ERRATA TO "POLARIZABLE TWISTOR $\mathscr{D}$-MODULES"

by

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(1) On page 21, line 5 and page 22, line 5, replace $\boldsymbol{L}^*_{z_o}$ with $\boldsymbol{L}i^*_{z_o}$.

(2) On page 30, Lemma 1.5.3 and in its proof, replace $\mathscr{C}^{\infty,\mathrm{an}}_{X|\mathbf{S}}$ with $\mathscr{C}^{\infty,0}_{\mathscr{X}|\mathbf{S}}$.

(3) On page 30, (1.5.5) reads

$$(t\eth_t - \beta \star z)u_{\beta,\ell} = -zu_{\beta,\ell-1} \tag{1.5.5}$$

and on page 31, (1.5.6) reads

$$(\overline{t\eth_t - \beta \star z})u_{\beta,\ell} = \frac{1}{z}u_{\beta,\ell-1}. \tag{1.5.6}$$

(4) On page 32, line 8, the isomorphism $\mathscr{T}^*(-k) \to \mathscr{T}(k)^*$ is not the morphism obtained by adjunction of (1.6.3), but the inverse morphism obtained from (1.6.3) where we replace $\mathscr{T}$ by $\mathscr{T}^*$. The choice of (1.6.3) is universal and holds for any $\mathscr{T}$.

(5) On page 33, 4th line of 1.6.b, replace $\mathscr{C}^{\infty,\mathrm{an}}_{X|\mathbf{S}}$ with $\mathscr{C}^{\infty,0}_{\mathscr{X}|\mathbf{S}}$.

(6) On page 48, the text of Remark 2.2.1 has to be replaced by the following text:

***Remark 2.2.1.* —** We have seen that the sesquilinear pairing C takes values in $\mathscr{C}^{\infty,0}_{\mathscr{X}|\mathbf{S}}$, according to Lemma 1.5.3. So the restriction to x_o of each component of the smooth twistor structure is well defined. Then, according to (2.1.1), C takes values in $\mathscr{C}^{\infty,\mathrm{an}}_{X|\mathbf{S}}$. It is also nondegenerate and gives a gluing of $\mathscr{H}'^*$ with $\overline{\mathscr{H}''}$, defining thus a $\mathscr{C}^{\infty,\mathrm{an}}_{X\times\mathbb{P}^1}$-bundle $\widetilde{\mathscr{H}}$ on $X \times \mathbb{P}^1$.

(7) On page 89, formulas (3.6.4)($*$) and (3.6.5)($*$), the exponent of the Γ factor is $-L$, not L.

(8) On page 90, second line after Remark 3.6.8, read "with respect to s" instead of "with respect to $\mathbf{S}$".

(9) The statement of Lemma 3.6.33 (which is not used in the text) has to be replaced with

$$\left\langle \phi_{t,0} C([m_0'], \overline{[m_0'']}), \bullet \right\rangle = \operatorname{Res}_{s=0} \frac{-1}{s} \left\langle (|t|^{2s} - s) C(m_0', \overline{m_0''}), \bullet \wedge \chi(t) \tfrac{i}{2\pi} dt \wedge d\bar{t} \right\rangle.$$

Proof. — We write $m_0'' = \eth_t m_{-1}'' + \mu_{<0}''$. By definition,

$$\begin{aligned} \left\langle \phi_{t,0} C([m_0'], \overline{[m_0'']}), \varphi \right\rangle &= \operatorname{Res}_{s=0} \left\langle C(m_0', \overline{\eth_t m_{-1}''}), \varphi \wedge I_{\widehat{\chi}} \chi \tfrac{i}{2\pi} dt \wedge d\bar{t} \right\rangle \\ &= \operatorname{Res}_{s=0} \left\langle C(m_0', \overline{m_{-1}''}), \varphi \wedge (\eth_t I_{\widehat{\chi}}) \chi \tfrac{i}{2\pi} dt \wedge d\bar{t} \right\rangle \\ &= -z^{-1} \operatorname{Res}_{s=-1} \left\langle C(m_0', \overline{m_{-1}''}), \varphi \wedge t|t|^{2s} \chi(t) \tfrac{i}{2\pi} dt \wedge d\bar{t} \right\rangle, \end{aligned}$$

