A TENTATIVE PROGRAMME FOR IrrMHM

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1. The programme

The general idea is that the irregular Hodge filtration of a $\mathcal{D}_X$-module underlying an object of $\mathsf{IrrMHM}(X)$ should have properties very similar to those of the Hodge filtration of a mixed Hodge module, as defined in [**Sai90**]. For example, we have already proved analogues of the Kodaira-Saito vanishing theorem [**Sab24**].

So the program should be to go further in the understanding of the category $\mathsf{IrrMHM}(X)$ in order to have enough properties similar to those of $\mathsf{MHM}(X)$.

A first possible application would be a theorem of the type Kollár-Saito [**Sai91**] (see Section 1.k) for the lowest piece of the irregular Hodge filtration. This could lead to theorems of the type Kawamata-Viehweg vanishing for this coherent sheaf.

One could also have a look at the generic vanishing results of Popa and Schnell [**PS13**]. Another possible application would be to consider the cohomology support loci of an object of $\mathsf{IrrMHM}(A)$, where A is an abelian variety and show their arithmeticity when the object of $\mathsf{IrrMHM}(A)$ has a $\mathbb{Z}$-structure (see [**Sch15b**]), on noting that their linearity is already proved in [**Sch15a**]. For the notion of $\mathbb{Q}$-structure on a holonomic $\mathscr{D}_X$-module, see [**Moc14**], and define the notion of $\mathbb{Z}$-structure correspondingly (see also [**Sab22**, §4]).

1.a. Specialization properties of the irregular Hodge filtration. Let X be a complex manifold and let $\mathcal{D}_X$ denote the sheaf of holomorphic differential operators. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module underlying an object of the category $\mathsf{IrrMHM}(X)$ of irregular mixed Hodge modules. It thus comes equipped with a finite nested family $F^{\mathrm{irr}}_{\alpha+\bullet}\mathcal{M}$ ($\alpha \in A \subset [0,1)$) of coherent F-filtrations, called its irregular Hodge filtration. The question we address is the description of the behaviour of the irregular Hodge filtration by specialization along a principal divisor.

The question is local and we can assume that the divisor is defined by a coordinate x on X, so that we can speak of the Kashiwara-Malgrange V-filtration with respect to x, the we denote by ${}^x\!V_\bullet$ (we use the normalization that $t_1\partial_{t_1} + \beta$ is nilpotent on ${}^x\!V_\beta/{}^x\!V_{<\beta}$). We then consider the nearby cycle and vanishing cycle functors $\psi_{x,\lambda}$ and $\phi_{x,1}$ for any $\lambda \in \mathbb{C}^*$ (in fact, $\lambda \in S^1$ is enough). They are realized as $\mathrm{gr}^{{}^x\!V}_\beta$ with $\beta \in [0,1)$ and $\lambda = \exp(2\pi \mathsf{i}\beta)$, and $\mathrm{gr}^{{}^x\!V}_1$ respectively.

We can then decompose the question in various steps.

(a) If $\mathcal{M}$ underlies an object of $\mathsf{IrrMHM}(X)$, do $\psi_{x,\lambda}\mathcal{M}$ and $\phi_{x,1}\mathcal{M}$ also underlie such objects?

(b) In general, given a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ with a coherent $F_\bullet\mathcal{D}_X$-filtration $F_\bullet\mathcal{M}$, how to a obtain from $F_\bullet\mathcal{M}$ a coherent F-filtration on $\psi_{x,\lambda}\mathcal{M}$ and $\phi_{x,1}\mathcal{M}$? More precisely,

> (i) is the naive definition $F_p\mathrm{gr}^V_\beta\mathcal{M} := F_p\mathcal{M} \cap {}^x\!V_\beta\mathcal{M}/F_p\mathcal{M} \cap {}^x\!V_{<\beta}\mathcal{M}$, that we call the *induced filtration*, a coherent F-filtration?
>
> (ii) if not, is there a general procedure to produce such a coherent F-filtration?

(c) In the setting of (a), if we have a positive answer to (ii), is the filtration obtained in (ii) from $F^{\mathrm{irr}}_\bullet\mathcal{M}$ equal to the irregular Hodge filtration of $\psi_{x,\lambda}\mathcal{M}$ and $\phi_{x,1}\mathcal{M}$?

1.b. What is known about Questions (a)–(c)**: the case of regular holonomic modules.** If $\mathcal{M}$ underlies a complex mixed Hodge module and if $F_\bullet\mathcal{M}$ is its Hodge filtration, which is then equal to its irregular Hodge filtration, the situation is well understood: the $\mathcal{D}_X$-modules $\psi_{x,\lambda}\mathcal{M}$ and $\phi_{x,1}\mathcal{M}$ are mixed Hodge modules and the Hodge filtration on them is that induced by $F_\bullet\mathcal{M}$, that is, $F_p\mathcal{M}\cap{}^x\!V_\beta\mathcal{M}/F_p\mathcal{M}\cap{}^x\!V_{<\beta}\mathcal{M}$. This behavior is encoded in the definition of a mixed Hodge module.

On the other hand, if $\mathcal{M}$ is regular holonomic in the neighborhood of the hypersurface $H = \{x = 0\}$ and underlies an object of $\mathsf{IrrMHM}(X)$, then T. Mochizuki [**Moc21**] has proved that $\psi_{x,\lambda}\mathcal{M}$ and $\phi_{x,1}\mathcal{M}$ also underlie objects of $\mathsf{IrrMHM}(X)$ and that the irregular Hodge filtration behaves as in the case of mixed Hodge modules (in such a case, this behavior is *not* encoded in the definition of irregular mixed Hodge modules). This generalizes the analysis of [**SY15**] near $v = 0$.

1.c. The case of exponential mixed Hodge modules. We recall that the category $\mathsf{IrrMHM}(X)$ contains a subcategory $\mathsf{EMHM}(X)$ of exponential mixed Hodge modules (see [**Moc21**]). This is the category of those mixed Hodge modules on $X \times \mathbb{A}^1_t$ whose pushforward to X is zero. More precisely, we consider a new variable t and the sheaf $\mathcal{D}_{X\times\mathbb{P}^1_t}$. Let $(\mathcal{N}, F_\bullet\mathcal{N})$ be a filtered regular holonomic $\mathcal{D}_{X\times\mathbb{P}^1_t}$-module underlying a mixed Hodge module on $X \times \mathbb{P}^1_t$. Let us denote by $\mathcal{E}^t$ the $\mathcal{D}_{X\times\mathbb{P}^1_t}$-module $(\mathcal{O}_{X\times\mathbb{P}^1_t}(*\infty), \mathrm{d} + \mathrm{d}t)$ and let π denote the projection $X \times \mathbb{P}^1_t \to \mathbb{P}^1_t$. The $\mathcal{D}_X$-module $\mathcal{M}$ defined as

$$\mathcal{M} = \mathcal{H}^0{}_{\mathrm{D}}\pi_*(\mathcal{N} \otimes \mathcal{E}^t)$$

underlies an object of $\mathsf{IrrMHM}(X)$. In such a case, we have the equality $\psi_{x,\lambda}(\mathcal{M}) = \mathcal{H}^0{}_{\mathrm{D}}\pi_*((\psi_{x,\lambda}\mathcal{N}) \otimes \mathcal{E}^t)$ and a similar equality for $\phi_{x,1}$, and since $\mathcal{N}$ is a mixed Hodge module, $\psi_{x,\lambda}\mathcal{N}$ underlies a mixed Hodge module, hence so does $\psi_{x,\lambda}(\mathcal{M})$ (and similarly with $\phi_{x,1}$), so that we have a positive answer to Question (a) in this case. However, this does not provide us with an answer to (c) (see Section 1.i).

1.d. Behavior of the irregular Hodge filtration with respect to exponential twists

If the nearby and vanishing cycles of $\mathcal{M}$ along (x) vanish identically, the previous discussion is empty. However, one can expect limiting properties for the irregular Hodge filtration. Let us recall that the definition of a mixed twistor D-module (which is the category where irregular Hodge modules live) makes use of nearby and vanishing cycles after a twist by the exponential of a function depending only of some root of x (called Deligne specializability property in [**Sab09**], see also [**Moc11**, §17.1]). However, even after such twists, the nearby and vanishing cycles may still be zero. It is thus necessary to allow twists by exponentials of arbitrary ramified meromorphic functions. This rises two more questions:

(4) Is the general exponential twist of an object of $\mathsf{IrrMHM}(X)$ still an object of this category?

(5) If so, how to describe the behavior of the irregular Hodge filtration after such an exponential twist?

1.e. Behavior of characteristic varieties. If we consider a general coherent F-filtration on a holonomic $\mathcal{M}$, we cannot expect a precise behavior with respect to specialization, but the behavior by specialization of the characteristic cycle $\mathrm{CCh}\,\mathcal{M}$, i.e., the Lagrangian cycle associated with $\mathrm{gr}^F\mathcal{M}$, gives us a hint of what can be expected.

Let us consider the specialization functor to the normal bundle T_HX instead of nearby and vanishing cycles. The specialization $\mathrm{Sp}_H(\mathcal{M}) = \bigoplus_\beta \mathrm{gr}^V_\beta\mathcal{M}$ of $\mathcal{M}$ is a $\mathcal{D}_{T_HX}$-module. If $\mathcal{M}$ is regular holonomic $\mathcal{D}_X$-module, then $\mathrm{CCh}(\mathrm{Sp}_H(\mathcal{M}))$ is obtained from $\mathrm{CCh}(\mathcal{M})$ by the geometric operation of specialization to the normal bundle of T^*_HX in T^*X (there is an identification between $T^*(T_HX)$ and the normal bundle of T^*_HX in T^*X). On the other hand, if $\mathcal{M}$ has irregular singularities, this property may fail: for example, it is possible that $\mathrm{Sp}_H(\mathcal{M}) = 0$ without $\mathcal{M}$ being zero, but the specialization of $\mathrm{CCh}(\mathcal{M})$ is never zero if $\mathcal{M}$ is not zero and H intersects the support of $\mathcal{M}$.

After the work of Y. Laurent on the second microlocalization, and in particular [**Lau87**], we are led to considering various specializations with slopes.

We consider the bi-filtration $FV_{p,k}\mathcal{D}_X = F_p\mathcal{D}_X \cap V_\ell\mathcal{D}_X$ and, for any linear form $L(p,k) = ap + bk$ with coprime positive integer coefficients a, b (whose quotient b/a is called the slope of L), the L-filtration

$$ {}^L\!FV_\ell(\mathcal{D}_X) = \sum_{L(p,j)\leqslant\ell} FV_{p,j}\mathcal{D}_X. \tag{1.1}$$

The notion of a coherent ${}^L\!FV$-filtration on a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is well-defined, and leads to a L-homogeneous characteristic cycle $\mathrm{CCh}^{(L)}\mathcal{M}$ in the cotangent space $T^*(T_HX)$ of the normal bundle T_HX, which is independent of the choice of the ${}^L\!FV$-filtration. Only a finite number of such linear forms are important: they are called the "slopes" of $\mathcal{M}$ along H. After the work of Y. Laurent and Z. Mebkhout [**LM99**], these slopes are the jumping indices of the Gevrey filtration of the irregularity complex of $\mathcal{M}$ along H.

1.f. Induced filtration. For any holonomic $\mathcal{D}_X$-module, one can define the notion of coherent FV-bi-filtration. There always exist locally such bi-filtrations, but they are far from being canonical or unique, hence may not exist globally. For such a local bi-filtration $FU_{\bullet,\bullet}\mathcal{M}$, we generate a V-filtration $U_\bullet\mathcal{M}$ by the formula $U_\alpha\mathcal{M} = \sum_p FU_{p,\alpha}\mathcal{M}$. It is a coherent V-filtration, hence admits a Bernstein polynomial, but there is no reason that we can find FU such that the associated U is the Kashiwara-Malgrange filtration.

Nevertheless, we can perform the following operation to obtain a coherent F-filtration on each $\mathrm{gr}^U_\alpha\mathcal{M}$. For each linear form L we can associate to $FU_{\bullet,\bullet}\mathcal{M}$ a coherent ${}^L\!FV$-filtration ${}^L\!FU_\bullet\mathcal{M}$ by the formula ${}^L\!FU_\lambda\mathcal{M} = \sum_{L(p,k)\leqslant\lambda} FU_{p,k}\mathcal{M}$. If the linear form L has a big enough slope and if its coefficient a is equal to 1, then one can expect that the induced filtration

$$\frac{{}^L\!FU_{L(p,k)}\mathcal{M}\cap U_k\mathcal{M}}{{}^L\!FU_{L(p,k)}\mathcal{M}\cap U_{k-1}\mathcal{M}}$$

is a coherent F-filtration of $U_k\mathcal{M}/U_{k-1}\mathcal{M}$.

If $\mathcal{M}$ underlies an object of $\mathsf{IrrMHM}(X)$, the question is then whether one can construct such a coherent bi-filtration (indexed by $A+\mathbb{Z}^2$ for some finite subset $A\subset[0,1)^2$) whose associated simple filtration for $a=0$ and $b=1$ is the canonical V-filtration ${}^x\!V_\bullet\mathcal{M}$ and that for $a=1$ and $b=0$ is $F^{\mathrm{irr}}_\bullet\mathcal{M}$. To answer this question, it is necessary to start from the construction of the irregular Hodge filtration. It is obtained from a Kashiwara-Malgrange filtration defined on the rescaled twistor

D-module that $\mathcal{M}$ underlies. So we are led to analyze the interaction of two Kashiwara-Malgrange filtrations, corresponding to two different functions, on an integrable mixed twistor D-module.

1.g. Strictly bi-specializable $\widetilde{\mathscr{R}}_{\mathscr{X}}$-modules. We use the notation of [**Sab24**, §1.e], which are similar to those of [**Moc21**]. Let x, y be two coordinate functions locally defined on X. We consider the bi-filtration $V_{\bullet,\bullet}(\widetilde{\mathscr{R}}_{\mathscr{X}})$ relative to these coordinates, defined by

$$V_{i,j}(\widetilde{\mathscr{R}}_{\mathscr{X}}) = {}^{x}V_i(\widetilde{\mathscr{R}}_{\mathscr{X}}) \cap {}^{y}V_j(\widetilde{\mathscr{R}}_{\mathscr{X}}), \quad i, j \in \mathbb{Z}.$$

Definition 1.2. We say that a coherent $\widetilde{\mathscr{R}}_{\mathscr{X}}$-module $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along (t_1, t_2) if there exist a finite set $A \subset [0,1)^2$ and a coherent bi-V-filtration $U_{\bullet,\bullet}\mathscr{M}$ indexed by $A + \mathbb{Z}^2$ such that

(1) $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along x and y separately with V-filtrations equal to ${}^{L}U_{\bullet}\mathscr{M}$ with L equal to $(1,0)$ and $(0,1)$ respectively, and indexed by the first (resp. second) projection A_{t_1} (resp. A_{t_2}) of A;

(2) for each L with positive coefficients and each $(\alpha, \beta) \in A + \mathbb{Z}^2$, the $\mathrm{gr}_0^{{}^{L}V}(\widetilde{\mathscr{R}}_{\mathscr{X}})$-module $\mathrm{gr}^{{}^{L}U}_{L(\alpha,\beta)}(\mathscr{M})$ is strict (i.e., has no z-torsion) and the operator

$$L(t_1\eth_{t_1} + \alpha z, t_2\eth_{t_2} + \beta z)$$

is nilpotent on it.

We do not claim that, if such a bi-filtration exists, it is unique, but each ${}^{L}V$-filtration satisfying (2) is unique (see Section 2.b). It is then denoted by ${}^{L}V_{\bullet}\mathscr{M}$. However, there exists a finite set Λ of slopes L (including $(1,0)$ and $(0,1)$), such that the saturated bi-filtration $V_{\alpha,\beta}\mathscr{M} = \bigcap_L {}^{L}V_{L(\alpha,\beta)}\mathscr{M}$ is a coherent bi-V-filtration indexed by $A + \mathbb{Z}^2$ and satisfies

$$V_{\alpha,\beta}\mathscr{M} = \bigcap_{L\in\Lambda} {}^{L}V_{L(\alpha,\beta)}\mathscr{M} \quad \text{(finite intersection)},$$

and the associated ${}^{L}V$-filtration is the unique ${}^{L}V$-filtration ${}^{L}V_{\bullet}\mathscr{M}$. In such a way, if $\mathscr{M}$ is locally strictly $\mathbb{R}$-specializable along a pair of functions (f, g), in the sense that the pushforward ${}_{\mathrm{D}}\iota_*\mathscr{M}$ by the graph inclusion $\iota : X \hookrightarrow X \times \mathbb{C}^2_{t_1,t_2}$ is locally strictly $\mathbb{R}$-specializable along (t_1, t_2), then a bi-V-filtration exists globally on X. We call it the canonical bi-V-filtration.

