MODERATE AND RAPID-DECAY NEARBY CYCLES FOR HOLONOMIC $\mathscr{D}$-MODULES

by

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Abstract. We introduce the notion of moderate and rapid decay nearby cycles relative to a holomorphic function f for an arbitrary holonomic $\mathscr{D}$-module. They are proved to be $\mathbb{R}$-constructible complexes on the product of the special fiber of the function and the circle S^1 parametrizing the values of $f/|f|$. Duality properties are conjectured in general, and proved in special cases. Relations with the irregularity complexes as defined by Z. Mebkhout are given.

Contents

1. Introduction

Let X be a complex manifold and let $f : X \to \mathbb{C}$ be a holomorphic function. We set $X_0 = f^{-1}(0)$ and we denote by $j_f : X \smallsetminus X_0 \hookrightarrow X$ and $i_f : X_0 \hookrightarrow X$ the open and closed inclusions respectively. Let $\varpi_f : \widetilde{X}(f) \to X$ be the real blowing up of $f^{-1}(0)$. One has $\partial\widetilde{X}(f) := \varpi_f^{-1}(X_0) \simeq X_0 \times S^1$ (see Section 2.a below). We also denote by $\widetilde{\jmath}_f : X \smallsetminus X_0 \hookrightarrow \widetilde{X}(f)$ and $\widetilde{\imath}_f : \partial\widetilde{X}(f) \hookrightarrow \widetilde{X}(f)$ the open and closed inclusions respectively. Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module.

On the one hand, the theory of the Kashiwara-Malgrange V-filtration enables one to define a holonomic $\mathscr{D}_X$-module supported on X_0, denoted here by $\psi_f\mathscr{M}$, equipped with an automorphism T.

On the other hand, the construction of Grothendieck-Deligne of the nearby cycle functor, applied to the perverse de Rham complex ${}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}$, produces a perverse complex ${}^{\mathrm{p}}\psi_f\,{}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}$ supported on X_0, equipped with an automorphism T.

If $\mathscr{M}$ has *regular singularities* along X_0, a theorem of Kashiwara and Malgrange identifies $({}^{\mathrm{p}}\mathrm{DR}\,\psi_f\mathscr{M},\mathrm{T})$ with $({}^{\mathrm{p}}\psi_f\,{}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M},\mathrm{T})$. For a general holonomic $\mathscr{M}$, both complexes may differ. If $X=\mathbb{C}$ and $f=\mathrm{Id}$, ${}^{\mathrm{p}}\mathrm{DR}\,\psi_f\mathscr{M}$ and ${}^{\mathrm{p}}\psi_f\,{}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}$ are finite-dimensional vector spaces with an automorphism T, that one can equivalently regard as local systems of finite rank on the circle S^1 (that we can interpret here as $\partial\widetilde{X}(f)$). The relation between both local systems is obtained by introducing the subsheaves of the latter consisting of sectorial germs of horizontal sections of $\mathscr{M}$ having moderate growth/rapid decay on $\partial\widetilde{X}(f)$. One recovers the former as the quotient sheaf of these two sheaves.

Our aim is to perform a similar construction in arbitrary dimension and for arbitrary f. Given a holonomic $\mathscr{D}_X$-module $\mathscr{M}$, we will construct an $\mathbb{R}$-constructible complex $\widetilde{{}^{\mathrm{p}}\psi_f}^{*}\mathscr{M}$ on $\partial\widetilde{X}(f)$, that we will compare with that coming from $({}^{\mathrm{p}}\psi_f\,{}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M},\mathrm{T})$. We will then construct the moderate growth and rapid-decay complexes in the category $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\partial\widetilde{X}(f))$, denoted by $\widetilde{{}^{\mathrm{p}}\psi_f}^{\mathrm{mod}}\mathscr{M}$ and $\widetilde{{}^{\mathrm{p}}\psi_f}^{\mathrm{rd}}\mathscr{M}$, and get from them a complex $\widetilde{{}^{\mathrm{p}}\psi_f}^{\mathrm{mod}/\mathrm{rd}}\mathscr{M}$ on $\partial\widetilde{X}(f)$ corresponding to $({}^{\mathrm{p}}\mathrm{DR}\,\psi_f\mathscr{M},\mathrm{T})$. This construction can also be performed for $\mathscr{M}$ in $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$. We have two distinguished triangles of $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\partial\widetilde{X}(f))$:

$$\widetilde{{}^{\mathrm{p}}\psi_f}^{\mathrm{mod}}\mathscr{M}\longrightarrow\widetilde{{}^{\mathrm{p}}\psi_f}^{*}\mathscr{M}\longrightarrow\widetilde{{}^{\mathrm{p}}\psi_f}^{>\mathrm{mod}}\mathscr{M}\xrightarrow{+1}$$
$$\widetilde{{}^{\mathrm{p}}\psi_f}^{\mathrm{rd}}\mathscr{M}\longrightarrow\widetilde{{}^{\mathrm{p}}\psi_f}^{*}\mathscr{M}\longrightarrow\widetilde{{}^{\mathrm{p}}\psi_f}^{>\mathrm{rd}}\mathscr{M}\xrightarrow{+1}.$$

If $\dim X=1$ and $\mathscr{M}$ is a holonomic $\mathscr{D}_X$-module, these triangles are in fact short exact sequences of sheaves, as follows from the Hukuhara-Turrittin theorem (see e.g. [**Mal91**, Th. 1, p. 205]) and $\widetilde{{}^{\mathrm{p}}\psi_f}^{*}\mathscr{M}$ is a local system.

Let us recall that, for a holonomic $\mathscr{D}_X$-module $\mathscr{M}$, the irregularity complexes

$${}^{\mathrm{p}}\mathrm{Irr}_{X_0}\,\mathscr{M}:=\mathrm{Irr}_{X_0}\,\mathscr{M}[\dim X]\quad\text{and}\quad{}^{\mathrm{p}}\mathrm{Irr}^{*}_{X_0}\,\mathscr{M}:=\mathrm{Irr}^{*}_{X_0}\,\mathscr{M}[\dim X]$$

are perverse (see [**Meb04**, Th. 3.5-2]). For a holonomic $\mathscr{D}_X$-module $\mathscr{M}$, we denote by $\mathscr{M}^\vee$ its dual, which is the left $\mathscr{D}_X$-module associated to the right $\mathscr{D}_X$-module $\mathscr{E}xt^{\dim X}_{\mathscr{D}_X}(\mathscr{M},\mathscr{D}_X)$.

Proposition 1.1. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. We have functorial isomorphisms*

$$\boldsymbol{R}\varpi_{f*}\widetilde{{}^{\mathrm{p}}\psi_f}^{\mathrm{rd}}\mathscr{M}[1]\simeq{}^{\mathrm{p}}\mathrm{Irr}^{*}_{X_0}\,\mathscr{M}^\vee,\qquad\boldsymbol{R}\varpi_{f*}\widetilde{{}^{\mathrm{p}}\psi_f}^{>\mathrm{mod}}\mathscr{M}\simeq{}^{\mathrm{p}}\mathrm{Irr}_{X_0}\,\mathscr{M}.$$

We conclude that both complexes $\boldsymbol{R}\varpi_{f*}\widetilde{{}^{\mathrm{p}}\psi_f}^{\mathrm{rd}}\mathscr{M}[1]$ and $\boldsymbol{R}\varpi_{f*}\widetilde{{}^{\mathrm{p}}\psi_f}^{>\mathrm{mod}}\mathscr{M}$ are perverse. Our main result is an analogous statement for objects on $\partial\widetilde{X}(f)$.

If $\mathscr{G}$ is an $\mathbb{R}$-constructible sheaf on $\partial\widetilde{X}(f)$, the X_0-*support of* $\mathscr{G}$ is by definition the smallest closed complex analytic subset of X_0 containing the image of the support of $\mathscr{G}$

by $\varpi:\partial\widetilde{X}(f)\to X_0$. We will say that a bounded complex $\mathscr{G}^\bullet$ with $\mathbb{R}$-constructible cohomology on $\partial\widetilde{X}(f)$ satisfies the *X_0-support condition* if

$$\forall\, j,\quad \dim X_0\text{-}\operatorname{Supp}\mathscr{H}^j\mathscr{G}^\bullet\leqslant -j.$$

Theorem 1.2. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Then the complexes ${}^{\mathrm{p}}\widetilde{\psi_f}{}^{*}\mathscr{M}$, ${}^{\mathrm{p}}\widetilde{\psi_f}{}^{>\mathrm{mod}}\mathscr{M}$, ${}^{\mathrm{p}}\widetilde{\psi_f}{}^{>\mathrm{rd}}\mathscr{M}$, ${}^{\mathrm{p}}\widetilde{\psi_f}{}^{\mathrm{mod}/\mathrm{rd}}\mathscr{M}$, ${}^{\mathrm{p}}\widetilde{\psi_f}{}^{\mathrm{mod}}\mathscr{M}[1]$ and ${}^{\mathrm{p}}\widetilde{\psi_f}{}^{\mathrm{rd}}\mathscr{M}[1]$ on $\partial\widetilde{X}(f)$ have $\mathbb{R}$-constructible cohomology and satisfy the X_0-support condition.*

The $\mathbb{R}$-constructibility part of this theorem follows from Remark 5.1 and Theorem 4.7.

We denote by $\boldsymbol{D}$ either the Poincaré-Verdier duality functor (see e.g. [**KS90**, Chap. 3]), or the duality functor for $\mathscr{D}_X$-modules (so that, for a holonomic $\mathscr{D}_X$-module, $\boldsymbol{D}\mathscr{M}\simeq\mathscr{M}^\vee$).

Conjecture 1.3 **(Behaviour with respect to duality).** *Let $\mathscr{M}$ be an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$. The Poincaré-Verdier dual in $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\partial\widetilde{X}(f))$*

$$\boldsymbol{D}{}^{\mathrm{p}}\widetilde{\psi_f}{}^{>\mathrm{mod}}\mathscr{M}\longrightarrow\boldsymbol{D}{}^{\mathrm{p}}\widetilde{\psi_f}{}^{*}\mathscr{M}\longrightarrow\boldsymbol{D}{}^{\mathrm{p}}\widetilde{\psi_f}{}^{\mathrm{mod}}\mathscr{M}\xrightarrow{+1}$$

of the distinguished triangle

$${}^{\mathrm{p}}\widetilde{\psi_f}{}^{\mathrm{mod}}\mathscr{M}\longrightarrow{}^{\mathrm{p}}\widetilde{\psi_f}{}^{*}\mathscr{M}\longrightarrow{}^{\mathrm{p}}\widetilde{\psi_f}{}^{>\mathrm{mod}}\mathscr{M}\xrightarrow{+1}$$

is functorially isomorphic to the distinguished triangle

$${}^{\mathrm{p}}\widetilde{\psi_f}{}^{\mathrm{rd}}\boldsymbol{D}\mathscr{M}[1]\longrightarrow{}^{\mathrm{p}}\widetilde{\psi_f}{}^{*}\boldsymbol{D}\mathscr{M}[1]\longrightarrow{}^{\mathrm{p}}\widetilde{\psi_f}{}^{>\mathrm{rd}}\boldsymbol{D}\mathscr{M}[1]\xrightarrow{+1}.$$

We prove Conjecture 1.3 in the special case in a local setting and when $\mathscr{M}$ is a meromorphic flat bundle on X, by applying results of [**Moc14**]. Functoriality of this isomorphism is lacking in order to obtain the general case, which would rely on triple compatibility pushforward, duality and moderate/rapid decay de Rham functors. We can gather these results as follows.