by (3.6.23). On the other hand,

$$\begin{aligned} &\operatorname{Res}_{s=0} \frac{-1}{s} \left\langle |t|^{2s} C(m_0', \overline{\eth_t m_{-1}''}), \varphi \wedge \chi(t) \tfrac{i}{2\pi} dt \wedge d\bar{t} \right\rangle \\ &\quad = \operatorname{Res}_{s=-1} \frac{-1}{s+1} \left\langle C(m_0', \overline{\eth_t m_{-1}''}), \varphi \wedge |t|^{2(s+1)} \chi(t) \tfrac{i}{2\pi} dt \wedge d\bar{t} \right\rangle \\ &\quad = \operatorname{Res}_{s=-1} \frac{1}{s+1} \left\langle C(m_0', \overline{m_{-1}''}), \varphi \wedge \eth_t (|t|^{2(s+1)} \chi(t)) \tfrac{i}{2\pi} dt \wedge d\bar{t} \right\rangle \\ &\quad = -z^{-1} \operatorname{Res}_{s=-1} \left\langle C(m_0', \overline{m_{-1}''}), \varphi \wedge t|t|^{2s} \chi(t) \tfrac{i}{2\pi} dt \wedge d\bar{t} \right\rangle \\ &\qquad + \operatorname{Res}_{s=-1} \frac{1}{s+1} \left\langle C(m_0', \overline{m_{-1}''}), \varphi \wedge |t|^{2(s+1)} \eth_t \chi(t) \tfrac{i}{2\pi} dt \wedge d\bar{t} \right\rangle \\ &\quad = -z^{-1} \operatorname{Res}_{s=-1} \left\langle C(m_0', \overline{m_{-1}''}), \varphi \wedge t|t|^{2s} \chi(t) \tfrac{i}{2\pi} dt \wedge d\bar{t} \right\rangle \\ &\qquad - \left\langle C(m_0', \overline{\eth_t m_{-1}''}), \varphi \wedge \chi \tfrac{i}{2\pi} dt \wedge d\bar{t} \right\rangle \end{aligned}$$

and

$$\operatorname{Res}_{s=0} \frac{-1}{s} \left\langle |t|^{2s} C(m_0', \overline{\mu_{<0}''}), \varphi \wedge \chi(t) \tfrac{i}{2\pi} dt \wedge d\bar{t} \right\rangle = -\left\langle C(m_0', \overline{\mu_{<0}''}), \varphi \wedge \chi \tfrac{i}{2\pi} dt \wedge d\bar{t} \right\rangle. \quad \square$$

(10) On page 119, in the statement of Corollary 4.2.9, replace $w+1$ with w.

(11) On page 121, the argument given on lines 10–14 is not correct, as the inverse image by the projection is not known to be a polarizable twistor $\mathscr{D}$-module. One can argue as follows.

Choose a finite morphism $\pi : Z \to Z'$ with Z' smooth and projective (a projective line, for instance) and consider the composed morphism $\nu \circ \pi : \widetilde{Z} \to Z'$. On $Z^o \subset \widetilde{Z}$, the object $(\mathscr{T}, \mathscr{S})$ defines a harmonic bundle (H, D_E'', θ_E, h) in the sense of C. Simpson **[3]**, according to the correspondence of Lemma 2.2.2 on Z^o. We can restrict Z^o so that $\pi : Z^o \to Z'^o$ is a finite covering. We wish to show that the eigenvalues of the Higgs field are (multivalued) meromorphic one-forms, with a pole of order at most one at each puncture, and a purely imaginary residue at any such punctures. Indeed, this will imply that the harmonic bundle (H, D_E'', θ_E, h) on Z^o is tame on $\widetilde{Z}$, and that its parabolic filtration at the punctures is the trivial one, so, by **[3]**, the corresponding local system is semisimple.

