The work of Andrey Bolibrukh

on isomonodromic deformations

"*Therefore, in essence, the invariant geometric language of vector bundles is unavoidable for the rigorous analysis of the inverse monodromy problem and isomonodromy deformations in the case of general linear systems.*

"*Therefore, in essence, the invariant geometric language of vector bundles is unavoidable for the rigorous analysis of the inverse monodromy problem and isomonodromy deformations in the case of general linear systems.*

At the same time, for specific linear systems related to the Painlevé equations, it is possible to perform a rigorous study of the inverse problem on the basis of analytic considerations only."

“*Therefore, in essence, the invariant geometric language of vector bundles is unavoidable for the rigorous analysis of the inverse monodromy problem and isomonodromy deformations in the case of general linear systems.*

At the same time, for specific linear systems related to the Painlevé equations, it is possible to perform a rigorous study of the inverse problem on the basis of analytic considerations only.”

A. A. BOLIBRUKH, A. R. ITS & A. A. KAPAEV

"*Therefore, in essence, the invariant geometric language of vector bundles is unavoidable for the rigorous analysis of the inverse monodromy problem and isomonodromy deformations in the case of general linear systems.*

At the same time, for specific linear systems related to the Painlevé equations, it is possible to perform a rigorous study of the inverse problem on the basis of analytic considerations only."

A. A. BOLIBRUKH, A. R. ITS & A. A. KAPAEV – "On the Riemann-Hilbert-Birkhoff inverse monodromy problem and the Painlevé equations", *Algebra i Analiz* **16** (2004), no. 1, p. 121–162.

Main themes

Main themes

- Isomonodromy and integrability.

Main themes

- Isomonodromy and integrability.
- Possible general form of an isomonodromic deformation of a Fuchsian system.

Main themes

- Isomonodromy and integrability.
- Possible general form of an isomonodromic deformation of a Fuchsian system.
- The Schlesinger system and the Painlevé property.

Main themes

- Isomonodromy and integrability.
- Possible general form of an isomonodromic deformation of a Fuchsian system.
- The Schlesinger system and the Painlevé property.
  - Equation for the “Theta divisor”.

Main themes

- Isomonodromy and integrability.
- Possible general form of an isomonodromic deformation of a Fuchsian system.
- The Schlesinger system and the Painlevé property.
  - Equation for the “Theta divisor”.
  - Bounds for the order of the pole of the solutions along the “Theta divisor”.

Main themes— continuation

- Isomonodromic confluences.

Main themes— continuation

- Isomonodromic confluences.
  - Preservation of regularity at the confluence point.

Main themes— continuation

- Isomonodromic confluences.
  - Preservation of regularity at the confluence point.
  - Any vector bundle with logarithmic connection on the Riemann sphere can be obtained from Fuchsian systems by confluence

Main themes— continuation

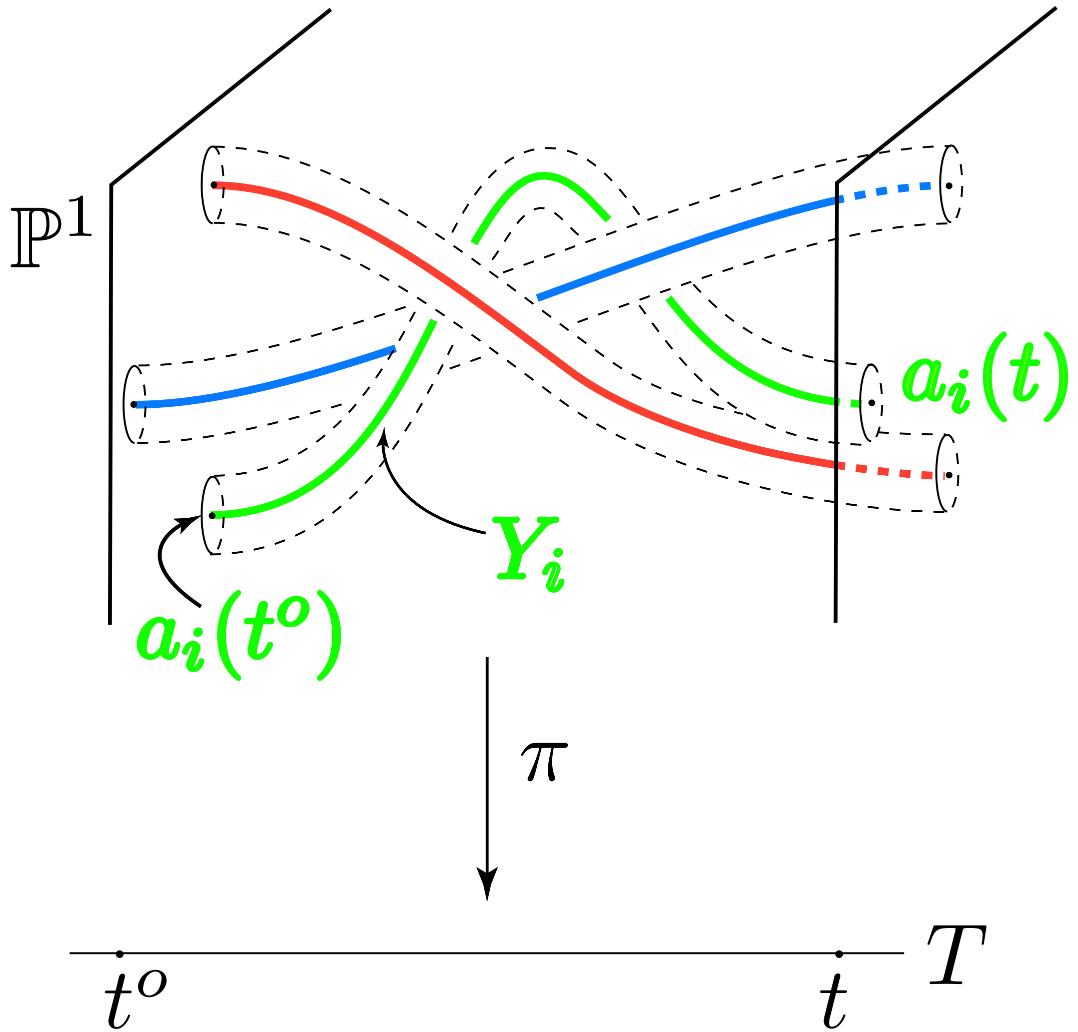
- Isomonodromic confluences.
  - Preservation of regularity at the confluence point.
  - Any vector bundle with logarithmic connection on the Riemann sphere can be obtained from Fuchsian systems by confluence (a dynamical version of the adjunction of an apparent singularity).

Main themes— continuation

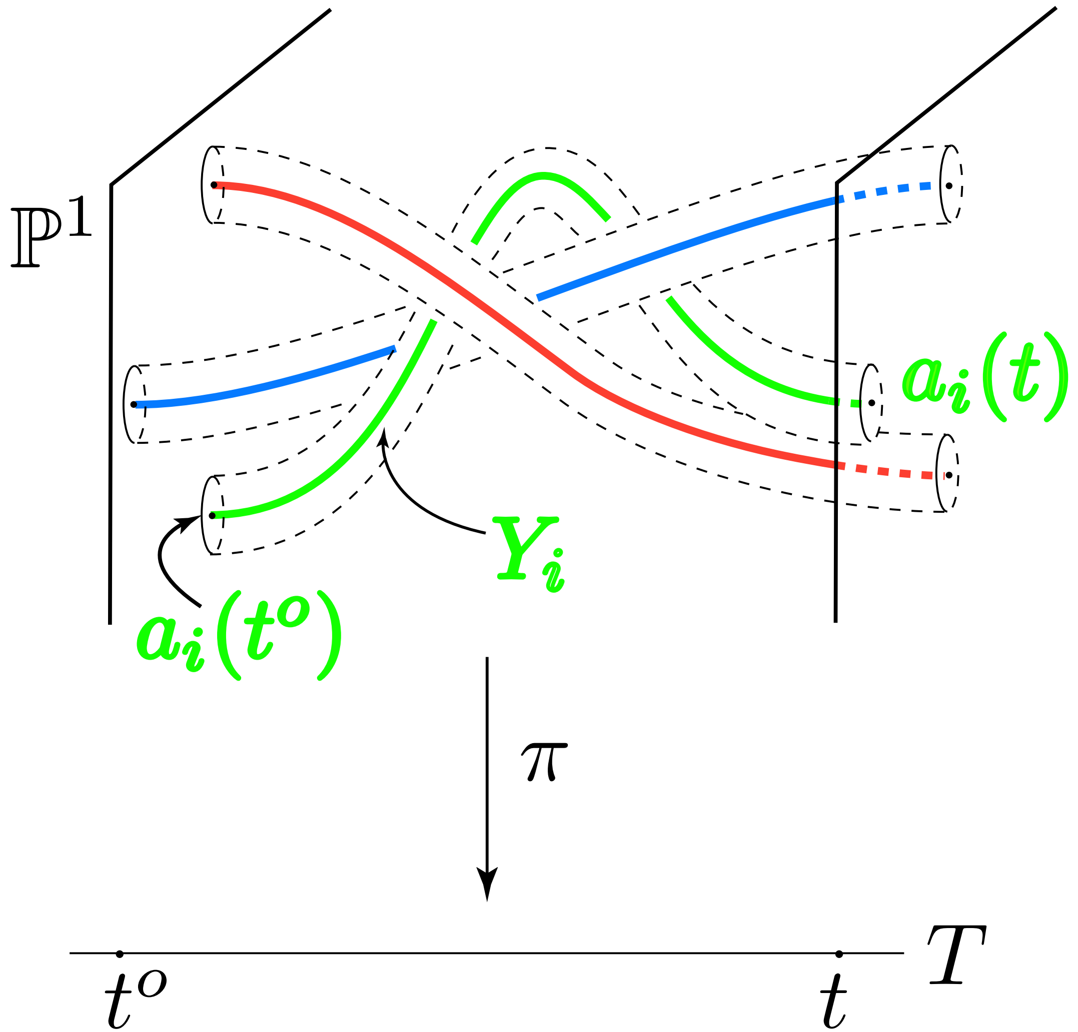
- Isomonodromic confluences.
  - Preservation of regularity at the confluence point.
  - Any vector bundle with logarithmic connection on the Riemann sphere can be obtained from Fuchsian systems by confluence
(a dynamical version of the adjunction of an apparent singularity).
- Isomonodromy and irregular singularities

What is an isomonodromic deformation?

What is an isomonodromic deformation?