Corollary 1.4. *If $\mathscr{M}$ is a holonomic $\mathscr{D}_X$-module for which Conjecture* 1.3 *holds, both ${}^{\mathrm{p}}\widetilde{\psi_f}{}^{>\mathrm{mod}}\mathscr{M}$ and $\boldsymbol{D}{}^{\mathrm{p}}\widetilde{\psi_f}{}^{>\mathrm{mod}}\mathscr{M}$ satisfy the X_0-support condition. Similarly, both ${}^{\mathrm{p}}\widetilde{\psi_f}{}^{>\mathrm{rd}}\mathscr{M}$ and $\boldsymbol{D}{}^{\mathrm{p}}\widetilde{\psi_f}{}^{>\mathrm{rd}}\mathscr{M}$, and both ${}^{\mathrm{p}}\widetilde{\psi_f}{}^{*}\mathscr{M}$ and $\boldsymbol{D}{}^{\mathrm{p}}\widetilde{\psi_f}{}^{*}\mathscr{M}$, satisfy the X_0-support condition, as well as both ${}^{\mathrm{p}}\widetilde{\psi_f}{}^{\mathrm{mod}}\mathscr{M}[1]$ and $\boldsymbol{D}({}^{\mathrm{p}}\widetilde{\psi_f}{}^{\mathrm{mod}}\mathscr{M}[1])$, and both ${}^{\mathrm{p}}\widetilde{\psi_f}{}^{\mathrm{rd}}\mathscr{M}[1]$ and $\boldsymbol{D}({}^{\mathrm{p}}\widetilde{\psi_f}{}^{\mathrm{rd}}\mathscr{M}[1])$.* $\square$

Remark 1.5. One can wonder whether a stronger property occurs in Theorem 1.2, namely, that ${}^{\mathrm{p}}\widetilde{\psi_f}{}^{\mathrm{mod}}\mathscr{M}$ and ${}^{\mathrm{p}}\widetilde{\psi_f}{}^{\mathrm{rd}}\mathscr{M}$ satisfy the X_0-support condition. This holds in the "good case" (see Subsection 6.d). Would this be the case, we could then replace the Poincaré duality functor in Corollary 1.4 with the shifted functor $\boldsymbol{D}[-1]:\mathscr{F}\mapsto(\boldsymbol{D}\mathscr{F})[-1]$.

If we consider the notion of generalized t-structure on $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_{\partial\widetilde{X}(f)})$ as defined in [**Kas16**] (which we refer to for the notation), we obtain:

Corollary 1.6. *If $\mathscr{M}$ is a holonomic $\mathscr{D}_X$-module for which Conjecture* 1.3 *holds, then the complexes* $\widetilde{{}^{\mathrm{p}}\psi_f}^{*}\mathscr{M}$, $\widetilde{{}^{\mathrm{p}}\psi_f}^{\mathrm{mod}}\mathscr{M}[1]$, $\widetilde{{}^{\mathrm{p}}\psi_f}^{\mathrm{rd}}\mathscr{M}[1]$, $\widetilde{{}^{\mathrm{p}}\psi_f}^{\mathrm{mod/rd}}\mathscr{M}$, $\widetilde{{}^{\mathrm{p}}\psi_f}^{>\mathrm{mod}}\mathscr{M}$ *and* $\widetilde{{}^{\mathrm{p}}\psi_f}^{>\mathrm{rd}}\mathscr{M}$ *are objects of the categories* ${}^{1/2}\mathsf{D}^{\leqslant -1/2}_{\mathbb{R}\text{-c}}(\mathbb{C}_{\partial\widetilde{X}(f)})$ *and of* ${}^{1/2}\mathsf{D}^{\geqslant -1/2}_{\mathbb{R}\text{-c}}(\mathbb{C}_{\partial\widetilde{X}(f)})$. □

For the complexes $\widetilde{{}^{\mathrm{p}}\psi_f}^{*}\mathscr{M}$ and $\widetilde{{}^{\mathrm{p}}\psi_f}^{\mathrm{mod/rd}}\mathscr{M}$, we have a more precise relation with the moderate or topological nearby cycles. We consider the diagram (with the identification $\partial\widetilde{X}(f) = X_0 \times S^1$ and $\varpi_{f_0} := \varpi_{f|\partial\widetilde{X}(f)}$)

$$\begin{tikzcd} X_0 \times \mathbb{R} \arrow[r, "\widetilde{\rho}_0"] \arrow[dr, "q_0"'] & X_0 \times S^1 \arrow[d, "\varpi_{f_0}"] \\ & X_0 \end{tikzcd}$$

Theorem 1.7.

(1) *There is a functorial isomorphism*

$$q_0^{-1}\,{}^{\mathrm{p}}\psi_f\,{}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M} \simeq \widetilde{\rho}_0^{-1}\widetilde{{}^{\mathrm{p}}\psi_f}^{*}\mathscr{M}.$$

(2) *There is a functorial isomorphism*

$$q_0^{-1}\,{}^{\mathrm{p}}\mathrm{DR}\,{}^{\mathrm{p}}\psi_f\,\mathscr{M} \simeq \widetilde{\rho}_0^{-1}\widetilde{{}^{\mathrm{p}}\psi_f}^{\mathrm{mod/rd}}\mathscr{M}.$$

Organization of the paper. In Section 2 we recall basic constructions involving real blow-up spaces and in Section 3 we introduce the various sheaves of functions that we will need on these spaces. Their fundamental properties (mainly, flatness) have been proved in [**Moc14**] and we review them in Appendix B. To any holonomic $\mathscr{D}_X$-module we associate various de Rham complexes on the real blow-up space $\widetilde{X}(f)$. We examine their relations and prove their $\mathbb{R}$-constructibility in Section 4. We also conjecture duality properties for these de Rham complexes, that are shown to hold for meromorphic flat bundles in Appendix C, according to results in [**Moc14**]. For further purpose, we also consider a relative statement in Section 4.c.

2. Real blow-up spaces and their stratifications

2.a. Real blow-up space along $f = 0$. Let $f : X \to \mathbb{C}$ be a holomorphic function on a complex manifold X and set $X_0 = f^{-1}(0)$, $X^* = X \smallsetminus X_0$. Recall (see [**Sab13**, §8.b]) that the real oriented blow-up $\widetilde{X}(f)$ of X along X_0 is the closure in $X \times S^1$ of the graph of $f/|f| : X^* \to S^1$. The map $\varpi_f : \widetilde{X}(f) \to X$ is the restriction to $\widetilde{X}(f)$ of the first projection. We have $\partial\widetilde{X}(f) := \varpi_f^{-1}(X_0) = X_0 \times S^1$. As a consequence, $\widetilde{X}(f)$ is the subset of the real analytic manifold $X \times S^1$ defined by the equation $f(x) - |f(x)|e^{i\theta} = 0$, if $e^{i\theta}$ is the coordinate on S^1. This endows $\widetilde{X}(f)$ with the structure of a semi-analytic subset of $X \times S^1$.

Let us consider the graph inclusion $\gamma_f : x \mapsto (x, f(x))$ in the following diagram:

$$\begin{tikzcd} X \arrow[r, hook, "\gamma_f"] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{Id}"] & X \times \mathbb{C} \arrow[r, "r"] \arrow[dl, "p"] & X \\ \mathbb{C} & & \end{tikzcd}$$

We then have the corresponding graph inclusion $\widetilde{\gamma}_f$ in the corresponding diagram:

$$\begin{tikzcd} \widetilde{X}(f) \arrow[r, hook, "\widetilde{\gamma}_f"] \arrow[d, "\widetilde{f}"'] \arrow[rrr, bend left, "\varpi_f"] & X \times \widetilde{\mathbb{C}} \arrow[r, "\varpi"] \arrow[dl, "\widetilde{p}"] & X \times \mathbb{C} \arrow[r, "r"] & X \\ \widetilde{\mathbb{C}} & & & \end{tikzcd} \tag{2.1}$$

where $\widetilde{\mathbb{C}} = S^1 \times \mathbb{R}_+$ is the oriented real blow-up of $\mathbb{C}$ at the origin, and $\widetilde{\gamma}_f(\widetilde{X}(f))$ is also identified with the closure of $\widetilde{\gamma}_f(X^*) = \gamma_f(X^*)$ in $X \times \widetilde{\mathbb{C}}$. We thus have $\widetilde{\gamma}_f = (\varpi_f, \widetilde{f})$.

For every morphism $\pi : Y \to X$ of complex manifold, setting $g = f \circ \pi$, we have natural morphism $\widetilde{\pi} : \widetilde{Y}(g) \to \widetilde{X}(f)$ induced by the real-analytic map $\pi \times \mathrm{Id}_{S^1}$.

We can make the construction of $\widetilde{X}(f)$ more global, and attached to the divisor defined by f. Let D be any divisor in X and let $L(D)$ be the bundle associated with D, having a section $f : \mathscr{O}_X \to \mathscr{O}_X(D)$ defining the divisor D as $f^*(0)$. Let $S^1(D)$ be the associated S^1-bundle on X. We thus have a section $f/|f| : X \smallsetminus D \to S^1(D)$, and the closure of its image is "the" real blow-up space of X along D. Locally, it is defined by a real-analytic equation in $S^1(D)$ as above, and this gives the structure of a semi-analytic subset of the real analytic manifold $S^1(D)$. If we change the section (by a unit $u \in \Gamma(X, \mathscr{O}_X^*)$), then the map $u/|u| : S^1(D) \to S^1(D)$ induces a real-analytic isomorphism between both real blown-up spaces.

2.b. The case of a normal crossing divisor. Let Y be a complex manifold equipped with a normal crossing divisor D with smooth components D_i $(i \in I)$. We choose sections f_i of $L(D_i)$.

The real blow-up space $\widetilde{Y}(D_{i\in I})$ of Y along the components D_i of D, that we simply denote here, and in the remaining part of this article, by $\widetilde{Y}(D)$, is the closure in the fibre product $\times_{Y,i\in I} S^1(D_i)$ of the image of $Y \smallsetminus D$ by the section $(f_i/|f_i|)_{i\in I}$. A local computation shows that $\widetilde{Y}(D) = \times_{Y,i\in I}\widetilde{Y}(D_i)$. Therefore, $\widetilde{Y}(D)$ is a semi-analytic subset of the real manifold $\times_{Y,i\in I} S^1(D_i)$, and this structure does not depend on the choices made (sections f_i, order on I to define the fibre product). Moreover, $\widetilde{Y}(D)$ is a complex manifold with a topologically smooth boundary, locally real-analytic isomorphic to $(S^1)^\ell \times (\mathbb{R}_+)^\ell \times \mathbb{C}^{n-\ell}$, in the neighbourhood of any point in the intersection of exactly ℓ components of D.

Assume now that $g : Y \to \mathbb{C}$ is a holomorphic function such that $g^{-1}(0) = D$, with D as above. Let m_i be the multiplicity of g along D_i. Then $\widetilde{Y}(g) = \widetilde{Y}(\sum_i m_i D_i)$.

We have a factorization of ϖ_D:

$$\widetilde{Y}(D) \xrightarrow{\varpi_{D,g}} \widetilde{Y}(g) \xrightarrow{\varpi_g} Y,$$

where $\varpi_{D,g}$ is induced by the natural real analytic map $\times_{Y,i\in I} S^1(m_i D_i) \to S^1(\sum_i m_i D_i)$.

