
THE WORK OF ANDREY BOLIBRUKH ON ISOMONODROMIC DEFORMATIONS

by

Claude Sabbah

Abstract. We give a description of the work of Andrey Bolibrukh on isomonodromic deformations and relate it to existing results in this domain.

Résumé (Les travaux d'Andrei Bolibroukh sur les déformations isomonodromiques)

Nous décrivons les travaux d'Andrei Bolibroukh sur les déformations isomonodromiques en les situant dans le contexte des résultats existant dans ce domaine.

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Introduction

Let me begin by quoting [8]:

“Therefore, in essence, the invariant geometric language of vector bundles is unavoidable for the rigorous analysis of the inverse monodromy problem and isomonodromy deformations in the case of general linear systems. At the same time, for specific linear systems related to the Painlevé equations, it is possible to perform a rigorous study of the inverse problem on the basis of analytic considerations only.”

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This phrase illustrates the approach of Andrey to differential equations: he was able to speak the language of vector bundles with algebraic/differential geometers and the classical language with analysts. His work on isomonodromy problems was much influenced by the algebraic geometry approach, through Malgrange's papers, but he was also much involved in applications to Painlevé equations from a more analytic and concrete point of view, working on explicit formulas, as in [8]. For instance, he worked on an algorithmic computation of the τ function of Miwa and Jimbo in [7].

In this article, we try to explain his results concerning isomonodromic deformations of systems with regular singularities or Fuchsian systems⁽¹⁾ and to relate them to other results in this domain. When necessary, we will use the language of vector bundles with connection and, in any case, we try to give translations of the results in this language.

1. What is an isomonodromic deformation?

Let X_t be a holomorphic family of connected complex manifolds parametrized by a complex connected manifold T with base point t^o , which have constant fundamental group $\pi_1(X_t, *)$. Let (E_{t^o}, ∇_{t^o}) be a vector bundle on X_{t^o} equipped with a flat holomorphic connection $\nabla_{t^o} : E_{t^o} \rightarrow \Omega_{X_{t^o}}^1 \otimes_{\mathcal{O}_{X_{t^o}}} E_{t^o}$.

An isomonodromic deformation of (E_{t^o}, ∇_{t^o}) is a holomorphic family E_t of holomorphic vector bundles equipped with a *flat* connection $\nabla_t : E_t \rightarrow \Omega_{X_t}^1 \otimes_{\mathcal{O}_{X_t}} E_t$ such that the conjugation class of the monodromy representation defined by horizontal sections of ∇_t is constant.

Such a situation often occurs in the following way. We start with a complex manifold \bar{X} with a smooth divisor Y and a holomorphic map $\pi : \bar{X} \rightarrow T$ which is assumed to be smooth on the pair (\bar{X}, Y) and therefore defines a C^∞ fibration. We put $X = \bar{X} \setminus Y$ and π still defines a C^∞ fibration on X , so that all fibres X_t have the same topological type (in fact, the topological type of the pair (\bar{X}_t, Y_t) is constant). Assume that we have a holomorphic vector bundle E on \bar{X} and a *flat meromorphic* connection ∇ on E with poles along Y . The π_1 of the fibres X_t is constant, and the monodromy representation of $\pi_1(X_t)$ defined by each ∇_t on $E|_{X_t}$ is constant up to conjugation.

Example 1.1. We will mainly consider below the case where T is the (universal cover of the) n -fold product $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ minus diagonals (a point in $a \in T$ is an ordered set of n distinct points a_1, \dots, a_n of \mathbb{P}^1), $\bar{X} = \mathbb{P}^1 \times T$ and $Y = \cup_{i=1}^n Y_i$, where $Y_i = \{x = a_i\}$ if x is the point in the first \mathbb{P}^1 . Then X is the $n+1$ -fold product $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ minus diagonals and the fibre X_a is $\mathbb{P}^1 \setminus \{a_1, \dots, a_n\}$.

⁽¹⁾this explains why we do not consider his joint article [8], which is concerned with a special example of irregular isomonodromic deformation and would necessitate the introduction of many other notions.

We will also consider the case where $\overline{X} = D \times T$, where D is a disc centered at 0 in \mathbb{C} , and $Y = \{0\} \times T$.

One has to be careful that the integrability property above (flatness of ∇) may be strictly stronger than isomonodromy, if one does not impose a supplementary condition, namely that ∇ has *regular singularity* along Y . In practice however, various authors use the word “isomonodromy” instead of “integrability” when irregular singularities occur. This is justified by an extension of what one calls “monodromy representation”: in the irregular singular case, one adds to the classical monodromy representation the Stokes data.

In [5, 3], Andrey considered the question of comparing precisely these two notions, namely isomonodromy and integrability. Although he only considers the example above, his results apply in a much more general situation.

Theorem 1.2 (mainly in [3] and [5]). *Let $\pi : \overline{X} \rightarrow T$ and $Y \subset \overline{X}$ be as above. Assume that the π_1 of some (or any) fibre X_t is finitely generated.*

(1) *Let $(E_{t^\circ}, \nabla_{t^\circ})$ be a vector bundle on \overline{X}_{t° equipped with a meromorphic flat connection with poles along Y_{t° , having regular singularities along Y_{t° . Then any isomonodromic deformation of $(E_{t^\circ}|_{X_{t^\circ}}, \nabla_{t^\circ})$ in the family $\pi : X \rightarrow T$ can be realized, locally near t° , by a meromorphic bundle $E'(*Y)$ equipped with a flat connection ∇' with regular singularities along Y , such that $(E'_{t^\circ}(*Y_{t^\circ}), \nabla'_{t^\circ}) \xrightarrow{\sim} (E_{t^\circ}(*Y_{t^\circ}), \nabla_{t^\circ})$.*

(2) *Let $(E, \nabla_{\overline{X}/T})$ be a vector bundle on \overline{X} equipped with a meromorphic relative flat connection $\nabla_{\overline{X}/T}$ with poles along Y , defining an isomonodromic deformation on X and such that each (E_t, ∇_t) has regular singularities along Y_t . Then, locally on T , there exists a meromorphic connection ∇ on E with poles on Y , having $\nabla_{\overline{X}/T}$ as associated relative connection, and with regular singularities along Y .*

Remarks 1.3

(1) Of course, the assumption on the fundamental group of fibres is satisfied in all usual examples.

