluent hypergeometric equation as a Fourier transform, up to ramification, of a regular hypergeometric equation, and reduce to the case considered by Fedorov.

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