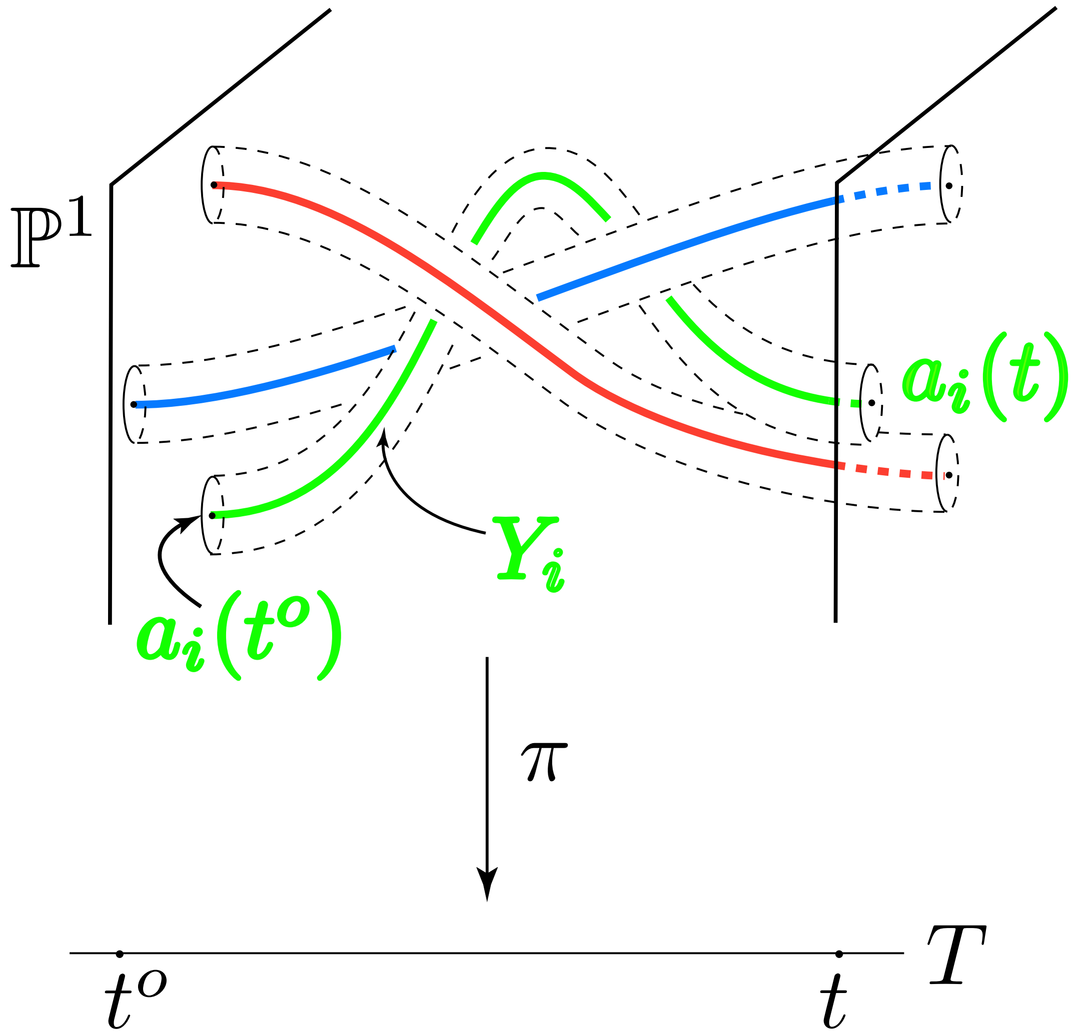


What is an isomonodromic deformation?



$$\overline{X} = \mathbb{P}^1 \times T$$

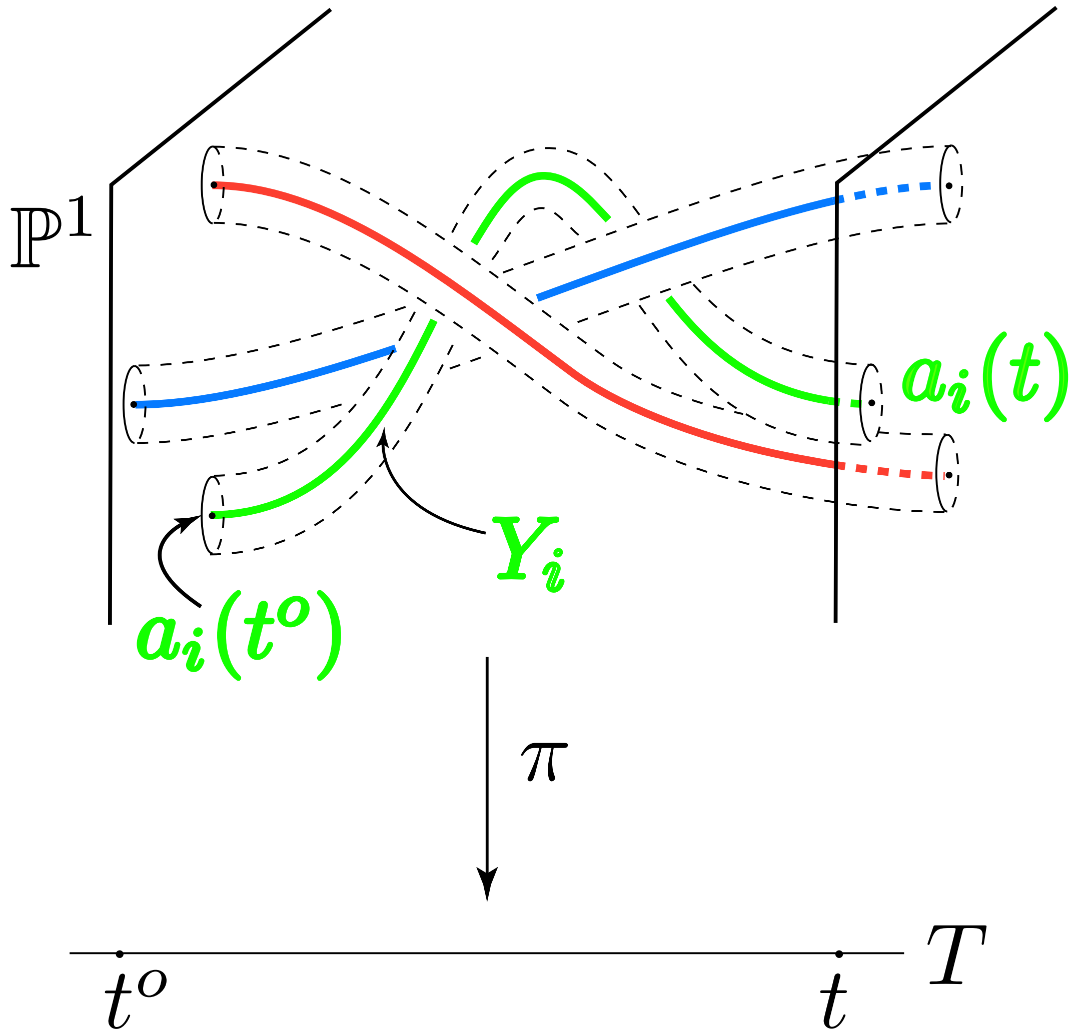
What is an isomonodromic deformation?



$$\overline{X} = \mathbb{P}^1 \times T$$

$$Y = \cup_i Y_i$$

What is an isomonodromic deformation?

