AN EXTENSION OF HODGE THEORY

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Claude Sabbah

Introduction

In order to give some information on the fundamental group of smooth complex projective varieties, one consider the linear representations of this group: $\rho : \pi_1(X, \star) \to \mathrm{GL}(d, \mathbb{C})$.

There is a "base-point-free" approach to such question: replace the linear representation ρ by the locally constant sheaf $\mathscr{L}_\rho$ of rank d that it defines (in the same way one uses to replace subgroups of $\pi_1(X, \star)$ by coverings of X).

So I will mainly consider locally constant sheaves of $\mathbb{C}$-vector spaces on X. Saying that the representation ρ is irreducible (resp. semisimple) is then equivalent to saying that the locally constant sheaf $\mathscr{L}_\rho$ does not possess any nontrivial locally constant subsheaf (resp. is the direct sum of irreducible locally constant sheaves).

It is classical that giving a locally constant sheaf $\mathscr{L}$ is equivalent to giving a holomorphic vector bundle L equipped with a holomorphic connection

$$\nabla : L \longrightarrow \Omega^1_X \otimes_{\mathscr{O}_X} L$$

which is *integrable*, that is, such that its curvature ∇^2 is zero. By GAGA, this is also equivalent to giving an algebraic vector bundle equipped with an algebraic connection (but be careful when using GAGA for the connection).

Example: the constant rank-one local system $\mathbb{C}_X$ corresponds to $\mathscr{O}_X$ equipped with the standard differential $d : \mathscr{O}_X \to \Omega^1_X$.

On the other hand, this is also equivalent to giving a C^∞ vector bundle H on X with an integrable C^∞ connection $D : H \to \mathscr{A}^1_X \otimes H$.

Let X be a smooth complex projective variety and let $\mathscr{L}$ be a locally constant sheaf of $\mathbb{C}$-vector spaces on X (also called a local system of coefficients).

Pull-back

***Theorem 0* (C. Simpson, 1992).** *Let $f : Y \to X$ be a morphism between smooth complex projective varieties. If $\mathscr{L}$ is semisimple, then $f^{-1}\mathscr{L}$ is also semisimple.*

Note that the property for a representation ρ to be irreducible (or semisimple) is a property of the image Γ of ρ. So for instance the theorem is clear if $f_* : \pi_1(Y) \to \pi_1(X)$ is onto, *e.g.* when Y is a general hyperplane section of X. But it is not clear otherwise.

Sketch of proof. The idea is to replace a global property like semisimplicity, which is difficult to manipulate, by a local one. The is a theorem of K. Corlette (1988): A C^∞ vector bundle with flat connection (H, D) corresponds to a semisimple representation of $\pi_1(X, \star)$ iff it admits a *harmonic metric*.

Given a metric, to check that it is harmonic is a local property. However, the existence of such a metric is a global property. The harmonicity property is preserved by pull-back. □

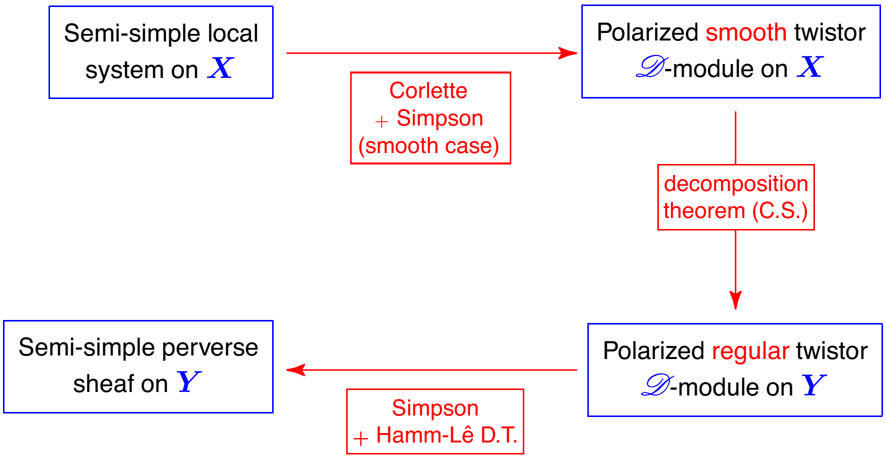
Push-forward. Let now $f : X \to Y$ be a morphism between smooth complex projective varieties. There exists a Zariski dense open set $V \subset Y$ such that $f : U := f^{-1}(V) \to V$ is smooth. For any $y \in V$, we get a local system $\mathscr{L}_y$ on the smooth projective variety $X_y = f^{-1}(y)$. Taking its cohomologies enables one to get various local systems on V (related to the Gauss-Manin connection of f), that is, for a fixed y_o in V, representations of $\pi_1(V, y_o)$ in the vector spaces $H^k(X_{y_o}, \mathscr{L}_{y_o})$, $k = 0, 1, \dots$ I will denote by $R^k f_* \mathscr{L}_U$ the corresponding local system on V.