(2) Assume that each fibre X_t is a curve. If we moreover assume that $(E_{t^\circ}, \nabla_{t^\circ})$ is *Fuchsian* (i.e., has only simple poles at each point of Y_{t°), then there exists locally on T a unique isomonodromic deformation (E, ∇) where ∇ has logarithmic poles along Y (cf. [11] or Theorem 3.1 below). There may exist other isomonodromic deformations. By the theorem, these deformations can be searched as meromorphic connections with regular singularities along Y . This will be used in § 2.

(3) Let us explain the difference between the two statements. In the second one, we fix a meromorphic structure of the bundle along Y and, knowing that the relative connection is meromorphic with respect to it, we show that the absolute connection is also meromorphic with respect to this structure. In the first one, such a structure is constructed simultaneously with the absolute connection, in such a way that the latter is meromorphic with respect to the former.

Proof. For the first part, the proof has 2 steps: the smooth step, where one forgets about the polar locus Y and the meromorphic step, where one shows that ∇ can be chosen to be meromorphic along Y .

Proof of 1.2(1), first step. Consider a holomorphic vector bundle E on X equipped with a flat relative connection $\nabla_{X/T} : E \rightarrow \Omega_{X/T}^1 \otimes_{\mathcal{O}_X} E$. By the Cauchy-Kowalevski theorem with parameters, $\text{Ker } \nabla_{X/T}$ is a locally constant sheaf of locally free $\pi^{-1}\mathcal{O}_T$ -modules (cf. [10, Th. 2.23]) and $(E, \nabla_{X/T}) \xrightarrow{\sim} (\mathcal{O}_X \otimes_{\pi^{-1}\mathcal{O}_T} \text{Ker } \nabla_{X/T}, d_{X/T} \otimes \text{Id})$. Under the isomonodromy condition, we want to show that there exists, locally with respect to T , a locally constant sheaf \mathcal{F} of finite dimensional \mathbb{C} -vector spaces on X such that $\text{Ker } \nabla_{X/T} = \pi^{-1}\mathcal{O}_T \otimes_{\mathbb{C}} \mathcal{F}$. We will then define ∇ so that $(E, \nabla) \xrightarrow{\sim} (\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{F}, d_X \otimes \text{Id})$, and by definition we will have $(E_{t^\circ}, \nabla_{t^\circ}) \xrightarrow{\sim} (\mathcal{O}_{X_{t^\circ}} \otimes_{\mathbb{C}} \mathcal{F}_{|X_{t^\circ}}, d)$.

We fix $t^\circ \in T$ and work in a neighbourhood of t° . The fundamental group $\pi_1(X_{t^\circ}, *)$ is generated by loops $\gamma_1, \dots, \gamma_p$. A linear representation of $\pi_1(X_{t^\circ}, *)$ in $\text{GL}_d(\mathbb{C})$ consists of the datum of p invertible matrices M_1, \dots, M_p which satisfy the same relations as the γ_i do. The set Rep of these is therefore the closed algebraic subset of $(\text{GL}_d(\mathbb{C}))^p$ defined by algebraic equations of the form $M_1^{n_1} \cdots M_p^{n_p} - \text{Id} = 0$. The group $\text{GL}_d(\mathbb{C})$ acts on $(\text{GL}_d(\mathbb{C}))^p$ by $P \cdot (M_1, \dots, M_p) = (PM_1P^{-1}, \dots, PM_pP^{-1})$ and leaves Rep invariant. The orbit of a given representation ρ° consists of the representations which are conjugate to ρ° .

The assumption of the theorem shows that there exists a neighbourhood V of t° in T and a holomorphic map $V \rightarrow (\text{GL}_d(\mathbb{C}))^p$, sending t° to ρ° , such that its image is contained in the orbit of ρ° . As the natural map $\text{GL}_d(\mathbb{C}) \rightarrow \text{GL}_d(\mathbb{C}) \cdot \rho^\circ$ has everywhere maximal rank, one can locally lift $V \rightarrow \text{GL}_d(\mathbb{C}) \cdot \rho^\circ$ to a holomorphic map $V \rightarrow \text{GL}_d(\mathbb{C})$. The holomorphic family ρ_t of representations of $\pi_1(X_{t^\circ}, *)$ is therefore conjugate to the constant family ρ_\circ .

Proof of 1.2(1), second step. By [10], the bundle $\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{F}$ with its flat connection d extends as a meromorphic bundle $E'(*Y)$ with a connection ∇' having regular singularities along Y . The isomorphism $(E_{t^\circ}, \nabla_{t^\circ}) \xrightarrow{\sim} (E'_{|X_{t^\circ}}, \nabla'_{t^\circ})$ that we constructed in the first step is a relative horizontal section σ on X_{t° of the meromorphic bundle $\mathcal{H}om_{\mathcal{O}_{\overline{X}_{t^\circ}}(*Y_{t^\circ})}(E_{t^\circ}(*Y_{t^\circ}), E'_{t^\circ}(*Y_{t^\circ}))$. The connection on this bundle (obtained from ∇_{t° and ∇'_{t°) has a regular singularity along Y_{t° , hence the section σ is meromorphic along Y_{t° . A similar argument applies to σ^{-1} , so that we have an isomorphism $(E_{t^\circ}(*Y_{t^\circ}), \nabla_{t^\circ}) \xrightarrow{\sim} (E'_{|\overline{X}_{t^\circ}}(*Y_{t^\circ}), \nabla'_{t^\circ})$.