If L has coefficients $(1, b)$ with b large enough, we expect that the induced filtration $({}^{L}V_{L(\alpha,\beta)}\mathscr{M} \cap {}^{y}V_{\beta}\mathscr{M})/{}^{L}V_{L(\alpha,\beta)}\mathscr{M} \cap {}^{y}V_{<\beta}\mathscr{M}$ is the filtration ${}^{x}V_{\alpha}(\mathrm{gr}^{{}^{y}V}_{\beta}\mathscr{M})$, and in particular is independent of b large enough. The following theorem should be true.

Theorem 1.3. *If $\mathscr{M}$ underlies an integrable mixed twistor D-module, then it is strictly $\mathbb{R}$-specializable along any pair of functions (f, g).*

T. Mochizuki has proved this theorem when the underlying $\mathscr{D}_X$-module $\mathcal{M}$ is regular holonomic. In such a case, the set of slopes Λ is reduced to $(1,0)$ and $(0,1)$.

1.h. Application to the irregular Hodge filtration. We apply this to the rescaled module of an integrable mixed twistor D-module $\mathscr{M}$, with the variable τ instead of y.

Corollary 1.4. *If $\mathscr{M}$ underlies an object of* $\mathsf{IrrMHM}(X)$*, then for each holomorphic function $f : X \to \mathbb{C}$, the specialized integrable mixed twistor D-modules with underlying $\widetilde{\mathscr{R}}_{\mathscr{X}}$-modules $\psi_{f,\lambda}\mathscr{M}$ and $\phi_{f,1}\mathscr{M}$ are objects of* $\mathsf{IrrMHM}(X)$.

We now show how to compute the irregular Hodge filtration on $\psi_{f,\lambda}\mathcal{M}$ and $\phi_{f,1}\mathcal{M}$. By restricting to $\tau = z$, we obtain a coherent FV-filtration (a bi-filtration) denoted $FV_{\bullet,\bullet}\mathcal{M}$ on the underlying $\mathcal{D}_X$-module $\mathcal{M}$, indexed by $A + \mathbb{Z}^2$. This bi-filtration possesses the following properties:

(1) The V-filtration it generates by the formula $V_\alpha\mathcal{M} = \sum_p FV_{p,\alpha}\mathcal{M}$ is the Kashiwara-Malgrange filtration of $\mathcal{M}$ along (x).

(2) The F-filtration it generates by the formula $F_p\mathcal{M} = \sum_\alpha FV_{p,\alpha}\mathcal{M}$ is the irregular Hodge filtration $F_p^{\mathrm{irr}}\mathcal{M}$.

(3) For any linear form $L(p,k) = p + bk$ with $b \gg 0$, the L-filtration ${}^L\!FV_\bullet\mathcal{M}$ it generates by the formula ${}^L\!FV_\lambda\mathcal{M} = \sum_{p,\alpha\mid L(p,\alpha)\leqslant\lambda} FV_{p,\alpha}\mathcal{M}$ induces on each $\mathrm{gr}_\alpha^V\mathcal{M}$ the irregular Hodge filtration $F_p^{\mathrm{irr}}\mathrm{gr}_\alpha^V\mathcal{M}$.

1.i. The case of an exponential Hodge module. We take up the setting of Section 1.c. Let $(\mathcal{N}, F_\bullet\mathcal{N})$ be the filtered $\mathcal{D}_X$-module underlying such a mixed Hodge module and let $R_F\mathcal{N}$ be the associated Rees module. In such a case, the Fourier transform ${}^{\mathrm{F}}(R_F\mathcal{N})$ plays the role of the rescaled integrable twistor D-module. It is a module over the ring $R_{X,\tau} = R_F\mathcal{D}_X[\tau]\langle\eth_\tau\rangle$. On the other hand, the $\mathcal{D}_X$-module we are interested in is $\mathcal{M}$, which is the pushforward to X of $\mathcal{N}\otimes\mathcal{E}^t$ or, equivalently, the restriction to $\tau = 1$ of the Fourier transform of $\mathcal{N}$, that we also regard as obtained as the restriction to $z = 1$ of the restriction to $\tau = z$ of ${}^{\mathrm{F}}(R_F\mathcal{N})$.

The construction of the irregular Hodge filtration is done in the following way. The Fourier transform ${}^{\mathrm{F}}(R_F\mathcal{N})$ is strictly $\mathbb{R}$-specializable along the divisor (τ) and is moreover regular along (τ), meaning that each term ${}^\tau V_\alpha({}^{\mathrm{F}}(R_F\mathcal{N}))$, which is a priori coherent over ${}^\tau V_0(R_{X,\tau}) := R_X[\tau]\langle\tau\eth_\tau\rangle$, is in fact coherent over the ring $R_X[\tau]$. The restriction ${}^\tau V_\alpha({}^{\mathrm{F}}(R_F\mathcal{N}))|_{\tau=z}$ produces a coherent $R_F\mathcal{D}_X$-module which is the Rees module of the irregular Hodge filtration $F^{\mathrm{irr}}_{\alpha+\bullet}\mathcal{M}$.

Before restricting to $\tau = z$, we see that the question of the behavior with respect to the functors $\psi_{x,\lambda}$ and $\phi_{x,1}$ is related to the the question of how to describe the ${}^\tau V$-filtration of $\psi_{x,\lambda}{}^{\mathrm{F}}(R_F\mathcal{N})$ in terms of that of ${}^{\mathrm{F}}(R_F\mathcal{N})$. Can we just 'induce' the ${}^\tau V$-filtration of ${}^{\mathrm{F}}(R_F\mathcal{N})$ on $\mathrm{gr}^\beta_{{}^xV}({}^{\mathrm{F}}(R_F\mathcal{N}))$? In other words, do we have the quality

$$
{}^\tau V_\bullet\big(\mathrm{gr}^\beta_{{}^xV}({}^{\mathrm{F}}(R_F\mathcal{N}))\big) = \frac{{}^\tau V_\bullet({}^{\mathrm{F}}(R_F\mathcal{N})) \cap {}^xV^\beta({}^{\mathrm{F}}(R_F\mathcal{N}))}{{}^\tau V_\bullet({}^{\mathrm{F}}(R_F\mathcal{N})) \cap {}^xV^{>\beta}({}^{\mathrm{F}}(R_F\mathcal{N}))}\ ? \tag{1.5}
$$

The answer is 'no' in general, as suggested by the similar question before taking Fourier transform.

1.j. The situation before the partial Fourier transformation. We regard $\mathcal{N}$ as a regular holonomic $\mathcal{D}_{X\times\mathbb{P}^1_t}$-module which is equal to its localization $\mathcal{N}(*(X\times\infty))$. Instead of the ${}^{\tau}V$-filtration, we consider the ${}^{t}V$-filtration *along* $t=\infty$. Omitting the Hodge filtration for a moment, the question is now whether the Kashiwara-Malgrange ${}^{t}V$-filtration of $\mathcal{N}$ along $(X\times\infty)$ induces that of $\mathrm{gr}^{\beta}_{{}^{x}V}(\mathcal{N})$ by a formula similar to (1.5).

In order to understand what happens, let us consider the case where $X=\Delta_{t_1}$ is a small disc with coordinate x. If the singular support of $\mathcal{N}$ in the neighborhood of $(0,\infty)$ is reduced to $\Delta_{t_1}\times\infty$, then $\mathcal{N}$ is of normal crossing type and the commutation occurs. More precisely, in that case, the hypersurface $x=0$ is non characteristic with respect to $\mathcal{N}$ near $(0,\infty)$. Putting now the Hodge filtration into the picture, if $(\mathcal{N},F_\bullet)$ underlies a mixed Hodge module, then the same argument extends to $R_F\mathcal{N}$.

On the other hand, if the singular support of $\mathcal{N}$, which is a curve, contains branches at $(0,\infty)$ distinct from $\Delta_{t_1}\times\infty$, commutation does not occur in general. The question of the relation between both V-filtrations ${}^{x}V$ and ${}^{t}V$ has been analyzed in [**Sab87**].

1.k. Application to an analogue of a theorem of Kollár-Saito for the irregular Hodge filtration

Assume that $\mathscr{M}$ underlies an object of $\mathsf{IrrMHM}(X)$ with associated filtered holonomic $\mathcal{D}_X$-module $(\mathcal{M},F^{\mathrm{irr}}_\bullet\mathcal{M})$. Let $p_o(\mathcal{M})\in\mathbb{R}$ be any index p such that $F^{\mathrm{irr}}_{p-1}\mathcal{M}=0$ (there may be finitely many such indices, all contained in a semi-closed interval of length 1). Recall that we have already proved the following consequence of the Kodaira-Saito vanishing theorem: if X is projective and L is an ample line bundle, then

$$H^k(X,\omega_X\otimes F^{\mathrm{irr}}_{p_o}\mathcal{M}\otimes L)=0\quad\text{for } k>0.$$

One should be able to prove that, if $\mathcal{M}$ is supported on a closed analytic subset $Z\subset X$, then for each $p_o=p_o(\mathcal{M})$, the sheaf $\omega_X\otimes F^{\mathrm{irr}}_{p_o}\mathcal{M}$ is the sheaf-theoretic pushforward by the embedding $\iota:Z\hookrightarrow X$ of an $\mathscr{O}_Z$-coherent module $\mathcal{S}^{\mathrm{irr}}_{p_o}(Z,\mathcal{M})$.

Assume now that $\mathscr{M}$ underlies a *pure* object of $\mathsf{IrrMHM}(X)$ with pure support an irreducible closed analytic subset $Z\subset X$. Since $\mathscr{M}$ has no nonzero submodule supported on a proper subset of Z, it follows that for each $p_o=p_o(\mathcal{M})$, the $\mathscr{O}_Z$-coherent module $\mathcal{S}^{\mathrm{irr}}_{p_o}(Z,\mathcal{M})$ is torsion-free.

Let $f:X\to Y$ be a projective morphism to a complex manifold Y. By the decomposition theorem applied to the integrable twistor D-module that $\mathscr{M}$ underlies, the direct image ${}_{\mathrm{D}}f_*(\mathcal{M},F^{\mathrm{irr}}_\bullet\mathcal{M})$ decomposes (in a non canonical way) as $\bigoplus_k {}_{\mathrm{D}}f^{(k)}_*(\mathcal{M},F^{\mathrm{irr}}_\bullet\mathcal{M})[-k]$. For each k, we denote by $(\mathcal{M}^{(k)},F^{\mathrm{irr}}_\bullet\mathcal{M}^{(k)})$ the component of ${}_{\mathrm{D}}f^{(k)}_*(\mathcal{M},F^{\mathrm{irr}}_\bullet\mathcal{M})$ (recall that it underlies a pure object of $\mathsf{IrrMHM}(Y)$) with pure support equal to the irreducible analytic subset $T:=f(Z)$ of Y. We still denote by f the restriction $f|_Z:Z\to T$.

Theorem 1.6. *Under these conditions, the following properties hold for each k:*

(1) $p_o({}_{\mathrm{D}}f_*^{(k)}(\mathcal{M}))=p_o(\mathcal{M}^{(k)})=p_o(\mathcal{M})$,

(2) *and for each $p_o=p_o(\mathcal{M})$, we have*

$$\boldsymbol{R}f_*\mathcal{S}^{\mathrm{irr}}_{p_o}(Z,\mathcal{M})\simeq\bigoplus_k R^kf_*\mathcal{S}^{\mathrm{irr}}_{p_o}(Z,\mathcal{M})[-k]\simeq\bigoplus_k\mathcal{S}^{\mathrm{irr}}_{p_o}(T,\mathcal{M}^{(k)})[-k].$$

In particular, each coherent $\mathcal{O}_T$-module $R^kf_*\mathcal{S}^{\mathrm{irr}}_{p_o}(Z,\mathcal{M})$ is torsion-free.

Idea of proof. For (2), we prove it for the sheaves on X and Y respectively. Let x be a local coordinate on X defining a smooth hypersurface H, and let $FV_{\bullet,\bullet}\mathcal{M}$ be the bi-filtration considered in Section 1.h. We note that, since $FV_{p,\alpha}\mathcal{M}\subset F_p^{\mathrm{irr}}\mathcal{M}$ for any α, we have $FV_{p_o-1,\alpha}\mathcal{M}=0$. It follows that, for $L(p,k)=p+bk$ with $b\gg0$, we have ${}^L\!FV_{L(p_o-1,\alpha)}\mathcal{M}\subset V_{<\alpha}\mathcal{M}$ for any α. We deduce that $F^{\mathrm{irr}}_{p_o-1}\mathrm{gr}^V_\alpha\mathcal{M}=0$. Since $\mathscr{M}$ has pure support Z, $\mathcal{M}$ is a minimal extension along (x) and can : $\mathrm{gr}^V_{-1}\mathcal{M}\to\mathrm{gr}^V_0\mathcal{M}$ is onto. Furthermore, as $\mathrm{gr}^V_{-1}\mathscr{M}$ and $\mathrm{gr}^V_0\mathscr{M}$ underlie objects of $\mathsf{IrrMHM}(H)$, can strictly shifts the irregular Hodge filtration by 1. It follows that $F^{\mathrm{irr}}_{p_o}\mathrm{gr}^V_0\mathcal{M}=0$.

The argument of Saito can now be applied to show that, for any k and any component $\mathcal{M}'^{(k)}$ of ${}_{\mathrm{D}}f_*^{(k)}(\mathcal{M})$ with pure support in $T'\neq T$, we have $p_o(\mathcal{M}'^{(k)})>p_o(\mathcal{M}^{(k)})=p_o(\mathcal{M})$. □

2. L-specializability of integrable $\mathscr{R}_{\mathscr{X}}$-modules

We fix two holomorphic functions g_1,g_2 on a complex manifold X. By a linear form L we will mean a linear form of two variables $L(s_1,s_2)=\ell_1s_1+\ell_2s_2$ with $\ell_1,\ell_2\in\mathbb{N}^*$ coprime (we also consider the cases $(\ell_1,\ell_2)=(1,0)$ and $(0,1)$).

2.a. The bi-V-filtration of $\widetilde{\mathscr{R}}_{\mathscr{Y}}$. We can work either with the category of filtered $\mathcal{D}_X$-modules, that we consider as modules over the Rees ring $R_F\mathcal{D}_X$, or with the category of integrable $\mathscr{R}_{\mathscr{X}}$-modules, that we consider as modules over the ring $\widetilde{\mathscr{R}}_{\mathscr{X}}$. For the sake of simplicity, we denote both rings by $\widetilde{\mathscr{R}}_{\mathscr{X}}$.

Let $\iota:X\hookrightarrow Y=X\times\mathbb{C}^2$ denote the graph embedding of g_1,g_2 with coordinates t_1,t_2 on $\mathbb{C}^2$. We consider the sheaf of rings $\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1,t_2]\langle\eth_{t_1},\eth_{t_2}\rangle$ on $\mathscr{X}$ and its analytification $\widetilde{\mathscr{R}}_{\mathscr{Y}}$ on $\mathscr{Y}$. On each of these rings we consider the V-filtrations along (t_1) and (t_2) indexed by $\mathbb{Z}$, that we denote by ${}^{t_1}V_\bullet$ and ${}^{t_2}V_\bullet$ respectively. We also consider the bi-V-filtration indexed by $\mathbb{Z}^2$ defined as $V_{k_1,k_2}={}^{t_1}V_{k_1}\cap{}^{t_2}V_{k_2}$. In particular, we have

$$V_{0,0}(\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1,t_2]\langle\eth_{t_1},\eth_{t_2}\rangle)=\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1,t_2]\langle t_1\eth_{t_1},t_2\eth_{t_2}\rangle,$$

and a similar expression for $\widetilde{\mathscr{R}}_{\mathscr{Y}}$.