***Theorem 1*.** *If $\mathscr{L}$ is semisimple on X, then any of the local systems $R^k f_* \mathscr{L}_U$ are semisimple.*

Two kinds of proofs. This theorem has various proofs and leads to various generalizations. The first important case is that of the *constant local system* $\mathbb{C}_X$. The previous statement is then a simplified version of the *Decomposition Theorem* of Beilinson, Bernstein, Deligne and Gabber [**1**]. The first proof of this Decomposition theorem was obtained through characteristic-p methods. Later (1986), a proof using Hodge theory and $\mathscr{D}$-modules was given by M. Saito [**8**].

Similarly, the previous theorem has been given two proofs. The first one, by V. Drinfeld [**3**], goes through characteristic-p methods, and relies on a conjecture by de Jong. That this conjecture is now proved in the generality needed for Drinfeld's argument is not clear for me, but D. Gaitsgory [**4**] on the one hand and Böckle–Khare [**2**] on the other hand, obtained far reaching results in this direction. The second one was obtained in [**7**] by using analytic methods in the spirit of Simpson's proof of the first theorem, as well as an extension of M. Saito's $\mathscr{D}$-module technique.

Sketch of the analytic proof



Proof of Theorem 1

Extension of the statement. The theorem can be extended to the case where one only assumes that X is smooth and quasiprojective. The vector spaces $H^k(X_{y_o}, \mathscr{L}_{y_o})$ have then to be replaced by the spaces $IH^k(\overline{X}_{y_o}, \mathscr{L}_{y_o})$, where $\overline{f} : \overline{X} \to Y$ is some projectivization of $f : X \to Y$, and IH denotes the Goresky-MacPherson Intersection cohomology (with coefficients in the local system $\mathscr{L}_{y_o}$). This extension of the theorem was completely taken into account in B-B-D-G and M. Saito's approaches when $\mathscr{L}$ is the constant local system (*i.e.* the trivial rank-one representation of the fundamental group of X). It is also taken into account in Drinfeld's approach. On the other hand, the analytic approach "à la Simpson" needs further highly non trivial improvements to get the general case. These were obtained, approximately at the same time I got the previous theorem, by T. Mochizuki [**6**].

The final result is expressed in the following way (Decomposition Theorem):

***Theorem 2* (V. Drinfeld, T. Mochizuki).** *Let f be a morphism between (smooth) complex projective varieties. If $\mathscr{F}$ is a semisimple perverse sheaf of $\mathbb{C}$-vector spaces on X, then the direct image complex $Rf_*\mathscr{F}$ decomposes as the direct sum of its perverse cohomology sheaves (conveniently shifted):*

$$Rf_*\mathscr{F} \simeq \bigoplus_k {}^p\!R^k f_*\mathscr{F}[-k]$$

and each perverse cohomology sheaf ${}^p\!R^k f_*\mathscr{F}$ is a semisimple perverse sheaf on Y.

This theorem was conjectured by M. Kashiwara (approximately in 1998, [**5**]), inspired by the theorem of C. Simpson stated at the beginning.

What is a semisimple perverse sheaf?