It is then enough to prove that such a property is satisfied for the direct image $\pi_*(H, D''_E, \theta_E, h)$ on Z'^o, as locally the covering is trivial (in a local coordinate t on $\widetilde{Z}$ and t' on Z' for which $\pi(t) = t' = t^q$, we have $dt'/t' = q\,dt/t$, and, if the eigenvalues of θ'_E are written as $\alpha(t)dt/t$, the eigenvalues of $\pi_*\theta'_E$ are of the form $\frac{1}{q}\alpha(\zeta t)\frac{dt'}{t'}$, with $\zeta^q = 1$; hence the condition on eigenvalues is satisfied for θ'_E if and only if it is satisfied for $\pi_*\theta'_E = \theta'_{\pi_* E}$).

Now, a particular case of Theorem 6.1.1 (the case when π is finite) implies that $\pi_+(\mathscr{T}, \mathscr{S})$ is an object of $\mathrm{MT}^{(\mathrm{r})}(Z, 0)^{(\mathrm{p})}$, and we apply the correspondence of Theorem 5.0.1.

(12) On page 127, line -7: replace "for some integers a_k" with "for some coefficients $a_k(z)$".

(13) On page 135, the line after (5.3.5), read $\mathscr{O}_{\mathscr{X}}$ instead of $\mathscr{O}_{\mathscr{Q}}$.

(14) On page 156, line -1 and page 157, line 1, replace $n_j + \beta_j = -1$ by $n_j + \beta_j = 0$, and $\ell_z(n_j + \beta_j) = -1$ by $\ell_z(n_j + \beta_j) = 0$. This does not affect thre reasoning.

(15) On page 167, line 2: it is implicitly understood that $\omega_{\beta,\ell,k}$ is holomorphic even at $t = 0$, although the previous reasoning only gives the holomorphy away from $t = 0$. The argument that $(\mathcal{D}'_z \eta_{\neq(0,0)})_{\neq(0,0)}$ is L^2 has to be corrected. I thank T. Mochizuki for pointing out the mistake and providing the following proof.

(a) Let us set $\widetilde{\omega}_{\beta,\ell,k} = t\omega_{\beta,\ell,k}$, which is holomorphic on $D^* \times \mathrm{nb}(z_o)$. Assume first (see (b) below) we have proved that $\widetilde{\omega}_{\beta,\ell,k} e'^{(z_o)}_{\beta,\ell,k}$ is L^2 when we fix z in $\mathrm{nb}(z_o)$. Then, if we expand $\widetilde{\omega}_{\beta,\ell,k} = \sum_{n\in\mathbb{Z}} \widetilde{\omega}_{\beta,\ell,k,n}(z)t^n$, we claim that the coefficients $\widetilde{\omega}_{\beta,\ell,k,n}(z)$ identically vanish when $n \leqslant -1$. In order to prove this, we can argue with z fixed. The L^2 condition we assume is that, for any $n \in \mathbb{Z}$, $|\widetilde{\omega}_{\beta,\ell,k,n}(z)|\, r^{n+\ell_z(q_{\beta,\zeta_o}+\beta)} \mathrm{L}(r)^{\ell/2-1} \in L^2_{\mathrm{loc}}(d\theta\, dr/r)$. But when $n \leqslant -1$ and $a < 1$ (as is $\ell_z(q_{\beta,\zeta_o}+\beta)$ for z near z_o), $r^{n+a}\mathrm{L}(r)^{k/2}$ does not belong to $L^2_{\mathrm{loc}}(d\theta, dr/r)$, hence the coefficients $\widetilde{\omega}_{\beta,\ell,k,n}(z)$ have to vanish when $n \leqslant -1$.

In order to conclude, we want to show that $\widetilde{\omega}_{\beta,\ell,k} e'^{(z_o)}_{\beta,\ell,k} dt/t$ is L^2_{loc}, while we have only assumed that $\widetilde{\omega}_{\beta,\ell,k} e'^{(z_o)}_{\beta,\ell,k}$ is so. If $\ell_{z_o}(q_{\beta,\zeta_o} + \beta) \neq 0$, multiplying by $\mathrm{L}(r)$ will not cause an escape from the L^2 space, as the L^2 condition is governed by terms like $r^{n+\ell_z(q_{\beta,\zeta_o}+\beta)}$. If $\ell_{z_o}(q_{\beta,\zeta_o} + \beta) = 0$, the previous argument is not valid if $n = 0$. But we precisely considered the $\neq (0,0)$ parts, so the corresponding coefficient $\widetilde{\omega}_{\beta,\ell,k,0}(z)$ is identically 0 by definition.