Given the linear form L, we also consider the ${}^LV$-filtration indexed by $\mathbb{Z}$ on these rings defined by ${}^LV_k=\sum_{L(k_1,k_2)\leqslant k}V_{k_1,k_2}$. In particular, if $(\ell_1,\ell_2)\neq(1,0),(0,1)$, the restriction of ${}^LV$ to $t_1\neq0$ (in the algebraic sense of inverting t_1 or the analytic sense) is equal to that of $\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1,t_2]\langle\eth_{t_1},\eth_{t_2}\rangle$, and similarly by exchanging 1 and 2, so that each graded object $\mathrm{gr}_k^{{}^LV}\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1,t_2]\langle\eth_{t_1},\eth_{t_2}\rangle$ is supported on $\{t_1=t_2=0\}$. Therefore, when considering a linear form L distinct from a coordinate linear form, we usually sheaf-theoretically restrict the sheaves to the subspace $\{t_1=t_2=0\}$. For example, ${}^LV_0(\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1,t_2]\langle\eth_{t_1},\eth_{t_2}\rangle)$ is the free $\widetilde{\mathscr{R}}_{\mathscr{X}}$-module having as a basis the monomials $t_1^{a_1}t_2^{a_2}\eth_{t_1}^{b_1}\eth_{t_2}^{b_2}$ for $(a_1,a_2,b_1,b_2)\in\mathbb{N}^4$ satisfying

$$\ell_1(b_1-a_1)+\ell_2(b_2-a_2)\leqslant 0.$$

As an example, $t_1\eth_{t_2}$ belongs to ${}^LV_0$ iff $\ell_1\geqslant\ell_2$. Furthermore,

$$\mathrm{gr}_0^{{}^LV}(\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1,t_2]\langle\eth_{t_1},\eth_{t_2}\rangle)=\widetilde{\mathscr{R}}_{\mathscr{X}}\langle t_1\eth_{t_1},t_2\eth_{t_2},t_2^{\ell_1}\partial_{t_1}^{\ell_2},t_1^{\ell_2}\partial_{t_2}^{\ell_1}\rangle,$$

and both t_1,t_2 act by zero on this graded ring.

We also consider pairs $\Gamma=(L',L'')$ of such linear forms such that $\det(L',L'')=1$, that is, $\ell'_1\ell''_2-\ell''_1\ell'_2=1$. We identify Γ with the cone of apex 0 in the first quadrant of $\mathbb{R}^2$ generated by the half-lines $\mathbb{R}_+L',\mathbb{R}_+L''$. We say that such a cone Γ is *smooth*. Then (L',L'') induces an isomorphism $\mathbb{Z}^2\xrightarrow{\sim}\mathbb{Z}^2$. We define the Γ-bifiltration on these rings, as the bi-filtration indexed by $\mathbb{Z}^2$ defined as

$${}^\Gamma V_{k',k''}(\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1,t_2]\langle\eth_{t_1},\eth_{t_2}\rangle)=\sum_{\substack{L'(k_1,k_2)\leqslant k'\\ L''(k_1,k_2)\leqslant k''}}V_{k_1,k_2}(\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1,t_2]\langle\eth_{t_1},\eth_{t_2}\rangle),$$

and a similar definition for $\widetilde{\mathscr{R}}_{\mathscr{Y}}$. In particular, ${}^\Gamma V_{0,0}(\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1,t_2]\langle\eth_{t_1},\eth_{t_2}\rangle)$ is the free $\widetilde{\mathscr{R}}_{\mathscr{X}}$-module having as a basis the monomials $t_1^{a_1}t_2^{a_2}\eth_{t_1}^{b_1}\eth_{t_2}^{b_2}$ for $(a_1,a_2,b_1,b_2)\in\mathbb{N}^4$ satisfying

$$\ell'_1(b_1-a_1)+\ell'_2(b_2-a_2)\leqslant 0\quad\text{and}\quad\ell''_1(b_1-a_1)+\ell''_2(b_2-a_2)\leqslant 0.$$

Lemma 2.1. *The ${}^\Gamma V$-filtration of the sheaves of rings $\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1,t_2]\langle\eth_{t_1},\eth_{t_2}\rangle$ and $\widetilde{\mathscr{R}}_{\mathscr{Y}}$ satisfies*

$${}^\Gamma V_{k_-,k_+}={}^{L'}V_{k_-}\cap{}^{L''}V_{k_+}\quad\forall\,(k_-,k_+)\in\mathbb{Z}^2.$$

Proof. The assertion follows from the description in terms of a basis. □

If we identify as above

$$\mathrm{gr}_0^{{}^{L'}V}(\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1,t_2]\langle\eth_{t_1},\eth_{t_2}\rangle)=\widetilde{\mathscr{R}}_{\mathscr{X}}\langle t_1\eth_{t_1},t_2\eth_{t_2},t_2^{\ell'_1}\partial_{t_1}^{\ell'_2},t_1^{\ell'_2}\partial_{t_2}^{\ell'_1}\rangle,$$

the L''-degree of the terms $t_1\eth_{t_1},t_2\eth_{t_2}$ is zero, that of $t_2^{\ell'_1}\partial_{t_1}^{\ell'_2}$ is -1 and that of $t_1^{\ell'_2}\partial_{t_2}^{\ell'_1}$ is 1. In particular,

$$\mathrm{gr}_{0,0}^{{}^\Gamma V}(\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1,t_2]\langle\eth_{t_1},\eth_{t_2}\rangle)=\mathrm{gr}_{0,0}^{V}(\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1,t_2]\langle\eth_{t_1},\eth_{t_2}\rangle)=\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1\eth_{t_1},t_2\eth_{t_2}].$$

2.b. Strict L-specializability along g_1, g_2. Let $\mathscr{M}$ be a coherent $\widetilde{\mathscr{R}}_{\mathscr{X}}$-module and let $\mathscr{M}_g := {}_{\mathrm{D}}\iota_*\mathscr{M}$ be its pushforward, either in the partial algebraic sense or in the analytic sense. A strict ${}^{L}V$-filtration on $\mathscr{M}_g$ is a coherent ${}^{L}V$-filtration ${}^{L}V_\bullet(\mathscr{M}_g)$ indexed by a subset ${}^{L}A+\mathbb{Z}$ with ${}^{L}A \subset [-1,0)$ finite, such that for each $\lambda \in {}^{L}A+\mathbb{Z}$, the coherent $\mathrm{gr}_0^{{}^{L}V}(\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1,t_2]\langle\eth_{t_1},\eth_{t_2}\rangle)$-module $\mathrm{gr}_\lambda^{{}^{L}V}(\mathscr{M}_g)$ is *strict* and $L(t_1\eth_{t_1}, t_2\eth_{t_2})+\lambda z$ is nilpotent on it. Due to the strictness property, if such a filtration exists, it is unique and we call it canonical. We then say that $\mathscr{M}$ is *strictly L-specializable along* (t_1,t_2). The full subcategory of $\mathsf{Mod}_{\mathrm{coh}}(\widetilde{\mathscr{R}}_{\mathscr{X}})$ consisting of strictly L-specializable modules is abelian and any morphism is strict with respect to the canonical ${}^{L}V$-filtrations.

Lemma 2.2. *In the above setting, $\mathscr{M}$ is strictly L-specializable along (t_1,t_2) in the partial algebraic sense if and only if it is so in the analytic sense.*

Proof. Assume that $\mathscr{M}$ is strictly L-specializable along (t_1,t_2) in the partial algebraic sense. Then tensoring the ${}^{L}V$-filtration with $\mathscr{O}_{\mathscr{Y}}$ yields a ${}^{L}V$-filtration in the analytic setting which satisfies the strictness property due to flatness of $\mathscr{O}_{\mathscr{Y}}$ over $\mathscr{O}_{\mathscr{X}}[t_1,t_2]$.

Conversely, let us denote by $\overline{\mathscr{Y}}$ the partial compactification $\mathscr{X}\times\mathbb{P}$ (with $\mathbb{P}=\mathbb{P}^2$), by ∞ the divisor at infinity $\mathscr{X}\times\{\infty\}$, and by $p:\overline{\mathscr{Y}}\to\mathscr{X}$ the projection. On noting that $p_*\mathscr{O}_{\overline{\mathscr{Y}}}(*\infty)=\mathscr{O}_{\mathscr{X}}[t_1,t_2]$, we can argue as in [**DS03**, App. A] to show that p_* induces an equivalence of categories $\mathsf{Mod}_{\mathrm{coh}}(\mathscr{O}_{\overline{\mathscr{Y}}}(*\infty))\to\mathsf{Mod}_{\mathrm{coh}}(\mathscr{O}_{\mathscr{X}}[t_1,t_2])$.

As ${}^{L}V_k(\mathscr{M}_g)^{\mathrm{an}}$ is supported on the graph of (g_1,g_2), it extends in an obvious way as a coherent ${}^{L}V_0(\widetilde{\mathscr{R}}_{\overline{\mathscr{Y}}})(*\infty)$-module and the image of the latter by p_* is a coherent ${}^{L}V_0(\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1,t_2]\langle\eth_{t_1},\eth_{t_2}\rangle)$-module, whose pullback by p^* gives back ${}^{L}V_k(\mathscr{M}_g)^{\mathrm{an}}$. One deduces that this image (when k varies in ${}^{L}A+\mathbb{Z}$) satisfies the characteristic properties of the canonical ${}^{L}V$-filtration in the partial algebraic sense, hence is equal to it. □

The main result of this section is:

Theorem 2.3. *Assume that $\mathscr{M}$ is a $\widetilde{\mathscr{R}}_{\mathscr{X}}$-module underlying an integrable mixed twistor D-module. Then $\mathscr{M}$ is strictly L-specializable along (g_1,g_2).*

2.c. A preliminary identification. We consider the partial algebraic version of the L-deformation to the normal bundle of $X\times\{(0,0)\}$ in $X\times\mathbb{A}^2_{t_1,t_2}$ and its extension to $\mathscr{X}$. It is described by the morphism $q: X\times\mathbb{A}^3_{u,v,w}\to X\times\mathbb{A}^2_{t_1,t_2}$ induced by the morphism of rings

$$\begin{aligned}\mathscr{O}_{\mathscr{X}}[t_1,t_2] &\longrightarrow \mathscr{O}_{\mathscr{X}}[u,v,w]\\ (t_1,t_2) &\longmapsto (u^{\ell_1}v, u^{\ell_2}w).\end{aligned}$$

Let $\mathscr{M}$ be a (left) $\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1,t_2]\langle\partial_{t_1},\partial_{t_2}\rangle$-module. The localized pullback ${}_{\mathrm{D}}q^*\mathscr{M}(*u)$ is the $\widetilde{\mathscr{R}}_{\mathscr{X}}[u,u^{-1},v,w]\langle\eth_u,\eth_v,\eth_w\rangle$-module

$$ {}_{\mathrm{D}}q^*\mathscr{M}(*u)=\mathscr{O}_{\mathscr{X}}[u,u^{-1},v,w]\otimes_{\mathscr{O}_{\mathscr{X}}[t_1,t_2]}\mathscr{M}$$

with the $\widetilde{\mathscr{R}}_{\mathscr{X}}[u,u^{-1},v,w]\langle\eth_u,\eth_v,\eth_w\rangle$-action defined by

$$\eth_v(1\otimes m)=u^{\ell_1}\otimes\eth_{t_1}m,\quad \eth_w(1\otimes m)=u^{\ell_2}\otimes\eth_{t_2}m,\quad u\eth_u(1\otimes m)=1\otimes L(t_1\eth_{t_1},t_2\eth_{t_2})m.$$

Since the u-localization $\mathscr{O}_{\mathscr{X}}[u,u^{-1},v,w]$ is isomorphic to the free $\mathscr{O}_{\mathscr{X}}[t_1,t_2]$-module $\bigoplus_{k\in\mathbb{Z}}(u^k\otimes\mathscr{O}_{\mathscr{X}}[t_1,t_2])$, the module ${}_{\mathrm{D}}q^*\mathscr{M}(*u)$ is isomorphic to $\bigoplus_{k\in\mathbb{Z}}(u^k\otimes\mathscr{M})$ and, *if $\mathscr{M}$ is strict* the degree-k, term is recovered as $\ker(u\eth_u-L(v\eth_v,w\eth_w)-kz)$. Furthermore, if we consider $\widetilde{\mathscr{R}}_{\mathscr{X}}[u,u^{-1},v,w]\langle\eth_u,\eth_v,\eth_w\rangle$ as a graded ring where u has degree one, $u\eth_u$ has degree zero, v has degree $-\ell_1$, etc., then the previous decomposition of ${}_{\mathrm{D}}q^*\mathscr{M}(*u)$ makes it a graded module over the graded ring $\widetilde{\mathscr{R}}_{\mathscr{X}}[u,u^{-1},v,w]\langle\eth_u,\eth_v,\eth_w\rangle$.

We consider the ${}^{L}V$-filtration of $\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1,t_2]\langle\partial_{t_1},\partial_{t_2}\rangle$ defined so that a monomial $t_1^at_2^b\eth_{t_1}^c\eth_{t_2}^d$ has ${}^{L}V$-order equal to $L(c-a,d-b)$, and any local section of $\widetilde{\mathscr{R}}_{\mathscr{X}}$ has ${}^{L}V$-order zero. The L-specialization ring is the Rees ring

$$R_{{}^{L}V}(\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1,t_2]\langle\eth_{t_1},\eth_{t_2}\rangle)=\bigoplus_{\ell\in\mathbb{Z}}u^\ell\otimes{}^{L}V_\ell(\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1,t_2]\langle\eth_{t_1},\eth_{t_2}\rangle).$$

The Rees construction induces an equivalence between (coherent) ${}^{L}V$-filtrations of an $\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1,t_2]\langle\partial_{t_1},\partial_{t_2}\rangle$-module $\mathscr{M}$ and graded (coherent) $R_{{}^{L}V}(\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1,t_2]\langle\eth_{t_1},\eth_{t_2}\rangle)$-modules which have no u-torsion and whose restriction to $u=1$ gives back $\mathscr{M}$.

As $\eth_{t_1}$ (resp. $\eth_{t_2}$) is of ${}^{L}V$-degree ℓ_1 (resp. ℓ_2), and opposite degrees for t_1 (resp. t_2), we can identify the graded ring $R_{{}^{L}V}(\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1,t_2]\langle\eth_{t_1},\eth_{t_2}\rangle)$ (where u has degree one and the other terms have degree zero) with the ring $\widetilde{\mathscr{R}}_{\mathscr{X}}[u,v,w]\langle\eth_v,\eth_w\rangle$ by setting $v=u^{-\ell_1}t_1$, etc., and the grading is such that u has degree one, v has degree $-\ell_1$, etc.

On the other hand, we consider the ${}^{u}V$-filtration ${}^{u}V_\bullet(\widetilde{\mathscr{R}}_{\mathscr{X}}[u,u^{-1},v,w]\langle\eth_u,\eth_v,\eth_w\rangle)$, with ${}^{u}V_0(\widetilde{\mathscr{R}}_{\mathscr{X}}[u,u^{-1},v,w]\langle\eth_u,\eth_v,\eth_w\rangle)=\widetilde{\mathscr{R}}_{\mathscr{X}}[u,v,w]\langle u\eth_u,\eth_v,\eth_w\rangle$.

Lemma 2.4. *Let $\mathscr{M}$ be a strict $\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1,t_2]\langle\partial_{t_1},\partial_{t_2}\rangle$-module and let $U_\bullet({}_{\mathrm{D}}q^*\mathscr{M}(*u))$ be an exhaustive ${}^{u}V$-filtration of ${}_{\mathrm{D}}q^*\mathscr{M}(*u)$ such that each $\mathrm{gr}_k^U({}_{\mathrm{D}}q^*\mathscr{M}(*u))$ is strict (i.e., with no z-torsion). Then,*

(1) *there exists a unique exhaustive ${}^{L}V$-filtration ${}^{L}U_\bullet(\mathscr{M})$ such that, with respect to the grading of ${}_{\mathrm{D}}q^*\mathscr{M}(*u)$, $U_0({}_{\mathrm{D}}q^*\mathscr{M}(*u))$ is graded and decomposes as the direct sum $\bigoplus_{k\in\mathbb{Z}}u^k\otimes{}^{L}U_k(\mathscr{M})$;*

(2) *if $U_\bullet({}_{\mathrm{D}}q^*\mathscr{M}(*u))$ is ${}^{u}V_0(\widetilde{\mathscr{R}}_{\mathscr{X}}[u,u^{-1},v,w]\langle\eth_u,\eth_v,\eth_w\rangle)$-coherent, then ${}^{L}U_\bullet\mathscr{M}$ is a coherent ${}^{L}V$-filtration of $\mathscr{M}$;*

(3) *if ${}_{\mathrm{D}}q^*\mathscr{M}(*u)$ is strictly $\mathbb{R}$-specializable along (u), then $\mathscr{M}$ is strictly L-specializable along (t_1,t_2).*

Proof.