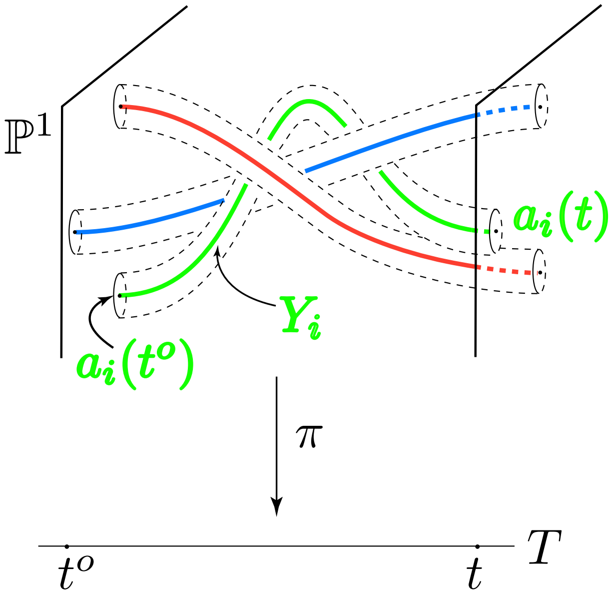


$$\overline{X} = \mathbb{P}^1 \times T$$

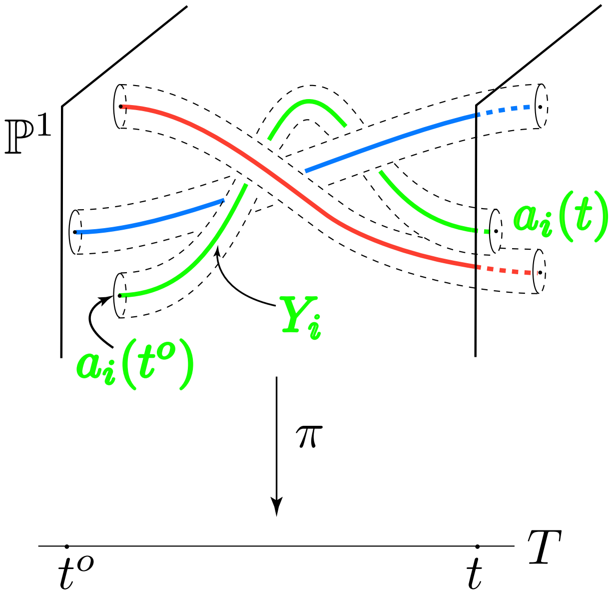
$$Y = \cup_i Y_i$$

$$X = \overline{X} \setminus Y$$

Equations



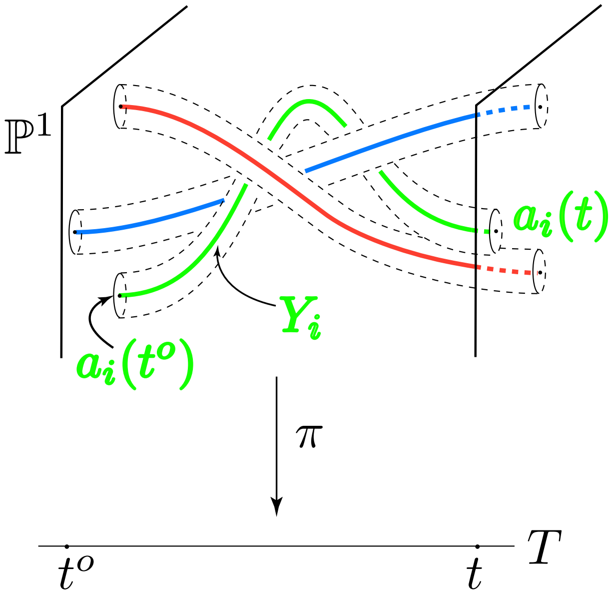
Equations



Fuchsian system

$$\frac{du}{dx} = \sum_{i=1}^{n} \frac{A_i^o}{x - a_i(t^o)} \cdot u$$

A_i^o: $d \times d$ constant matrices



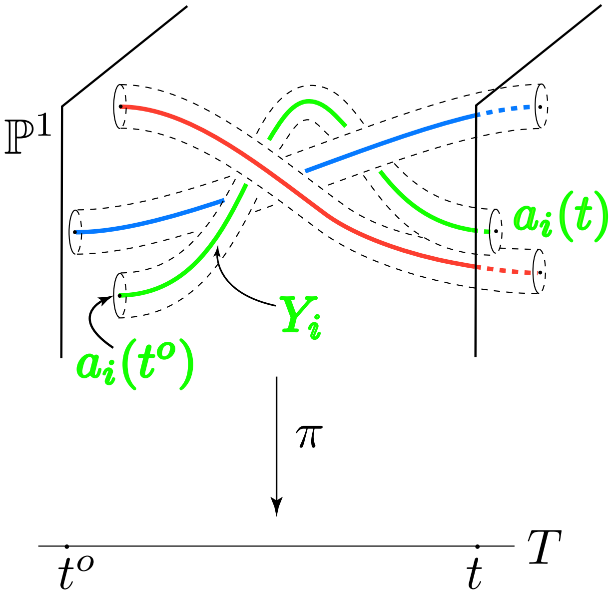
Equations

Matrix of 1-forms

$$\Omega^o = \sum_{i=1}^{n} \frac{A_i^o}{x - a_i(t^o)} \cdot dx$$

A_i^o: $d \times d$ constant matrices

Equations



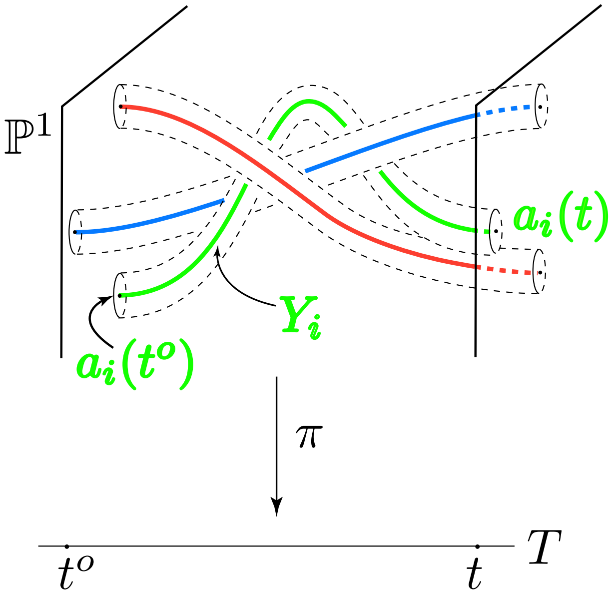
Matrix of 1-forms

$$\Omega^o = \sum_{i=1}^{n} \frac{A_i^o}{x - a_i(t^o)} \cdot dx$$

A_i^o: $d \times d$ constant matrices

Isomonodromic deformation parametrized by T:

Equations



Matrix of 1-forms

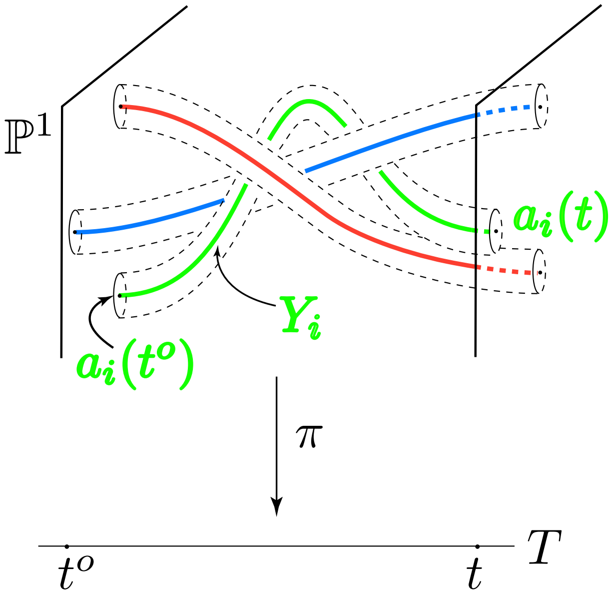
$$\Omega^o = \sum_{i=1}^{n} \frac{A_i^o}{x - a_i(t^o)} \cdot dx$$

A_i^o: $d \times d$ constant matrices

Isomonodromic deformation parametrized by T:

$$\Omega = \sum_{i=1}^{n} \frac{A_i(t)}{x - a_i(t)} \cdot dx + \sum_j \Omega_j(x,t)\, dt_j,$$

Equations



Matrix of 1-forms

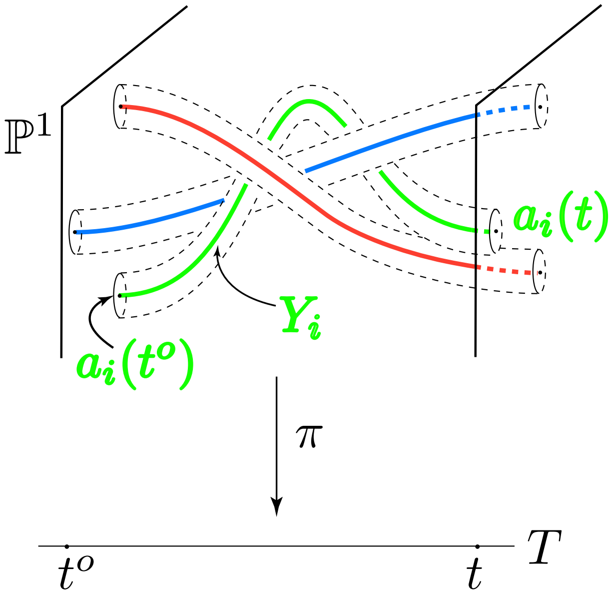
$$\Omega^o = \sum_{i=1}^{n} \frac{A_i^o}{x - a_i(t^o)} \cdot dx$$

A_i^o: $d \times d$ constant matrices

Isomonodromic deformation parametrized by T:

$$\Omega = \sum_{i=1}^{n} \frac{A_i(t)}{x - a_i(t)} \cdot dx + \sum_j \Omega_j(x,t)\, dt_j, \qquad d\Omega + \Omega \wedge \Omega = 0$$

Equations



Matrix of 1-forms

$$\Omega^o = \sum_{i=1}^{n} \frac{A_i^o}{x - a_i(t^o)} \cdot dx$$

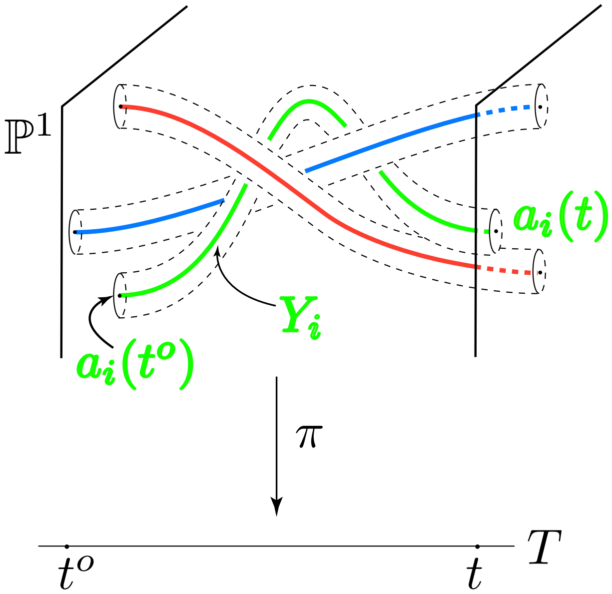
A_i^o: $d \times d$ constant matrices

Isomonodromic deformation parametrized by T:

$$\Omega = \sum_{i=1}^{n} \frac{A_i(t)}{x - a_i(t)} \cdot dx + \sum_j \Omega_j(x,t)\, dt_j, \qquad d\Omega + \Omega \wedge \Omega = 0$$

$A_i(t^o) = A_i^o$ $\qquad$ $A_i(t)$ holomorphic

Equations



Matrix of 1-forms

$$\Omega^o = \sum_{i=1}^{n} \frac{A_i^o}{x - a_i(t^o)} \cdot dx$$

A_i^o: $d \times d$ constant matrices

Isomonodromic deformation parametrized by T:

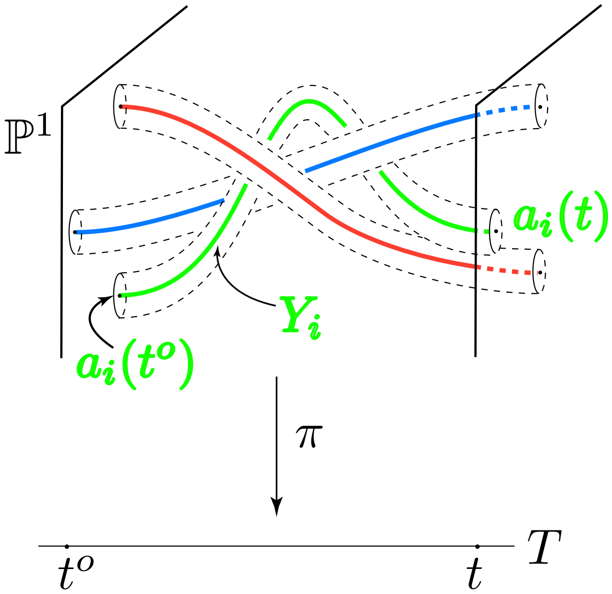
$$\Omega = \sum_{i=1}^{n} \frac{A_i(t)}{x - a_i(t)} \cdot dx + \sum_j \Omega_j(x,t)\, dt_j, \qquad d\Omega + \Omega \wedge \Omega = 0$$

$A_i(t^o) = A_i^o$ $\qquad$ $A_i(t)$ holomorphic

$\Omega_j(x,t)$ is meromorphic with poles along Y

(regular deformation)

Equations



Matrix of 1-forms

$$\Omega^o = \sum_{i=1}^{n} \frac{A_i^o}{x - a_i(t^o)} \cdot dx$$

A_i^o: $d \times d$ constant matrices

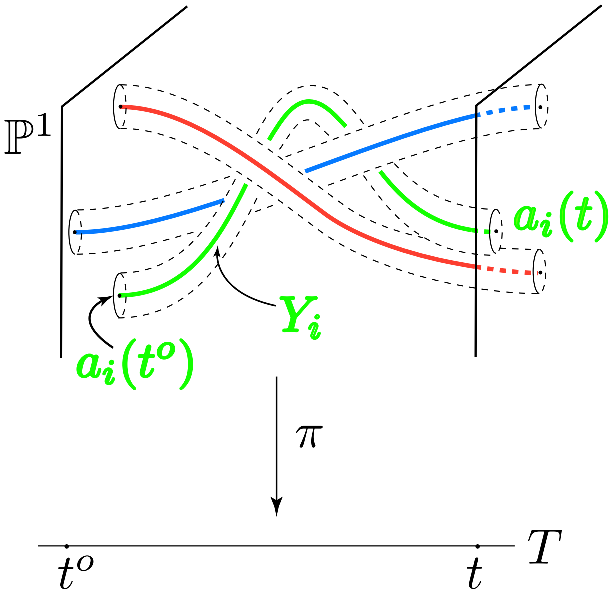
Isomonodromic deformation parametrized by T:

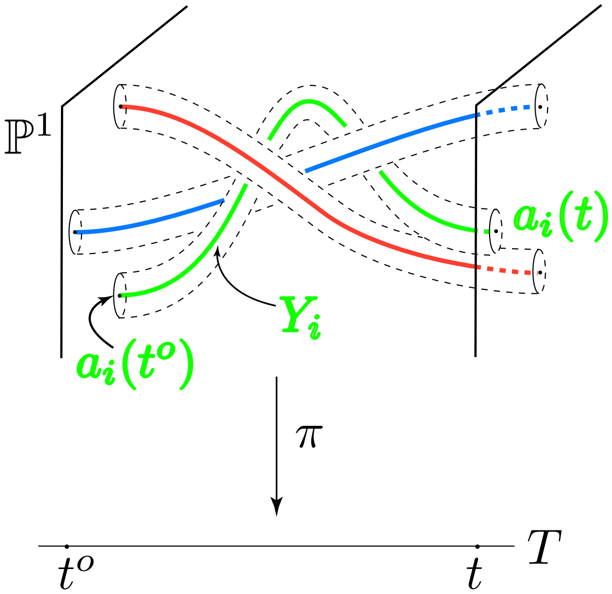
$$\Omega = \sum_{i=1}^{n} \frac{A_i(t)}{x - a_i(t)} \cdot dx + \sum_j \Omega_j(x,t)\, dt_j, \quad d\Omega + \Omega \wedge \Omega = 0$$