(b) Let us now fix $z \in \mathrm{nb}(z_o)$, that we still denote by z_o for simplicity. The operator $D_E + z_o\theta''_E - \overline{z_o}\theta'_E = \mathcal{D}''_{z_o} + \delta'_{z_o}$ is compatible the harmonic metric h on H by definition, and we have $\mathcal{D}'_{z_o} = z_o\delta'_{z_o} + (1 + |z_o|^2)\theta'_E$. If we know (*cf.* (c) below) that $(\mathcal{D}''_{z_o} + \delta'_{z_o})(\eta_{\neq(0,0)})$ is a section of $\mathscr{L}^1_{(2)}(H, h)$ then, by the definition of η, the same property holds for $\delta'_{z_o}(\eta_{\neq(0,0)})$. On the other hand, by the expression of Θ'_{z_o} given before (6.2.7), $\mathrm{L}(t)^{-1}\theta'_{z_o}(\eta_{\neq(0,0)})$ is also in $\mathscr{L}^1_{(2)}(H,h)$ (the term $\mathrm{L}(t)^{-1}$ is here to compensate the norm of dt/t). Therefore, we find that $\mathrm{L}(t)^{-1}\mathcal{D}'_{z_o}(\eta_{\neq(0,0)})$ is in $\mathscr{L}^1_{(2)}(H,h)$ and finally, by definition of ω, that $\mathrm{L}(t)^{-1}\omega$ is in $\mathscr{L}^1_{(2)}(H,h)$, so the assumption in (a) above is fulfilled.

(c) As $D_{z_o} \overset{\mathrm{def}}{=} \mathcal{D}''_{z_o} + \delta'_{z_o}$ is compatible with h, we have, for a C^∞_c section e of H on D^*:

$$0 = d^2h(e,\overline{e}) = 2\|D_{z_o}e\|^2_h + h(R_{z_o}e,\overline{e}) + h(e,\overline{R_{z_o}e}),$$

where R_{z_o} denotes the curvature operator of D_{z_o}, and where the (fiberwise) norm of $D_{z_o}e$ is computed with the metric h and the Poincaré metric (for the 1-form components). Arguing as in [**3**, page 737], we find the the L^2 norm of the operator R_{z_o} with respect to the metric h and the Poincaré metric is bounded by a constant. It follows that $\|D_{z_o}e\|_h \leqslant C\|e\|_h$ and therefore, if e moreover is a local section of $\mathscr{L}^0_{(2)}(H,h)$, then $D_{z_o}e$ is a local section of $\mathscr{L}^1_{(2)}(H,h)$. By density, we conclude that this holds for any local section of $\mathscr{L}^0_{(2)}(H,h)$. We apply this to $\eta_{\neq(0,0)}$ to get (b).

(16) On page 172, step (2) of the proof: the argument is not correct, since the spectral sequence is not as indicated, and the indices are not correct. A correct proof of this step has later been given [**1**, §18.4] by T. Mochizuki in the more general case of wild twistor $\mathscr{D}$-modules, by using moreover the weak Lefschetz theorem and Gysin morphisms, as originally does by M. Saito [**2**, §5.3.8].

References

[1] T. Mochizuki – *Wild harmonic bundles and wild pure twistor D-modules*, Astérisque, vol. 340, Société Mathématique de France, Paris, 2011.

[2] M. Saito – "Modules de Hodge polarisables", *Publ. RIMS, Kyoto Univ.* **24** (1988), p. 849–995.

[3] C. Simpson – "Harmonic bundles on noncompact curves", *J. Amer. Math. Soc.* **3** (1990), p. 713–770.

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