Proof of 1.2(2). The proof is similar to that of 1.2(1). We first construct $(E'(*Y), \nabla')$ as above. We now have an isomorphism $(E_{|X}, \nabla) \xrightarrow{\sim} (E'_{|X}, \nabla')$ constructed in the first step: it is a horizontal section σ on X of the meromorphic bundle $\mathcal{H}om_{\mathcal{O}_{\overline{X}}(*Y)}(E(*Y), E'(*Y))$. Restricting to each fibre \overline{X}_t , the connection on this bundle (obtained from $\nabla_{\overline{X}/T}$ and $\nabla'_{\overline{X}/T}$) has a regular singularity along Y_t , hence the section $\sigma_{|X_t}$ is meromorphic along Y_t . The order of its pole is locally bounded by a

constant computed from the matrices of $\nabla_{\overline{X}/T}$ and $\nabla'_{\overline{X}/T}$ in local meromorphic bases of $E(*Y)$ and $E'(*Y)$. Hence σ is meromorphic along Y . A similar argument applies to σ^{-1} . We therefore have an isomorphism $(E(*Y), \nabla_{\overline{X}/T}) \xrightarrow{\sim} (E'(*Y), \nabla'_{\overline{X}/T})$. \square

2. Isomonodromic deformations: the local setting

In this section, we consider a disc D centered at the origin in the complex plane, with coordinate x , and a parameter space T , which is a neighbourhood of the origin in \mathbb{C}^n , with coordinates $t = (t_1, \dots, t_n)$.

We consider a linear differential system of size d in the variable x , which is Fuchsian, that is

$$(*) \quad x \cdot \frac{du}{dx} = A(x) \cdot u(x),$$

where $u(x)$ is a vector of size d of unknown functions, and $A(x)$ is a matrix of size d with holomorphic entries.

In other words, we consider the trivial holomorphic vector bundle E° (free \mathcal{O}_D -module) of rank d on D , with a meromorphic connection $\nabla : E^\circ \rightarrow \Omega_D^1(\log\{0\}) \otimes_{\mathcal{O}_D} E^\circ$ having a pole of order at most one at the origin.

A Fuchsian isomonodromic deformation of $(*)$ parametrized by T is a system

$$(*_t) \quad x \cdot \frac{du(x,t)}{dx} = A(x,t) \cdot u(x,t),$$

such that $A(x,t)$ is holomorphic and that, for any $t^\circ \in T$, the monodromy at the origin of the system $(*_{t^\circ})$ is independent of t° (up to conjugation). By Theorem 1.2, the isomonodromy condition can also be stated by saying that there exists a matrix

$$(2.1) \quad \Omega = A(x,t) \frac{dx}{x} + \sum_{i=1}^n \Omega_i(x,t) dt_i,$$

where $\Omega_i(x,t)$ are holomorphic on $D^* \times T$, such that Ω satisfies the integrability condition

$$d\Omega + \Omega \wedge \Omega = 0.$$

We say that the isomonodromic deformation is *regular* if Ω is *meromorphic* along $x = 0$ (equivalently, each Ω_i is so). By Theorem 1.2(2), any Fuchsian isomonodromic deformation is regular.

Equivalently, we are given a trivial vector bundle E on $D \times T$ with an *integrable* meromorphic connection

$$\nabla : E \longrightarrow \Omega_{D \times T}[* (\{0\} \times T)] \otimes_{\mathcal{O}_{D \times T}} E$$

having poles along $x = 0$ at most, and such that its restriction to each $D \times \{t^\circ\}$ is logarithmic. In particular, the meromorphic connection ∇ has *regular singularities* along $x = 0$. But it may or may not be *logarithmic* along $x = 0$, that is, some Ω_i may not

be *holomorphic* at $x = 0$. In the following, we consider only Fuchsian isomonodromic deformations, even if some statements hold in a more general situation.

Proposition 2.2. *In an isomonodromic deformation, the eigenvalues of $A(0, t)$ are independent of t .*

Proof. Indeed, for t fixed, the characteristic polynomial of $A(0, t)$ determines the characteristic polynomial of the monodromy of the corresponding system by the following rule: to any term $(X - \alpha)^{\mu_\alpha}$, associate $(S - e^{-2i\pi\alpha})^{\mu_\alpha}$. As the latter is constant by isomonodromy, the eigenvalues of $A(0, t)$ can only vary by integral jumps, so, by continuity, they are constant. \square

The datum of a Fuchsian system of rank d on the disc D , with pole at 0 only, is equivalent to the datum of a \mathbb{C} -vector space H of dimension d equipped with an automorphism M (monodromy) and a decreasing filtration $F^\bullet H$ stable by M (called the *Levelt filtration*). This filtration takes into account the resonances (nonzero integral differences of eigenvalues) in the matrix of the connection.

Corollary 2.3 ([3, Theorem 2]). *In an isomonodromic deformation, the Levelt normal form can be achieved locally holomorphically with respect to the parameters.*

Proof. According to the previous proposition, there exists, locally with respect to t (say near t^0), a base change such that the matrix $A(0, t)$ is block-diagonal, one (constant) eigenvalue per block. We order the blocks in such a way that the integral parts of eigenvalues are decreasing, hence we get a diagonal matrix $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_d)$ where the integers λ_j satisfy $\lambda_j \geq \lambda_{j+1}$. Put $\lambda = \max_{i,j} |\lambda_i - \lambda_j|$. Then $A_0(t) := A(0, t)$ commutes with Λ . One looks for a formal power series $\widehat{P}(x, t) = \text{Id} + \sum_{k \geq 1} x^k P_k(t)$ and matrices $B_1(t), \dots, B_\lambda(t)$ such that $[\Lambda, B_j(t)] = -jB_j(t)$ ($j = 1, \dots, \lambda$) and, putting $B(x, t) = A_0(t) + xB_1(t) + \dots + x^\lambda B_\lambda(t)$, we have

$$x\widehat{P}'(x, t) = \widehat{P}(x, t)B(x, t) - A(x, t)\widehat{P}(x, t).$$

The formal solution P is obtained by solving successively, for $k \geq 1$,

$$(\text{ad } A_0(t) + k \text{Id})P_k(t) = B_k(t) + \Phi_k(A_{\leq k}(t), B_{< k}(t), P_{< k}(t))$$

where Φ_k depends on the previous coefficients. As $A_0(t)$ commutes with Λ , we can decompose this equation on the eigenspaces of the semisimple endomorphism $\text{ad } \Lambda$. For the eigenvalue μ , we have then to solve, for $k \geq 1$,

$$(\text{ad } A_0(t) + k \text{Id})P_k^{(\mu)}(t) = \begin{cases} B_k(t) + \Phi_k^{(\mu)}(A_{\leq k}(t), B_{< k}(t), P_{< k}(t)) & \text{if } \mu + k = 0, \\ \Phi_k^{(\mu)}(A_{\leq k}(t), B_{< k}(t), P_{< k}(t)) & \text{if } \mu + k \neq 0, \end{cases}$$

by assumption on B_k . If $k \neq -\mu$, the endomorphism $(\text{ad } A_0(t) + k \text{Id})$ is invertible on $\text{Ker}(\text{ad } \Lambda - \mu \text{Id})$, hence we can solve in a unique way the second line. If $k = -\mu$, we choose $B_k(t)$ so that the right-hand term is in the image of $(\text{ad } A_0(t) + k \text{Id})$.