$A_i(t^o) = A_i^o$ $\qquad$ $A_i(t)$ holomorphic

$\Omega_j(x,t)$ is holomorphic (logarithmic deformation)

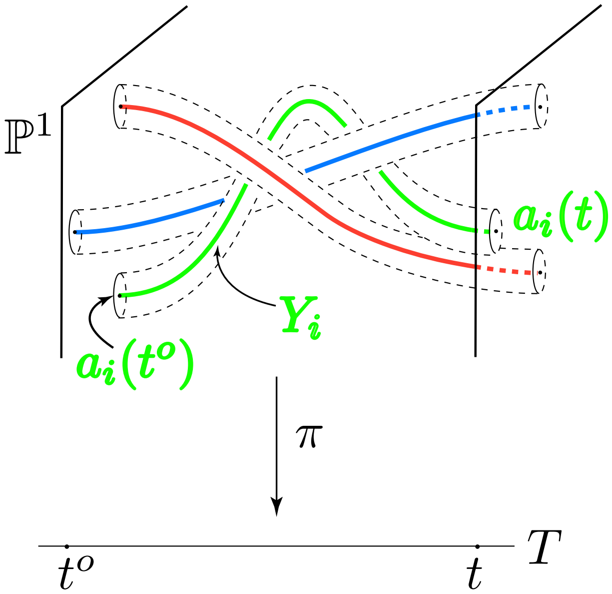
Vector bundles





Vector bundles

E^o holomorphic vector bundle on $\mathbb{P}^1$

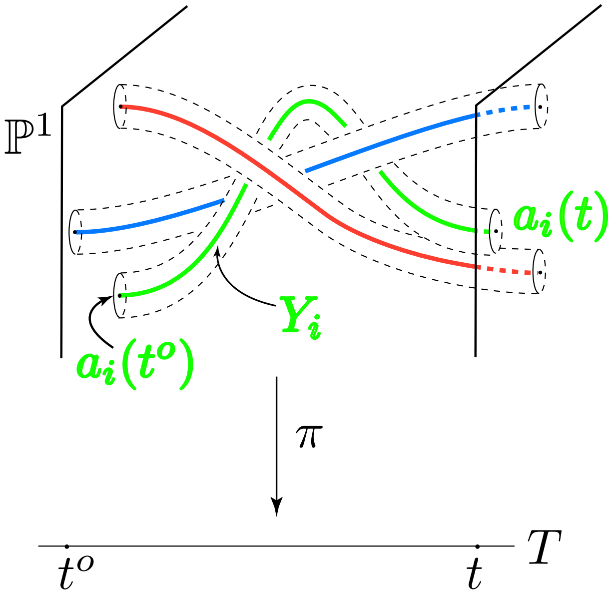


Vector bundles

E^o holomorphic vector bundle on $\mathbb{P}^1$

$\nabla^o : E^o \to \Omega^1_{\mathbb{P}^1}(*Y^o) \otimes E^o$
integrable meromorphic connection
with regular singularities along Y^o

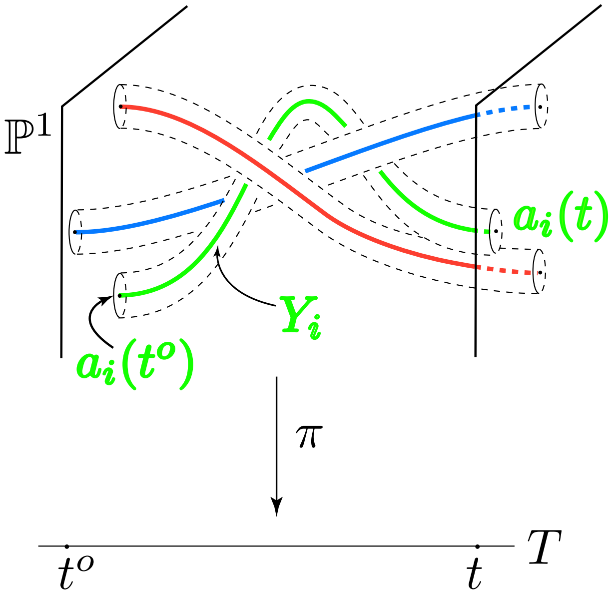
Vector bundles



$\boldsymbol{E^o}$ holomorphic vector bundle on $\mathbb{P}^1$

$\boldsymbol{\nabla^o : E^o \to \Omega^1_{\mathbb{P}^1}(\log Y^o) \otimes E^o}$
integrable meromorphic connection
with logarithmic poles along $\boldsymbol{Y^o}$

Vector bundles

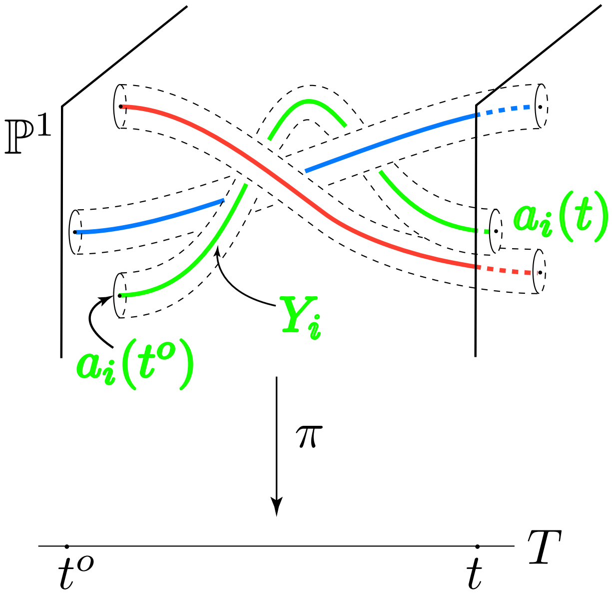


E^o holomorphic vector bundle on $\mathbb{P}^1$

$\nabla^o : E^o \to \Omega^1_{\mathbb{P}^1}(*Y^o) \otimes E^o$

integrable meromorphic connection with regular singularities along Y^o

Vector bundles

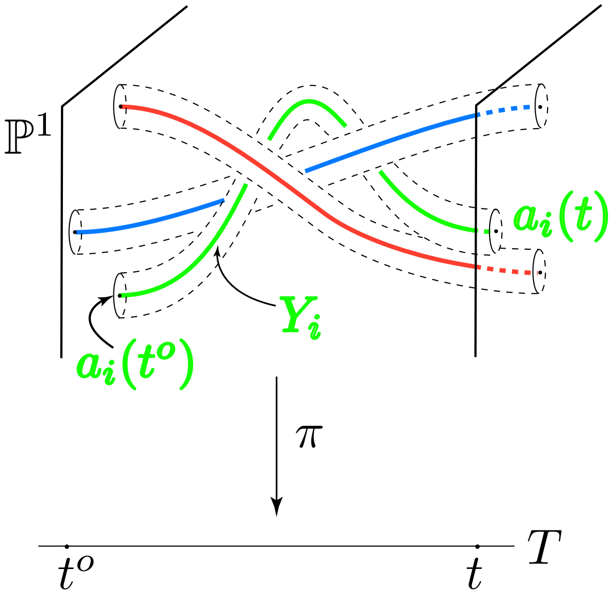


E^o holomorphic vector bundle on $\mathbb{P}^1$

$\nabla^o : E^o \to \Omega^1_{\mathbb{P}^1}(*Y^o) \otimes E^o$
integrable meromorphic connection
with regular singularities along Y^o

Isomonodromic deformation parametrized by T:

Vector bundles



E^o holomorphic vector bundle on $\mathbb{P}^1$

$\nabla^o : E^o \to \Omega^1_{\mathbb{P}^1}(*Y^o) \otimes E^o$

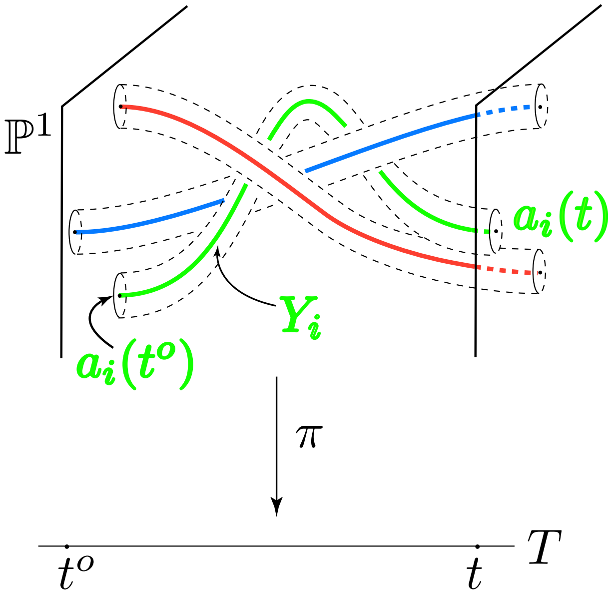
integrable meromorphic connection with regular singularities along Y^o

Isomonodromic deformation parametrized by T:

$$\nabla : E \to \Omega^1_{X}(*Y) \otimes E, \quad \nabla \circ \nabla = 0, \quad \nabla_{|E^o} = \nabla^o$$

integrable meromorphic connection with regular singularities along Y

Vector bundles



E^o holomorphic vector bundle on $\mathbb{P}^1$

$\nabla^o : E^o \to \Omega^1_{\mathbb{P}^1}(\log Y^o) \otimes E^o$

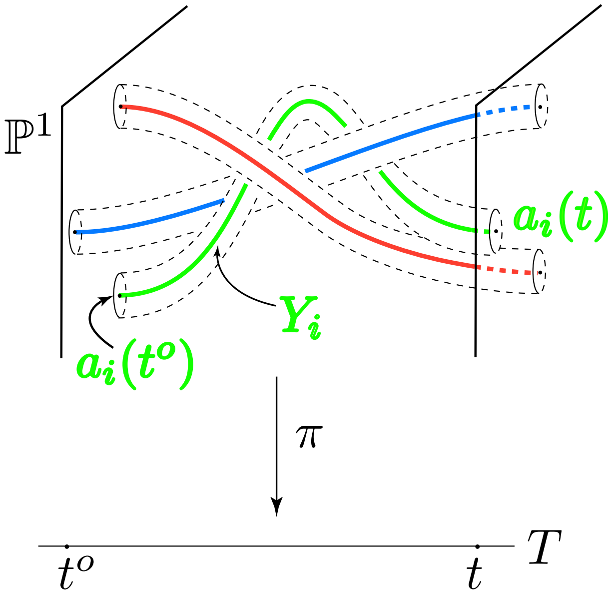
integrable meromorphic connection with logarithmic poles along Y^o

Isomonodromic deformation parametrized by T:

$$\nabla : E \to \Omega^1_X(*Y) \otimes E, \quad \nabla \circ \nabla = 0, \quad \nabla_{|E^o} = \nabla^o$$

integrable meromorphic connection with regular singularities along Y and each ∇_t on E_t has logarithmic poles along Y_t

Vector bundles



E^o holomorphic vector bundle on $\mathbb{P}^1$

$\nabla^o : E^o \to \Omega^1_{\mathbb{P}^1}(\log Y^o) \otimes E^o$

integrable meromorphic connection with logarithmic poles along Y^o

Isomonodromic deformation parametrized by T:

$$\nabla : E \to \Omega^1_X(\log Y) \otimes E, \quad \nabla \circ \nabla = 0, \quad \nabla_{|E^o} = \nabla^o$$

integrable meromorphic connection with logarithmic poles along Y

The Schlesinger system

- a finite set of distinct points $a^o = \{a_1^o, \dots, a_n^o\}$ on $\mathbb{P}^1 \setminus \{\infty\}$,

The Schlesinger system

- a finite set of distinct points $a^o = \{a_1^o, \ldots, a_n^o\}$ on $\mathbb{P}^1 \setminus \{\infty\}$,
- a system $\dfrac{du}{dx} = \sum_{i=1}^n \dfrac{A_i^o}{x - a_i^o} \cdot u, \quad \sum_i A_i^o = 0$.