A semisimple perverse sheaf is the direct sum of simple perverse sheaves. A simple perverse sheaf on $X$is obtained by the following recipe:

(1) Take an irreducible closed subvariety Z of X.

(2) Take a smooth Zariski dense open set Z^o of Z.

(3) Take an irreducible linear representation of the fundamental group of Z^o, that is, an irreducible local system $\mathscr{L}$ on Z^o.

(4) Take the Goresky-MacPherson Intersection complex on Z with coefficients in $\mathscr{L}$. This is a simple perverse sheaf.

In order to prove the theorem, there are three locks to unlock:

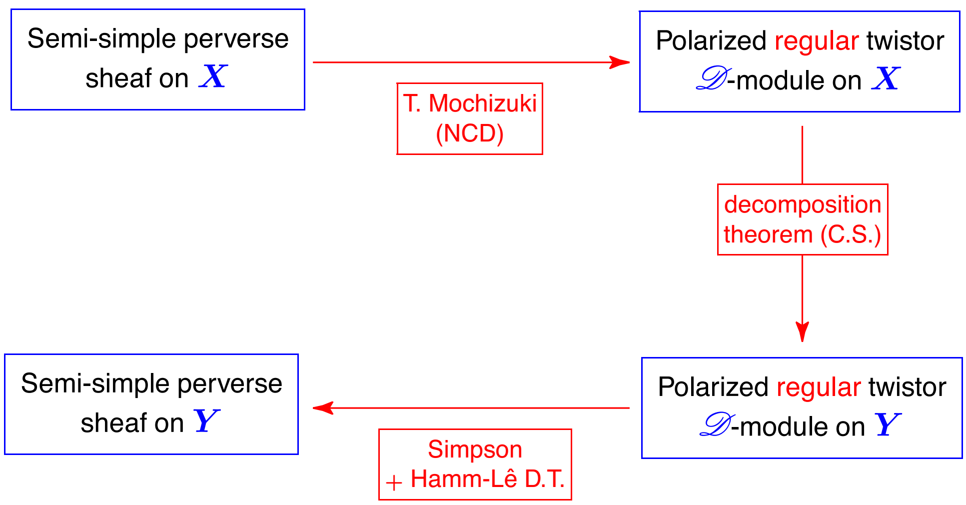
(1) To get the decomposition of the complex $Rf_*\mathscr{F}$ as the direct sum of its perverse cohomology subcomplexes: this uses an argument of Deligne, (Hard Lefschetz Theorem) going back to 1968.

(2) For any perverse cohomology complex $\mathscr{G}$, to get the decomposition with respect to the irreducible components of the support. This is a geometric statement, which uses the analysis of vanishing cycles. It is obtained simultaneously with the property that $\mathscr{G}$ is the "intermediate extension" of its restriction to a smooth Zariski dense open subset of its support.

(3) To show the semisimplicity of the representation on this Zariski open set.

All three points are shown simultaneously, and are interdependent, while being of a different nature.

Sketch of the analytic proof



Proof of Theorem 2

The conjecture of Kashiwara. The original conjecture of Kashiwara is much more ambitious than what has been already proved. It is stated at the level of $\mathscr{D}$-modules.

***Conjecture* (M. Kashiwara, [5]).** *Let $f : X \to Y$ be a morphism between smooth complex projective varieties. If $\mathscr{M}$ is any semisimple holonomic $\mathscr{D}_X$-module, then the direct image complex $f_+\mathscr{M}$ decomposes as the direct sum of its cohomology modules $\mathscr{H}^k f_+\mathscr{M}$ (which are known to be holonomic $\mathscr{D}_Y$-modules) and each $\mathscr{H}^k f_+\mathscr{M}$ is a semisimple holonomic $\mathscr{D}_Y$-module.*

The point in this conjecture is to be able to treat holonomic $\mathscr{D}$-modules with irregular singularities, going beyond the Riemann-Hilbert correspondence

$$\text{Regular holonomic } \mathscr{D}\text{-modules} \longleftrightarrow \text{Perverse sheaves}$$

Applications. At the moment, there are no direct applications of the decomposition theorem for semisimple perverse sheaves (or holonomic $\mathscr{D}$-modules). For instance, we are lacking numerical invariants to get numerical consequences (in Hodge theory, one would have the Hodge numbers).