Then, by a standard argument for regular singularities, one shows that the matrix \widehat{P} is convergent in some neighbourhood of $(0, t^o)$ and we denote it by P . Therefore, after the base change given by P , the matrix of the connection can be written as

$$(2.4) \quad \Omega' = (A_0(t) + xB_1(t) + \dots + x^\lambda B_\lambda(t)) \frac{dx}{x} + \sum_{i=1}^n \Omega'_i(x, t) dt_i$$

with $\Omega'_i(x, t)$ meromorphic and having a pole of order less than or equal to that of Ω_i along $x = 0$, as the base change is holomorphically invertible. \square

In terms of filtrations, this result means that the family $F_t^\bullet H$ of filtrations of H parametrized by T is holomorphic, *i.e.*, defines a filtration of the bundle $\mathcal{O}_T \otimes_{\mathbb{C}} H$ by holomorphic subbundles, in such a way that the graded pieces are vector bundles (*i.e.*, the rank does not jump with t).

Corollary 2.5 ([3, Theorem 3]). *In an isomonodromic deformation, the pole of each matrix Ω_i along $x = 0$ has order at most λ .*

Proof. By Corollary 2.3, we can assume that we start with a matrix Ω as in (2.1) such that $A(x, t)$ has the Levelt normal form (2.4). The eigenvalues of $A(0, t) - \Lambda$ do not differ by a nonzero integer and the monodromy matrix is $\exp -2i\pi(A(0, t) - \Lambda)$. Then there exists a holomorphic invertible matrix $C(t)$ such that $\exp -2i\pi(A(0, t) - \Lambda) = C(t)^{-1} \cdot \exp -2i\pi(A(0, 0) - \Lambda) \cdot C(t)$. Therefore, after the base change of matrix $C(t)$, the connection can be written as $d + (A(0, 0) - \Lambda)dx/x$. Putting $P(x, t) = x^\Lambda C(t)$, we therefore have $\Omega_i = x^\Lambda \partial_{t_i} C(t) C(t)^{-1} x^{-\Lambda}$, which has a pole of order $\leq \lambda$ along $x = 0$. \square

This corollary implies, in particular that, under some circumstances, any regular isomonodromic deformation is in fact logarithmic. This occurs for instance when $A(0)$ in (*) is nonresonant, that is, if its eigenvalues do not differ by a nonzero integer.

3. Logarithmic isomonodromic deformations of Fuchsian systems on the Riemann sphere: the Schlesinger system

3.a. The Painlevé property, after Malgrange. Let me recall the proof of the Painlevé property of the Schlesinger system given by Malgrange in [11].

We fix a finite set of distinct points $a^o = \{a_1^o, \dots, a_n^o\}$ in the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ and we consider a vector bundle E^o on \mathbb{P}^1 equipped with a connection ∇^o having logarithmic poles at a^o and no other pole.

Our parameter space T is now global: it is the universal covering of $(\mathbb{P}^1)^n \setminus \text{diagonals}$ (one can reduce the dimension by 3, if we fix by a homography three points among the a_i to 0, 1, ∞ , say). We view a^o as point of $(\mathbb{P}^1)^n \setminus \text{diagonals}$ and we choose a lift \tilde{a}^o of a^o in T . On $\mathbb{P}^1 \times T$ we have natural hypersurfaces Y_i defined by the equation $x = \tilde{a}_i$ (\tilde{a}_i is the lift to T of the function a_i).

Theorem 3.1 ([11]). *There exists a unique vector bundle E on $\mathbb{P}^1 \times T$ equipped with an integrable logarithmic connection ∇ having poles along the hypersurfaces Y_i , and with an identification $(E, \nabla)_{\mathbb{P}^1 \times \{\tilde{a}^\circ\}} \xrightarrow{\sim} (E^\circ, \nabla^\circ)$. \square*

Assume now that E° is trivial. If we fix a basis of this bundle and a coordinate x on $\mathbb{P}^1 \setminus \{\infty\}$, the datum of ∇° is equivalent to the datum of a Fuchsian system

$$(3.2) \quad \frac{du}{dx} = \sum_{i=1}^n \frac{A_i^\circ}{x - a_i^\circ} \cdot u,$$

where A_i° are $d \times d$ matrices. Then there exists a divisor Θ in T consisting of points \tilde{a} where $E_{\mathbb{P}^1 \times \{\tilde{a}\}}$ is not trivial. More precisely, there exists a meromorphic (along $\mathbb{P}^1 \times \Theta$) trivialization of E which extends the trivialization of E° . In other words, there exists a basis of $E(*\Theta)$ extending the given basis of E° . The matrix of ∇ in this basis takes the form

$$\sum_{i=1}^n A_i(\tilde{a}) \frac{d(x - \tilde{a}_i)}{(x - \tilde{a}_i)} + \sum_{i=1}^n B_i(\tilde{a}_i) d\tilde{a}_i.$$

The basis can moreover be chosen in such a way that all the B -terms vanish identically: this is obtained by imposing flatness with respect to the residual connection on Y_n , say. To simplify notation, it is simpler (but not less general) to assume that $Y_n = \{\infty\} \times T$. In such a basis, the matrix of ∇ thus takes the form

$$(3.3) \quad \sum_{i=1}^n A_i(\tilde{a}) \frac{d(x - \tilde{a}_i)}{(x - \tilde{a}_i)}$$

and the integrability property is equivalent to the fact that the A_i are solutions of the *Schlesinger system*

$$(Schl) \quad dA_i = \sum_{j \neq i} [A_i, A_j] \frac{d(\tilde{a}_i - \tilde{a}_j)}{(\tilde{a}_i - \tilde{a}_j)}, \quad i = 1, \dots, n.$$

These equations imply in particular that the residue $-\sum_i A_i(\tilde{a})$ along $\{\infty\} \times T$ is constant (the basis is chosen precisely so that this property is satisfied).