The Schlesinger system

- a finite set of distinct points $a^o = \{a_1^o, \ldots, a_n^o\}$ on $\mathbb{P}^1 \smallsetminus \{\infty\}$,
- a system $\frac{du}{dx} = \sum_{i=1}^n \frac{A_i^o}{x - a_i^o} \cdot u$, $\quad \sum_i A_i^o = 0$.

$\Longleftrightarrow$ a logarithmic connection ∇^o on the trivial bundle E^o of rank d on $\mathbb{P}^1$, with poles at a^o.

The Schlesinger system

- a finite set of distinct points $a^o = \{a^o_1, \ldots, a^o_n\}$ on $\mathbb{P}^1 \smallsetminus \{\infty\}$,
- a system $\dfrac{du}{dx} = \sum_{i=1}^n \dfrac{A_i^o}{x - a_i^o} \cdot u$, $\quad \sum_i A_i^o = 0$.

  $\Longleftrightarrow$ a logarithmic connection ∇^o on the trivial bundle E^o of rank d on $\mathbb{P}^1$, with poles at a^o.
- T: universal cover of $(\mathbb{P}^1)^n \smallsetminus$ diagonals, with base point $\widetilde{a}^o$.

The Schlesinger system

- a finite set of distinct points $a^o = \{a_1^o, \dots, a_n^o\}$ on $\mathbb{P}^1 \smallsetminus \{\infty\}$,
- a system $\dfrac{du}{dx} = \sum_{i=1}^n \dfrac{A_i^o}{x - a_i^o} \cdot u$, $\quad \sum_i A_i^o = 0$.

  $\Longleftrightarrow$ a logarithmic connection ∇^o on the trivial bundle E^o of rank d on $\mathbb{P}^1$, with poles at a^o.
- T: universal cover of $(\mathbb{P}^1)^n \smallsetminus$ diagonals, with base point $\widetilde{a}^o$.

Theorem (Malgrange). *There exists a unique vector bundle E on $\mathbb{P}^1 \times T$ equipped with an integrable logarithmic connection ∇ having poles along the hypersurfaces Y_i, and with an identification $(E, \nabla)_{|\mathbb{P}^1 \times \{\widetilde{a}^o\}} \xrightarrow{\sim} (E^o, \nabla^o)$.*

The Schlesinger system— continuation

- Divisor $\Theta = \{\widetilde{a} \in T \mid E_{\widetilde{a}} \text{ is not trivial}\}$,

The Schlesinger system— continuation

- Divisor $\Theta = \{\widetilde{a} \in T \mid E_{\widetilde{a}} \text{ is not trivial}\}$,
- The matrix of ∇ in a basis of $E(*\Theta)$ extending that of E^o is:

$$\sum_{i=1}^{n} A_i(\widetilde{a}) \frac{d(x - \widetilde{a}_i)}{(x - \widetilde{a}_i)} + \sum_{i=1}^{n} B_i(\widetilde{a}_i) d\widetilde{a}_i$$

The Schlesinger system— continuation

- Divisor $\Theta = \{\widetilde{a} \in T \mid E_{\widetilde{a}} \text{ is not trivial}\}$,
- The matrix of ∇ in a basis of $E(*\Theta)$ extending that of E^o is:

$$\sum_{i=1}^{n} A_i(\widetilde{a}) \frac{d(x - \widetilde{a}_i)}{(x - \widetilde{a}_i)} + \sum_{i=1}^{n} B_i(\widetilde{a}_i) d\widetilde{a}_i$$

- On can choose the basis such that $B_i \equiv 0$.

The Schlesinger system— continuation

- Divisor $\Theta = \{\widetilde{a} \in T \mid E_{\widetilde{a}} \text{ is not trivial}\}$,
- The matrix of ∇ in a basis of $E(*\Theta)$ extending that of E^o is:

$$\sum_{i=1}^{n} A_i(\widetilde{a}) \frac{d(x - \widetilde{a}_i)}{(x - \widetilde{a}_i)}$$

- On can choose the basis such that $B_i \equiv 0$.

The Schlesinger system— continuation

- Divisor $\Theta = \{\widetilde{a} \in T \mid E_{\widetilde{a}}$ is not trivial$\}$,
- The matrix of ∇ in a basis of $E(*\Theta)$ extending that of E^o is:

$$\sum_{i=1}^{n} A_i(\widetilde{a}) \, \frac{d(x - \widetilde{a}_i)}{(x - \widetilde{a}_i)}$$

- On can choose the basis such that $B_i \equiv 0$.
- The Schlesinger system (integrability condition):

$$dA_i = \sum_{j \neq i} [A_i, A_j] \, \frac{d(\widetilde{a}_i - \widetilde{a}_j)}{(\widetilde{a}_i - \widetilde{a}_j)}, \quad i = 1, \ldots, n.$$

Corollary. *The solutions of the Schlesinger system with initial value A_i^o at $\widetilde{a}^o$ are meromorphic on T with poles along Θ at most.*

Corollary. *The solutions of the Schlesinger system with initial value* A_i^o *at* $\widetilde{a}^o$ *are meromorphic on* T *with poles along* Θ *at most.*

Behaviour of the solutions to the Schlesinger system near the polar set Θ?

Corollary. *The solutions of the Schlesinger system with initial value A_i^o at $\widetilde{a}^o$ are meromorphic on T with poles along Θ at most.*

Behaviour of the solutions to the Schlesinger system near the polar set Θ?

Andrey has given a method to produce examples and describe in concrete terms this behaviour.

Local equation for Θ

Local equation for Θ

Initial data $\widetilde{a}^o, A_i^o$

Local equation for Θ

Initial data $\widetilde{a}^o, A_i^o \rightsquigarrow (E, \nabla)$ on $\mathbb{P}^1 \times T$,

Local equation for Θ

Initial data $\widetilde{a}^o, A_i^o \rightsquigarrow (E, \nabla)$ on $\mathbb{P}^1 \times T$, $\Theta \subset T$.

Local equation for Θ

Initial data $\widetilde{a}^o, A_i^o \rightsquigarrow (E, \nabla)$ on $\mathbb{P}^1 \times T$, $\Theta \subset T$.

Take $a^* \in \Theta$.

Local equation for Θ

Initial data $\widetilde{a}^o, A_i^o \rightsquigarrow (E, \nabla)$ on $\mathbb{P}^1 \times T$, $\Theta \subset T$.

Take $a^* \in \Theta$. Hence $E_{a^*} \simeq \oplus_{j=1}^d \mathcal{O}(-k_j)$,

Local equation for Θ

Initial data $\widetilde{a}^o, A_i^o \rightsquigarrow (E, \nabla)$ on $\mathbb{P}^1 \times T$, $\Theta \subset T$.

Take $a^* \in \Theta$. Hence $E_{a^*} \simeq \bigoplus_{j=1}^{d} \mathcal{O}(-k_j)$, with $k_1 \leqslant \cdots \leqslant k_d$ and $\deg E_{a^*} = -(k_1 + \cdots + k_d) = 0$.

Local equation for Θ

Initial data $\widetilde{a}^o, A_i^o \rightsquigarrow (E, \nabla)$ on $\mathbb{P}^1 \times T$, $\Theta \subset T$.

Take $a^* \in \Theta$. Hence $E_{a^*} \simeq \oplus_{j=1}^{d} \mathcal{O}(-k_j)$, with $k_1 \leqslant \cdots \leqslant k_d$ and $\deg E_{a^*} = -(k_1 + \cdots + k_d) = 0$.
Typical example: $E_{a^*} \simeq \mathcal{O}(1) \oplus \mathcal{O}(-1)$

Local equation for Θ

Initial data $\widetilde{a}^o, A_i^o \rightsquigarrow (E, \nabla)$ on $\mathbb{P}^1 \times T$, $\Theta \subset T$.

Take $a^* \in \Theta$. Hence $E_{a^*} \simeq \bigoplus_{j=1}^{d} \mathcal{O}(-k_j)$, with $k_1 \leqslant \cdots \leqslant k_d$ and $\deg E_{a^*} = -(k_1 + \cdots + k_d) = 0$. Hence $\exists\, \ell, m \in \{1, \ldots, d\}$ such that $k_\ell - k_m \geqslant 2$.

Local equation for Θ

Initial data $\widetilde{a}^o, A_i^o \rightsquigarrow (E, \nabla)$ on $\mathbb{P}^1 \times T$, $\Theta \subset T$.

Take $a^* \in \Theta$. Hence $E_{a^*} \simeq \oplus_{j=1}^d \mathcal{O}(-k_j)$, with $k_1 \leqslant \cdots \leqslant k_d$ and $\deg E_{a^*} = -(k_1 + \cdots + k_d) = 0$. Hence $\exists\, \ell, m \in \{1, \ldots, d\}$ such that $k_\ell - k_m \geqslant 2$.

There exists a holomorphic subbundle $E_{a^*}^{(0)}$ of the meromorphic bundle $E_{a^*}[*\infty]$ which is trivial and on which the connection ∇ has only logarithmic poles.

Local equation for Θ

Initial data $\widetilde{a}^o, A_i^o \rightsquigarrow (E, \nabla)$ on $\mathbb{P}^1 \times T$, $\Theta \subset T$.

Take $a^* \in \Theta$. Hence $E_{a^*} \simeq \oplus_{j=1}^{d} \mathcal{O}(-k_j)$, with $k_1 \leqslant \cdots \leqslant k_d$ and $\deg E_{a^*} = -(k_1 + \cdots + k_d) = 0$. Hence $\exists\, \ell, m \in \{1, \ldots, d\}$ such that $k_\ell - k_m \geqslant 2$.

There exists a holomorphic subbundle $E_{a^*}^{(0)}$ of the meromorphic bundle $E_{a^*}[*\infty]$ which is trivial and on which the connection ∇ has only logarithmic poles.

Matrix of the connection: $\sum_{i=1}^{n} \dfrac{B_i^{(0)}(a^*)}{x - a_i^*}\, dx$

Local equation for Θ

Initial data $\widetilde{a}^o, A_i^o \rightsquigarrow (E, \nabla)$ on $\mathbb{P}^1 \times T$, $\Theta \subset T$.

Take $a^* \in \Theta$. Hence $E_{a^*} \simeq \bigoplus_{j=1}^{d} \mathcal{O}(-k_j)$, with $k_1 \leqslant \cdots \leqslant k_d$ and $\deg E_{a^*} = -(k_1 + \cdots + k_d) = 0$. Hence $\exists\, \ell, m \in \{1, \dots, d\}$ such that $k_\ell - k_m \geqslant 2$.

There exists a holomorphic subbundle $E_{a^*}^{(0)}$ of the meromorphic bundle $E_{a^*}[*\infty]$ which is trivial and on which the connection ∇ has only logarithmic poles.