2.c. Semi-analytic stratification attached to a stratified $\mathcal{I}$-covering

Local study. Let us keep the setting as in §2.b and let us fix local coordinates $(y_1, \dots, y_n)$ on Y centered at a point of D such that $D = \{y_1 \cdots y_\ell = 0\}$ in the neighbourhood of this point, that we still denote by Y. Let $\boldsymbol{d} = (d_1, \dots, d_\ell, 1, \dots, 1)$ be an n-multi-index consisting of positive integers and let

$$\begin{aligned} Y_{\boldsymbol{d}} &\xrightarrow{\rho_{\boldsymbol{d}}} Y \\ (y'_1, \dots, y'_\ell, y'_{>\ell}) &\longmapsto (y'^{d_1}_1, \dots, y'^{d_\ell}_\ell, y'_{>\ell}) = (y'^{\boldsymbol{d}}, y'_{>\ell}) \end{aligned}$$

be the corresponding ramified covering. Then $D \simeq D_{\boldsymbol{d}} := \rho_{\boldsymbol{d}}^{-1}(D) = \{y'_1 \cdots y'_\ell = 0\}$. We have $\widetilde{Y}_{\boldsymbol{d}} \simeq (S^1)^\ell \times \mathbb{R}_+^\ell \times \mathbb{C}^{n-\ell}$ with coordinates $(e^{i\theta'}, r', y'_{>\ell})$, $\partial\widetilde{Y}_{\boldsymbol{d}}$ is defined by $\prod_{i=1}^\ell r'_i = 0$, and the map $\rho_{\boldsymbol{d}}$ lifts as the map $\widetilde{\rho}_{\boldsymbol{d}} : \widetilde{Y}_{\boldsymbol{d}} \to \widetilde{Y}$ given in coordinates by $(e^{i\theta'}, r', y'_{>\ell}) \mapsto (e^{i\boldsymbol{d}\theta'}, r'^{\boldsymbol{d}}, y'_{>\ell})$.

Let $\varphi \in \mathscr{O}_{Y_{\boldsymbol{d}}}(*D_{\boldsymbol{d}})/\mathscr{O}_{Y_{\boldsymbol{d}}}$ which is *purely monomial*, that is, $\varphi = u(y')y'^{-\boldsymbol{m}}$ for some $\boldsymbol{m} \in \mathbb{N}^\ell$ and u' invertible. It defines a semi-analytic stratification of $\partial\widetilde{Y}_{\boldsymbol{d}}$ formed by

- the closed subsets of $(S^1)^\ell \times \partial\mathbb{R}_+^\ell \times \mathbb{C}^{n-\ell}$ defined by the equation

$$\arg u(y') - \sum_i m_i \theta'_i = \pm\pi/2$$

(these are the product of $2\gcd(\boldsymbol{m})$ disjoint subtori $(S^1)^{\ell-1}$ with $\partial\mathbb{R}_+^\ell \times \mathbb{C}^{n-\ell}$),
- their open complements.

Similarly, any finite family $\Phi_{\boldsymbol{d}} \subset \mathscr{O}_{Y_{\boldsymbol{d}}}(*D_{\boldsymbol{d}})/\mathscr{O}_{Y_{\boldsymbol{d}}}$ whose elements are purely monomial defines a semi-analytic stratification $\partial\widetilde{\mathscr{Y}}_{\boldsymbol{d}}$ of $\partial\widetilde{Y}_{\boldsymbol{d}}$ which is finer than that defined by any element of $\Phi_{\boldsymbol{d}}$. There exists then a semi-analytic stratification $\partial\widetilde{\mathscr{Y}}$ of $\partial\widetilde{Y}$ whose pull-back by $\rho_{\boldsymbol{d}}$ is finer than the latter. Let us finally notice that, if $\mathscr{F}$ is an object of $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{\partial\widetilde{Y}})$ and if $\rho_{\boldsymbol{d}}^{-1}\mathscr{F}$ is a sheaf (in degree zero) which is constructible with respect to $\partial\widetilde{\mathscr{Y}}_{\boldsymbol{d}}$, then $\mathscr{F}$ is a also sheaf (in degree zero), and it is constructible with respect to $\partial\widetilde{\mathscr{Y}}$.

Global study. The notion of a subset $\Phi_{\boldsymbol{d}}$ can be globalized along D as the notion of stratified $\mathcal{I}$-covering (see [**Sab13**, Def. 1.46 & §9.c]). It corresponds to that of a system of irregular values as defined in [**Moc11**, Def. 2.4.2]. One usually adds a goodness condition for $\Phi_{\boldsymbol{d}} \cup \{0\}$ (see [**Sab13**, Def. 9.12]), which ensures the pure monomiality of its local sections. Recall that a meromorphic flat bundle $\mathscr{M}$ on Y with poles along D and having good formal structure along D determines a good stratified $\mathcal{I}$-covering of $\partial\widetilde{Y}(D)$ (see loc. cit.).

One then obtains the following lemma in an obvious way.

Lemma 2.2. *Let $\widetilde{\Sigma}$ be a good stratified $\mathfrak{I}$-covering of $\partial\widetilde{Y}(D)$. Then the locally defined semi-analytic stratifications $\partial\widetilde{\mathscr{Y}_{\mathfrak{I}_y}}$ $(y \in D)$ glue together and define a semi-analytic stratification, denoted by $\partial\widetilde{\mathscr{Y}_{\mathfrak{I}}}$, of $\partial\widetilde{Y}$.* □

3. Sheaves on the real blow-up spaces

In this section, we recall various results of [**Moc14**]. We add some easy complements, whose proof is given in Appendix B, following the same lines as in loc. cit.

3.a. Sheaves of functions on the real blow-up space along $f = 0$. We keep the notation as in §2.a and we implicitly refer to Diagram (2.1). We will be mainly interested in the following two sheaves of functions on $\widetilde{X}(f)$, whose restriction to X^* is equal to $\mathscr{O}_{X^*}$:

- $\mathscr{A}^{\mathrm{mod}X_0}_{\widetilde{X}(f)}$, that we simply denote by $\mathscr{A}^{\mathrm{mod}}_{\widetilde{X}(f)}$, is the sheaf of functions which are holomorphic on X^* and have moderate growth along $\partial\widetilde{X}(f)$,
- $\mathscr{A}^{\mathrm{rd}X_0}_{\widetilde{X}(f)}$, that we simply denote by $\mathscr{A}^{\mathrm{rd}}_{\widetilde{X}(f)}$, is the sheaf of functions which are holomorphic on X^* and have rapid decay along $\partial\widetilde{X}(f)$.

We thus have natural inclusions

$$\mathscr{A}^{\mathrm{rd}}_{\widetilde{X}(f)} \subset \mathscr{A}^{\mathrm{mod}}_{\widetilde{X}(f)} \subset \widetilde{\jmath}_{f*}\mathscr{O}_{X^*} =: \mathscr{A}^{*}_{\widetilde{X}(f)}.$$

- We will moreover set

$$\begin{aligned}\mathscr{A}^{>\mathrm{mod}}_{\widetilde{X}(f)} &:= \widetilde{\jmath}_{f*}\mathscr{O}_{X^*}/\mathscr{A}^{\mathrm{mod}}_{\widetilde{X}(f)},\\ \mathscr{A}^{>\mathrm{rd}}_{\widetilde{X}(f)} &:= \widetilde{\jmath}_{f*}\mathscr{O}_{X^*}/\mathscr{A}^{\mathrm{rd}}_{\widetilde{X}(f)},\\ \mathscr{A}^{\mathrm{mod}/\mathrm{rd}}_{\widetilde{X}(f)} &:= \mathscr{A}^{\mathrm{mod}}_{\widetilde{X}(f)}/\mathscr{A}^{\mathrm{rd}}_{\widetilde{X}(f)} = \mathscr{A}^{>\mathrm{rd}}_{\widetilde{X}(f)}/\mathscr{A}^{>\mathrm{mod}}_{\widetilde{X}(f)},\end{aligned}$$

which are sheaves supported on $\partial\widetilde{X}(f)$.

Notation 3.1. We will use the notation $\mathscr{A}^{\star}_{\widetilde{X}(f)}$ to denote any of the previous sheaves, with $\star = *, \mathrm{mod}, \mathrm{rd}, {>}\mathrm{mod}, {>}\mathrm{rd}, \mathrm{mod}/\mathrm{rd}$.

Similarly, we will consider the special case of a projection, i.e., the space $X\times\mathbb{C}$ with divisor $X\times\{0\}$ and real blown-up space $X\times\widetilde{\mathbb{C}}$, and the corresponding sheaves $\mathscr{A}^{\star}_{X\times\widetilde{\mathbb{C}}}$. Let us first notice the following properties.

(a) Multiplication by f is invertible on $\mathscr{A}^{\star}_{\widetilde{X}(f)}$ (obvious).

(b) We have $\boldsymbol{R}\widetilde{\jmath}_{f*}\mathscr{O}_{X^*} = \widetilde{\jmath}_{f*}\mathscr{O}_{X^*}$.

Indeed, each point of $X\times S^1 \subset X\times\widetilde{\mathbb{C}}$ as a fundamental system of open neighbourhoods in $X\times\widetilde{\mathbb{C}}$ whose trace on $X\times\mathbb{C}^*$ is a Stein open set, since each open sector centered at the origin in $\mathbb{C}^*$ is convex. By taking the trace of these neighbourhoods on the graph of $\widetilde{\gamma}_f$ and by applying Cartan-Serre's Theorem B, we obtain Assertion (b).

Let us recall basic results concerning these sheaves in the present setting, proved in [**Moc14**, Th. 4.1.5] with bigger generality. (See Appendix B for details.)

(1) The sheaves $\mathscr{A}^{\star}_{\widetilde{X}(f)}$ are $\varpi_f^{-1}\mathscr{O}_X$-flat.

(2) $\mathscr{A}^{\star}_{X\times\widetilde{\mathbb{C}}}\otimes_{\varpi^{-1}\mathscr{O}_{X\times\mathbb{C}}}\varpi^{-1}\mathscr{O}_{\gamma_f(X)}\simeq \boldsymbol{R}\widetilde{\gamma}_{f*}\mathscr{A}^{\star}_{\widetilde{X}(f)}$.

On the other hand, we have (see e.g. [**Sab00**, §II.1.1])

(3) $\boldsymbol{R}\varpi_*\mathscr{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}}=\varpi_*\mathscr{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}}\simeq\mathscr{O}_{X\times\mathbb{C}}(*(X\times 0))$,

(4) $\boldsymbol{R}\varpi_*\mathscr{A}^{\mathrm{rd}}_{X\times\widetilde{\mathbb{C}}}\simeq\{0\to\mathscr{O}_{X\times\mathbb{C}}\to\mathscr{O}_{\widehat{X\times\mathbb{C}|X\times 0}}\to 0\}$, where the latter sheaf is the formal completion of $\mathscr{O}_{X\times\mathbb{C}}$ along $X\times 0$ (it is zero on $X\times\mathbb{C}^*$).