Corollary 3.4. *The solutions of the Schlesinger system (Schl) with initial value A_i° at \tilde{a}° are meromorphic on T with poles along Θ at most. \square*

The behaviour of the solutions to (Schl) near the polar set Θ is hard to analyze. In [4] and [7], Andrey has given a method to produce examples and describe in concrete terms this behaviour. I will try to explain his approach.

3.b. Equation for the “theta divisor”. Starting from a solution of the Schlesinger system with initial values A_i° at \tilde{a}° , we obtain a hypersurface Θ in T , which is the set of points \tilde{a} where the bundle $E_{\mathbb{P}^1 \times \{\tilde{a}\}}$ is not trivial (however its degree remains equal to 0). Let \tilde{a}^* be a point on Θ . We will work locally near \tilde{a}^* , and therefore we will not distinguish between \tilde{a} and $a \in (\mathbb{P}^1)^n \setminus \text{diagonals}$. Moreover, we now forget

about the initial data which have produced this isomonodromic deformation and the corresponding Θ , and we denote by T a small neighbourhood of a^* . We also assume, for convenience, that none of the points a_i ($i = 1, \dots, n$ and $a \in T$) is equal to ∞ (it is enough to assume that this is true for a_i^* and take T small enough).

The bundle E_{a^*} is not trivial. By the Birkhoff-Grothendieck theorem, it is decomposed as a sum of rank-one vector bundles $\oplus_{j=1}^d \mathcal{O}(k_j)$ with some $k_j \neq 0$ and $k_1 \geq \dots \geq k_d$ (and $\deg E_{a^*} = k_1 + \dots + k_d = 0$, so that there exists $\ell, m \in \{1, \dots, d\}$ such that $k_\ell - k_m \geq 2$). The typical example when $d = 2$ is $E_{a^*} = \mathcal{O}(1) \oplus \mathcal{O}(-1)$.

The meromorphic bundle $E[*](\infty \times T)$ is trivializable. It contains E as a holomorphic subbundle. The bundle with connection (E, ∇) can be characterized as the unique extension of $(E[*](\infty \times T), \nabla)$ which is holomorphic at infinity.

On the other hand, there exists a holomorphic subbundle $E_{a^*}^{(0)}$ of the meromorphic bundle $E_{a^*}[*\infty]$ which is trivial and on which the connection ∇ has only logarithmic poles. One can choose $E_{a^*}^{(0)}$ such that, in any trivialization, the matrix of the connection ∇ has residue $-K^{(0)}$ at ∞ , with $K^{(0)} := \text{diag}(k_1^{(0)}, \dots, k_d^{(0)})$ (with $k_i^{(0)} = k_i$): Andrey uses Sauvage Lemma to do so; in terms of vector bundles, remark that there exists a basis of $\mathcal{O}(k_j)[*\infty] = \mathcal{O}[*\infty]$ in which the matrix of the differential d has a pole at ∞ only, which is logarithmic with residue $-k_j$; using the splitting of E_{a^*} given by the Birkhoff-Grothendieck theorem, one gets the desired basis of E_{a^*} .

Denote by $B_i^{(0)}(a^*)$ the residue of the connection ∇ at the pole a_i^* . The matrix of the connection in the chosen basis is then written as

$$\sum_{i=1}^n \frac{B_i^{(0)}(a^*)}{x - a_i^*} dx$$

The point at infinity is an apparent (logarithmic) singularity and we have $\sum_i B_i^{(0)}(a^*) = K^{(0)}$.

Apply now Theorem 3.1 starting with $(E_{a^*}^{(0)}, \nabla)$ to construct a holomorphic subbundle $E^{(0)}$ of $E[*](\infty \times T)$ with a logarithmic connection having poles at $Y_1 \cup \dots \cup Y_n \cup (\{\infty\} \times T)$ and, maybe after taking a smaller T , choose the canonical trivialization by extending flatly the basis along $\{\infty\} \times T$. In this basis of $E^{(0)}$, the matrix of ∇ is written in the form

$$\sum_i B_i^{(0)}(a) \frac{d(x - a_i)}{(x - a_i)}$$

and the $B_i^{(0)}(a)$ satisfy the Schlesinger system.

Lemma 3.5 ([7, Lemma 1]). *There exists $\ell, m \in \{1, \dots, d\}$ such that $k_m^{(0)} - k_\ell^{(0)} \geq 2$ and $i \in \{1, \dots, n\}$ such that the (ℓ, m) -entry $B_{i, \ell m}^{(0)}(a)$ does not vanish identically.*

Proof. Otherwise, the base change near $\{\infty\} \times T$ with matrix $x^{K^{(0)}}$ would simultaneously (with respect to a) eliminate the apparent singularity. We would obtain

the bundle E as a result, as mentioned above. This would mean that E_a has splitting type $k_d^{(0)}, \dots, k_1^{(0)}$ for any $a \in T$. But E_a is known to be trivial for $a \notin \Theta$, a contradiction. \square

Lemma 3.6 ([7, § 2]). *Fix $\ell, m \in \{1, \dots, d\}$ such that $k_m^{(0)} - k_\ell^{(0)} \geq 2$ and $B_{i, \ell m}^{(0)}(a) \neq 0$. Put $\tau^{(0)}(a) = \sum_i B_{i, \ell m}^{(0)}(a)a_i = 0$ and let $\Theta^{(0)}$ be the support of $\text{Div}(\tau^{(0)})$. Then there exists an holomorphic extension $E^{(1)}[*\Theta^{(0)}]$ of $E[* (\infty \times T) \cup \Theta^{(0)}]$ such that, out of $\Theta^{(0)}$,*

- (1) *for any $a \in T \setminus \Theta^{(0)}$, the bundle $E^{(1)}[*\Theta^{(0)}]_a$ is trivial,*
- (2) *the connection ∇ is logarithmic on $E^{(1)}[*\Theta^{(0)}]$ with poles on $Y_1 \cup \dots \cup Y_n \cup (\infty \times T)$ and its residue along $\infty \times T$ is $-K^{(1)} = -\text{diag}(k_1^{(1)}, \dots, k_d^{(1)})$ with*

$$\sum_{j=1}^d (k_j^{(1)})^2 \leq \sum_{j=1}^d (k_j^{(0)})^2 - 2. \quad \square$$

Notice that these lemmas implies that the stratum of Θ consisting of points $a \in \Theta$ where the splitting type of E_a is the same as that of E_{a^*} is defined by the equations $B_{i, \ell m}^{(0)}(a) = 0$ for all $i = 1, \dots, n$ and all pairs ℓ, m with $k_m - k_\ell \geq 2$.