Matrix of the connection: $\sum_{i=1}^n \dfrac{B_i^{(0)}(a^*)}{x - a_i^*}\, dx$

$\infty =$ apparent singularity and

$\sum_i B_i^{(0)}(a^*) = \mathrm{diag}(k_1, \dots, k_d) =: K^{(0)}$

Local equation for Θ— continuation

Malgrange's theorem applied to $E^{(0)}_{a^*}$ near a^*

Local equation for Θ— continuation

Malgrange's theorem applied to $\boldsymbol{E^{(0)}_{a^*}}$ near $\boldsymbol{a^*}$
$\rightsquigarrow$ $\boldsymbol{(E^{(0)}, \nabla)}$ trivial on $\mathbb{P}^{\mathbf{1}} \times \mathbf{nb}(\boldsymbol{a^*})$,

Local equation for Θ— continuation

Malgrange's theorem applied to $E^{(0)}_{a^*}$ near a^*
$\rightsquigarrow (E^{(0)}, \nabla)$ trivial on $\mathbb{P}^1 \times \mathbf{nb}(a^*)$,

matrix of ∇:

$$\sum_i B_i^{(0)}(a) \frac{d(x - a_i)}{(x - a_i)}, \qquad \sum_i B_i^{(0)}(a) \equiv K^{(0)}$$

and the $B_i^{(0)}(a)$ satisfy the Schlesinger system.

Local equation for Θ— continuation

Malgrange's theorem applied to $E_{a^*}^{(0)}$ near a^*
$\rightsquigarrow (E^{(0)}, \nabla)$ trivial on $\mathbb{P}^1 \times \mathbf{nb}(a^*)$,

matrix of ∇:

$$\sum_i B_i^{(0)}(a)\,\frac{d(x-a_i)}{(x-a_i)}, \qquad \sum_i B_i^{(0)}(a) \equiv K^{(0)}$$

and the $B_i^{(0)}(a)$ satisfy the Schlesinger system.

Lemma 1. *There exists $\ell, m \in \{1, \ldots, d\}$ such that $k_m - k_\ell \geqslant 2$ and $i \in \{1, \ldots, n\}$ such that the (ℓ, m)-entry $B_{i,\ell m}^{(0)}(a)$ does not vanish identically.*

Local equation for Θ— continuation

Lemma 2. *Fix* $\ell, m \in \{1, \ldots, d\}$ *such that* $k_m - k_\ell \geqslant 2$ *and* $B^{(0)}_{i,\ell m}(a) \not\equiv 0$.

Local equation for Θ— continuation

Lemma 2. *Fix* $\ell, m \in \{1, \dots, d\}$ *such that* $k_m - k_\ell \geqslant 2$ *and* $B^{(0)}_{i,\ell m}(a) \not\equiv 0$.

Put $\tau^{(0)}(a) = \sum_i B^{(0)}_{i,\ell m}(a) a_i$ *and* $\Theta^{(0)} = \{\tau^{(0)} = 0\}$.

Local equation for Θ— continuation

Lemma 2. *Fix* $\ell, m \in \{1, \dots, d\}$ *such that* $k_m - k_\ell \geqslant 2$ *and* $B^{(0)}_{i,\ell m}(a) \not\equiv 0$.

Put $\tau^{(0)}(a) = \sum_i B^{(0)}_{i,\ell m}(a) a_i$ *and* $\Theta^{(0)} = \{\tau^{(0)} = 0\}$.

Then there exists an extension $E^{(1)}[*\Theta^{(0)}]$ *of* $E[*(\infty \times T) \cup \Theta^{(0)}]$ *such that, out of* $\Theta^{(0)}$,

Local equation for Θ— continuation

Lemma 2. *Fix $\ell, m \in \{1, \ldots, d\}$ such that $k_m - k_\ell \geqslant 2$ and $B^{(0)}_{i,\ell m}(a) \not\equiv 0$.*

Put $\tau^{(0)}(a) = \sum_i B^{(0)}_{i,\ell m}(a) a_i$ and $\Theta^{(0)} = \{\tau^{(0)} = 0\}$.

Then there exists an extension $E^{(1)}[*\Theta^{(0)}]$ of $E[*(\infty \times T) \cup \Theta^{(0)}]$ such that, out of $\Theta^{(0)}$,

- *for any $a \in T \smallsetminus \Theta^{(0)}$, the bundle $E^{(1)}[*\Theta^{(0)}]_a$ is trivial,*

Local equation for Θ— continuation

Lemma 2. *Fix* $\ell, m \in \{1, \ldots, d\}$ *such that* $k_m - k_\ell \geqslant 2$ *and* $B^{(0)}_{i,\ell m}(a) \not\equiv 0$.

Put $\tau^{(0)}(a) = \sum_i B^{(0)}_{i,\ell m}(a) a_i$ *and* $\Theta^{(0)} = \{\tau^{(0)} = 0\}$.

Then there exists an extension $E^{(1)}[*\Theta^{(0)}]$ *of* $E[*(\infty \times T) \cup \Theta^{(0)}]$ *such that, out of* $\Theta^{(0)}$,

- *for any* $a \in T \smallsetminus \Theta^{(0)}$, *the bundle* $E^{(1)}[*\Theta^{(0)}]_a$ *is trivial,*
- *the connection* ∇ *is logarithmic on* $E^{(1)}[*\Theta^{(0)}]$ *with poles on* $Y_1 \cup \cdots \cup Y_n \cup (\infty \times T)$ *and its residue along* $\infty \times T$ *is* $-K^{(1)} = -\mathrm{diag}(k^{(1)}_1, \ldots, k^{(1)}_d)$ *with*
$$\sum_{j=1}^d (k^{(1)}_j)^2 \leqslant \sum_{j=1}^d (k^{(0)}_j)^2 - 2.$$

Local equation for Θ— continuation

- If $K^{(1)} = 0$, then $E^{(1)}[*\Theta^{(0)}] = E[*\Theta^{(0)}]$ and $\Theta \subset \Theta_0$.

Local equation for Θ— continuation

- If $K^{(1)} = 0$, then $E^{(1)}[*\Theta^{(0)}] = E[*\Theta^{(0)}]$ and $\Theta \subset \Theta_0$.
- If $K^{(1)} \neq 0$, the matrix of ∇: $\sum_i B_i^{(1)}(a) \dfrac{d(x - a_i)}{(x - a_i)}$,

  $\sum_i B_i^{(1)}(a) \equiv K^{(1)}$ and the $B_i^{(1)}(a)$ satisfy the Schlesinger system.

Local equation for Θ— continuation

- If $K^{(1)} = 0$, then $E^{(1)}[*\Theta^{(0)}] = E[*\Theta^{(0)}]$ and $\Theta \subset \Theta_0$.
- If $K^{(1)} \neq 0$, the matrix of ∇: $\sum_i B_i^{(1)}(a) \dfrac{d(x-a_i)}{(x-a_i)}$,

  $\sum_i B_i^{(1)}(a) \equiv K^{(1)}$ and the $B_i^{(1)}(a)$ satisfy the Schlesinger system.
  - Apply Lemma 1.

Local equation for Θ— continuation

- If $K^{(1)} = 0$, then $E^{(1)}[*\Theta^{(0)}] = E[*\Theta^{(0)}]$ and $\Theta \subset \Theta_0$.
- If $K^{(1)} \neq 0$, the matrix of ∇: $\sum_i B_i^{(1)}(a) \dfrac{d(x-a_i)}{(x-a_i)}$, $\sum_i B_i^{(1)}(a) \equiv K^{(1)}$ and the $B_i^{(1)}(a)$ satisfy the Schlesinger system.
  - Apply Lemma 1.

Lemma 1. *There exists $\ell, m \in \{1, \dots, d\}$ such that $k_m^{(1)} - k_\ell^{(1)} \geqslant 2$ and $i \in \{1, \dots, n\}$ such that the (ℓ, m)-entry $B_{i,\ell m}^{(1)}(a)$ does not vanish identically.*

Local equation for Θ— continuation

- If $K^{(1)}=0$, then $E^{(1)}[*\Theta^{(0)}]=E[*\Theta^{(0)}]$ and $\Theta\subset\Theta_0$.
- If $K^{(1)}\neq 0$, the matrix of ∇: $\sum_i B_i^{(1)}(a)\,\dfrac{d(x-a_i)}{(x-a_i)}$,
  $\sum_i B_i^{(1)}(a)\equiv K^{(1)}$ and the $B_i^{(1)}(a)$ satisfy the Schlesinger system.
  - Apply Lemma 1.
  - Get $\tau^{(1)}$ and $\Theta^{(1)}\supset\Theta^{(0)}$.

Local equation for Θ— continuation

- If $K^{(1)} = 0$, then $E^{(1)}[*\Theta^{(0)}] = E[*\Theta^{(0)}]$ and $\Theta \subset \Theta_0$.
- If $K^{(1)} \neq 0$, the matrix of ∇: $\sum_i B_i^{(1)}(a) \dfrac{d(x - a_i)}{(x - a_i)}$,

  $\sum_i B_i^{(1)}(a) \equiv K^{(1)}$ and the $B_i^{(1)}(a)$ satisfy the Schlesinger system.
  - Apply Lemma 1.
  - Get $\tau^{(1)}$ and $\Theta^{(1)} \supset \Theta^{(0)}$.
  - Apply Lemma 2 and get $E^{(2)}[*\Theta^{(1)}]$ and $K^{(2)}$.