We then deduce

(5) $\boldsymbol{R}\varpi_{f*}\mathscr{A}^{\mathrm{mod}}_{\widetilde{X}(f)}=\varpi_{f*}\mathscr{A}^{\mathrm{mod}}_{\widetilde{X}(f)}\simeq\mathscr{O}_X(*X_0)$. Indeed,

$$\begin{aligned}\boldsymbol{R}\gamma_{f*}\boldsymbol{R}\varpi_{f*}\mathscr{A}^{\mathrm{mod}}_{\widetilde{X}(f)}&\simeq\boldsymbol{R}\varpi_*\boldsymbol{R}\widetilde{\gamma}_{f*}\mathscr{A}^{\mathrm{mod}}_{\widetilde{X}(f)}\\&\simeq\boldsymbol{R}\varpi_*(\mathscr{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}}\otimes_{\varpi^{-1}\mathscr{O}_{X\times\mathbb{C}}}\varpi^{-1}\mathscr{O}_{\gamma_f(X)})\quad\text{(after (2))}\\&=\boldsymbol{R}\varpi_*(\mathscr{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}}\overset{\boldsymbol{L}}{\otimes}_{\varpi^{-1}\mathscr{O}_{X\times\mathbb{C}}}\varpi^{-1}\mathscr{O}_{\gamma_f(X)})\quad\text{(after (1))}\\&\simeq\boldsymbol{R}\varpi_*\mathscr{A}^{\mathrm{mod}}_{X\times\widetilde{\mathbb{C}}}\overset{\boldsymbol{L}}{\otimes}_{\mathscr{O}_{X\times\mathbb{C}}}\mathscr{O}_{\gamma_f(X)}\quad\text{(projection formula)}\\&\simeq\mathscr{O}_{X\times\mathbb{C}}(*(X\times 0))\overset{\boldsymbol{L}}{\otimes}_{\mathscr{O}_{X\times\mathbb{C}}}\mathscr{O}_{\gamma_f(X)}\simeq\boldsymbol{R}\gamma_{f*}\mathscr{O}_X(*X_0)\quad\text{(after (3))}.\qquad\square\end{aligned}$$

(6) $\boldsymbol{R}\varpi_{f*}\mathscr{A}^{\mathrm{rd}}_{\widetilde{X}(f)}\simeq\{0\to\mathscr{O}_X\to\mathscr{O}_{\widehat{X|X_0}}\to 0\}$ or, equivalently,

$$\boldsymbol{R}\varpi_{f*}\mathscr{A}^{\mathrm{rd}}_{\widetilde{X}(f)}\simeq\{0\to\mathscr{O}_X(*X_0)\to\mathscr{O}_{\widehat{X|X_0}}(*X_0)\to 0\}.$$

Indeed, we have similarly

$$\boldsymbol{R}\gamma_{f*}\boldsymbol{R}\varpi_{f*}\mathscr{A}^{\mathrm{rd}}_{\widetilde{X}(f)}\simeq\{0\to\mathscr{O}_{X\times\mathbb{C}}\to\mathscr{O}_{\widehat{X\times\mathbb{C}|X\times 0}}\to 0\}\overset{\boldsymbol{L}}{\otimes}_{\mathscr{O}_{X\times\mathbb{C}}}\mathscr{O}_{\gamma_f(X)}$$

and by flatness of $\mathscr{O}_{\widehat{X\times\mathbb{C}|X\times 0}}$ over $\mathscr{O}_{X\times\mathbb{C}}$, the assertion is reduced to the identification (see e.g. [**Ser65**, Cor. II.2])

$$\mathscr{O}_{\widehat{X\times\mathbb{C}|X\times 0}}\otimes_{\mathscr{O}_{X\times\mathbb{C}}}\mathscr{O}_{\gamma_f(X)}\simeq\gamma_{f*}\mathscr{O}_{\widehat{X|X_0}}.\qquad\square$$

We set $\mathscr{Q}_{X_0}=i_f^{-1}(\mathscr{O}_{\widehat{X|X_0}}/\mathscr{O}_X)=i_f^{-1}\mathscr{O}_{\widehat{X|X_0}}/i_f^{-1}\mathscr{O}_X$ (notation of [**Meb90**]). Then

$$i_f^{-1}\boldsymbol{R}\varpi_{f*}\mathscr{A}^{\mathrm{rd}}_{\widetilde{X}(f)}\simeq\mathscr{Q}_{X_0}[-1].\qquad\square$$

(7) We also conclude that

$$\boldsymbol{R}\varpi_{f*}\mathscr{A}^{\mathrm{mod/rd}}_{\widetilde{X}(f)}\simeq\mathscr{O}_{\widehat{X|X_0}}(*X_0).$$

For every projective morphism $\pi:Y\to X$, setting $g=f\circ\pi$, we also have, according to [**Moc14**, Th. 4.1.5],

(8) $\forall\mathscr{C}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(Y),\quad\boldsymbol{R}\widetilde{\pi}_*(\mathscr{A}^{\star}_{\widetilde{Y}(g)}\overset{\boldsymbol{L}}{\otimes}_{\varpi_g^{-1}\mathscr{O}_Y}\varpi_g^{-1}\mathscr{C})\simeq\mathscr{A}^{\star}_{\widetilde{X}(f)}\overset{\boldsymbol{L}}{\otimes}_{\varpi_f^{-1}\mathscr{O}_X}\varpi_f^{-1}\boldsymbol{R}\pi_*\mathscr{C}$.

For example, if $e:Y\to X$ is a proper modification which is an isomorphism above $X\smallsetminus X_0$, we deduce from (8):

(9) $\mathscr{A}^{\star}_{\widetilde{X}(f)}=\widetilde{e}_*\mathscr{A}^{\star}_{\widetilde{Y}(g)}=\boldsymbol{R}\widetilde{e}_*\mathscr{A}^{\star}_{\widetilde{Y}(g)}$.

3.b. The case of a normal crossing divisor. The sheaves $\mathscr{A}^{\star}_{\widetilde{Y}(D)}$ are defined on $\widetilde{Y}(D)$ in a way similar to the case of a smooth divisor (§3.a). The following results hold, according to [**Moc14**, Th. 4.1.5, Prop. 4.2.4, Th. 4.5.1].

(1) The sheaves $\mathscr{A}^{\star}_{\widetilde{Y}(D)}$ are $\varpi_D^{-1}\mathscr{O}_Y$-flat.

(2) $\boldsymbol{R}\varpi_{D,g*}\mathscr{A}^{\star}_{\widetilde{Y}(D)} = \varpi_{D,g*}\mathscr{A}^{\star}_{\widetilde{Y}(D)} = \mathscr{A}^{\star}_{\widetilde{Y}(g)}$.

3.c. Localization and formalization of $\mathscr{D}_X$-modules. For a coherent $\mathscr{D}_X$-module $\mathscr{M}$, the localized $\mathscr{D}_X$-module $\mathscr{M}(*X_0)$ is defined as $\mathscr{O}_X(*X_0)\otimes_{\mathscr{O}_X}\mathscr{M}$. On the other hand, the formalized $\mathscr{D}_X$-module $\mathscr{M}_{\widehat{X|X_0}}$ along X_0 is defined as $\mathscr{O}_{\widehat{X|X_0}}\otimes_{\mathscr{O}_X}\mathscr{M}$. There are natural morphisms

$$\mathscr{M}\longrightarrow\mathscr{M}(*X_0)\quad\text{and}\quad\mathscr{M}\longrightarrow\mathscr{M}_{\widehat{X|X_0}}.$$

Let us denote by $\mathscr{Q}_{\widehat{X|X_0}}$ the complex $\mathscr{O}_X\to\mathscr{O}_{\widehat{X|X_0}}$ with terms in degrees 0 and 1 respectively. With the previous notation, we have $i_f^{-1}\mathscr{H}^1\mathscr{Q}_{\widehat{X|X_0}}=\mathscr{Q}_{X_0}$. The sheaf-theoretic restriction of $\mathscr{Q}_{\widehat{X|X_0}}$ to X^* is $\mathscr{O}_{X^*}$ and that to X_0 is $\mathscr{O}_{\widehat{X|X_0}}/\mathscr{O}_X[-1]$. This complex is isomorphic to the complex $\mathscr{O}_X(*X_0)\to\mathscr{O}_{\widehat{X|X_0}}(*X_0)$. For a bounded complex of $\mathscr{D}_X$-modules, there is a distinguished triangle in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$:

$$\mathscr{Q}_{\widehat{X|X_0}}\otimes^L_{\mathscr{O}_X}\mathscr{M}\longrightarrow\mathscr{M}(*X_0)\longrightarrow\mathscr{M}_{\widehat{X|X_0}}(*X_0)\xrightarrow{+1}. \tag{3.2}$$

Since $\varpi_f^{-1}\mathscr{D}_X$ acts in a natural way on $\mathscr{A}^{\star}_{\widetilde{X}(f)}$ (i.e., the condition $\star$ is preserved by derivation), the sheaf

$$\mathscr{D}^{\star}_{\widetilde{X}(f)}:=\mathscr{A}^{\star}_{\widetilde{X}(f)}\otimes_{\varpi_f^{-1}\mathscr{O}_X}\varpi_f^{-1}\mathscr{D}_X=\varpi_f^{-1}\mathscr{D}_X\otimes_{\varpi_f^{-1}\mathscr{O}_X}\mathscr{A}^{\star}_{\widetilde{X}(f)}$$

is a sheaf of rings on $\widetilde{X}(f)$. Any $\mathscr{D}_X$-module $\mathscr{M}$ gives rise to a $\mathscr{D}^{\star}_{\widetilde{X}(f)}$-module

$$\varpi_f^{\star}\mathscr{M}:=\mathscr{D}^{\star}_{\widetilde{X}(f)}\otimes_{\varpi_f^{-1}\mathscr{D}_X}\varpi_f^{-1}\mathscr{M}=\mathscr{A}^{\star}_{\widetilde{X}(f)}\otimes_{\varpi_f^{-1}\mathscr{O}_X}\varpi_f^{-1}\mathscr{M}.$$

Flatness of $\mathscr{A}^{\star}_{\widetilde{X}(f)}$ over $\varpi_f^{-1}\mathscr{O}_X$ immediately implies:

Proposition 3.3. *Let $\mathscr{M}$ be a $\mathscr{D}_X$-module. The pushforward by ϖ_f of the exact sequence*

$$0\longrightarrow\varpi_f^{\mathrm{rd}}\mathscr{M}\longrightarrow\varpi_f^{\mathrm{mod}}\mathscr{M}\longrightarrow\varpi_f^{\mathrm{mod/rd}}\mathscr{M}\longrightarrow 0$$

is isomorphic to the distinguished triangle (3.2). □

4. The moderate and rapid-decay de Rham complexes on $\widetilde{X}(f)$

4.a. Moderate and rapid-decay de Rham complexes for $\mathscr{D}_X$-modules

We still use Notation 3.1. Let $\mathscr{M}$ be a $\mathscr{D}_X$-module. Since $\mathscr{A}^{\star}_{\widetilde{X}(f)}$ is a left $\varpi_f^{-1}\mathscr{D}_X$-module, we can set

$$\mathrm{DR}^{\star}_{\widetilde{X}(f)}\mathscr{M}:=\big(\mathscr{A}^{\star}_{\widetilde{X}(f)}\otimes_{\varpi_f^{-1}\mathscr{O}_X}(\Omega^{\bullet}_X\otimes\mathscr{M}),\nabla\big)$$

and

$${}^{\mathrm{p}}\mathrm{DR}^{\star}_{\widetilde{X}(f)}\mathscr{M}:=\mathrm{DR}^{\star}_{\widetilde{X}(f)}\mathscr{M}[\dim X].$$

We can replace $\Omega_X^\bullet$ with $\Omega_X^\bullet(*X_0)$ since f is invertible on $\mathscr{A}_{\widetilde{X}(f)}^\star$. Recall that the Spencer complex $\mathrm{Sp}(\mathscr{D}_X)$ is a resolution of $\mathscr{O}_X$ by locally free left $\mathscr{D}_X$-modules. Then $\mathrm{Sp}(\mathscr{D}_{\widetilde{X}(f)}^\star) := \mathscr{A}_{\widetilde{X}(f)}^\star \otimes_{\varpi_f^{-1}\mathscr{O}_X} \varpi_f^{-1}\mathrm{Sp}(\mathscr{D}_X)$ is a resolution of $\mathscr{A}_{\widetilde{X}(f)}^\star$ by locally free left $\mathscr{D}_{\widetilde{X}(f)}^\star$-modules. The following result is obtained in a standard way, and can be used for the definition of ${}^{\mathrm{p}}\mathrm{DR}_{\widetilde{X}(f)}^\star\mathscr{M}$ for $\mathscr{M}$ in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$.