(1) We first show that $U_0({}_{\mathrm{D}}q^*\mathscr{M}(*u))$ is graded. Let n be a local section of $U_0({}_{\mathrm{D}}q^*\mathscr{M}(*u))$, that we decompose as a finite sum $\sum_{k\in K}u^k\otimes m_k$ with respect to the grading of ${}_{\mathrm{D}}q^*\mathscr{M}(*u)$. It is a matter of showing that each $n_k=u^k\otimes m_k$ is a local section of $U_0({}_{\mathrm{D}}q^*\mathscr{M}(*u))$. Applying $\prod_{k'\in K\smallsetminus\{k\}}(u\eth_u-L(v\eth_v,w\eth_w)-k'z)$ to n yields

$\prod_{k'\in K\smallsetminus\{k\}}((k-k')z)\cdot n_k$, which is thus a local section of $U_0({}_{\mathrm{D}}q^*\mathscr{M}(*u))$. Assume that n_k is a local section of some $U_\ell({}_{\mathrm{D}}q^*\mathscr{M}(*u))$ with $\ell>0$. Then $z^{\#K-1}n_k$ induces zero in $\mathrm{gr}^U_\ell({}_{\mathrm{D}}q^*\mathscr{M}(*u))$, so n_k is also zero in $\mathrm{gr}^U_\ell({}_{\mathrm{D}}q^*\mathscr{M}(*u))$ by the strictness assumption. By decreasing induction on ℓ we deduce that n_k is a local section of $U_0({}_{\mathrm{D}}q^*\mathscr{M}(*u))$. We can thus write

$$U_0({}_{\mathrm{D}}q^*\mathscr{M}(*u))=\bigoplus_k u^k\otimes{}^L U_k(\mathscr{M})=R_{{}^LU}(\mathscr{M}). \tag{2.5}$$

for some ${}^LV$-filtration of $\mathscr{M}$.

(2) If $U_0({}_{\mathrm{D}}q^*\mathscr{M}(*u))$ is ${}^uV_0$-coherent, we deduce by the previous identifications that $R_{{}^LU}(\mathscr{M})$ is $R_{{}^LV}(\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1,t_2]\langle\partial_{t_1},\partial_{t_2}\rangle)\langle u\partial_u\rangle$-coherent. As the action of $u\partial_u$ on $u^k\otimes{}^LU_k(\mathscr{M})$ is by $kz\,\mathrm{Id}+(1\otimes L(t_1\eth_{t_1},t_2\eth_{t_2})$, we conclude that $R_{{}^LU}(\mathscr{M})$ is $R_{{}^LV}(\widetilde{\mathscr{R}}_{\mathscr{X}}[t_1,t_2]\langle\partial_{t_1},\partial_{t_2}\rangle)$-coherent, so ${}^LU_\bullet\mathscr{M}$ is a coherent ${}^LV$-filtration.

(3) From the expression (2.5) we deduce that strictness of the V-graded objects of ${}_{\mathrm{D}}q^*\mathscr{M}(*u)$ passes to the ${}^LV$-graded objects of $\mathscr{M}$. Similarly, the relation between $u\eth_u$ and $L(t_1\eth_{t_1},t_2\eth_{t_2})$ shows that the Bernstein relation passes from one side to the other. □

Proposition 2.6 **(Stability by pushforward).** *Let $f:X\to X'$ be a projective morphism, let g_1',g_2' be holomorphic functions on X' and set $g_i=g_i'\circ f$ ($i=1,2$). Let $\mathscr{M}$ be an $\widetilde{\mathscr{R}}_{\mathscr{X}}$-module which is strictly L-specializable along (g). Assume that for each i, ${}_{\mathrm{D}}f_*^{(i)}\mathrm{gr}^{{}^LV}\mathscr{M}_g$ is strict. Then each ${}_{\mathrm{D}}f_*^{(i)}\mathscr{M}$ is strictly L-specializable along (g') and for each $k\in L(A)+\mathbb{Z}$, there exists a natural morphism*

$${}_{\mathrm{D}}f_*^{(i)}\,{}^LV_k\mathscr{M}_g\longrightarrow{}^LV_k\,{}_{\mathrm{D}}f_*^{(i)}\mathscr{M}_g$$

which is an isomorphism.

Sketch. One can reduce to the case of the usual V-filtration by using Lemma 2.4. Or one can adapt the proof for the usual V-filtration. □

Corollary 2.7 **(of Theorem 2.3 and Proposition 2.6).** *In the setting of Proposition 2.6, assume that $\mathscr{M}$ underlies an object of $\mathsf{MTM}(X)$. Then, for each $k\in L(A)+\mathbb{Z}$, the natural morphism*

$${}_{\mathrm{D}}f_*^{(i)}\,{}^LV_k\mathscr{M}_g\longrightarrow{}^LV_k({}_{\mathrm{D}}f_*^{(i)}\mathscr{M}_g)={}^LV_k({}_{\mathrm{D}}f_*^{(i)}\mathscr{M})_{g'}$$

is an isomorphism.

Proof. Since $\mathrm{gr}^V_0({}_{\mathrm{D}}q^*\mathscr{M}_g(u^*))$ underlies a mixed twistor D-module on $X\times\mathbb{A}^2_{v,w}$, its pushforward by f is strict, hence so is the pushforward of $\mathrm{gr}^V\mathscr{M}_g$, and Proposition 2.6 applies. □

2.d. Proof of Theorem 2.3. Let $\mathscr{M}$ be a $\widetilde{\mathscr{R}}_{\mathscr{X}}$-module and let ${}_{\mathrm{D}}\iota_*$ its pushforward by the graph inclusion ι in $\mathscr{X}\times\mathbb{C}^2$. It also underlies a mixed twistor D-module on the later space. Let $q:{}^L\mathscr{X}\to\mathscr{X}\times\mathbb{C}^2$ be the analytification of the L-deformation to the normal bundle considered in Section 2.c. Then ${}_{\mathrm{D}}q^*(\mathscr{M}_g)$ underlies a mixed twistor D-module on ${}^L\mathscr{X}$, according to [**Moc15**, Def. 7.2.1 & Prop. 11.4.6]. We note that, since $\mathscr{M}_g$ is the analytification of a partially algebraic object as in Section 2.b, ${}_{\mathrm{D}}q^*(\mathscr{M}_g)(*u)$ is the analytification of the partially algebraic object as in Section 2.c. Since ${}_{\mathrm{D}}q^*(\mathscr{M}_g)(*u)$ is strictly $\mathbb{R}$-specializable along (u) because ${}_{\mathrm{D}}q^*(\mathscr{M}_g)$ is so by [**Moc15**, Def. 7.2.1], we conclude that its partial algebraic version is so by arguing in a way similar to that of Section 2.b. Then Lemma 2.4 allows us to conclude. □

2.e. Iterated L-specializability. Given two linear forms L',L'' which are on the boundary of a smooth cone Γ as in Section 2.a, and given $\mathscr{M}$ which is strictly L'-specializable along (g), then each $\mathrm{gr}_{k'}^{L'V}\mathscr{M}_g$ ($k'\in L'(A)+\mathbb{Z}$) is a $\mathrm{gr}_0^{L'V}\widetilde{\mathscr{R}}_X[t_1,t_2]\langle\eth_{t_1},\eth_{t_2}\rangle$-module. As the latter ring is equipped with an ${}^{L''}V$-filtration, we can define the notion of canonical coherent ${}^{L''}V$-filtration on $\mathrm{gr}_{k'}^{L'V}\mathscr{M}_g$ indexed by $L''(A)+\mathbb{Z}$, so that there exists $N''\in\mathbb{N}^*$ such that, for each k'', the operator $(L''(t\eth_t)-k''z)^{N''}$ sends ${}^{L''}V_{k''}\mathrm{gr}_{k'}^{L'V}\mathscr{M}_g$ into ${}^{L''}V_{<k''}\mathrm{gr}_{k'}^{L'V}\mathscr{M}_g$, and correspondingly that of strict L''-specializability. Note that, at this point, we do not expect that ${}^{L''}V_{k''}\mathrm{gr}_{k'}^{L'V}\mathscr{M}_g$ should be induced by a filtration on $\mathscr{M}_g$.

We can complement Theorem 2.3 and Corollary 2.7 as follows.

Theorem 2.8. *In the above setting, if $\mathscr{M}$ underlies an object of $\mathsf{MTM}(X)$, then $\mathrm{gr}_{k'}^{L'V}\mathscr{M}_g$ is strictly L''-specializable, ${}_{\mathrm{D}}f_*^{(i)}(\mathrm{gr}_{k''}^{L''V}\mathrm{gr}_{k'}^{L'V}\mathscr{M}_g)$ is strict, and the natural morphism*

$$ {}_{\mathrm{D}}f_*^{(i)}({}^{L''}V_{k''}\mathrm{gr}_{k'}^{L'V}\mathscr{M}_g)\longrightarrow{}^{L''}V_{k''}\mathrm{gr}_{k'}^{L'V}({}_{\mathrm{D}}f_*^{(i)}\mathscr{M}_g) $$

is an isomorphism.

3. Strict bi-specializability

We keep the setting and notation of Section 2 and we refer to Appendix A for the various notions used below.

***Definition 3.1* (Strict $\mathbb{R}$-bi-specializability along (g)).** We say that a coherent $\widetilde{\mathscr{R}}_{\mathscr{X}}$-module $\mathscr{M}$ is *strictly bi-specializable along $(g)=(g_1,g_2)$ at $x_o\in g_1^{-1}(0)\cap g_2^{-1}(0)$* if there exists, locally at $(x_o,0,0)\in X\times\mathbb{C}^2$, a coherent bi-$V$-filtration $U_{\bullet,\bullet}\mathscr{M}_g$ index by $A+\mathbb{Z}^2$ for some finite subset $A\subset[-1,0)^2$ which admits a flattening smooth fan Σ and such that, for each $L\in\Sigma(1)$ there exists $N\geqslant 1$ so that, for each $\boldsymbol{a}=(a_1,a_2)\in A+\mathbb{Z}^2$, the operator $L(t_1\eth_{t_1}-a_1z,t_2\eth_{t_2}-a_2z)^N$ vanishes on $\mathrm{gr}_{L(\boldsymbol{a})}^{{}^LU}\mathscr{M}_g$.

We say that $\mathscr{M}$ is strictly bi-specializable along $(g)=(g_1,g_2)$ if it is so at any point $x_o\in g_1^{-1}(0)\cap g_2^{-1}(0)$.

We do not claim uniqueness of such a bi-filtration $U_{\bullet,\bullet}\mathscr{M}_g$, so the question we will have to consider is the global existence of such a bi-filtration if we assume strict bi-specializability at each point of $g_1^{-1}(0)\cap g_2^{-1}(0)$. However, we have a uniqueness statement in each L-direction. On the other hand, let us already notice that, if $A'\subset[-1,0)^2$ contains A, the bi-filtration $U_{\bullet,\bullet}\mathscr{M}_g$ indexed by $A+\mathbb{Z}^2$ can naturally be extended as a bi-filtration indexed by $A'+\mathbb{Z}^2$, and the latter satisfies the strict L-specializability property for each L: this follows from the fact that the associated ${}^L\!V$-filtration is nothing but the filtration naturally extended from ${}^L\!U_\bullet\mathscr{M}_g$ by the inclusion $L(A)+\mathbb{Z}\subset L(A')+\mathbb{Z}$.

Lemma 3.2. *Assume that $\mathscr{M}$ is strictly bi-specializable along $(g)=(g_1,g_2)$ at $x_o\in g_1^{-1}(0)\cap g_2^{-1}(0)$. Then $\mathscr{M}$ is strictly L-specializable along (g_1,g_2) in the neighborhood of x_o and, for any bi-V-filtration $U_{\bullet,\bullet}\mathscr{M}_g$ as in Definition 3.1 and every linear form L, the associated ${}^L\!V$-filtration ${}^L\!U_\bullet\mathscr{M}_g$ is the canonical ${}^L\!V$-filtration of $\mathscr{M}_g$.*

Proof. Due to uniqueness of the canonical ${}^L\!V$-filtration and Definition 3.1, the assertion is clear for any $L\in\Sigma(1)$. What about the other linear forms L? Assume that L is interior to a cone $\Gamma\in\Sigma(2)$, that we write (L',L''), so that $L=\ell'L'+\ell''L''$ with $\ell',\ell''\in\mathbb{N}^*$. On the one hand, Proposition A.9(1) implies the strictness of the L-graded objects for any linear form L. On the other hand, by Remark A.4, we can express ${}^L\!U_\bullet\mathscr{M}_g$ only in terms of ${}^\Gamma U_{\bullet,\bullet}\mathscr{M}_g$:

$$
{}^L\!U_k\mathscr{M}_g=\sum_{\ell'k'+\ell''k''\leqslant k}{}^\Gamma U_{k',k''}\mathscr{M}_g,\quad k\in L(A)+\mathbb{Z},\ k'\in L'(A)+\mathbb{Z},\ k''\in L''(A)+\mathbb{Z},
$$

and, by the flatness property, ${}^\Gamma U_{k',k''}\mathscr{M}_g={}^{L'}U_{k'}\mathscr{M}_g\cap{}^{L''}U_{k''}\mathscr{M}_g$. By assumption, there exists N such that $(L^*(t\eth_t)-L(k^*)z)^N$ sends ${}^{L^*}U_{k^*}\mathscr{M}_g$ in ${}^{L^*}U_{<k^*}\mathscr{M}_g$ (with $*={}'$ or ${}''$). It follows that $(L(t\eth_t)-L(k)z)^{2N}$ sends ${}^L\!U_k\mathscr{M}_g$ into ${}^L\!U_{<k}\mathscr{M}_g$. □

Proposition 3.3. *Assume that $\mathscr{M}$ is strictly bi-specializable along (g) at $x_o\in g_1^{-1}(0)\cap g_2^{-1}(0)$, and let $U_{\bullet,\bullet}\mathscr{M}_g$ be a bi-filtration as in Definition 3.1. Then the saturated bi-filtration $\overline{U}_{\bullet,\bullet}\mathscr{M}_g$, as defined in Proposition A.9(3), is a coherent bi-V-filtration of $\mathscr{M}_g$.*

We denote the saturated bi-V-filtration as $V_{\bullet,\bullet}\mathscr{M}_g$. It is canonical, as being equal to an intersection of the canonical ${}^L\!V$-filtrations for all linear forms L. In particular, the local saturated bi-V-filtrations glue together along $g_1^{-1}(0)\cap g_2^{-1}(0)$ by uniqueness.

Corollary 3.4. *Assume that $\mathscr{M}$ is strictly bi-specializable along (g) at each point of $g_1^{-1}(0)\cap g_2^{-1}(0)$. Then there exists a coherent bi-V-filtration $V_{\bullet,\bullet}\mathscr{M}_g$ defined in the neighborhood of $g_1^{-1}(0)\cap g_2^{-1}(0)$ and satisfying the properties of Definition 3.1 at each x_o.* □

Proposition 3.5. *Let $f:X\to X'$ be a projective morphism, let g_1',g_2' be holomorphic functions on X' and set $g_i=g_i'\circ f$ $(i=1,2)$. Let $\mathscr{M}$ be an $\widetilde{\mathscr{R}}_{\mathscr{X}}$-module underlying* *an object of* $\mathsf{MTM}(X)$*, and which is strictly bi-specializable along* (g) *at each point of* $g_1^{-1}(0)\cap g_2^{-1}(0)$*. Then, for each* i*, the pushforward* ${}_{\mathrm{D}}f_*^{(i)}\mathscr{M}$ *is strictly bi-specializable along* (g') *at each point of* $g_1'^{-1}(0)\cap g_2'^{-1}(0)$*.*

4. Strict bi-$\mathbb{R}$-specializability in the normal crossing case

As a warm-up, we start with the case of functions g_1, g_2 which are powers of a single coordinate, corresponding to the case where the normal crossing divisor is smooth.

4.a. Strict bi-specializability along powers of a coordinate

Let $\mathscr{M}$ be a holonomic $\widetilde{\mathscr{R}}_X$-module which is strictly $\mathbb{R}$-specializable along (x) with Kashiwara-Malgrange filtration indexed by $A+\mathbb{Z}$. We set $g=(g_1,g_2)$ with $g_1(x,y)=x^{e_1}$ and $g_2(x,y)=\eta_2(x,y)x^{e_2}$, where $\eta_2(0,y)\equiv 1$. We consider the direct image $\mathscr{M}_g$ of $\mathscr{M}$ by the graph $\iota_g: X\hookrightarrow X\times\mathbb{C}^2$, $(y,x)\mapsto(y,x,x^{e_1},x^{e_2})$ of g. Let us set $A_e^2=(A/e_1)\times(A/e_2)$.