Local equation for Θ— continuation

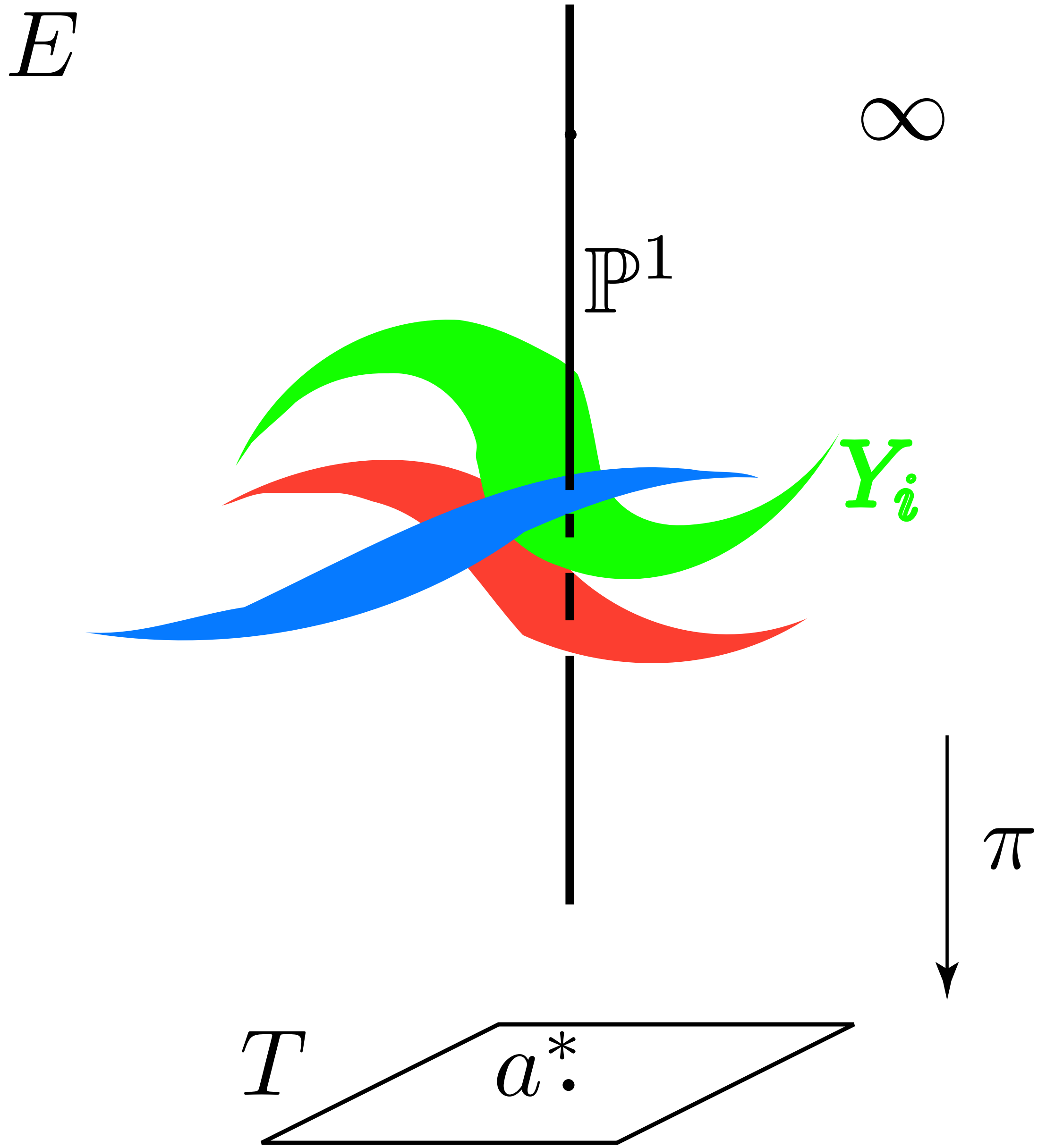
- If $K^{(1)} = 0$, then $E^{(1)}[*\Theta^{(0)}] = E[*\Theta^{(0)}]$ and $\Theta \subset \Theta_0$.
- If $K^{(1)} \neq 0$, the matrix of ∇: $\sum_i B_i^{(1)}(a) \dfrac{d(x-a_i)}{(x-a_i)}$, $\sum_i B_i^{(1)}(a) \equiv K^{(1)}$ and the $B_i^{(1)}(a)$ satisfy the Schlesinger system.
  - Apply Lemma 1.
  - Get $\tau^{(1)}$ and $\Theta^{(1)} \supset \Theta^{(0)}$.
  - Apply Lemma 2 and get $E^{(2)}[*\Theta^{(1)}]$ and $K^{(2)}$.
- etc. Get $\tau^{(\nu)}$, $\Theta^{(\nu)} \supset \Theta^{(\nu-1)}$, $E^{(\nu+1)}[*\Theta^{(\nu)}]$, $K^{(\nu+1)} = 0$.

Local equation for Θ— continuation

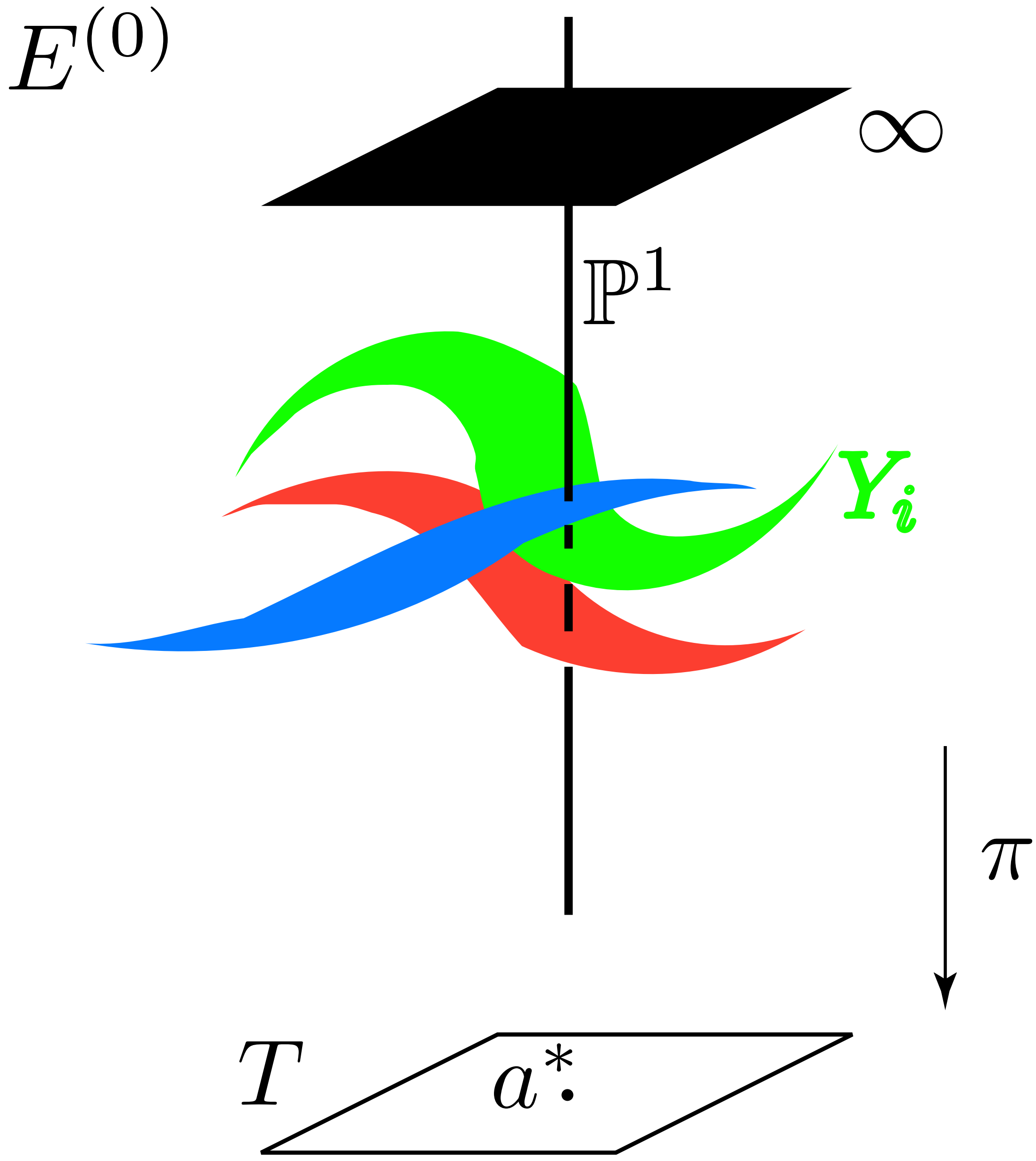
- If $K^{(1)} = 0$, then $E^{(1)}[*\Theta^{(0)}] = E[*\Theta^{(0)}]$ and $\Theta \subset \Theta_0$.
- If $K^{(1)} \neq 0$, the matrix of ∇: $\sum_i B_i^{(1)}(a) \dfrac{d(x-a_i)}{(x-a_i)}$,
  $\sum_i B_i^{(1)}(a) \equiv K^{(1)}$ and the $B_i^{(1)}(a)$ satisfy the Schlesinger system.
  - Apply Lemma 1.
  - Get $\tau^{(1)}$ and $\Theta^{(1)} \supset \Theta^{(0)}$.
  - Apply Lemma 2 and get $E^{(2)}[*\Theta^{(1)}]$ and $K^{(2)}$.
- etc. Get $\tau^{(\nu)}$, $\Theta^{(\nu)} \supset \Theta^{(\nu-1)}$, $E^{(\nu+1)}[*\Theta^{(\nu)}]$, $K^{(\nu+1)} = 0$.
- Then $E^{(\nu+1)}[*\Theta^{(\nu)}] = E[*\Theta^{(\nu)}]$ and $\Theta \subset \Theta^{(\nu)}$.