Lemma 4.1. *For a left $\mathscr{D}_X$-module $\mathscr{M}$ we have*

$$\begin{aligned}{}^{\mathrm{p}}\mathrm{DR}_{\widetilde{X}(f)}^\star\mathscr{M} &\simeq \mathscr{H}\!om_{\mathscr{D}_{\widetilde{X}(f)}^\star}\big(\mathrm{Sp}(\mathscr{D}_{\widetilde{X}(f)}^\star), \varpi_f^\star\mathscr{M}\big)\\ &\simeq \boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_{\widetilde{X}(f)}^\star}\big(\mathscr{A}_{\widetilde{X}(f)}^\star, \varpi_f^\star\mathscr{M}\big)\\ &\simeq \boldsymbol{R}\mathscr{H}\!om_{\varpi_f^{-1}\mathscr{D}_X}\big(\varpi_f^{-1}\mathscr{O}_X, \varpi_f^\star\mathscr{M}\big). \qquad \square\end{aligned}$$

The de Rham complexes with $\star = {>}\mathrm{mod}, {>}\mathrm{rd}, \mathrm{mod/rd}$ are supported on $\partial\widetilde{X}(f)$. We have the following natural distinguished triangles

$$\begin{aligned}&{}^{\mathrm{p}}\mathrm{DR}_{\widetilde{X}(f)}^{\mathrm{mod}}\mathscr{M} \longrightarrow {}^{\mathrm{p}}\mathrm{DR}_{\widetilde{X}(f)}^{*}\mathscr{M} \longrightarrow {}^{\mathrm{p}}\mathrm{DR}_{\widetilde{X}(f)}^{>\mathrm{mod}}\mathscr{M} \xrightarrow{+1}\\ &{}^{\mathrm{p}}\mathrm{DR}_{\widetilde{X}(f)}^{\mathrm{rd}}\mathscr{M} \longrightarrow {}^{\mathrm{p}}\mathrm{DR}_{\widetilde{X}(f)}^{*}\mathscr{M} \longrightarrow {}^{\mathrm{p}}\mathrm{DR}_{\widetilde{X}(f)}^{>\mathrm{rd}}\mathscr{M} \xrightarrow{+1}\end{aligned}\tag{4.2}$$

and

$$ {}^{\mathrm{p}}\mathrm{DR}_{\widetilde{X}(f)}^{\mathrm{rd}}\mathscr{M} \longrightarrow {}^{\mathrm{p}}\mathrm{DR}_{\widetilde{X}(f)}^{\mathrm{mod}}\mathscr{M} \longrightarrow {}^{\mathrm{p}}\mathrm{DR}_{\widetilde{X}(f)}^{\mathrm{mod/rd}}\mathscr{M} \xrightarrow{+1}. \tag{4.3}$$

Proposition 4.4. *If $\mathscr{M}$ is an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$, then the natural morphism*

$${}^{\mathrm{p}}\mathrm{DR}_{\widetilde{X}(f)}^{*}\mathscr{M} \longrightarrow \boldsymbol{R}\widetilde{\jmath}_{f*}j_f^{-1}\,{}^{\mathrm{p}}\mathrm{DR}_X\mathscr{M}$$

is an isomorphism.

Proof. It is enough to prove the result for a coherent $\mathscr{D}_X$-module, and the assertion is local. We will work with the unshifted de Rham complex DR. We first notice, in a way similar to 3.a(b), that for each $\mathscr{O}_X$-coherent sheaf $\mathscr{F}$, we have $\boldsymbol{R}\widetilde{\jmath}_{f*}\widetilde{\jmath}_f^{-1}\mathscr{F} = \widetilde{\jmath}_{f*}\widetilde{\jmath}_f^{-1}\mathscr{F}$. Then, by using a local resolution of $\mathscr{F}$ by free $\mathscr{O}_X$-modules of finite rank, one checks that the natural morphism

$$\widetilde{\jmath}_{f*}\mathscr{O}_{X^*}\otimes_{\varpi_f^{-1}\mathscr{O}_X}\varpi_f^{-1}\mathscr{F} \longrightarrow \widetilde{\jmath}_{f*}\widetilde{\jmath}_f^{-1}\mathscr{F}$$

is an isomorphism. Let us choose a local good filtration $F_\bullet\mathscr{M}$ and filter the de Rham complex by

$$F_p\,\mathrm{DR}_X\mathscr{M} := \{F_p\mathscr{M} \to \Omega_X^1\otimes F_{p+1}\mathscr{M} \to \cdots\},$$

that we also denote by $\mathrm{DR}_X F_{p+\bullet}\mathscr{M}$. It is known that, for $p \gg 0$ locally, the natural morphism $\mathrm{DR}_X F_{p+\bullet}\mathscr{M} \to \mathrm{DR}_X\mathscr{M}$ is an isomorphism. We thus have a commutative diagram

$$\begin{tikzcd}
\mathrm{DR}_{\widetilde{X}(f)}(\widetilde{\jmath}_{f*}\widetilde{\jmath}_f^{\,-1}\mathscr{O}_{X^*}\otimes\varpi_f^{-1}F_{p+\bullet}\mathscr{M}) \arrow[r] \arrow[d, "\wr"'] & \mathrm{DR}_{\widetilde{X}(f)}(\widetilde{\jmath}_{f*}\widetilde{\jmath}_f^{\,-1}\mathscr{O}_{X^*}\otimes\varpi_f^{-1}\mathscr{M}) \arrow[dd] \\
\mathrm{DR}_{\widetilde{X}(f)}(\widetilde{\jmath}_{f*}j_f^{-1}F_{p+\bullet}\mathscr{M}) \arrow[d, "\wr"'] & \\
\boldsymbol{R}\widetilde{\jmath}_{f*}j_f^{-1}\,\mathrm{DR}_X\,F_{p+\bullet}\mathscr{M} \arrow[r, "\sim"] & \boldsymbol{R}\widetilde{\jmath}_{f*}j_f^{-1}\,\mathrm{DR}_X\,\mathscr{M}
\end{tikzcd}$$

and the right vertical morphism is an isomorphism if and only if the upper horizontal one is so. For the latter, it is enough to show that for $p\gg 0$,

$$\mathrm{DR}_{\widetilde{X}(f)}(\widetilde{\jmath}_{f*}\widetilde{\jmath}_f^{\,-1}\mathscr{O}_{X^*}\otimes\varpi_f^{-1}\,\mathrm{gr}^F_{p+\bullet}\mathscr{M})\simeq 0.$$

Since $\mathrm{gr}^F_q\mathscr{M}$ is $\mathscr{O}_X$-coherent for each q, the latter complex is isomorphic to $\boldsymbol{R}\widetilde{\jmath}_{f*}j_f^{-1}\,\mathrm{DR}_X\,\mathrm{gr}^F_{p+\bullet}\mathscr{M}$, hence is zero for $p\gg 0$ locally. □

As a consequence, for $\mathscr{M}$ in $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$, we can identify the natural distinguished triangles (4.2) with the distinguished triangles

$$(4.5)\qquad \begin{aligned}
&{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}(f)}\mathscr{M}\longrightarrow \boldsymbol{R}\widetilde{\jmath}_{f*}\widetilde{\jmath}_f^{\,-1}\,{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}(f)}\mathscr{M}\longrightarrow \boldsymbol{R}\widetilde{\imath}_{f*}\widetilde{\imath}_f^{\,!}\,{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}(f)}\mathscr{M}[1]\xrightarrow{+1}\\
&{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}(f)}\mathscr{M}\longrightarrow \boldsymbol{R}\widetilde{\jmath}_{f*}\widetilde{\jmath}_f^{\,-1}\,{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}(f)}\mathscr{M}\longrightarrow \boldsymbol{R}\widetilde{\imath}_{f*}\widetilde{\imath}_f^{\,!}\,{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}(f)}\mathscr{M}[1]\xrightarrow{+1}.
\end{aligned}$$

From 3.a(5) and (6) we obtain the following, due to the flatness property 3.a(1).

Lemma 4.6. *For a $\mathscr{D}_X$-module $\mathscr{M}$ we have*

$$\begin{aligned}
\boldsymbol{R}\varpi_{f*}\,{}^{\mathrm{p}}\mathrm{DR}^{*}_{\widetilde{X}(f)}\mathscr{M}&\simeq \boldsymbol{R}j_{f*}j_f^{-1}\,{}^{\mathrm{p}}\mathrm{DR}_X\mathscr{M},\\
\boldsymbol{R}\varpi_{f*}\,{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}(f)}\mathscr{M}&\simeq {}^{\mathrm{p}}\mathrm{DR}_X(\mathscr{M}(*X_0)),\\
\boldsymbol{R}\varpi_{f*}\,{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}(f)}\mathscr{M}&\simeq \mathrm{Cone}\big[{}^{\mathrm{p}}\mathrm{DR}_X\mathscr{M}\to{}^{\mathrm{p}}\mathrm{DR}(\mathscr{O}_{\widehat{X|X_0}}\otimes_{\mathscr{O}_X}\mathscr{M})\big][-1]\\
&=\mathrm{Cone}\big[{}^{\mathrm{p}}\mathrm{DR}_X(\mathscr{M}(*X_0))\to{}^{\mathrm{p}}\mathrm{DR}(\mathscr{O}_{\widehat{X|X_0}}\otimes_{\mathscr{O}_X}\mathscr{M}(*X_0))\big][-1]\\
\boldsymbol{R}\varpi_{f*}\,{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{mod/rd}}_{\widetilde{X}(f)}\mathscr{M}&\simeq{}^{\mathrm{p}}\mathrm{DR}(\mathscr{O}_{\widehat{X|X_0}}\otimes_{\mathscr{O}_X}\mathscr{M}(*X_0)),\\
\boldsymbol{R}\varpi_{f*}\,{}^{\mathrm{p}}\mathrm{DR}^{>\mathrm{mod}}_{\widetilde{X}(f)}\mathscr{M}&\simeq\boldsymbol{R}i_{f*}i_f^{!}\,{}^{\mathrm{p}}\mathrm{DR}_X(\mathscr{M}(*X_0))[1].
\end{aligned}$$
□

Theorem 4.7. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Then ${}^{\mathrm{p}}\mathrm{DR}^{\star}_{\widetilde{X}(f)}\mathscr{M}$ belongs to $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_{\widetilde{X}(f)})$.*

Proof. The case $\star=*$ will follow from Theorem 1.7 and we postpone its proof. It is then enough to prove the cases $\star=\mathrm{mod}$ and $\star=\mathrm{rd}$. By a standard "dévissage", we can assume that $\mathscr{M}$ is a meromorphic flat bundle on X with pole divisor P containing X_0. We can work locally on X, so we can find a projective modification such that the pole divisor of the pull-back connection has only normal crossings with smooth components. Moreover, according to the theorem of Kedlaya and Mochizuki (see [**Ked11**], and [**Moc09**] in the algebraic case), up to blowing-up more, we can also assume that the pull-back connection has a *good formal structure* along its normal crossing pole set D. We denote by $e:(Y,D)\to(X,P)$ the projective modification thus obtained and we set $g=f\circ e$, $D_g=g^{-1}(0)=e^{-1}(X_0)\subset D$ (it is the union of some components of D). We consider the commutative diagram

$$
\begin{tikzcd}
\widetilde{Y}(D) \arrow[r, "\varpi_{D,D_g}"] \arrow[rr, bend right, "\varpi_{D,g}"'] \arrow[rrr, bend left, "\widetilde{\varepsilon}"] \arrow[drr, bend right, "\varpi_D"'] & \widetilde{Y}(D_g) \arrow[r, "\varpi_{D_g,g}"] & \widetilde{Y}(g) \arrow[r, "\widetilde{e}"] \arrow[d, "\varpi_g"'] & \widetilde{X}(f) \arrow[d, "\varpi_f"] \\
& & Y \arrow[r, "e"] & X
\end{tikzcd}
\tag{4.8}
$$

The spaces $\widetilde{Y}(D)$ and $\widetilde{Y}(D_g)$ are complex manifolds with corners. We set $D=D_g\cup D'$, where D' has no common component with D_g. It will be useful to distinguish between the behaviors along D_g and D'.