If $K^{(1)} = 0$, then $E^{(1)}[*\Theta^{(0)}]$ coincides with $E[*\Theta^{(0)}]$, by the uniqueness of the extension of $E[*\{\infty\} \times T]$ which is smooth along $\{\infty\} \times T$. We therefore have $\Theta \subset \Theta^{(0)}$.

If $K^{(1)} \neq 0$, we are in the situation of Lemma 3.5, except the fact that all the coefficients are meromorphic along $\Theta^{(0)}$ and maybe not holomorphic. Then, applying Lemmas 3.5, we construct the meromorphic function $\tau^{(1)}$ (with poles on $\Theta^{(0)}$ at most) and we define $\Theta^{(1)}$ as the union of the support of $\text{Div}(\tau^{(1)})$ and $\Theta^{(0)}$. We then construct $E^{(2)}[*\Theta^{(1)}]$, etc.

In a finite number of applications of Lemmas 3.5 and 3.6, we get a divisor $\tilde{\Theta}$ in T and an extension $\tilde{E}[*\tilde{\Theta}]$ of $E[* (\infty \times T) \cup \tilde{\Theta}]$ on which the connection has no pole along $\{\infty\} \times T$ and which is trivial. In particular, it coincides with E out of $\tilde{\Theta}$ and, more precisely, we have $\tilde{E}[*\tilde{\Theta}] = E[*\tilde{\Theta}]$. By definition of Θ , we have the inclusion

$$\Theta \subset \tilde{\Theta}.$$

In some sense, this inductive procedure transfers the apparent polar locus $\{\infty\} \times T$ to a polar locus $\tilde{\Theta}$ contained in T .

We also have a finite sequence of meromorphic functions $\tau^{(0)}, \tau^{(1)}, \dots$ ($\tau^{(0)}$ being holomorphic) and we put $\tilde{\tau} = \prod_\nu \tau^{(\nu)}$.

On the other hand, denote by τ the function of Miwa and Jimbo defining the divisor Θ , after the theorem of Miwa (see [11]). We have near a^*

$$\frac{d\tau}{\tau} = \frac{1}{2} \sum_{i \neq j} \text{tr}(B_i(a)B_j(a)) \frac{d(a_i - a_j)}{(a_i - a_j)}.$$

Theorem 3.7 ([7, § 3]). *The functions τ and $\tilde{\tau}$ define the same divisor.*

Proof. At each step of the previous procedure, the coefficients $B_i^{(\nu)}(a)$ satisfy a Schlesinger system. Therefore, the form

$$\omega^{(\nu)} := \frac{1}{2} \sum_{i \neq j} \operatorname{tr} (B_i^{(\nu)}(a) B_j^{(\nu)}(a)) \frac{d(a_i - a_j)}{(a_i - a_j)}$$

is *closed*. Notice that $\omega^{(0)}$ is holomorphic on T (as we have $a_i \neq a_j$ if $i \neq j$), so that in particular $\omega^{(0)} = d \log h$ for h holomorphic and nonvanishing near a^* , but $\omega^{(\nu)}$ is only meromorphic for $\nu \geq 1$. More precisely we have:

Lemma 3.8 ([7, § 3]). *For any $\nu \geq 1$, we have $\omega^{(\nu)} - \omega^{(\nu-1)} = d \log \tau^{(\nu-1)}$, where $\tau^{(\nu-1)}$ is the equation obtained by the previous procedure when going from the step $\nu - 1$ to the step ν . \square*

At the final step ν_{final} , the form $\omega^{(\nu_{\text{final}})}$ is the form ω_{MJ} of Miwa and Jimbo for the original system (3.3). Putting $\tilde{\tau} = \prod_{\nu=0}^{\nu_{\text{final}}-1} \tau^{(\nu)}$, we find

$$\omega_{\text{MJ}} - \omega^{(0)} = d \log \tilde{\tau}.$$

As we know, by a theorem of Miwa, that ω_{MJ} represents Θ (in the sense of [11, Def. 6.1]), we obtain the equality $\operatorname{Div} \tau = \operatorname{Div} \tilde{\tau}$. \square

Remark 3.9 (effectivity). Although it is in general difficult to compute the functions $B_{i,\ell m}^{(0)}$, and then the functions $\tau^{(\nu)}$, hence the function $\tilde{\tau}$, it is possible, in some examples, to compute the k -jets of these functions for k large enough, and to get information on the geometry of Θ as well as on the order of poles of the solutions of the Schlesinger system.

3.c. The order of the pole along Θ of the solutions to the Schlesinger system. We start again with the situation of § 3.a with a system (3.2). Assume now that the size of the matrices A_i^o is 2 (*i.e.*, $d = 2$ above). We have initial data $(A_i^o, a_i^o)_{i=1, \dots, n}$, and an isomonodromy deformation (3.3) of Schlesinger type (*i.e.*, the matrices A_i satisfy (Schl)) with a corresponding polar divisor Θ for the matrices A_i .

Make moreover the following assumptions:

- (1) The monodromy representation defined by ∇^o on E^o is *irreducible*;
- (2) at a point a^* of Θ , the splitting type of E_{a^*} is $(1, -1)$.

Remark 3.10. As E_{a^*} has degree 0 and is not trivial, its splitting type is $(k, -k)$ with $k \geq 1$. As the monodromy representation is irreducible and as $d = 2$, one has the bound $2k \leq n - 2$. When $n = 4$, Assumption (2) is therefore implied by Assumption (1).