Proposition 4.1. *Under these conditions, for any linear form* L*,* $\mathscr{M}_g$ *is strictly* L*-specializable, and there exists a coherent bi-filtration* $U_{\bullet,\bullet}\mathscr{M}_g$ *indexed by* $A_e^2+\mathbb{Z}^2$ *such that, for any such* L*,* ${}^LU_\bullet\mathscr{M}_g={}^LV_\bullet\mathscr{M}_g$*.*

This result will be a consequence of the definition (4.3)–(4.4) of the bi-filtration and of Lemmas 4.5–4.13. We can write

$$\mathscr{M}_g=\bigoplus_{i,j\geqslant 0}\mathscr{M}\otimes\eth_u^i\otimes\eth_v^j,$$

with the action of $\mathscr{D}_{\times\mathbb{C}^2}$ given by

$$\begin{aligned}(m\otimes\eth_u^i\otimes\eth_v^j)\eth_u^{i'}\eth_v^{j'}&=m\otimes\eth_u^{i+i'}\otimes\eth_v^{j+j'},\\ (m\otimes 1\otimes 1)\cdot\varphi(x,u,v)&=m\varphi(x,x^{e_1},x^{e_2})\otimes 1\otimes 1,\\ (m\otimes 1\otimes 1)\eth_x&=(m\delta_x\otimes 1\otimes 1)-e_1(mx^{e_1-1}\otimes\eth_u\otimes 1)\\ &\qquad -e_2(mx^{e_2-1}\eta\otimes 1\otimes\eth_v),\end{aligned}\tag{4.2}$$

where $\eta(x,y):=\eta_2+(x/e_2)\partial\eta_2/\partial x$ satisfies $\eta(0,y)\equiv 1$.

We define the bi-filtration $U_{\bullet,\bullet}\mathscr{M}_g$ by

$$U_{a,b}(\mathscr{M}_g)=\sum_{k\geqslant 0}\widetilde{U}_{a,b}(\mathscr{M}_h)\cdot\eth_x^k,\tag{4.3}$$

with

$$\widetilde{U}_{a,b}(\mathscr{M}_g)=\bigoplus_{i,j\geqslant 0}(V_{e_1\min(0,a-i)+e_2\min(0,b-j)}\mathscr{M}\otimes\eth_u^i\otimes\eth_v^j).\tag{4.4}$$

We note that $U_{a,b}(\mathscr{M}_g)$ is a coherent $V_{0,0}\mathscr{D}_{X\times\mathbb{C}^2}$-module. For example, stability by $u\eth_u$ follows from the inequality

$$e_1\min(0,a-i)-e_1\leqslant e_1\min(0,a-(i+1)).$$

Lemma 4.5. *The bi-filtration $U_{\bullet,\bullet}(\mathscr{M}_g)$ is a coherent bi-V-filtration indexed by $A_e^2+\mathbb{Z}^2$, and which is $\mathscr{D}_{X\times\mathbb{C}^2}$-generated by $U_{a,b}(\mathscr{M}_g)$ for $a,b\leqslant 0$.*

Proof. For $a,b\leqslant 0$, $\widetilde{U}_{a,b}(\mathscr{M}_g)$ reads $\bigoplus_{i,j\geqslant 0}(V_{e_1(a-i)+e_2(b-j)}\mathscr{M}\otimes\eth_u^i\otimes\eth_v^j)$ and jumps in the first (resp. second) index only occur for $a\in A/e_1$ (resp. $b\in A/e_2$). It also satisfies $\widetilde{U}_{a,b}(\mathscr{M}_g)\cdot u=\widetilde{U}_{a-1,b}(\mathscr{M}_g)$ and $\widetilde{U}_{a,b}(\mathscr{M}_g)\cdot v=\widetilde{U}_{a,b-1}(\mathscr{M}_g)$. Let us check the last assertion. We will show for example that, for $a>0$ and $b\leqslant 0$, we have

$$\widetilde{U}_{a,b}(\mathscr{M}_g)=\sum_{k=0}^{\lceil a\rceil-1}\widetilde{U}_{0,b}(\mathscr{M}_g)\eth_u^k+\widetilde{U}_{a-\lceil a\rceil,b}(\mathscr{M}_g)\cdot\eth_u^{\lceil a\rceil}. \tag{4.6}$$

On the one hand, we write

$$\begin{aligned}\widetilde{U}_{a-\lceil a\rceil,b}(\mathscr{M}_g)\cdot\eth_u^{\lceil a\rceil}&=\bigoplus_{i,j\geqslant 0}(V_{e_1(a-\lceil a\rceil-i)+e_2(b-j)}\mathscr{M}\otimes\eth_u^{i+\lceil a\rceil}\otimes\eth_v^j)\\&=\bigoplus_{\substack{i\geqslant\lceil a\rceil\\ j\geqslant 0}}(V_{e_1(a-i)+e_2(b-j)}\mathscr{M}\otimes\eth_u^i\otimes\eth_v^j).\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{k=0}^{\lceil a\rceil-1}\widetilde{U}_{0,b}(\mathscr{M}_g)\eth_u^k&=\sum_{k=0}^{\lceil a\rceil-1}\bigoplus_{i,j\geqslant 0}(V_{-e_1i+e_2(b-j)}\mathscr{M}\otimes\eth_u^{i+k}\otimes\eth_v^j)\\&=\bigoplus_{i,j\geqslant 0}\sum_{k=0}^{\min(i,\lceil a\rceil-1)}(V_{-e_1(i-k)+e_2(b-j)}\mathscr{M}\otimes\eth_u^i\otimes\eth_v^j)\\&=\bigoplus_{\substack{0\leqslant i\leqslant\lceil a\rceil-1\\ j\geqslant 0}}(V_{e_2(b-j)}\mathscr{M}\otimes\eth_u^i\otimes\eth_v^j)\\&\qquad\oplus\bigoplus_{\substack{i\geqslant\lceil a\rceil\\ j\geqslant 0}}(V_{e_1(\lceil a\rceil-1-i)+e_2(b-j)}\mathscr{M}\otimes\eth_u^i\otimes\eth_v^j),\end{aligned}$$

and we notice that the second sum is included in $\widetilde{U}_{a-\lceil a\rceil,b}(\mathscr{M}_g)\cdot\eth_u^{\lceil a\rceil}$. This yields (4.6). □

We let ${}^1U_a(\mathscr{M}_g)=\bigcup_b U_{a,b}(\mathscr{M}_g)$ and ${}^2U_b(\mathscr{M}_g)=\bigcup_a U_{a,b}(\mathscr{M}_g)$.

Lemma 4.7. *We have ${}^1U_a(\mathscr{M}_g)={}^uV_a(\mathscr{M}_g)$ and ${}^2U_b(\mathscr{M}_g)={}^vV_b(\mathscr{M}_g)$.*

Proof. For example, we have $\widetilde{{}^1U}_a(\mathscr{M}_g)=\bigoplus_{i,j\geqslant 0}V_{e_1\min(0,a-i)}\mathscr{M}\otimes\eth_u^i\otimes\eth_v^j$ and the assertion is then straightforward, by omitting the variable v first. □

Let L be the linear form $L(s_1,s_2)=\ell_1s_1+\ell_2s_2$ with $\ell_1,\ell_2\in\mathbb{N}^*$ and $\gcd(\ell_1,\ell_2)=1$. We let $\varepsilon=\min(e_1/\ell_1,e_2/\ell_2)$ and, for $\lambda\in\mathbb{R}$, ${}^LU_\lambda(\mathscr{M}_g)=\sum_{\ell_1a+\ell_2b\leqslant\lambda}U_{a,b}(\mathscr{M}_g)$.

Lemma 4.8. *We have ${}^LU_\lambda(\mathscr{M}_g)=\sum_{k\geqslant 0}\widetilde{{}^LU}_\lambda(\mathscr{M}_g)\cdot\eth_x^k$, with*

$$\widetilde{{}^LU}_\lambda(\mathscr{M}_g)=\bigoplus_{\substack{i,j\geqslant 0\\ \lambda-\ell_1i-\ell_2j\leqslant 0}}(V_{(\lambda-\ell_1i-\ell_2j)\varepsilon}\mathscr{M}\otimes\eth_u^i\otimes\eth_v^j).$$

Proof. Let us focus on ${}^L\widetilde{U}_\lambda(\mathscr{M}_g)$. We first prove, by means of (4.4), the inclusion of $\widetilde{U}_{a,b}(\mathscr{M}_g)$ in the right-hand side for all $a,b\in\mathbb{R}$ such that $\ell_1a+\ell_2b=\lambda$. As we already know that $e_1\min(0,a-i)+e_2\min(0,b-j)\leqslant 0$, it is enough to prove the inequality $e_1\min(0,a-i)+e_2\min(0,b-j)\leqslant(\lambda-\ell_1i-\ell_2j)\varepsilon$. Recall that $\ell_i\varepsilon\leqslant e_i$ $(i=1,2)$. We write

$$\begin{aligned}(\lambda-\ell_1i-\ell_2j)\varepsilon&=\ell_1\varepsilon(a-i)+\ell_2\varepsilon(b-j)\\&\geqslant e_1\min(0,a-i)+e_2\min(0,b-j),\end{aligned}$$

hence the conclusion.

For the reverse inclusion, we fix $i,j\geqslant 0$ and λ such that $\lambda-\ell_1i-\ell_2j\leqslant 0$, and we will prove the inclusion $(V_{(\lambda-\ell_1i-\ell_2j)\varepsilon}\mathscr{M}\otimes\eth_u^i\otimes\eth_v^j)\subset{}^LU_\lambda(\mathscr{M}_g)$. Let us assume for example that $\varepsilon=e_1/\ell_1\leqslant e_2/\ell_2$.

- Let us first consider the case $i=j=0$. Then $\lambda\leqslant 0$, and choosing $a=\lambda/\ell_1$ and $b=0$ in (4.4) yields $V_{e_1\lambda/\ell_1}\mathscr{M}\otimes 1\otimes 1\subset\widetilde{U}_{\lambda/\ell_1,0}\subset{}^L\widetilde{U}_\lambda(\mathscr{M}_g)$, as desired, since $e_1\lambda/\ell_1=\lambda\varepsilon$.
- For $i,j\geqslant 0$ arbitrary such that $\lambda-\ell_1i-\ell_2j\leqslant 0$, the previous point implies

$$(V_{(\lambda-\ell_1i-\ell_2j)\varepsilon}\mathscr{M}\otimes 1\otimes 1)\subset{}^L\widetilde{U}_{\lambda-\ell_1i-\ell_2j}(\mathscr{M}_g),$$

and since $\eth_u$ (resp. $\eth_v$) has L-degree ℓ_1 (resp. ℓ_2), we deduce

$$(V_{(\lambda-\ell_1i-\ell_2j)\varepsilon}\mathscr{M}\otimes\eth_u^i\otimes\eth_v^j)\subset{}^L\widetilde{U}_{\lambda-\ell_1i-\ell_2j}(\mathscr{M}_g)\cdot\eth_u^i\eth_v^j\subset{}^L\widetilde{U}_\lambda(\mathscr{M}_g).\qquad\square$$

***Lemma 4.9* (Bernstein relation).** *For each $\lambda\in\mathbb{R}$, the operator $L(u\eth_u,v\eth_v)-\lambda z$ is nilpotent on $\mathrm{gr}_\lambda^{{}^LU}\mathscr{M}_g$.*

Proof. We first notice that $x\eth_x$ acts by zero on $\mathrm{gr}_\lambda^{{}^LU}\mathscr{M}_g$. Therefore, since $\eta(0,y)\equiv 1$, $e_1u\eth_u+e_2v\eth_v$ acts as $x\delta_x$. Let $a,b\in\mathbb{R}$ be such that $\lambda=\ell_1a+\ell_2b$. For each $i,j\geqslant 0$, we set $\lambda_{i,j}=\lambda-\ell_1i-\ell_2j$. From the formula of Lemma 4.8, it is enough to show that for each $i,j\geqslant 0$, some power of $L(u\eth_u,v\eth_v)-\lambda_{i,j}z$ sends $V_{\lambda_{i,j}\varepsilon}\mathscr{M}\otimes 1\otimes 1$ to ${}^L\widetilde{U}_{<\lambda}(\mathscr{M}_h)$, and that this power can be chosen independent of i,j,λ.

If $\varepsilon=e_1/\ell_1<e_2/\ell_2$, we have $\lambda_{i,j}\varepsilon-e_2<\lambda_{i,j}-\ell_2\varepsilon=\lambda_{i,j+1}$, and the formula of Lemma 4.8 shows that $v\eth_v$ sends ${}^L\widetilde{U}_\lambda(\mathscr{M}_g)$ to ${}^L\widetilde{U}_{<\lambda}(\mathscr{M}_g)$. We can then neglect the action of $v\eth_v$, and $L(u\eth_u,v\eth_v)-\lambda_{i,j}z$ acts as $\ell_1u\eth_u-\lambda_{i,j}z$ on $V_{\lambda_{i,j}\varepsilon}\mathscr{M}\otimes 1\otimes 1$ modulo ${}^L\widetilde{U}_{<\lambda}(\mathscr{M}_g)$, while $e_1u\eth_u+e_2v\eth_v$ acts as $x\delta_x$. Therefore, $L(u\eth_u,v\eth_v)-\lambda_{i,j}z$ acts as $\varepsilon^{-1}x\eth_x-\lambda_{i,j}z$ modulo ${}^L\widetilde{U}_{<\lambda}(\mathscr{M}_g)$ and some power of it sends $V_{\lambda_{i,j}\varepsilon}\mathscr{M}\otimes 1\otimes 1$ to $V_{<\lambda_{i,j}\varepsilon}\mathscr{M}\otimes 1\otimes 1$. This power is bounded by the nilpotency index of $x\delta_x-\lambda_{ij}\varepsilon$ on $\mathrm{gr}^V_{\lambda_{i,j}\varepsilon}\mathscr{M}$, that is bounded by a fixed constant independent of i,j,λ locally on X.

We argue similarly if $e_1/\ell_1>e_2/\ell_2$ by switching the roles of u and v. It remains to consider the case where $e_1/\ell_1=e_2/\ell_2$. In such a case, $\varepsilon^{-1}L(u\eth_u,v\eth_v)=e_1u\eth_u+e_2v\eth_v$ acts as $x\delta_x$, and the result follows similarly. $\square$

***Lemma 4.10* (Strictness of $\mathrm{gr}_\lambda^{{}^LU}\mathscr{M}_g$).** *For each $\lambda\in\mathbb{R}$, $\mathrm{gr}_\lambda^{{}^LU}\mathscr{M}_g$ is strict.*

The assertion follows from the next lemma and from strictness of $\mathrm{gr}^V_\bullet\mathscr{M}$, since $\mathrm{gr}^{\widetilde{^LU}}_\lambda\mathscr{M}_g$ is a direct sum of terms $\mathrm{gr}^V_\alpha\mathscr{M}\otimes\eth_u^i\otimes\eth_v^j$.

Lemma 4.11. *The natural surjective morphism $\mathrm{gr}^{\widetilde{^LU}}_\lambda\mathscr{M}_g\otimes_{\mathbb{C}}\mathbb{C}[\eth_x]\to\mathrm{gr}^{^LU}_\lambda\mathscr{M}_g$, defined by $[\widetilde{m}]\otimes\eth_x^k\mapsto[\widetilde{m}\eth_x^k]$ is injective.*

Proof. We consider a local section m of ${}^LU_\lambda(\mathscr{M}_g)$ that we write as,

$$m=\sum_{k\geqslant0}\sum_{i,j}(m_{i,j,k}\otimes\eth_u^i\otimes\eth_v^j)\eth_x^k, \tag{4.12}$$

where, setting as above $\lambda_{i,j}=\lambda-\ell_1i-\ell_2k$, the sum is over $i,j\geqslant0$ such that $\lambda_{i,j}\leqslant0$, and $m_{i,j,k}$ is a local section of $V_{\lambda_{i,j}\varepsilon}\mathscr{M}$. We aim at proving that, if m is a local section of ${}^LU_{<\lambda}\mathscr{M}_g$, then $m_{i,j,k}$ is a local section of $V_{<\lambda_{i,j}\varepsilon}\mathscr{M}$ for every i,j,k.