Let $\mathscr{M}':=e^{+}\mathscr{M}$ be the pull-back meromorphic flat bundle. Then one can check that $\mathscr{M}=e_{+}\mathscr{M}'$.

Remark 4.9. In [**Moc14**], the de Rham functors ${}^{\mathrm{p}}\mathrm{DR}^{\leqslant D}_{\widetilde{Y}(D)}$, ${}^{\mathrm{p}}\mathrm{DR}^{<D_g,\leqslant D'}_{\widetilde{Y}(D)}$ etc. are considered, where the symbol $\leqslant D$ refers to coefficients in the Nilsson class, and $<D$ to rapid decay coefficients. On the other hand, we will consider the de Rham functors ${}^{\mathrm{p}}\mathrm{DR}^{\mathrm{mod}\,D}_{\widetilde{Y}(D)}$, ${}^{\mathrm{p}}\mathrm{DR}^{\mathrm{rd}\,D_g,\mathrm{mod}\,D'}_{\widetilde{Y}(D)}$ etc., with the more general condition of moderate growth. This is no problem since we only considered these functors when applied to meromorphic flat bundles which have a *good formal structure* along (Y,D). In this case, the natural morphisms between the corresponding de Rham functors is an isomorphism: this follows from [**Moc14**, Prop. 5.1.3] and Theorem 4.11 below.

We then have:

$$
\begin{aligned}
{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{mod}\,D_g}_{\widetilde{Y}(g)}\,\mathscr{M}' &\simeq \boldsymbol{R}\varpi_{D,g*}\,{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{mod}\,D}_{\widetilde{Y}(D)}\,\mathscr{M}',\\
{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{rd}\,D_g}_{\widetilde{Y}(g)}\,\mathscr{M}' &\simeq \boldsymbol{R}\varpi_{D,g*}\,{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{rd}\,D_g,\mathrm{mod}\,D'}_{\widetilde{Y}(D)}\,\mathscr{M}'.
\end{aligned}
\tag{4.10}
$$

Indeed, this is obtained by applying first [**Moc14**, Lem. 5.1.6] to the morphism ϖ_{D,D_g}, according to the remark above, and then [**Moc14**, Prop. 4.7.4] to $\rho=\varpi_{D_g,g}$.

Since $\mathbb{R}$-constructibility is stable by proper push-forward by a real analytic map between real analytic manifolds, we conclude from (4.10) and Corollary B.7 (due to the remarks in §2), that it is enough to prove the $\mathbb{R}$-constructibility of ${}^{\mathrm{p}}\mathrm{DR}^{\mathrm{mod}\,D}_{\widetilde{Y}(D)}\,\mathscr{M}'$ and ${}^{\mathrm{p}}\mathrm{DR}^{\mathrm{rd}\,D_g,\mathrm{mod}\,D'}_{\widetilde{Y}(D)}\,\mathscr{M}'$. The generalized Hukuhara-Turrittin theorem gives the following consequence (see [**Sab13**, §12.d]):

Theorem 4.11. *Let $\mathscr{M}'$ be a meromorphic flat bundle on Y with poles along D. Assume moreover that $\mathscr{M}'$ has a good formal structure along D. Then the complexes $\mathrm{DR}^{\mathrm{rd}\,D}_{\widetilde{Y}(D)}\,\mathscr{M}'$, $\mathrm{DR}^{\mathrm{rd}\,D_g,\mathrm{mod}\,D'}_{\widetilde{Y}(D)}\,\mathscr{M}'$ and $\mathrm{DR}^{\mathrm{mod}\,D}_{\widetilde{Y}(D)}\,\mathscr{M}'$ have cohomology in degree zero at* *most, and their $\mathscr{H}^0$ are nested subsheaves of the local system $\mathscr{H}^0\,\mathrm{DR}^*_{\widetilde{Y}(D)}\,\mathscr{M}'$, which are $\mathbb{R}$-constructible with respect to the stratification on $\partial\widetilde{Y}(D)$ determined by the good stratified $\mathfrak{I}$-covering associated with $\mathscr{M}'$ (see* §2.c*).* $\square$

This concludes the proof of Theorem 4.7 (modulo the case $\star=*$). $\square$

***Example 4.12* (Regular singularities).** Assume that $\mathscr{M}$ is a regular holonomic $\mathscr{D}_X$-module. Then, in the "dévissage" aforementioned, the meromorphic bundle with flat connection $\mathscr{M}'$ has regular singularities along D. In Theorem 4.11, one finds that the sheaf $\mathscr{H}^0\,\mathrm{DR}^{\mathrm{rd}D_g,\mathrm{mod}D'}_{\widetilde{Y}(D)}\,\mathscr{M}'$ vanishes along $\varpi_D^{-1}(D_g)$ and is equal to the local system $\mathscr{H}^0\,\mathrm{DR}^*_{\widetilde{Y}(D)}\,\mathscr{M}'$ when restricted to $\varpi_D^{-1}(D\smallsetminus D_g)$. Similarly, $\mathscr{H}^0\,\mathrm{DR}^{\mathrm{mod}D}_{\widetilde{Y}(D)}\,\mathscr{M}'$ is equal to $\mathscr{H}^0\,\mathrm{DR}^*_{\widetilde{Y}(D)}\,\mathscr{M}'$. One concludes that

$$\begin{aligned}
&{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}(f)}\,\mathscr{M}\simeq{}^{\mathrm{p}}\mathrm{DR}^{*}_{\widetilde{X}(f)}\,\mathscr{M}=\boldsymbol{R}\widetilde{\jmath}_{f*}\,\widetilde{\jmath}_f^{-1}\,{}^{\mathrm{p}}\mathrm{DR}_X\,\mathscr{M},\\
&{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}(f)}\,\mathscr{M}\simeq\boldsymbol{R}\widetilde{\jmath}_{f!}\,\widetilde{\jmath}_f^{-1}\,{}^{\mathrm{p}}\mathrm{DR}_X\,\mathscr{M},\\
&{}^{\mathrm{p}}\mathrm{DR}^{>\mathrm{mod}}_{\widetilde{X}(f)}\,\mathscr{M}=0.
\end{aligned}$$

4.b. Duality properties

Conjecture 4.13. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module.*

(1) *We have a functorial isomorphism*

$$\boldsymbol{D}\,{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}(f)}\,\mathscr{M}\simeq{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}(f)}\,\boldsymbol{D}\mathscr{M},$$

so that the dual of the distinguished triangle

$${}^{\mathrm{p}}\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}(f)}\,\mathscr{M}\longrightarrow{}^{\mathrm{p}}\mathrm{DR}^{*}_{\widetilde{X}(f)}\,\mathscr{M}\longrightarrow{}^{\mathrm{p}}\mathrm{DR}^{>\mathrm{mod}}_{\widetilde{X}(f)}\,\mathscr{M}\xrightarrow{+1}$$

is functorially isomorphic to the natural triangle

$$\boldsymbol{R}\widetilde{\imath}_{f*}\widetilde{\imath}_f^{-1}\,{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}(f)}\,\boldsymbol{D}\mathscr{M}[-1]\longrightarrow\boldsymbol{R}\widetilde{\jmath}_{f\,!}\,j_f^{-1}\,{}^{\mathrm{p}}\mathrm{DR}_X\,\boldsymbol{D}\mathscr{M}\longrightarrow{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}(f)}\,\boldsymbol{D}\mathscr{M}\xrightarrow{+1}.$$

(2) *We have a functorial isomorphism*

$$\boldsymbol{D}\,{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}(f)}\,\mathscr{M}\simeq{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}(f)}\,\boldsymbol{D}\mathscr{M},$$

so that the dual of the distinguished triangle

$${}^{\mathrm{p}}\mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}(f)}\,\mathscr{M}\longrightarrow{}^{\mathrm{p}}\mathrm{DR}^{*}_{\widetilde{X}(f)}\,\mathscr{M}\longrightarrow{}^{\mathrm{p}}\mathrm{DR}^{>\mathrm{rd}}_{\widetilde{X}(f)}\,\mathscr{M}\xrightarrow{+1}$$

is functorially isomorphic to the natural triangle

$$\boldsymbol{R}\widetilde{\imath}_{f*}\widetilde{\imath}_f^{-1}\,{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}(f)}\,\boldsymbol{D}\mathscr{M}[-1]\longrightarrow\boldsymbol{R}\widetilde{\jmath}_{f\,!}\,j_f^{-1}\,{}^{\mathrm{p}}\mathrm{DR}_X\,\boldsymbol{D}\mathscr{M}\longrightarrow{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}(f)}\,\boldsymbol{D}\mathscr{M}\xrightarrow{+1}.$$

Corollary 4.14. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. We have a natural isomorphism*

$$\boldsymbol{D}\,{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{mod/rd}}_{\widetilde{X}(f)}\,\mathscr{M}\simeq{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{mod/rd}}_{\widetilde{X}(f)}\,\boldsymbol{D}\mathscr{M}[-1].$$

Remark 4.15. We show in Appendix C how to use the results of [**Moc14**] to obtain a local version of Conjecture 4.13 when $\mathscr{M}$ is a meromorphic flat bundle on X (see Proposition C.1).

4.c. Non-characteristic properties. Let $p : X \to S$ be a smooth holomorphic map to a disc S with coordinate t. For $s \in S$, we denote by $f_s : X_s \to \mathbb{C}$ the function induced by f on $X_s := p^{-1}(s)$, by $i_s : X_s \hookrightarrow X$ and $\widetilde{\imath}_s : \widetilde{X}_s(f_s) \hookrightarrow \widetilde{X}(f)$ the inclusions.