Theorem 3.11 ([4, Theorem 2]). *Under these assumptions, for $i = 1, \dots, n$, the matrix $A_i(a)$ has a pole of order ≤ 2 on Θ near a^* .*

We assume for convenience that none of the numbers a_i^* ($i = 1, \dots, n$) is 0 or ∞ . We denote now by T a neighbourhood of a^* .

By the same⁽²⁾ procedure as in § 3.b, introduce an apparent singularity (now at $x = 0$, not at $x = \infty$) to get a trivial bundle with connection E'_{a^*} .

The procedure described in § 3.b has only one step, because of Assumption (2).

The matrix of ∇ in a basis of E'_{a^*} is written as

$$B'(x) dx = \left(\frac{B'_0}{x} + \sum_{i=1}^n \frac{B'_i}{x - a_i} \right) dx$$

with $B'_0 = \text{diag}(1, -1)$ and $\sum_{i=0}^n B'_i = 0$. Hence the entry $b'_{12}(x)$ of $B'(x)$ is holomorphic and vanishes at $x = 0$.

Lemma 3.12 ([4], p. 68). *Under Assumption (1), the valuation (order of vanishing) of $b'_{12}(x)$ at $x = 0$ is $< n - 1$.*

Proof. Indeed, the coefficient of x^m ($m \geq 1$) in $b'_{12}(x)$ is $-\sum_{i=1}^n b'_{i,12}/a_i^{*(m-1)}$. If the valuation of $b'_{12}(x)$ is $\geq n - 1$, this implies that all $b'_{i,12}$ are zero, and (E'_{a^*}, ∇) is reducible, hence its monodromy too, in contradiction with Assumption (1). \square

Applying the procedure (with one step) described in § 3.b, Andrey computes the equation of $\tilde{\Theta}$ and finds that $\tilde{\Theta}$ is smooth at a^* . This clearly implies that $\Theta = \tilde{\Theta}$.

On the other hand, the original system (3.3) can be obtained by a simple base change from the system obtained after the previous procedure. A detailed computation shows that the original matrices $A_i(a)$ have a pole of order ≤ 2 along Θ .

Remarks 3.13

(1) In [4], there is an explicit example where the order of the pole is 2.

(2) In [6, § 3], Andrey indicates that, without Assertion (2), a result similar to Theorem 3.11 still holds, but the order of the pole is $\leq 2k$, if E_{a^*} has splitting type $(k, -k)$.

4. Isomonodromic confluences

It is well known that a family of linear differential equations of one variable having only regular singularities may acquire, for some values of the parameter, an irregular singularity when various singular points for the generic value of the parameter merge together. In the algebraic or analytic setting, we have a vector bundle E on \overline{X} as in § 1 and we moreover assume that

⁽²⁾However, Theorem 3.11 appeared in a Nice preprint dated July 1995, and the results of § 3.b were obtained later.

- $\pi : \bar{X} \rightarrow T$ is smooth of relative dimension one,
- $\pi : Y \rightarrow T$ is finite (but Y is not necessarily smooth).

Given a relative connection $\nabla_{\bar{X}/T}$ on E , such that a generic fibre ∇_t on E_t has regular singularities on Y_t , it may happen that a special fibre ∇_{t° has an irregular singularity at some point of Y_{t° .

Example 4.1. Let $\bar{X} = \mathbb{C} \times \mathbb{C}$ with coordinates x, t and $Y = \{x^2 - t^2 = 0\}$. Take the trivial rank one bundle on \bar{X} with the relative connection having the matrix $dx/(x^2 - t^2)$. For $t \neq 0$, we have a regular singularity at $x = \pm t$, and, for $t = 0$, we have an irregular singularity at $x = 0$.

The results below say that, under an isomonodromy condition, such a phenomenon does not appear.

A setting more general than the previous one happens to be useful. This occurs for instance when one considers confluence in a Schlesinger family parametrized by the universal covering of $(\mathbb{P}^1)^n \setminus \text{diagonals}$: the confluence takes place in the inverse image in T of a neighbourhood of the diagonals. Near a generic point of the diagonals, when only two points coincide, such an open set looks like the product of a upper half plane by an open set in \mathbb{C}^{n-1} , and one studies the confluence in vertical strips in this upper half plane.

4.a. The algebraic/analytic case. This case was studied by Deligne [10] (see also [12]). Deligne used the full strength of Hironaka’s theorem on resolution of singularities. His approach has been much simplified by Z. Mebkhout, who gives a proof using resolution of singularities in dimension two only (cf. [13, 14], see also [15]).

We put here $\bar{X} = D \times T$, where D is a disc in \mathbb{C} . Let Y be a divisor in \bar{X} on which the projection $\pi : \bar{X} \rightarrow T$ is finite. Let E be a holomorphic vector bundle on \bar{X} equipped with a meromorphic *integrable* connection $\nabla : E \rightarrow \Omega_{\bar{X}}^1(*Y) \otimes_{\mathcal{O}_{\bar{X}}} E$ with poles along Y at most. It defines a meromorphic connection ∇_t on E_t with poles at the finite set of points Y_t for any $t \in T$.

Theorem 4.2 (Deligne). *Assume that, for generic $t \in T$, the connection ∇_t has only regular singularities at Y_t . Then this holds for any $t \in T$.*

Sketch. The proof uses a variant of the Riemann existence Theorem. One constructs a meromorphic bundle with a connection having regular singularities along Y at most, and having the same monodromy as the original connection. This auxiliary system satisfies the property of the theorem. Once such a system is constructed, one proves that it is isomorphic to the original one: one has an isomorphism between both bundles with connection out of Y ; due to the generic regular singularity of the original system, such an isomorphism is generically meromorphic along Y ; by Hartogs, it is meromorphic. \square

Moreover, it is then clear by a topological argument that, for $t^\circ \in T$, the monodromy of ∇_{t° on E_{t° around some point in Y_{t° can be computed as the product of well-chosen representatives of the monodromy operators of ∇_t on E_t near those points in Y_t which tend to the chosen point in Y_{t° when $t \rightarrow 0$.