Lemma 4.13. *The morphism*

$$\bigoplus_{\substack{i,j,k\geqslant0\\ \lambda_{i,j}\leqslant0}}(\mathrm{gr}^V_{\lambda_{i,j}\varepsilon}\mathscr{M}\otimes\eth_u^i\otimes\eth_v^j\otimes\eth_x^k)\xrightarrow{\ \varphi\ }\mathrm{gr}^V_\bullet\mathscr{M}\otimes_{\mathbb{C}}\mathbb{C}[\eth_u,\eth_v]$$
$$\mu_k\otimes\eth_x^k\longmapsto\mu_k\cdot(\delta_x-e_1x^{e_1-1}\eth_u-e_2x^{e_2-1}\eta\eth_v)^k$$

is injective.

Proof. We assume that there exists $\mu\neq0$ in the left-hand side such that $\varphi(\mu)=0$, and we will obtain a contradiction. We write

$$\sum_{\substack{i,j,k\geqslant0\\ \lambda_{i,j}\leqslant0}}(\mu_{i,j,k}\otimes\eth_u^i\otimes\eth_v^j\otimes\eth_x^k),$$

with $\mu_{i,j,k}\neq0$. For the proof, we assume that $\varepsilon=e_1/\ell_1$ and we switch the roles of u and v if $\varepsilon=e_2/\ell_2$. We then write $\lambda_{i,j}\varepsilon=\lambda'_j-e_1i$ with $\lambda'_j=(\lambda-\ell_2j)\varepsilon$.

We set $i_o=\max\{i+k\mid\exists\,j,\ \mu_{i,j,k}\neq0\}$ and we now fix j such that there exists i,k with $i+k=i_o$ and $\mu_{i,j,k}\neq0$. After developing $\varphi(\mu)$, we find that the coefficient of $\varphi(\mu)$ on the basis vector $\eth_u^{i_o}\otimes\eth_v^j$ writes (we neglect the nonzero constants before each term)

$$0=\sum_{\substack{k=0\\ \lambda_{i,j}\leqslant0}}^{i_o}\mu_{i,j,i_o-i}x^{(e_1-1)(i_o-k)}. \tag{4.14}$$

Let us set $i_j=\min\{i\in[0,i_o]\mid\mu_{i,j,i_o-i}\neq0\}$. We will show that $\mu_{i_j,j,i_o-i_j}=0$, leading to the desired contradiction. As $\lambda_{i_j,j}\leqslant0$, we also have $\lambda_{i,j}<0$ for $i\in[i_j+1,i_o]$ and (4.14) implies

$$\sum_{i=i_j}^{i_o}\mu_{i,j,i_o-i}x^{(e_1-1)(i_o-i)}=0.$$

Since

$$\forall\,i\in[i_j,i_o],\quad\mu_{i,j,i_o-i}x^{(e_1-1)(i_o-i)}\in\mathrm{gr}^V_{\lambda'_j-ei_o+i_o-i}\mathscr{M},$$

the above equation holds termwise, that is, $\mu_{i,j,i_o-i}x^{(e_1-1)(i_o-i)}=0$ for each $i\in[i_j,i_o]$. Furthermore, we have $\lambda'_j-e_1i<0$ for any $i\in[i_j+1,i_o]$ and thus $\mu_{i,j,i_o-i}=0$, as we have an isomorphism

$$x^{(e_1-1)(i_o-i)}:\mathrm{gr}^V_{\lambda'_j-e_1i}\mathscr{M}\xrightarrow{\ \sim\ }\mathrm{gr}^V_{\lambda'_j-e_1i_o+i_o-i}\mathscr{M}.$$

If $\lambda'_j-e_1i_j<0$, then the same argument implies that $\mu_{i_j,j,i_o-i}=0$, as was to be proved.

If $\lambda'_j-e_1i_j=0$, then $\mu_{i_j,j,i_o-i}\in\mathrm{gr}^V_0\mathscr{M}$ and we wish to prove that $\mu_{i_j,j,i_o-i_j}=0$. Let us consider the coefficient of $\varphi(\mu)$ on the basis element $\eth_u^i\otimes\eth_v^j$ for $i\geqslant i_j$. It is a sum of terms $\mu_{i',j',k'}\delta_x^k$ with $i'\leqslant i$, $j'\leqslant j$, and $k'=(i-i')+(j-j')+k$. Furthermore, if $\mu_{i',j',k'}\neq0$, we have $i'+k'\leqslant i_o$ by definition of i_o, so that $i+k=i'+k'-(j-j')\leqslant i_o$, that is, $k\leqslant i_o-i$, hence $k\leqslant i_o-i_j$. As the V-order of $\mu_{i',j',k'}$ is a priori $\leqslant 0$, that of $\mu_{i',j',k'}\delta_x^k$ is thus $\leqslant i_o-i_j$, with equality only if $j'=j$, $i'=i=i_j$ and $k'=k=i_o-i_j$. Therefore, the possible maximal V-order of the coefficients of $\varphi(\mu)$ on $\eth_u^i\otimes\eth_v^j$ for $i\geqslant i_j$ (and j fixed as above) is i_o-i_j, and it is achieved by the unique coefficient $\mu_{i_j,j,i_o-i_j}\delta_x^{i_o-i_j}$. Since $\varphi(\mu)=0$, we thus have $\mu_{i_j,j,i_o-i_j}\delta_x^{i_o-i_j}=0$ in $\mathrm{gr}^V_{i_o-i_j}\mathscr{M}$. Recall now that $\delta_x^{i_o-i_j}:\mathrm{gr}^V_0\mathscr{M}\to\mathrm{gr}^V_{i_o-i_j}\mathscr{M}$ is an isomorphism, so that $\mu_{i_j,j,i_o-i_j}=0$, as was to be proved. □

***End of the proof of Lemma* 4.11.** Let $m\in{}^{\mathrm{L}}U_{<\lambda}\mathscr{M}_g$ decomposed as in (4.12). It also has a similar decomposition with coefficients $m'_{i,j,k}$ being local sections of $V_{<\lambda_{ij}\varepsilon}\mathscr{M}$. After making the difference of both decompositions, we are led to proving that if any decomposition (4.12) of 0 has coefficients $m_{i,j,k}$ in $V_{<\lambda_{ij}\varepsilon}\mathscr{M}$. This assertion is provided by Lemma 4.13. □

5. Exponential twist of the canonical bi-V-filtration

In this section, we consider the complex manifold $X=Y\times\Delta^2$, where Δ^2 is a small polydisc with coordinates (x,t). We are given a integrable coherent $\mathscr{R}_{\mathscr{X}}$-module $\mathscr{N}$ and we assume that it admits a canonical bi-V-filtration $V_{\bullet,\bullet}\mathscr{N}$ relative to (x,t) indexed by $A+\mathbb{Z}^2$ with $A\subset[-1,0)^2$, and we denote by ${}^xA$, resp. ${}^tA$, the projection of A to the first, resp. second, factor. We also assume that each $V_{a,b}\mathscr{N}$ is coherent over ${}^xV_0(\mathscr{R}_{\mathscr{X}/\Delta_t})=\mathscr{O}_{\mathscr{X}}\langle\eth_y,x\eth_x\rangle$ (i.e., $t\eth_t$ is not needed to ensure coherence).

The bi-V-filtration on the naive localization. We set $\mathscr{M}=\mathscr{N}(*t)$ (naive localization). We will only consider the case where $t:\mathscr{N}\to\mathscr{N}$ is injective, i.e., $\mathscr{N}\subset\mathscr{N}(*t)$. Since $V_{\bullet,\bullet}\mathscr{N}$ is a coherent bi-V-filtration, locally on $\mathscr{X}$ there exists $b_o\in{}^tA+\mathbb{Z}$ such that, for each $(a,b)\in A+\mathbb{Z}^2$ with $b\leqslant b_o$, multiplication by t induces a ${}^xV_0(\mathscr{R}_{\mathscr{X}/\Delta_t})$-linear isomorphism $t:V_{a,b}\mathscr{N}\xrightarrow{\ \sim\ }V_{a,b-1}\mathscr{N}$.

We consider the bi-V-filtration on the t-localization $\mathscr{M}$ of $\mathscr{N}$ defined, for each fixed $(a,b)\in A+\mathbb{Z}^2$, by

$$V_{a,b}\mathscr{M}=t^kV_{a,b+k}\mathscr{N}\quad\text{for any }k\in\mathbb{Z}\text{ such that }b+k\leqslant b_o.$$

In such a way, for any $(a,b)\in A+\mathbb{Z}^2$, multiplication by t induces a ${}^x V_0(\mathscr{R}_{\mathscr{X}/\Delta_t})$-linear isomorphism $t: V_{a,b}\mathscr{M}\xrightarrow{\sim}V_{a,b-1}\mathscr{M}$.

Then $V_{\bullet,\bullet}\mathscr{M}$ is a $V_{\bullet,\bullet}\mathscr{R}_{\mathscr{X}}(*t)$-filtration (defined similarly) and each $V_{a,b}\mathscr{M}$ is coherent over ${}^x V_0(\mathscr{R}_{\mathscr{X}/\Delta_t})=\mathscr{O}_{\mathscr{X}}\langle\eth_y,x\eth_x\rangle$. Furthermore, away from $\{t=0\}$, we have $\mathscr{M}=\mathscr{N}$ and $V_{a,b}\mathscr{M}={}^x V_a\mathscr{N}$ for any $(a,b)\in A+\mathbb{Z}^2$.

We have ${}^x V_a(\mathscr{M})=({}^x V_a\mathscr{N})(*t)$ and ${}^t V_b(\mathscr{M})={}^t V_b(\mathscr{N}(*t))$. In particular, the corresponding graded modules are strict. Let L be any linear form as in Section 2. Then, for any $c\in{}^L A+\mathbb{Z}$, t induces an isomorphism ${}^L V_c\mathscr{M}\xrightarrow{\sim}{}^L V_{c-\ell_2}\mathscr{M}$. For any local section m of ${}^L V_c\mathscr{M}$, there exists an integer $k(m)=k$ such that $t^k m$ is a local section of ${}^L V_{c-\ell_2 k}\mathscr{N}$. It follows that each $\mathrm{gr}_c^{{}^L V}\mathscr{M}$ is strict: indeed, if m is a local section of ${}^L V_c\mathscr{M}$ and zm a local section of ${}^L V_{<c}\mathscr{M}$ then, for k big enough, $t^k m$ is a local section of ${}^L V_c\mathscr{N}$ such that $zt^k m$ is a local section of ${}^L V_{<c}\mathscr{N}$; by the strictness property of ${}^L V_\bullet\mathscr{N}$, we deduce that $t^k m$ is a local section of ${}^L V_{<c}\mathscr{N}$, and so m is a local section of ${}^L V_{<c}\mathscr{M}$. On the other hand, we deduce with a similar argument that, for each $c\in{}^L A+/\mathbb{Z}$, the operator $L(x\eth_x,t\eth_t)+cz$ is nilpotent on $\mathrm{gr}_c^{{}^L V}\mathscr{M}$. In other words, $V_{\bullet,\bullet}\mathscr{M}$ is a canonical $V_{\bullet,\bullet}\mathscr{R}_{\mathscr{X}}(*t)$-filtration of $\mathscr{M}$.

Exponential twist of $\mathscr{M}$. Let τ be a new variable and let $p:\mathscr{X}\times\mathbb{C}_\tau\to\mathscr{X}$ denote the projection. We consider the exponentially twisted $\mathscr{R}_{\mathscr{X}\times\mathbb{C}_\tau}$-module

$$ {}^{\mathscr{F}}\!\mathscr{M}=p^*\mathscr{M}\otimes\mathscr{E}^{-\tau/tz} $$

with action of $\eth_t,\eth_\tau$ defined as in [**Sab06**, (3.3)], denoting by $\mathrm{e}^{-\tau/tz}$ the generator 1 of $\mathscr{E}^{-\tau/tz}$ whose underlying $\mathscr{O}_{\mathscr{X}\times\mathbb{C}_\tau}$-module is $\mathscr{O}_{\mathscr{X}\times\mathbb{C}_\tau}(*t)$:

$$ \begin{aligned} \eth_t(m\otimes\mathrm{e}^{-\tau/tz})&=[(\eth_t+\tau/t^2)m]\otimes\mathrm{e}^{-\tau/tz},\\ \eth_\tau(m\otimes\mathrm{e}^{-\tau/tz})&=-m/t\otimes\mathrm{e}^{-\tau/tz}. \end{aligned} $$

It is integrable and that we equip with the bi-filtration $((a,b)\in A+\mathbb{Z}^2)$

$$ U_{a,b}({}^{\mathscr{F}}\!\mathscr{M})=\sum_{k\geqslant0}\eth_t^k\big[(p^*V_{a,b}\mathscr{M})\otimes\mathscr{E}^{-\tau/tz}\big]. $$

Proposition 5.1. *The bi-filtration $U_{\bullet,\bullet}({}^{\mathscr{F}}\!\mathscr{M})$ is a canonical bi-V-filtration of ${}^{\mathscr{F}}\!\mathscr{M}$ relative to (x,τ), indexed by $A+\mathbb{Z}^2$.*

Proof. We will adapt the proof of [**Sab06**, Prop. 4.1(ii)].

(1) It is clear that $U_{\bullet,\bullet}({}^{\mathscr{F}}\!\mathscr{M})$ is an exhaustive increasing bi-filtration of ${}^{\mathscr{F}}\!\mathscr{M}$ indexed by $A+\mathbb{Z}^2$. Furthermore, each $U_{a,b}({}^{\mathscr{F}}\!\mathscr{M})$ is coherent over ${}^x V_0(\mathscr{R}_{\mathscr{X}\times\mathbb{C}_\tau/\mathbb{C}_\tau})=\mathscr{O}_{\mathscr{X}\times\mathbb{C}_\tau}\langle\eth_y,\eth_t,x\eth_x\rangle$.

(2) Each $U_{a,b}({}^{\mathscr{F}}\!\mathscr{M})$ is stable by $\tau\eth_\tau$: this is proved as in [**Sab06**, Proof of 4.1, (ii)(2)]. In the same way, one shows the equality $U_{a,b+1}({}^{\mathscr{F}}\!\mathscr{M})=U_{a,b}({}^{\mathscr{F}}\!\mathscr{M})+\eth_\tau U_{a,b}({}^{\mathscr{F}}\!\mathscr{M})$ (see [**Sab06**, Proof of 4.1, (ii)(3)]).

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Appendix A. Multi-filtrations and flatness

A.a. Multi-filtrations indexed by $\mathbb{Z}\times\mathbb{Z}^2$. We work in an abelian category A like in [**MHMP**, Convention 10.2.1]. We consider the ring $\mathrm{A}=\mathbb{C}[z,z_1,z_2]$ and the category of tri-graded A-modules $M=\bigoplus_{\boldsymbol{k}\in\mathbb{Z}^3}\mathcal{M}_{\boldsymbol{k}}$, where each $\mathcal{M}_{\boldsymbol{k}}$ is an object of A and which satisfy $\mathcal{M}_{\boldsymbol{k}}=0$ whenever the first component k of $\boldsymbol{k}=(k,k_1,k_2)$ is small enough.

Lemma A.1. *The graded* A*-module M is torsion-free if and only if $(\mathcal{M}_{\boldsymbol{k}})_{\boldsymbol{k}}$ forms an increasing filtration of the object $\mathcal{M}:=\varinjlim_{\boldsymbol{k}}\mathcal{M}_{\boldsymbol{k}}$ of A indexed by $\mathbb{Z}^3$.*

When M is torsion-free, we regard it as the Rees module of a graded module $R_F\mathcal{M}$ with respect to a bi-filtration $U_{\bullet,\bullet}(R_F\mathcal{M})$ indexed by $\mathbb{Z}^2$. In other words, we now write $M=\bigoplus_{\boldsymbol{k}}\mathcal{M}_{\boldsymbol{k}}z^kz_1^{k_1}z_2^{k_2}$ to make explicit the A-action. We set $F_k\mathcal{M}=\varinjlim_{(k_1,k_2)}\mathcal{M}_{\boldsymbol{k}}$ and $R_F\mathcal{M}=\bigoplus_{k\in\mathbb{Z}}F_k\mathcal{M}z^k$. Then M reads $\bigoplus_{k_1,k_2}U_{k_1,k_2}(R_F\mathcal{M})=:R_U(R_F\mathcal{M})$.