Proposition 4.16. *For each $x_o \in p^{-1}(0)$ there exist an open neighbourhood of x_o in X such that, up to shrinking S, the following holds for each $s \in S \smallsetminus \{0\}$:*

(1) $\widetilde{X}_s(f_s) = \widetilde{X}(f)_{|X_s}$,
(2) $\mathscr{A}^{\star}_{\widetilde{X}_s(f_s)} = \widetilde{\imath}_s^* \mathscr{A}^{\star}_{\widetilde{X}(f)} = \boldsymbol{L}\widetilde{\imath}_s^* \mathscr{A}^{\star}_{\widetilde{X}(f)}$.

Proof. We consider the setting and notation of the proof of Theorem 4.7, and X still denotes a small neighbourhood of x_o. Since e is proper, we can shrink X and S so that, on each stratum of the natural stratification of D, $p \circ e$ has maximal rank over $S \smallsetminus \{0\}$. Locally near a point of $Y \smallsetminus (p \circ e)^{-1}(0)$, we can find local coordinates $(y_1, \dots, y_n)$ such that $D = \{y_1 \cdots y_\ell = 0\}$ $(\ell < n)$ and $p \circ e = y_n$. Let us check that he assertions corresponding to (1) and (2) for $\widetilde{Y}(D)$ and $p \circ e$ are true in this local setting, hence all over $\widetilde{Y}(D)$. This is clear for (1). For (2), this is clear for $\mathscr{A}^{*}_{\widetilde{Y}(D)} = \widetilde{\jmath}_* \mathscr{O}_{Y^*}$. We can argue with the maximum principle as in [**Moc14**, Lem. 4.4.1] for $\mathscr{A}^{\mathrm{mod}}_{\widetilde{Y}(D)}$ and $\mathscr{A}^{\mathrm{rd}}_{\widetilde{Y}(D)}$. It remains to show the injectivity of $t - s$ on $\widetilde{\jmath}_* \mathscr{O}_{Y^*} / \mathscr{A}^{\mathrm{mod}}_{\widetilde{Y}(D)}$, $\widetilde{\jmath}_* \mathscr{O}_{Y^*} / \mathscr{A}^{\mathrm{rd}}_{\widetilde{Y}(D)}$ and $\mathscr{A}^{\mathrm{mod}}_{\widetilde{Y}(D)} / \mathscr{A}^{\mathrm{rd}}_{\widetilde{Y}(D)}$. This is obtained by the same argument using the maximum principle.

We note that, for $s \neq 0$, $\widetilde{Y}_s(D_s)$ is the closure of Y_s^* in $\widetilde{Y}(D)$, and we have a similar property for $\widetilde{Y}_s(g_s)$. Since $\varpi_{D,g}$ is proper, we conclude that (1) holds for $\widetilde{Y}(g)$. Using now the properness of $\widetilde{e}$, we obtain similarly (1) for $\widetilde{X}(f)$.

Now, (2) for $\widetilde{X}(f)$ is obtained from (2) for $\widetilde{Y}(D)$ by using 3.b(2) and 3.a(9). $\square$

We consider the sheaf $\mathscr{D}_{X/S}$ of relative differential operators, which is a subsheaf of $\mathscr{D}_X$ and, for a holonomic $\mathscr{D}_X$-module $\mathscr{M}$, the relative de Rham complex $\mathrm{DR}_{X/S}\,\mathscr{M}$, which is a complex of $p^{-1}\mathscr{O}_S$-modules. By pulling it back to $\widetilde{X}(f)$ and tensoring the terms with $\mathscr{A}^{\star}_{\widetilde{X}(f)}$, we obtain the relative $\star$de Rham complex $\mathrm{DR}^{\star}_{\widetilde{X}(f)/S}\,\mathscr{M}$, which is a complex of $(p \circ \pi_f)^{-1}\mathscr{O}_S$-modules.

On the other hand, $\mathrm{DR}^{\star}_{\widetilde{X}(f)}\,\mathscr{M}$ is the single complex associated with the double complex

$$\mathrm{DR}^{\star}_{\widetilde{X}(f)/S}\,\mathscr{M} \xrightarrow{\partial_t} \mathrm{DR}^{\star}_{\widetilde{X}(f)/S}\,\mathscr{M},$$

and the natural morphism

$$\begin{tikzcd}
\mathrm{DR}^{\star}_{\widetilde{X}(f)/S}\,\mathscr{M} \arrow[r, "\partial_t"] \arrow[d, equal] & \mathrm{DR}^{\star}_{\widetilde{X}(f)/S}\,\mathscr{M} \arrow[d] \\
\mathrm{DR}^{\star}_{\widetilde{X}(f)/S}\,\mathscr{M} \arrow[r] & 0
\end{tikzcd}$$

induces a $(p \circ \varpi_f)^{-1}\mathscr{O}_S$-linear morphism $(p \circ \varpi_f)^{-1}\mathscr{O}_S \otimes_{\mathbb{C}} \mathrm{DR}^{\star}_{\widetilde{X}(f)}\,\mathscr{M} \to \mathrm{DR}^{\star}_{\widetilde{X}(f)/S}\,\mathscr{M}$.

Proposition 4.17. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Up to shrinking X, and restricting away from $p^{-1}(0)$, the natural morphism $(p\circ\varpi_f)^{-1}\mathscr{O}_S\otimes_{\mathbb{C}}\mathrm{DR}^{\star}_{\widetilde{X}(f)}\mathscr{M}\to\mathrm{DR}^{\star}_{\widetilde{X}(f)/S}\mathscr{M}$ is a quasi-isomorphism.*

Corollary 4.18. *With the assumptions above, for $s\neq 0$ we have*

$$\mathscr{H}^k i^+_{X_s}\mathscr{M}=0\quad\text{if } k\neq 0,$$

and a functorial isomorphism

$$i^{-1}_{X_s}\mathrm{DR}^{\star}_{\widetilde{X}(f)}\mathscr{M}\simeq\mathrm{DR}^{\star}_{\widetilde{X}_s(f_s)}\,i^+_{X_s}\mathscr{M}.$$

Proof. For X small enough, X_s is non-characteristic for $\mathscr{M}$ if $s\neq 0$, hence the first point. Then, according to Proposition 4.16,

$$\mathrm{DR}^{\star}_{\widetilde{X}_s(f_s)}\,i^+_{X_s}\mathscr{M}\simeq\boldsymbol{L}i^*_{X_s}\mathrm{DR}^{\star}_{\widetilde{X}(f)/S}\mathscr{M},\quad\text{if } s\neq 0.$$

We then conclude the proof by using Proposition 4.17. □

Remark 4.19. By definition, ${}^{\mathrm{p}}\mathrm{DR}^{\star}_{\widetilde{X}(f)}\mathscr{M}$ has nonzero cohomology in non-positive degrees at most. On the other hand, we claim that $\mathscr{H}^0\,{}^{\mathrm{p}}\mathrm{DR}^{\star}_{\widetilde{X}(f)}\mathscr{M}=0$ away from the pull-back by ϖ_f of a discrete set of points in X. Indeed, Let $x_o\in X$ and let $p:\mathrm{nb}(x_o)\to S$ be a smooth function defined in a neighbourhood of x_o, that we still denote by X. Then Corollary 4.18 reads

$$i^{-1}_{X_s}\,{}^{\mathrm{p}}\mathrm{DR}^{\star}_{\widetilde{X}(f)}\mathscr{M}\simeq{}^{\mathrm{p}}\mathrm{DR}^{\star}_{\widetilde{X}_s(f_s)}\,\mathscr{H}^0 i^+_{X_s}\mathscr{M}[1]\quad\text{for } s\neq 0,$$

hence the vanishing of $i^{-1}_{X_s}\mathscr{H}^0\,{}^{\mathrm{p}}\mathrm{DR}^{\star}_{\widetilde{X}(f)}\mathscr{M}$ for $s\neq 0$. One obtains the assertion by applying this to the projections along all coordinate hyperplanes centered at x_o.

Proof of Proposition 4.17. Since the argument for proving Corollary B.7 relies on [**Moc14**, Th. 4.1.5], one obtains that it holds for the relative $\star$de Rham complex, provided $p\circ\pi$ is smooth. Similarly, (4.10) holds in the relative case provided $p\circ e$ is smooth. The smoothness assumption holds when we restrict to $S\smallsetminus\{0\}$ if (X,x_o) is small enough.

We take up the setting and notation of the proof of Theorem 4.7, in particular as indicated in (4.8). Then, according to the preliminary remark above, we have

$$\begin{aligned}\boldsymbol{R}\widetilde{\varepsilon}_*\mathrm{DR}^{\star}_{\widetilde{Y}(D)}\mathscr{M}'&\simeq\mathrm{DR}^{\star}_{\widetilde{X}(f)}\,\varepsilon_+\mathscr{M}',\\ \boldsymbol{R}\widetilde{\varepsilon}_*\mathrm{DR}^{\star}_{\widetilde{Y}(D)/S}\mathscr{M}'&\simeq\mathrm{DR}^{\star}_{\widetilde{X}(f)/S}\,\varepsilon_+\mathscr{M}'.\end{aligned}$$

On the other hand,

$$\begin{aligned}\boldsymbol{R}\widetilde{\varepsilon}_*\Big[(p\circ e\circ\varpi_D)^{-1}\mathscr{O}_S\otimes_{\mathbb{C}}\mathrm{DR}^{\star}_{\widetilde{Y}(D)}\mathscr{M}'\Big]&\simeq\boldsymbol{R}\widetilde{\varepsilon}_*\Big[(p\circ\varpi_f\circ\widetilde{\varepsilon})^{-1}\mathscr{O}_S\otimes_{\mathbb{C}}\mathrm{DR}^{\star}_{\widetilde{Y}(D)}\mathscr{M}'\Big]\\ &\simeq(p\circ\varpi_f)^{-1}\mathscr{O}_S\otimes_{\mathbb{C}}\boldsymbol{R}\widetilde{\varepsilon}_*\mathrm{DR}^{\star}_{\widetilde{Y}(D)}\mathscr{M}'\\ &\simeq(p\circ\varpi_f)^{-1}\mathscr{O}_S\otimes_{\mathbb{C}}\mathrm{DR}^{\star}_{\widetilde{X}(f)}\,\varepsilon_+\mathscr{M}',\end{aligned}$$

so that it is enough to prove the proposition for $\widetilde{Y}(D)$ and $p\circ e$. The case when $\star=*$ being easy, we are reduced to checking the cases when $\star=\mathrm{rd}$ and $\star=\mathrm{mod}$. We can now work locally on (Y,D) near a point $y_o\in e^{-1}(x_o)$, due to the properness of e. We then take the notation $\mathscr{M}$ instead of $\mathscr{M}'$. We choose local coordinates near a point of a neighbourhood of y_o not in $(p\circ e)^{-1}(0)$, as in the proof of Proposition 4.16.

Let $\rho_{\boldsymbol{d}}$ be a local ramification along the components of D. Then $\mathscr{M}$ is a direct summand of $\rho_{\boldsymbol{d}+}\rho_{\boldsymbol{d}}^{+}\mathscr{M}$ so, by the push-forward argument already used, we can assume that $\mathscr{M}$ has a good formal decomposition along D. According to the generalized Hukuhara-Turrittin theorem already used in Theorem 4.11 (see e.g. [**Sab13**, Th. 12.5] and the references given therein), we can reduce to the case where $\mathscr{M}=\mathscr{E}^{\varphi}\otimes\mathscr{R}$, where $\mathscr{E}^{\varphi}=(\mathscr{O}_Y(*D),\mathrm{d}+\mathrm{d}\varphi)$ and φ is purely monomial, and $\mathscr{R}$ has a regular singularity. By induction on the rank of $\mathscr{R}$, we can assume that $\mathscr{R}$ has rank one as an $\mathscr{O}_Y(*D)$-module.