Nevertheless, there are applications of the techniques or of intermediate results. Here are some, which I am aware of.

T. Mochizuki has given restrictions to the fundamental group of quasiprojective varieties analogous to that given for projective varieties by C. Simpson (rigid discrete subgroups of real algebraic groups which are not of "Hodge type" cannot be the fundamental group of a smooth quasiprojective variety).

The dream of understanding the analogies between $\mathscr{D}$-module theory and ℓ-adic perverse sheaves has become more realistic, by better understanding Fourier transform of variations of Hodge structures.

The following is a first step toward an analogue of results of Katz and Laumon on the Fourier-Deligne transform of ℓ-adic perverse sheaves.

***Theorem 3*.** *The Fourier-Laplace transform of a variation of polarized Hodge structure is an integrable variation of polarized twistor structure.*

Similar objects were considered by physicists (Cecotti and Vafa) at the beginning of the 90'. They were introduce the expression 'topological-antitopological fusion'. Mathematical proofs of some of there results are now possible by using the tools developed for proving (in the analytic way, and in the case of regular singularities, *i.e.* for perverse sheaves) the conjecture of Kashiwara.

Variation of twistor structure after C. Simpson

Hodge structure. H is a $\mathbb{C}$-vector space equipped with a decomposition $H = \oplus_p H^{p,w-p}$. It is equipped with a sesquilinear pairing $k : H \otimes_{\mathbb{C}} \overline{H} \to \mathbb{C}$ for which the decomposition is orthogonal and $(-1)^p i^{-w} k$ is a metric (*i.e.* positive definite) on $H^{p,w-p}$. In particular, k is nondegenerate. We set $h = \oplus_p (-1)^p i^{-w} k$ and

$$F'^p = \bigoplus_{q\geqslant p} H^{q,w-q} \quad \text{and} \quad F''^p = \bigoplus_{q\geqslant p} H^{w-q,q}.$$

Twistor structures.

Hodge structure	Twistor structure
Filtered vect. sp. $(H, F'^\bullet H, F''^\bullet H)$	Holom. vect. bundle on $\mathbb{P}^1$
Conjugation $H \to \overline{H}$	Twistor conjugation $\mathscr{H} \to \overline{\mathscr{H}} := \sigma^* \overline{\mathscr{H}}$
	$\sigma : z \mapsto -1/\overline{z}$
Pure Hodge structure of weight $w = 0$	$\mathscr{H} \simeq \mathscr{O}_{\mathbb{P}^1}^d$
Underlying vector space H	$\Gamma(\mathbb{P}^1, \mathscr{H})$
Nondeg. sesqu. pairing $k : H \simeq H^*$	$\mathscr{H} \simeq \mathscr{H}^* := \overline{\mathscr{H}}^\vee$
	$(\Rightarrow\ \Gamma(\mathbb{P}^1, \mathscr{H}) \simeq \Gamma(\mathbb{P}^1, \mathscr{H})^*)$
Positivity of h	Positivity on $\Gamma(\mathbb{P}^1, \mathscr{H})$
Tate twist (ℓ), $\ell \in \mathbb{Z}$	$\otimes\, \mathscr{O}_{\mathbb{P}^1}(2\ell)$, $\ell \in \frac{1}{2}\mathbb{Z}$.

Triples. We set $\mathbf{S} = \{z \mid |z| = 1\}$.