We denote by ${}^1U_{\bullet}R_F\mathcal{M}$ the filtration ${}^1U_{k_1}R_F\mathcal{M}=\bigcup_{k_2}U_{k_1,k_2}R_F\mathcal{M}$, and similarly ${}^2U_{\bullet}R_F\mathcal{M}$. We have the identifications

$$R_{{}^1U}(R_F\mathcal{M})=R_U(R_F\mathcal{M})/(z_2-1),\qquad R_{{}^2U}(R_F\mathcal{M})=R_U(R_F\mathcal{M})/(z_1-1),$$
$$\mathrm{gr}^{{}^1U}(R_F\mathcal{M})=R_{{}^1U}(R_F\mathcal{M})/(z_1),\qquad \mathrm{gr}^{{}^2U}(R_F\mathcal{M})=R_{{}^2U}(R_F\mathcal{M})/(z_2).$$

Lemma A.2. *Assume that the graded* A*-module M is torsion-free. Then it is flat if and only if*

(1) *it is $\mathbb{C}[z_1,z_2]$-flat, i.e., $U_{k_1,k_2}R_F\mathcal{M}={}^1U_{k_1}R_F\mathcal{M}\cap{}^2U_{k_2}R_F\mathcal{M}$ for all $k_1,k_2\in\mathbb{Z}^2$,*

(2) *and each graded $\mathbb{C}[z]$-module $\mathrm{gr}^{{}^1U}_{k_1}R_F\mathcal{M}$, $\mathrm{gr}^{{}^2U}_{k_2}R_F\mathcal{M}$, as well as $\mathrm{gr}^{{}^1U}_{k_1}(\mathrm{gr}^{{}^2U}_{k_2}R_F\mathcal{M})=\mathrm{gr}^{{}^2U}_{k_2}(\mathrm{gr}^{{}^1U}_{k_1}R_F\mathcal{M})$, is strict, i.e., has no $\mathbb{C}[z]$-torsion (equivalently, is flat).*

Proof. In one direction, assume A-flatness of M. Then M is also $\mathbb{C}[z_1,z_2]$-flat. Furthermore, since flatness is preserved by base change, $M/(z_1,z_2-1)$, $M/(z_1-1,z_2)$ and $M/(z_1,z_2$ are $\mathbb{C}[z]$-flat, hence the second point.

Conversely, by [**MHMP**, Prop. 15.2.3], A-flatness is equivalent to the vanishing of the negative cohomology of the Koszul complexes attached to subsets of $\{z,z_1,z_2\}$. Subsets of cardinal one are ruled out by torsion-freeness. The vanishing of the negative cohomology of the Koszul complex attached to (z_1,z_2) is equivalent to the first condition. More precisely, it is equivalent to one of the following equivalent conditions (due to the symmetry in the Koszul complex), where we omit the underlying object $R_F\mathcal{M}$:

- for any $(k_1,k_2)\in\mathbb{Z}^2$, the natural morphism
$$U_{k_1,k_2-1}/U_{k_1-1,k_2-1}\longrightarrow U_{k_1,k_2}/U_{k_1-1,k_2}$$
is injective,
- for any $(k_1,k_2)\in\mathbb{Z}^2$, the natural morphism
$$U_{k_1-1,k_2}/U_{k_1-1,k_2-1}\longrightarrow U_{k_1,k_2}/U_{k_1,k_2-1}$$

is injective.

The first condition reads $U_{k_1,k_2-1}\cap U_{k_1-1,k_2}\subset U_{k_1-1,k_2-1}$, i.e., both terms are equal. By repeating this equality on the left-hand side and by using the exhaustivity property ${}^1U_{k_1}=\bigcup_{k_2}U_{k_1,k_2}$ and ${}^2U_{k_2}=\bigcup_{k_1}U_{k_1,k_2}$, we obtain the desired equality $U_{k_1,k_2}={}^1U_{k_1}\cap{}^2U_{k_2}$. Conversely, if the latter equality holds for any k_1,k_2, then each of the above conditions is clearly satisfied.

The first condition also means that the family $(U_{k_1,k_2}/U_{k_1-1,k_2})_{k_2}$ defines an exhaustive increasing filtration on

$$\varinjlim_{k_2}(U_{k_1,k_2}/U_{k_1-1,k_2})=(\varinjlim_{k_2}U_{k_1,k_2})/(\varinjlim_{k_2}U_{k_1-1,k_2})=\mathrm{gr}^{{}^1U}_{k_1},$$

and this filtration is nothing but the filtration naturally induced by ${}^2U_\bullet$.

Knowing the above condition, the vanishing for the other Koszul complexes follow from the second condition. For example, for that attached to (z_1,z), it means that z is injective on $R_U(R_F\mathcal{M})/z_1R_U(R_F\mathcal{M})$. As $U_{k_1,k_2}/U_{k_1-1,k_2}={}^2U_{k_2}(\mathrm{gr}^{{}^1U}_{k_1})$ by the first condition, we can write

$$R_U(R_F\mathcal{M})/z_1R_U(R_F\mathcal{M})=\bigoplus_{k_1,k_2}{}^2U_{k_2}(\mathrm{gr}^{{}^1U}_{k_1}R_F\mathcal{M}).$$

By the second condition, z is injective on each $\mathrm{gr}^{{}^1U}_{k_1}R_F\mathcal{M}$, hence on the $\mathbb{C}[z,z_2]$-module $R_U(R_F\mathcal{M})/z_1R_U(R_F\mathcal{M})$. □

In the following, we consider linear forms $L(s_1,s_2)=\ell_1s_1+\ell_2s_2$ with either $\ell_1,\ell_2\in\mathbb{N}^*$ and $(\ell_1,\ell_2)=1$, or $(\ell_1,\ell_2)=(1,0)$ or $(0,1)$. We also consider pairs $\Gamma=(L',L'')$ of such linear forms such that $\det(L',L'')=1$. We identify Γ with the cone of apex 0 in the first quadrant of $\mathbb{R}^2$ generated by the half-lines $\mathbb{R}_+L',\mathbb{R}_+L''$. We say that such a cone Γ is *smooth*. Then (L',L'') induces an isomorphism $\mathbb{Z}^2\xrightarrow{\sim}\mathbb{Z}^2$.

We consider the rings $\mathrm{A}_L,\mathrm{A}_\Gamma$ with morphisms from A, defined as

$$\begin{aligned}\mathrm{A}&\longrightarrow\mathrm{A}_L=\mathbb{C}[z,\zeta] &\qquad \mathrm{A}&\longrightarrow\mathrm{A}_\Gamma=\mathbb{C}[z,\zeta_-,\zeta_+]\\ p(z,z_1,z_2)&\longmapsto p(z,\zeta^{\ell_1},\zeta^{\ell_2}) &\qquad p(z,z_1,z_2)&\longmapsto p(z,\zeta_-^{\ell'_1}\zeta_+^{\ell''_1},\zeta_-^{\ell'_2}\zeta_+^{\ell''_2}).\end{aligned}$$

Lemma A.3. *Assume that the graded* A*-module* M *is torsion-free, hence equal to the Rees module* $R_U(R_F\mathcal{M})$ *of a bi-filtration* $U_{\bullet,\bullet}(R_F\mathcal{M})$ *indexed by* $\mathbb{Z}^2$*. Then the graded modules*

$$M_L=(\mathrm{A}_L\otimes_A M)/\mathrm{A}_L\text{-}\mathit{torsion}\quad\mathit{and}\quad M_\Gamma=(\mathrm{A}_\Gamma\otimes_A M)/\mathrm{A}_\Gamma\text{-}\mathit{torsion}$$

are respectively the Rees modules of the $\mathbb{Z}$-indexed filtration, resp. the $\mathbb{Z}^2$-indexed bi-filtration

$$ {}^{L}U_k(R_F\mathcal{M}) = \sum_{L(k_1,k_2)\leqslant k} U_{k_1,k_2}(R_F\mathcal{M}), $$

resp.

$$ {}^{\Gamma}U_{k',k''}(R_F\mathcal{M}) = \sum_{\substack{L'(k_1,k_2)\leqslant k'\\ L''(k_1,k_2)\leqslant k''}} U_{k_1,k_2}(R_F\mathcal{M}). $$

Remark A.4. If L is in the cone Γ (that we denote abusively by $L \in \Gamma$), that is, can be written as $\ell' L' + \ell'' L''$ with $\ell', \ell'' \in \mathbb{N}$, then we have

$$ \begin{pmatrix}\ell_1\\ \ell_2\end{pmatrix} = \begin{pmatrix}\ell_1' & \ell_1''\\ \ell_2' & \ell_2''\end{pmatrix} \cdot \begin{pmatrix}\ell'\\ \ell''\end{pmatrix} $$

and we have

$$ {}^{L}U_k(R_F\mathcal{M}) = \sum_{\ell' k' + \ell'' k'' \leqslant k} {}^{\Gamma}U_{k',k''}(R_F\mathcal{M}). $$

Similarly, if Γ' is a cone contained in Γ, we can write ${}^{\Gamma'}U_{\bullet,\bullet}$ in terms of ${}^{\Gamma}U_{\bullet,\bullet}$. This follows from the fact that the morphism $\mathrm{A} \to \mathrm{A}_L$ factorizes as $\mathrm{A} \to \mathrm{A}_\Gamma \to \mathrm{A}_L$, and similarly with Γ'.

Lemma A.2 applies as well to M_Γ, if we replace ${}^{1}U, {}^{2}U$ respectively with ${}^{L'}U, {}^{L''}U$. Let us make this explicit.

Proposition A.5. *Assume that the graded A_Γ-module M_Γ is torsion-free, hence is the Rees module of a bi-filtration ${}^{\Gamma}U_{\bullet,\bullet}(R_F\mathcal{M})$ indexed by $\mathbb{Z}^2$.*

(1) *Then M_Γ is A_Γ-flat if and only if*

(a) *it is $\mathbb{C}[\zeta_-,\zeta_+]$-flat, i.e., ${}^{\Gamma}U_{k',k''}R_F\mathcal{M} = {}^{L'}U_{k'}R_F\mathcal{M} \cap {}^{L''}U_{k''}R_F\mathcal{M}$ for all $k', k'' \in \mathbb{Z}^2$,*

(b) *the graded $\mathbb{C}[z]$-modules $\mathrm{gr}_{k'}^{{}^{L'}U} R_F\mathcal{M}$, $\mathrm{gr}_{k''}^{{}^{L''}U} R_F\mathcal{M}$, and $\mathrm{gr}_{k'}^{{}^{L'}U}(\mathrm{gr}_{k''}^{{}^{L''}U} R_F\mathcal{M}) = \mathrm{gr}_{k''}^{{}^{L''}U}(\mathrm{gr}_{k'}^{{}^{L'}U} R_F\mathcal{M})$, are strict, i.e., have no $\mathbb{C}[z]$-torsion (equivalently, are $\mathbb{C}[z]$-flat).*

(2) *If $\Gamma' = (L'^-, L'^+)$ is a smooth cone contained in Γ, then if both properties* (a) *and* (b) *hold for Γ and L', L'', they also hold for Γ' and L'^-, L'^+. In particular, for each L in Γ, the graded $\mathbb{C}[z]$-module $\mathrm{gr}_k^{{}^{L}U} R_F\mathcal{M}$ is strict.*

(3) *Assume that M_Γ is A_Γ-flat. Then we have, for any $(k', k'') \in \mathbb{Z}^2$,*

$$ {}^{\Gamma}U_{k',k''}(R_F\mathcal{M}) = {}^{L'}U_{k'}(R_F\mathcal{M}) \cap {}^{L''}U_{k''}(R_F\mathcal{M}) = \bigcap_{L\in\Gamma} {}^{L}U_{\ell' k' + \ell'' k''}(R_F\mathcal{M}). $$

Proof. Assertion (1) is nothing but Lemma A.2. For (2), we interpret the properties in terms of flatness, according to (1). Then the first assertion follows from the fact that flatness is preserved by base change. For the second assertion, we note that any L in Γ can be regarded as L'^- or L'^+ of a smooth cone Γ' contained in Γ.

In (3), the first equality is provided by (1)(a), and the inclusion $\supset$ of the second equality is clear. The inclusion $\subset$ follows from the expression of ${}^{L}U_k(R_F\mathcal{M})$ in Remark A.4, which implies in particular that ${}^{\Gamma}U_{k',k''}(R_F\mathcal{M}) \subset {}^{L}U_{\ell'k'+\ell''k''}(R_F\mathcal{M})$. □

***Definition A.6* (Flattening fan).** Let M be a torsion-free graded A-module. We say that a fan Σ subdividing the first quadrant in $\mathbb{R}^2$ is a *smooth flattening fan* for M if Σ is smooth, i.e., each two-dimensional cone Γ of Σ is smooth, and if each M_Γ is A_Γ-flat (hence M_L is A_L-flat for any L).

Proposition A.7. *Let M be a torsion-free graded* A*-module, corresponding to a $\mathbb{Z}^2$-indexed bi-filtration $U_{\bullet,\bullet}(R_F\mathcal{M})$. Assume that M admits a smooth flattening fan Σ. Then the* saturated bi-filtration

$$\overline{U}_{k_1,k_2}(R_F\mathcal{M}) := \bigcap_L {}^{L}U_{L(k_1,k_2)}(R_F\mathcal{M})$$

satisfies the following properties for each $(k_1,k_2)\in\mathbb{Z}^2$:

(1) *$\overline{U}_{k_1,k_2}(R_F\mathcal{M})$ is equal to the finite intersection $\bigcap_{L\in\Sigma(1)} {}^{L}U_{L(k_1,k_2)}(R_F\mathcal{M})$, where $\Sigma(1)$ is the set of primitive elements on the one-dimensional cones in Σ.*

(2) *$U_{k_1,k_2}(R_F\mathcal{M}) \subset \overline{U}_{k_1,k_2}(R_F\mathcal{M})$.*

(3) *For each L, ${}^{L}\overline{U}_k(R_F\mathcal{M}) = {}^{L}U_k(R_F\mathcal{M})$, so that $\overline{M}_L$ is flat. Furthermore, $\overline{\overline{U}}_{k_1,k_2}(R_F\mathcal{M}) = \overline{U}_{k_1,k_2}(R_F\mathcal{M})$.*

(4) *For each cone $\Gamma\in\Sigma(2)$, ${}^{\Gamma}\overline{U}_k(R_F\mathcal{M}) = {}^{\Gamma}U_k(R_F\mathcal{M})$, so that each $\overline{M}_\Gamma$ is A_Γ-flat.*

Proof. The first point follows from Proposition A.5(3). The second point follows from the inclusion $U_{k_1,k_2}(R_F\mathcal{M}) \subset {}^{L}U_{L(k_1,k_2)}(R_F\mathcal{M})$ for any (k_1,k_2) and L. For the third point, we first note that (2) yields the inclusion $\supset$. On the other hand, $\overline{U}_{k_1,k_2}(R_F\mathcal{M}) \subset {}^{L}U_{L(k_1,k_2)}(R_F\mathcal{M})$ for any L, by definition, and this implies the inclusion $\subset$. For (4), in the same way, (2) yields the inclusion $\supset$. On the other hand,

$$\overline{U}_{k_1,k_2}(R_F\mathcal{M}) \subset \bigcap_{L\in\Gamma} {}^{L}U_{L(k_1,k_2)}(R_F\mathcal{M}) = {}^{\Gamma}U_{(L'(k_1,k_2),L''(k_1,k_2))}(R_F\mathcal{M})$$

and thus the inclusion $\subset$ holds. □

A.b. Multi-filtrations indexed by $\mathbb{Z}\times(A+\mathbb{Z}^2)$. Let $A\subset[-1,0)^2$ be a finite set. We consider increasing bi-filtrations $U_{\alpha,\beta}(R_F\mathcal{M})$ indexed by $A+\mathbb{Z}^2$. We start with a preliminary remark.