4.b. Other confluences. Denote by D an open disc centered at 0 in the complex plane and by Δ the open disc $\{|t-1| < 1\}$. Denote by Y the intersection of $\bar{X} = D \times \Delta$ with the lines $x = a_i^\circ t$, with $a_i^\circ \in D$ for $i = 1, \dots, n$.

Consider an integrable (or isomonodromic) meromorphic system of differential equations on \bar{X} with poles on Y , with matrix $\Omega = A(x, t)dx + C(x, t)dt$ having poles along Y at most. Assume that the limits of $A(x, t), C(x, t)$ when $t \rightarrow 0$ exist and are meromorphic on D with pole at 0 at most.

Theorem 4.3 ([3, theorem 4]). *Assume that, for generic $t \in \Delta$, the system with matrix $A(x, t)dx$ has regular singularities at the points $a_i^\circ t$ ($i = 1, \dots, n$). Then the limiting system at $t = 0$ has a regular singularity at $x = 0$.*

In such a situation, Theorem 4.2 does not apply. The method of proof given by Andrey is nevertheless similar: to construct a system with the same monodromy satisfying the property of the theorem, and then show, by an argument using Hartogs, that this system is isomorphic to the original one. The existence result uses the particular form of the polar divisor Y , by solving explicitly the Schlesinger system. We only give details of the existence part.

Proof. For simplicity, let us first consider, as in *loc. cit.*, the case where the monodromy around the boundary of D (*i.e.*, the product of well-chosen representatives of the monodromies around each a_i°) is equal to the identity.

For the value $t = 1$ extend the system as a system on \mathbb{C} . Choose a point a_0° distinct from the other a_i° . There exists therefore, according to Plemelj, a Fuchsian system on \mathbb{P}^1 with no singularity at ∞ and an apparent singularity at a_0° , having the monodromy of the original system. Let us write the matrix of this Fuchsian system as

$$\sum_{i=0}^n B_i \frac{dx}{x - a_i^\circ}, \quad \text{with } \sum_i B_i = 0.$$

The Schlesinger system (Schl) with respect to the parameter t describing the isomonodromic deformation with pole on $\tilde{Y} = Y \cup \{x = a_0^\circ t\}$ can be written as

$$dB_i(t) = \sum_{j \neq i} [B_i(t), B_j(t)] \frac{dt}{t}$$

and therefore $\sum_i B_i(t)$ is constant, hence 0. The system can then be written as

$$dB_i(t) = [B_i(t), \sum_j B_j(t)] \frac{dt}{t} = 0,$$

that is, the $B_i(t)$ are constant. The matrix of the connection (3.3) is written as $\sum_{i=0}^n B_i d(x - a_i^o t)/(x - a_i^o t)$, its limit when $t \rightarrow 0$ does exist and is equal to 0, hence the regular singularity of the system restricted at $t = 0$. The original system is then shown to be meromorphically isomorphic to the previous model⁽³⁾.

If the monodromy around the boundary of D is not the identity, the previous construction can still be done, but we now have $\sum_{i=0}^n B_i = -B_\infty \neq 0$. In the Schlesinger system, we still have B_∞ constant, and $B_i(t)$ is a solution to the Fuchsian linear system

$$\frac{dB_i(t)}{dt} = \frac{\text{ad } B_\infty}{t} \cdot B_i(t),$$

hence $B_i(t) = t^{B_\infty} B_i t^{-B_\infty}$. The matrix of the connection, which is as in (3.3):

$$\Omega = \sum_{i=0}^n B_i(t) \frac{d(x - a_i^o t)}{(x - a_i^o t)}$$

satisfies, out of $x = 0$,

$$\lim_{t \rightarrow 0} x\Omega = B_\infty dx,$$

hence the regular singularity at the limit. \square

4.c. Confluence as a dynamical version of apparent singularities. Let us begin with preliminary remarks. Let a^* be a set of n distinct points in \mathbb{P}^1 and let E_{a^*} be a *nontrivial* holomorphic bundle of degree 0 with a connection ∇_{a^*} having logarithmic poles at a^* . If T is as in § 3.a, we have, after Theorem 3.1 applied to the initial condition (E_{a^*}, ∇_{a^*}) , a vector bundle E on $\mathbb{P}^1 \times T$ with a logarithmic connection having poles on Y , which restricts to the initial condition at $a = a^*$.

The bundle with connection (E_{a^*}, ∇_{a^*}) is contained in a meromorphic bundle with connection $(E_{a^*}(*a^*), \nabla_{a^*})$, in which the Riemann-Hilbert problem may or may not have a solution.

One can ask the question: *Is E_a trivial for generic a ?*

Andrey has given examples of a monodromy representation for which the R-H problem has no solution, whatever the choice of a position a of the poles could be (cf. [1, Prop. 5.2.1, p. 126]). In particular, the answer to the previous question may be negative.

On the other hand, if one allows confluence, he has obtained the following result in [5].

⁽³⁾Some eigenvalues of some B_i may differ by a nonzero integer: this usually happens at the apparent singularity; this would not happen in Deligne's method where the choice of a "Deligne extension" allows one, by its uniqueness, to construct a Fuchsian system in a global situation (after resolution of singularities) by a local procedure.

Let E'_{a^*} be a trivial holomorphic subbundle of $E_{a^*}(*a_n)$ on which the connection has a pole of Poincaré rank $r \geq 1$ at a_n . It is then possible to construct an isomonodromic confluence (as in § 4.b) of trivial bundles E'_t with logarithmic connection ∇_t (*i.e.*, Fuchsian systems) having poles at a_1, \dots, a_n and at a finite number of distinct points $b_j(t)$ which converge to a_n when $t \rightarrow 0$, so that the limit bundle is E'_{a^*} . The points $b_j(t)$ are apparent singularities for ∇_t and their number can be bounded by $(rd(d-1)/2)^2$, using a result of E. Corel [9].

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C. SABBAB, UMR 7640 du CNRS, Centre de Mathématiques Laurent Schwartz, École polytechnique,
F-91128 Palaiseau cedex, France • *E-mail* : sabbah@math.polytechnique.fr
Url : <http://www.math.polytechnique.fr/cmat/sabbah/sabbah.html>