Hodge structure	Twistor structure
Filtered vect. sp. $(H, F'^\bullet H, F''^\bullet H)$	Holom. vect. bundles $\mathscr{H}', \mathscr{H}''$ on $\mathbb{A}^1$
	plus a sesq. pairing $C : \mathscr{H}'_{\vert\mathbf{S}} \otimes_{\mathscr{O}_\mathbf{S}} \overline{\mathscr{H}''_{\vert\mathbf{S}}} \to \mathscr{O}_\mathbf{S}$
Adjoint H^*	Twistor adjoint $(\mathscr{H}', \mathscr{H}'', C)^* = (\mathscr{H}'', \mathscr{H}', C^*)$
	$C^*(v, \overline{u}) \overset{\text{def}}{=} \overline{C(u, \overline{v})}$
Nondeg. sesqu. form $H \simeq H^*$	$(\mathscr{H}', \mathscr{H}'', C) \simeq (\mathscr{H}', \mathscr{H}'', C)^*$
Tate twist (ℓ), $\ell \in \mathbb{Z}$	$(\mathscr{H}', \mathscr{H}'', C)(\ell) = (\mathscr{H}', \mathscr{H}'', (iz)^{-2\ell} C)$, $\ell \in \frac{1}{2}\mathbb{Z}$.

Example. To any Hodge structure we associate a twistor structure in the following way:

$$\mathscr{H}'^\vee = \bigoplus_p \overline{F}'^p z^{-p}, \quad \mathscr{H}'' = \bigoplus_p F''^p z^{-p}$$

and C is induced by the natural $\mathbb{C}$-duality pairing $\langle\,,\rangle$:

$$\begin{aligned}(\overline{H}^{p,w-p})^\vee z^p \otimes \overline{H}^{p,w-p} z^{w-p} &\longrightarrow z^w\mathbb{C}[z]\\ x^\vee z^p \otimes y z^{w-p} &\longmapsto \langle x^\vee, y\rangle z^w.\end{aligned}$$

Variation of twistor structures. Let X be a complex manifold. We set $\mathscr{X} = X \times \mathbb{A}^1$. The twistor conjugation will now be the ordinary conjugation on X and the twistor conjugation with respect to z. We introduce triples $(\mathscr{H}', \mathscr{H}'', C)$, where $\mathscr{H}', \mathscr{H}''$ are holomorphic bundle on $X \times \mathbb{A}^1$, and C is a "gluing":

$$C : \mathscr{H}'_{|X\times\mathbf{S}} \otimes_{\mathscr{O}_\mathbf{S}} \overline{\mathscr{H}''_{|X\times\mathbf{S}}} \longrightarrow \mathscr{C}^{\infty,\mathrm{an}}_{X\times\mathbf{S}}.$$

We assume that we have *flat* relative connections ∇', ∇'':

$$\mathscr{H}'^{(\prime\prime)} \longrightarrow \frac{1}{z}\Omega^1_{\mathscr{X}/\mathbb{A}^1} \otimes \mathscr{H}'^{(\prime\prime)}$$

which are compatible with the pairing:

$$d'_X C(u,\overline{v}) = C(\nabla' u, \overline{v}), \quad d''_X C(u,\overline{v}) = C(u, \overline{\nabla'' v}).$$

Adjunction and Tate twist are similar to the case $X = \mathrm{pt}$.

A Hermitian pairing in weight 0 is an isomorphism $(\mathscr{H}', \mathscr{H}'', C) \simeq (\mathscr{H}', \mathscr{H}'', C)^*$. It is a *polarization* if, when restricted to any $x \in X$, it gives a polarization of the corresponding twistor structure.

***Lemma 4* (C. Simpson).** *Variations of polarized twistor structures of weight* 0
$\overset{z=1}{\longleftrightarrow}$ *holom. vector bundle on X with flat connection ∇ and Hermitian metric h which is* harmonic
$\overset{z=0}{\longleftrightarrow}$ *holom. vector bundle on X with a Higgs field θ, and Hermitian metric h which is* harmonic.

In fact, if X is compact Kähler, then, according to a result of K. Corlette, the category for $z = 1$ is equivalent to that of semisimple representations of the fundamental group of X. Similarly, according to a result of C. Simpson, the category for $z = 0$ is equivalent to that of polystable Higgs bundles with vanishing Chern classes.