Lemma A.8. *Assume $A\subset A'\subset[-1,0)$ and let $U_{\bullet,\bullet}(R_F\mathcal{M})$ be a bi-filtration indexed by $A+\mathbb{Z}^2$. Let $U'_{\bullet,\bullet}(R_F\mathcal{M})$ be the extended bi-filtration indexed by $A'+\mathbb{Z}^2$, defined by*

$$U'_{\alpha',\beta'}(R_F\mathcal{M}) = \sum_{\substack{(\alpha,\beta)\in A+\mathbb{Z}^2\\ (\alpha,\beta)\leqslant(\alpha',\beta')}} U_{\alpha,\beta}(R_F\mathcal{M}).$$

The for any linear form L, resp. any smooth cone Γ, the filtration ${}^L\!U'_\bullet$, resp. the bi-filtration ${}^\Gamma U'_{\bullet,\bullet}$ is that naturally extended from ${}^L\!U_\bullet$, resp. ${}^\Gamma U_{\bullet,\bullet}$, by the inclusion $L(A+\mathbb{Z}^2)\subset L(A'+\mathbb{Z}^2)$, resp. $(L',L'')(A+\mathbb{Z}^2)\subset(L',L'')(A'+\mathbb{Z}^2)$.

Proof. We argue with L, the case of Γ being similar. We wish to prove that, for any $\lambda'\in L(A'+\mathbb{Z}^2)$, we have

$$ {}^L\!U'_{\lambda'}(R_F\mathcal{M})=\sum_{\substack{\lambda\in L(A+\mathbb{Z}^2)\\ \lambda\leqslant\lambda'}}{}^L\!U_\lambda(R_F\mathcal{M}). $$

By definition, we have

$$ {}^L\!U'_{\lambda'}(R_F\mathcal{M})=\sum_{\substack{(\alpha',\beta')\in A'+\mathbb{Z}^2\\ L(\alpha',\beta')\leqslant\lambda'}}U'_{\alpha',\beta'}(R_F\mathcal{M})=\sum_{\substack{(\alpha',\beta')\in A'+\mathbb{Z}^2\\ L(\alpha',\beta')\leqslant\lambda'}}\ \sum_{\substack{(\alpha,\beta)\in A+\mathbb{Z}^2\\ (\alpha,\beta)\leqslant(\alpha',\beta')}}U_{\alpha,\beta}(R_F\mathcal{M}). $$

The rightmost term is obviously contained in

$$ \sum_{\substack{(\alpha,\beta)\in A+\mathbb{Z}^2\\ L(\alpha,\beta)\leqslant\lambda'}}U_{\alpha,\beta}(R_F\mathcal{M}). $$

Summing first by fixing $L(\alpha,\beta)=\lambda\in L(A+\mathbb{Z}^2)$, the right-hand terms writes

$$ \sum_{\substack{\lambda\in L(A+\mathbb{Z}^2)\\ \lambda\leqslant\lambda'}}{}^L\!U_\lambda(R_F\mathcal{M}), $$

hence the inclusion $\subset$. For the reverse inclusion, it is enough to prove that, for $(\alpha,\beta)\in A+\mathbb{Z}^2$, if $L(\alpha,\beta)\leqslant\lambda'$, then $U_{(\alpha,\beta)}(R_F\mathcal{M})\subset{}^L\!U'_{\lambda'}(R_F\mathcal{M})$, which is obvious. □

The case where $A\subset\mathbb{Q}^2$**.** If $A\subset[-1,0)^2\cap\mathbb{Q}^2$, there exists N' such that $A+\mathbb{Z}^2\subset A'+\mathbb{Z}^2=((1/N')\mathbb{Z})^2$. Given a bi-filtration $U_{\bullet,\bullet}(R_F\mathcal{M})$ indexed by $A+\mathbb{Z}^2$, the extended filtration $U'_{\bullet,\bullet}(R_F\mathcal{M})$ indexed by $((1/N')\mathbb{Z})^2$ has an associated Rees module $R_{U'}(R_F\mathcal{M})$ which is a $\mathbb{C}[z][((1/N')\mathbb{Z})^2]$-module (with $\mathbb{C}[z][((1/N')\mathbb{Z})^2]=\mathbb{C}[z,z'_1,z'_2]$, where $z_i'^{N'}$ is the original variable z_i for $i=1,2$). If N'' is a multiple of N', then

$$ R_{U''}(R_F\mathcal{M})=\mathbb{C}[z,z''_1,z''_2]\otimes_{\mathbb{C}[z,z'_1,z'_2]}R_{U'}(R_F\mathcal{M}), $$

and $R_{U''}(R_F\mathcal{M})$ is $\mathbb{C}[z,z''_1,z''_2]$-flat if and only if $R_{U'}(R_F\mathcal{M})$ is $\mathbb{C}[z,z'_1,z'_2]$-flat. It follows from this remark and from Lemma A.8 that we can repeat all results of Section A.a by replacing everywhere the bi-filtration U indexed by $\mathbb{Z}^2$ with the bi-filtration U indexed by $A+\mathbb{Z}^2$, and the Rees module $R_U(R_F\mathcal{M})$ with the Rees module $R_{U'}(R_F\mathcal{M})$ for any N' as above.

The case where $A\not\subset\mathbb{Q}^2$**.** Let $U_{\bullet,\bullet}(R_F\mathcal{M})$ be a bi-filtration of $R_F\mathcal{M}$ indexed by $A+\mathbb{Z}^2$. One defines the filtrations ${}^L\!U_\bullet(R_F\mathcal{M})$ indexed by $L(A+\mathbb{Z}^2)=L(A)+\mathbb{Z}$ and the bi-filtrations ${}^\Gamma U_{\bullet,\bullet}(R_F\mathcal{M})$ indexed by $(L',L'')(A+\mathbb{Z}^2)=(L',L'')(A)+\mathbb{Z}^2$ by the formulas in Lemma A.3. On the other hand, one cannot immediately define the Rees modules attached to these (bi)filtrations. We will prove analogues of Proposition A.5 and A.7.

Proposition A.9. *Let $U_{\bullet,\bullet}(R_F\mathcal{M})$ be a bi-filtration of $R_F\mathcal{M}$ indexed by $A+\mathbb{Z}^2$ with $A\subset[-1,0)^2$ finite. Assume that there exists a smooth fan Σ subdividing the first quadrant in $\mathbb{R}^2$ such that, for each two-dimensional cone $\Gamma=(L',L'')$ of Σ, the "flatness properties"* (1)(a) *and* (b) *of Proposition* A.5 *hold (with indices $k'\in L'(A)+\mathbb{Z}$ and $k''\in L''(A)+\mathbb{Z}$, and where e.g. $\mathrm{gr}_{k'}^{L'U}R_F\mathcal{M}$ means ${}^{L'}U_{k'}R_F\mathcal{M}/{}^{L'}U_{<k'}R_F\mathcal{M}$). Then*

(1) *for any smooth fan Σ' subdividing Σ, the "flatness properties" with respect to (each cone of) this fan also hold; in particular, for any linear form L and any $k\in L(A)+\mathbb{Z}$, $\mathrm{gr}_k^{LU}R_F\mathcal{M}$ is strict;*

(2) *for any cone $\Gamma\in\Sigma(2)$ and any $k'\in L'(A)+\mathbb{Z},k''\in L''(A)+\mathbb{Z}$, we have*

$$ {}^{\Gamma}U_{k',k''}(R_F\mathcal{M})=\bigcap_{L\in\Gamma}{}^{L}U_{\ell'k'+\ell''k''}(R_F\mathcal{M}); $$

(3) *the* saturated bi-filtration

$$ \overline{U}_{k_1,k_2}(R_F\mathcal{M}):=\bigcap_{L}{}^{L}U_{L(k_1,k_2)}(R_F\mathcal{M})\quad((k_1,k_2)\in A+\mathbb{Z}^2) $$

is equal to the finite intersection $\bigcap_{L\in\Sigma(1)}{}^{L}U_{L(k_1,k_2)}(R_F\mathcal{M})$; it satisfies ${}^{L}\overline{U}={}^{L}U$ for any L and ${}^{\Gamma}\overline{U}={}^{\Gamma}U$ for any $\Gamma\in\Sigma'(2)$ for any smooth fan Σ' subdividing Σ.

Proof. Property (2), and Property (3) for the fan Σ, are proved as the similar properties in Proposition A.5(3) and in Proposition A.7. Once (1) is proved, then Property (3) for the fan Σ' can be proved similarly. We are thus reduced to proving (1).

We fix a cone $\Gamma\in\Sigma(2)$. We note that the assertion only depends on the bi-filtration ${}^{\Gamma}U_{\bullet,\bullet}R_F\mathcal{M}$ so that, in order to simplify notation, we assume that Γ is the first quadrant and the bi-filtration $U_{\bullet,\bullet}R_F\mathcal{M}$ itself satisfies the flatness properties (in other words, we work with the index set $(L',L'')(A+\mathbb{Z}^2)$, that we rename $A+\mathbb{Z}^2$).

Let $\varepsilon>0$ be small enough such that, for any two pair $(\alpha,\beta),(\alpha',\beta')\in A$, we have

$$ \alpha'\neq\alpha\implies|\alpha'-\alpha|>2\varepsilon\quad\text{and}\quad\beta'\neq\beta\implies|\beta'-\beta|>2\varepsilon. $$

Then, for each $(\alpha,\beta)\in A$, there exists $(\widetilde{\alpha},\widetilde{\beta})\in[-1,0)^2\cap\mathbb{Q}$ satisfying $0\leqslant\widetilde{\alpha}-\alpha<\varepsilon$ and $0\leqslant\widetilde{\beta}-\beta<\varepsilon$. The correspondence $(\alpha,\beta)\mapsto(\widetilde{\alpha},\widetilde{\beta})$ is a bijection preserving the order from A to a finite subset $\widetilde{A}$ of $[-1,0)^2\cap\mathbb{Q}$ that we extend as a bijection

$$ \varphi=(\varphi_1,\varphi_2):A+\mathbb{Z}^2\xrightarrow{\sim}\widetilde{A}+\mathbb{Z}^2 $$

by requiring that $\varphi(\alpha+k_1,\beta+k_2)=\varphi(\alpha,\beta)+(k_1,k_2)$ for each $(k_1,k_2)\in\mathbb{Z}^2$. We define an increasing bi-filtration $\widetilde{U}_{\bullet,\bullet}R_F\mathcal{M}$ indexed by $\widetilde{A}+\mathbb{Z}^2$ by setting

$$ \widetilde{U}_{\varphi(\alpha,\beta)+(k_1,k_2)}R_F\mathcal{M}=U_{\alpha+k_1,\beta+k_2}R_F\mathcal{M}. $$

The following property is clear by construction:

Lemma A.10. *For any $(\alpha,\beta)\in A$, we have*

$$ {}^{1}\widetilde{U}_{\varphi_1(\alpha)}R_F\mathcal{M}={}^{1}U_{\alpha}R_F\mathcal{M}\quad\textit{and}\quad{}^{2}\widetilde{U}_{\varphi_2(\beta)}R_F\mathcal{M}={}^{2}U_{\beta}R_F\mathcal{M}.\qquad\square $$

It follows that the flatness properties (1) and (2) of Lemma A.2 are satisfied for $U_{\bullet,\bullet}R_F\mathcal{M}$ (indexed by $A+\mathbb{Z}^2$) if and only if they are satisfied for $\widetilde{U}_{\bullet,\bullet}R_F\mathcal{M}$ (indexed by $\widetilde{A}+\mathbb{Z}^2$).

Let $\Gamma=(L',L'')$ be a smooth cone in the first quadrant of $\mathbb{R}^2$ (this cone corresponds to Γ' before the change of coordinates). If $\varepsilon>0$ is chosen small enough (depending on Γ), we can achieve the following property for $L=L',L''$: for each $\lambda\in L(A)+\mathbb{Z}$, set $A_\lambda=\{(a,b)\in A+\mathbb{Z}^2\mid L(a,b)=\lambda\}$; then the finite set $L(\varphi(A_\lambda))=\{\lambda_1,\dots,\lambda_n\}$ (with n depending on λ) is ordered as $\lambda\leqslant\lambda_1\leqslant\cdots\leqslant\lambda_n$ and if μ is the predecessor of λ in $L(A)+\mathbb{Z}$ with associated sequence $\mu\leqslant\mu_1\leqslant\cdots\leqslant\mu_m$, we have the inequalities

$$\mu\leqslant\mu_1\leqslant\cdots\leqslant\mu_m<\lambda\leqslant\lambda_1\leqslant\cdots\leqslant\lambda_n.$$

Lemma A.11. *For $L=L',L''$ and $\lambda\in L(A+\mathbb{Z}^2)$ with associated sequence $\lambda\leqslant\lambda_1\leqslant\cdots\leqslant\lambda_n$, we have*

$$^{L}U_\lambda R_F\mathcal{M}={}^{L}\widetilde{U}_{\lambda_n}R_F\mathcal{M},$$

and for $\lambda^*\in L^*(A+\mathbb{Z}^2)$ (with $*={}'$ or $''$) whose associated sequences can be assumed to be indexed by the same set $\{1,\dots,n\}$,

$$^{\Gamma}U_{\lambda',\lambda''}R_F\mathcal{M}={}^{\Gamma}\widetilde{U}_{\lambda'_n,\lambda''_n}R_F\mathcal{M}.$$

***End of the proof of Proposition* A.9(1).** We assume that $U_{\bullet,\bullet}R_F\mathcal{M}$ satisfies the flatness properties. By Lemma A.10, so does $\widetilde{U}_{\bullet,\bullet}R_F\mathcal{M}$. As the latter bi-filtration is discretely indexed by $\mathbb{Q}^2$, Proposition A.5(1) can be applied to it, as we have already seen, and the flatness properties hold for $^{\Gamma}\widetilde{U}_{\bullet,\bullet}R_F\mathcal{M}$.

On the one hand, by A.5(1)(a) for $^{\Gamma}\widetilde{U}_{\bullet,\bullet}R_F\mathcal{M}$ and Lemma A.11, we deduce

$$\begin{aligned}^{\Gamma}U_{\lambda',\lambda''}R_F\mathcal{M}&={}^{\Gamma}\widetilde{U}_{\lambda'_n,\lambda''_n}R_F\mathcal{M}\\&={}^{L'}\widetilde{U}_{\lambda'_n}R_F\mathcal{M}\cap{}^{L'}\widetilde{U}_{\lambda''_n}R_F\mathcal{M}={}^{L'}U_{\lambda'}R_F\mathcal{M}\cap{}^{L''}U_{\lambda''}R_F\mathcal{M},\end{aligned}$$

hence A.5(1)(a) for $^{\Gamma}U_{\bullet,\bullet}R_F\mathcal{M}$.

On the other hand, applying Lemma A.11 to $\lambda\in L(A+\mathbb{Z}^2)$, $\lambda^*\in L^*(A+\mathbb{Z}^2)$ and their predecessors μ, μ^*, yields

$$\mathrm{gr}^{^LU}_\lambda R_F\mathcal{M}:={}^{L}U_\lambda R_F\mathcal{M}/{}^{L}U_\mu R_F\mathcal{M}={}^{L}\widetilde{U}_{\lambda_n}R_F\mathcal{M}/{}^{L}\widetilde{U}_{\mu_m}R_F\mathcal{M}$$

and

$$\begin{aligned}\mathrm{gr}^{^\Gamma U}_{\lambda',\lambda''}R_F\mathcal{M}&:={}^{\Gamma}U_{\lambda',\lambda''}R_F\mathcal{M}/\big({}^{\Gamma}U_{\mu',\lambda''}R_F\mathcal{M}+{}^{\Gamma}U_{\lambda',\mu''}R_F\mathcal{M}\big)\\&={}^{\Gamma}\widetilde{U}_{\lambda'_n,\lambda''_n}R_F\mathcal{M}/\big({}^{\Gamma}\widetilde{U}_{\mu'_{m'},\lambda''_n}R_F\mathcal{M}+{}^{\Gamma}\widetilde{U}_{\lambda'_n,\mu''_{m''}}R_F\mathcal{M}\big).\end{aligned}$$

This implies that the graded or bi-graded objects associated to $^{\Gamma}U_{\bullet,\bullet}R_F\mathcal{M}$ have a finite filtration whose graded objects are (bi)graded objects associated to $^{\Gamma}\widetilde{U}_{\bullet,\bullet}R_F\mathcal{M}$, and therefore are $\mathbb{C}[z]$-flat. Then the (bi)graded objects associated to $^{\Gamma}U_{\bullet,\bullet}R_F\mathcal{M}$ are also $\mathbb{C}[z]$-flat. In other words, $^{\Gamma}U_{\bullet,\bullet}R_F\mathcal{M}$ satisfy the flatness properties A.5(1)(b).

For the last assertion we argue as follows. Given any linear form L, it belongs to some cone $\Gamma\in\Sigma(2)$ and there exists a smooth cone Γ' contained Γ such that L is on the boundary of Γ'. The flatness property of Proposition A.5(1)(b) yields the desired strictness. □

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