A picture illustrating the method

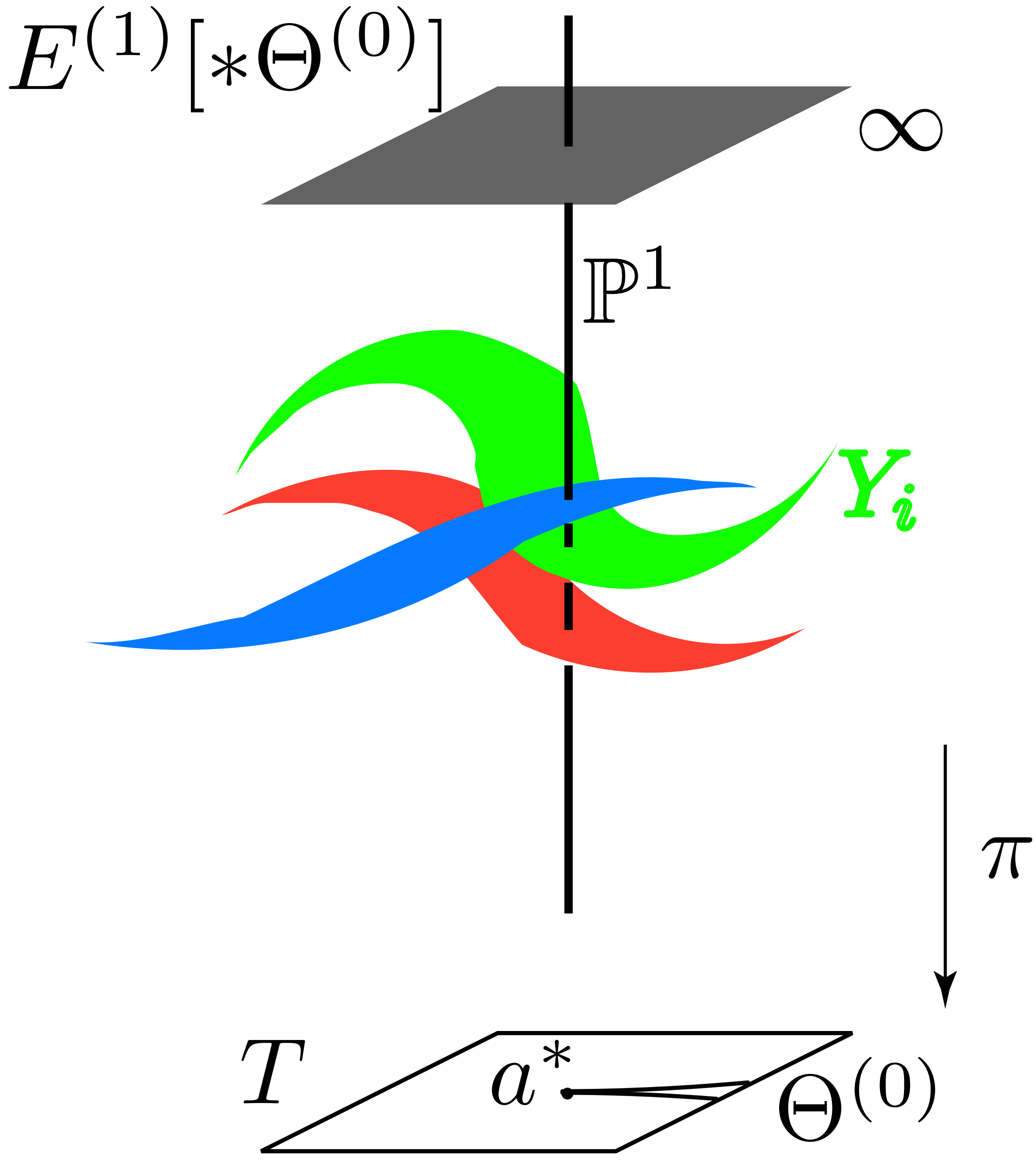
A picture illustrating the method



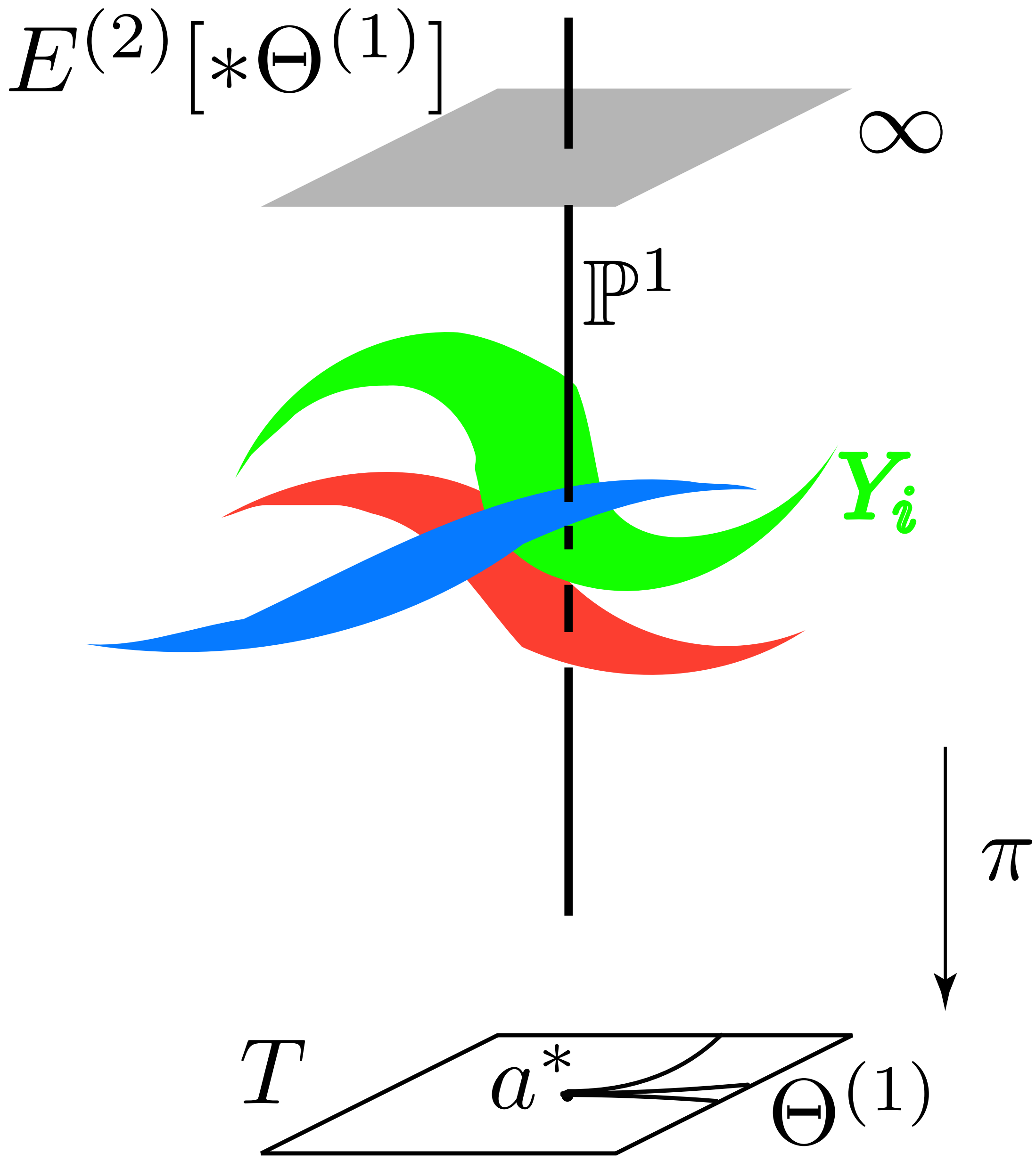
A picture illustrating the method



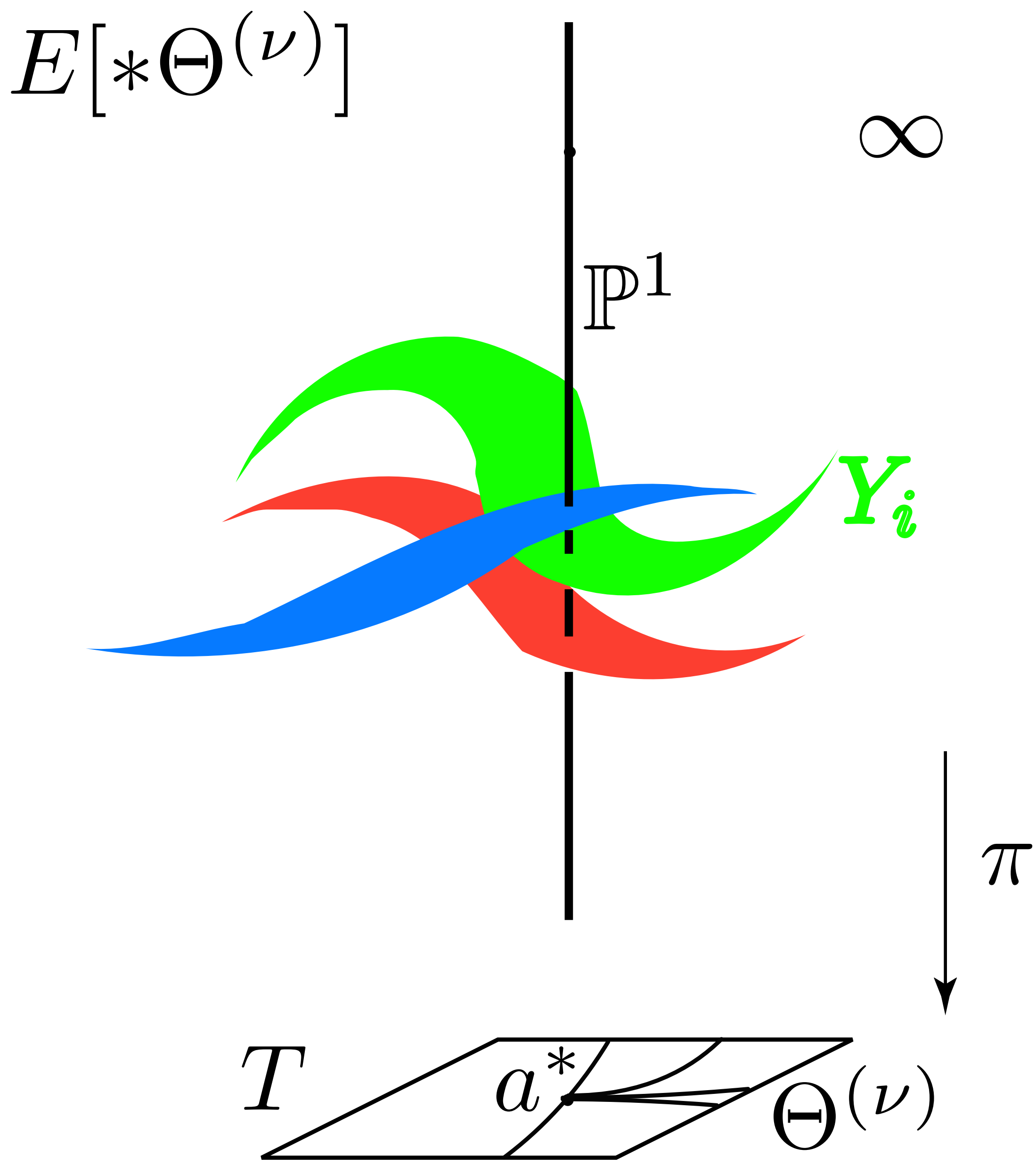
A picture illustrating the method



A picture illustrating the method



A picture illustrating the method



Theorem. *Set* $\widetilde{\tau} = \tau^{(0)} \cdot \tau^{(1)} \ldots \tau^{(\nu)}$.
Then $\widetilde{\tau}$ *is a local equation for* Θ.

Theorem. *Set* $\widetilde{\boldsymbol{\tau}} = \boldsymbol{\tau}^{(0)} \cdot \boldsymbol{\tau}^{(1)} \cdots \boldsymbol{\tau}^{(\nu)}$.
Then $\widetilde{\boldsymbol{\tau}}$ *is a local equation for* $\boldsymbol{\Theta}$.

Sketch of the proof.

Theorem. *Set* $\widetilde{\tau} = \tau^{(0)} \cdot \tau^{(1)} \dots \tau^{(\nu)}$.

Then $\widetilde{\tau}$ *is a local equation for* Θ.

Sketch of the proof.

$$\omega^{(\mu)} = \frac{1}{2} \sum_{i \neq j} \mathrm{tr}\big(B_i^{(\mu)}(a) B_j^{(\mu)}(a)\big) \frac{d(a_i - a_j)}{(a_i - a_j)}.$$

Theorem. *Set* $\widetilde{\tau} = \tau^{(0)} \cdot \tau^{(1)} \ldots \tau^{(\nu)}$.

Then $\widetilde{\tau}$ *is a local equation for* Θ.

Sketch of the proof.

$$\omega^{(\mu)} = \frac{1}{2} \sum_{i \neq j} \mathrm{tr}\big(B_i^{(\mu)}(a) B_j^{(\mu)}(a)\big) \frac{d(a_i - a_j)}{(a_i - a_j)}.$$

- $\omega^{(\nu)} = \dfrac{d\tau}{\tau}$ with $\Theta = \{\tau = 0\}$ (Theorem of Miwa).

Theorem. *Set* $\widetilde{\tau} = \tau^{(0)} \cdot \tau^{(1)} \ldots \tau^{(\nu)}$*.*

Then $\widetilde{\tau}$ *is a local equation for* Θ*.*

Sketch of the proof.

$$\omega^{(\mu)} = \frac{1}{2} \sum_{i \neq j} \mathrm{tr}\big(B_i^{(\mu)}(a) B_j^{(\mu)}(a)\big) \frac{d(a_i - a_j)}{(a_i - a_j)}.$$

- $\omega^{(\nu)} = \dfrac{d\tau}{\tau}$ with $\Theta = \{\tau = 0\}$ (Theorem of Miwa).
- $\omega^{(0)}$ is holomorphic and closed (Schlesinger).

Theorem. *Set* $\widetilde{\tau} = \tau^{(0)} \cdot \tau^{(1)} \dots \tau^{(\nu)}$*.*

Then $\widetilde{\tau}$ *is a local equation for* Θ*.*

Sketch of the proof.

$$\omega^{(\mu)} = \frac{1}{2} \sum_{i \neq j} \mathrm{tr}\big(B_i^{(\mu)}(a) B_j^{(\mu)}(a)\big) \, \frac{d(a_i - a_j)}{(a_i - a_j)}.$$

- $\omega^{(\nu)} = \dfrac{d\tau}{\tau}$ with $\Theta = \{\tau = 0\}$ (Theorem of Miwa).
- $\omega^{(0)}$ is holomorphic and closed (Schlesinger).
- $\omega^{(\mu)} - \omega^{(\mu-1)} = \dfrac{d\tau^{(\mu-1)}}{\tau^{(\mu-1)}}$.