Explanation of Theorem 3

On the punctured affine line $\mathbb{A}^1 \smallsetminus \{p_1, \dots, p_r\}$ with complex coordinate t we consider a variation of polarized Hodge structure (of weight $w = 0$, say). We associate to it a variation of polarized twistor structure $(\mathscr{H}', \mathscr{H}'', C)$ of weight 0.

Exponential twist. We add a new variable τ and we want to apply Fourier-Laplace transform with respect to the kernel $e^{-t\tau}$. This operation is possible at the level of variations of twistor structures.

Start with $(\mathscr{H}',\mathscr{H}'',C)$ and add a new variable τ trivially (*i.e.* consider the pull-back of $(\mathscr{H}',\mathscr{H}'',C)$ by the projection $(t,\tau)\mapsto t$) to get $(\mathscr{H}'[\tau],\mathscr{H}''[\tau],C)$ which is a variation of twistor structure on $(\mathbb{A}^1\smallsetminus\{p_1,\dots,p_r\})\times\mathbb{A}^1$.

The exponential twist $(\mathscr{H}'[\tau],\mathscr{H}''[\tau],C)\otimes e^{-t\tau/z}$ is defined as follows:

- Its first (resp. second) component is $\mathscr{H}'[\tau]$ (resp. $\mathscr{H}''[\tau]$) with connection $\nabla-\dfrac{1}{z}(\tau dt+td\tau)$;
- the sesquilinear pairing is $e^{\overline{t\tau}z-t\tau/z}\cdot C$.

Note that, when $z\in\mathbf{S}$, then $z=1/\overline{z}$ and $e^{\overline{t\tau}z-t\tau/z}=e^{-2i\operatorname{Im}(t\tau/z)}$, hence we are applying a Fourier kernel to C and a Laplace kernel to $\mathscr{H}',\mathscr{H}''$.

Integration. The next step is to integrate $(\mathscr{H}'[\tau],\mathscr{H}''[\tau],C)\otimes e^{t\tau/z}$ along $\mathbb{A}^1\smallsetminus\{p_1,\dots,p_r\}$. The theorem tells us that we get a polarized variation of twistor structure. In general, this is not a variation of Hodge structure, because the underlying connection (by restricting to $z=1$) has in general an irregular singularity at infinity.

References

[1] A.A. Beilinson, J.N. Bernstein & P. Deligne – "Faisceaux pervers", in *Analyse et topologie sur les espaces singuliers*, Astérisque, vol. 100, Société Mathématique de France, 1982, p. 7–171.

[2] G. Böckle & C. Khare – "Mod ℓ representations of arithmetic fundamental groups. II. A conjecture of A.J. de Jong", *Compositio Math.* **142** (2006), no. 2, p. 271–294, arXiv: math.NT/0312490.

[3] V. Drinfeld – "On a conjecture of Kashiwara", *Math. Res. Lett.* **8** (2001), p. 713–728.

[4] D. Gaitsgory – "On de Jong's conjecture", arXiv: math.AG/0402184, 2004.

[5] M. Kashiwara – "Semisimple holonomic $\mathscr{D}$-modules", in *Topological Field Theory, Primitive Forms and Related Topics* (M. Kashiwara, K. Saito, A. Matsuo & I. Satake, eds.), Progress in Math., vol. 160, Birkhäuser, Basel, Boston, 1998, p. 267–271.

[6] T. Mochizuki – *Asymptotic behaviour of tame harmonic bundles and an application to pure twistor D-modules*, vol. 185, Mem. Amer. Math. Soc., no. 869-870, American Mathematical Society, Providence, RI, 2007.

[7] C. Sabbah – *Polarizable twistor $\mathscr{D}$-modules*, Astérisque, vol. 300, Société Mathématique de France, Paris, 2005.

[8] M. Saito – "Modules de Hodge polarisables", *Publ. RIMS, Kyoto Univ.* **24** (1988), p. 849–995.

C. Sabbah, UMR 7640 du CNRS, Centre de Mathématiques Laurent Schwartz, École polytechnique, F–91128 Palaiseau cedex, France • *E-mail :* sabbah@math.polytechnique.fr
Url : http://www.math.polytechnique.fr/~sabbah