By the theorems of Majima [**Maj84**] (see also [**Sab93**, App.] for the rapid-decay case and [**Hie07**, App.] for the case with moderate growth), one proves that both the relative and the absolute de Rham complexes (in the variants rd and mod) have cohomology in degree zero only. Due to the special form of $\mathscr{M}$, computing the $\mathscr{H}^0$ of these complexes is easy, by twisting with $e^{-\varphi}$, and the desired isomorphism is then straightforward to obtain, as it is clear to decide whether $e^{-\varphi}x^{\alpha}$ ($\alpha\in\mathbb{C}^{\ell}$) has rapid decay (resp. moderate growth) in any given small multi-sector. $\square$

5. The sheaf of nearby cycles as a sheaf on the real blow-up space

5.a. The functor ${}^{\mathrm{p}}\widetilde{\psi}_f{}^*$. Let $\mathscr{F}$ be an object of $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_X)$. We set

$$\widetilde{\psi}_f{}^*\mathscr{F}:=\widetilde{\imath}_f{}^{-1}\boldsymbol{R}\widetilde{\jmath}_{f*}\widetilde{\jmath}_f{}^{-1}\mathscr{F},$$

and ${}^{\mathrm{p}}\widetilde{\psi}_f{}^*\mathscr{F}:=\widetilde{\psi}_f{}^*\mathscr{F}[-1]$ (recall that, similarly, ${}^{\mathrm{p}}\psi_f\,\mathscr{F}:=\psi_f\mathscr{F}[-1]$).

Remark 5.1. If $\mathscr{F}$ is an object of $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$, then $\widetilde{\psi}_f{}^*\mathscr{F}$ is an object of $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_{\partial\widetilde{X}})$:

- the pullback $\varpi_f^{-1}\mathscr{F}$ is $\mathbb{R}$-constructible, as follows from [**KS90**, Prop. 8.4.10(i)];
- weak $\mathbb{R}$-constructibility $\boldsymbol{R}\widetilde{\jmath}_{f\,!}\widetilde{\jmath}_f{}^{-1}\mathscr{F}$ follows from the existence of a subanalytic refinement compatible with the pair $(\widetilde{X},\partial\widetilde{X})$ of a given subanalytic stratification of $\widetilde{X}$, and the finiteness property for obtaining $\mathbb{R}$-constructibility is clear since $\boldsymbol{R}\widetilde{\jmath}_{f\,!}\widetilde{\jmath}_f{}^{-1}\mathscr{F}$ is zero on $\partial\widetilde{X}$;
- by duality (see [**KS90**, Prop. 8.4.9]), $\widetilde{\jmath}_{f*}\widetilde{\jmath}_f{}^{-1}\mathscr{F}$ is $\mathbb{R}$-constructible, and applying once more [**KS90**, Prop. 8.4.10(i)] one obtains the $\mathbb{R}$-constructibility of $\widetilde{\psi}_f{}^*\mathscr{F}$.

Lemma 5.2. *Let $\pi:Y\to X$ be a proper morphism between complex manifolds and set $g=f\circ\pi$. For $\mathscr{G}$ in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_Y)$, we have a functorial isomorphism*

$$\boldsymbol{R}\pi_*{}^{\mathrm{p}}\widetilde{\psi}_g{}^*\mathscr{G}\simeq{}^{\mathrm{p}}\widetilde{\psi}_f{}^*\boldsymbol{R}\pi_*\mathscr{F}.$$

Proof. The lemma immediately follows from the base change theorem for a proper morphism and the property that $\widetilde{Y}(g)=Y\times_X\widetilde{X}(f)$. $\square$

As a consequence, one can reduce the computation of ${}^{\mathrm{p}}\widetilde{\psi_f}^*\mathscr{F}$ to the case where f is the projection $X = X_0 \times \mathbb{C} \to \mathbb{C}$, by applying the lemma to the graph embedding of f.

For an object $\mathscr{M}$ of $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$, we set

$$
{}^{\mathrm{p}}\widetilde{\psi_f}^*\mathscr{M} := {}^{\mathrm{p}}\widetilde{\psi_f}^*\,{}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}. \tag{5.3}
$$

Lemma 5.4. *Same setting as in Lemma* 5.2. *For $\mathscr{M}$ in $\mathsf{D}^{\mathrm{b}}_{\pi\text{-good}}(\mathscr{D}_Y)$, we have a functorial isomorphism*

$$
\boldsymbol{R}\pi_*{}^{\mathrm{p}}\widetilde{\psi_g}^*\mathscr{M} \simeq {}^{\mathrm{p}}\widetilde{\psi_f}^*(\pi_+\mathscr{M}). \qquad \square
$$

5.b. Proof of Theorem 1.7(1)**.** This theorem follows from Proposition 5.5 below, which applies to any $\mathbb{C}$-constructible complex $\mathscr{F}$. We will use the notation of the diagram in Figure 1, where all squares are cartesian, $\widetilde{X} := \widetilde{X}(f)$, and all maps ρ are defined from the universal covering $\mathbb{R} \to S^1$.

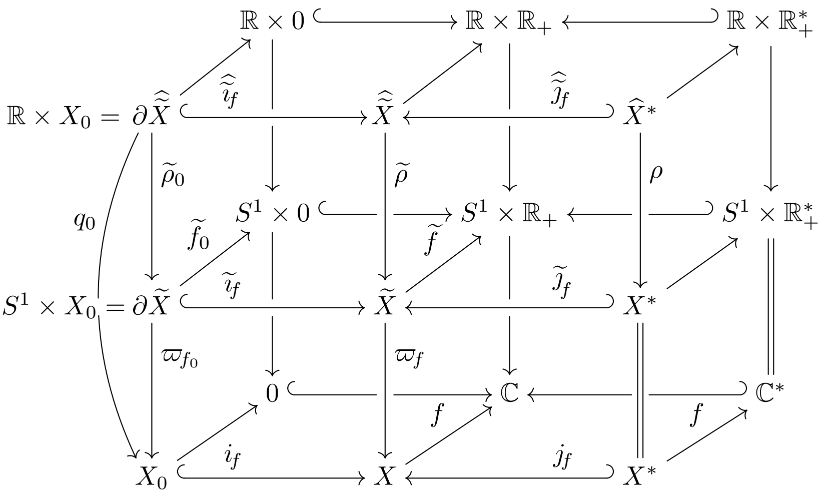

FIGURE 1.

Let us set $\mathscr{F}^* = j_f^{-1}\mathscr{F}$. Our first aim is to express the complex of nearby cycles $(\psi_f\mathscr{F}, \mathrm{T})$ as defined by Deligne [**Del73**] in terms of $\widetilde{\psi_f}^*\mathscr{F}$. Let $\sigma_0 : \partial\widehat{\widetilde{X}} \xrightarrow{\sim} \partial\widehat{\widetilde{X}}$ be the automorphism induced by $\theta \mapsto \theta + 1$ on $\mathbb{R}$. Since $\widetilde{\rho}_0 \circ \sigma_0 = \widetilde{\rho}_0$, we have an isomorphism

$$
\widetilde{\rho}_0^{-1}\widetilde{\psi_f}^*\mathscr{F} \xrightarrow{\sim} \sigma_0^{-1}\widetilde{\rho}_0^{-1}\widetilde{\psi_f}^*\mathscr{F},
$$

hence an isomorphism

$$
\widetilde{\mathrm{T}} : \boldsymbol{R}\widetilde{\rho}_{0*}\widetilde{\rho}_0^{-1}\widetilde{\psi_f}^*\mathscr{F} \longrightarrow \boldsymbol{R}\widetilde{\rho}_{0*}\widetilde{\rho}_0^{-1}\widetilde{\psi_f}^*\mathscr{F},
$$

and thus an automorphism $\widetilde{\mathrm{T}}$ of $\boldsymbol{R}q_{0*}\widetilde{\rho}_0^{-1}\widetilde{\psi_f}^*\mathscr{F}$.

Proposition 5.5. *We have a functorial isomorphism* $(\psi_f\mathscr{F}, \mathrm{T}) \simeq (\boldsymbol{R}q_{0*}\widetilde{\rho}_0^{-1}\widetilde{\psi_f}^*\mathscr{F}, \widetilde{\mathrm{T}})$ *and the morphism*

$$q_0^{-1}\psi_f\mathscr{F} \longrightarrow \widetilde{\rho}_0^{-1}\widetilde{\psi_f}^*\mathscr{F} \tag{5.5 $*$}$$

induced by the adjunction $q_0^{-1}\boldsymbol{R}q_{0*} \to \mathrm{Id}$ *is an isomorphism.*

Lemma 5.6. *Let us set* $\mathscr{F}^* := j_f^{-1}\mathscr{F}$. *We have a functorial isomorphism*

$$(\boldsymbol{R}\widetilde{\rho}_{0*}\widetilde{\rho}_0^{-1}\widetilde{\psi_f}^*\mathscr{F}, \widetilde{\mathrm{T}}) \simeq (\widetilde{\imath}_f^{\,-1}\boldsymbol{R}\widetilde{\jmath}_{f*}(\boldsymbol{R}\rho_*\rho^{-1}\mathscr{F}^*), \mathrm{T}).$$

Proof. We have

$$\begin{aligned}
\widetilde{\imath}_f^{\,-1}\boldsymbol{R}\widetilde{\jmath}_{f*}(\boldsymbol{R}\rho_*\rho^{-1}\mathscr{F}^*) &\simeq \widetilde{\imath}_f^{\,-1}\boldsymbol{R}\widetilde{\rho}_*\boldsymbol{R}\widehat{\widetilde{\jmath}}_{f*}\rho^{-1}\mathscr{F}^*\\
&\simeq \widetilde{\imath}_f^{\,-1}\boldsymbol{R}\widetilde{\rho}_*\widetilde{\rho}^{-1}\boldsymbol{R}\widetilde{\jmath}_{f*}\mathscr{F}^* \quad (\rho^{-1} = \rho^!,\ \widetilde{\rho}^{-1} = \widetilde{\rho}^!)\\
&\simeq \boldsymbol{R}\widetilde{\rho}_{0*}\,\widetilde{\widetilde{\imath}}_f^{-1}\,\widetilde{\rho}^{-1}\boldsymbol{R}\widetilde{\jmath}_{f*}\mathscr{F}^* \quad \text{(Example A.3)}\\
&= \boldsymbol{R}\widetilde{\rho}_{0*}\widetilde{\rho}_0^{-1}\widetilde{\imath}_f^{\,-1}\boldsymbol{R}\widetilde{\jmath}_{f*}\mathscr{F}^* = \boldsymbol{R}\widetilde{\rho}_{0*}\widetilde{\rho}_0^{-1}\widetilde{\psi_f}^*\mathscr{F}.
\end{aligned}$$

The compatibility with $\widetilde{\mathrm{T}}, \mathrm{T}$ is then clear. □

Proof of Proposition 5.5 *(first part).* We have

$$\begin{aligned}
\psi_f\mathscr{F} &= i_f^{-1}\boldsymbol{R}j_{f*}\boldsymbol{R}\rho_*\rho^{-1}\mathscr{F}^* \quad \text{(by definition)}\\
&= i_f^{-1}\boldsymbol{R}\varpi_{f*}\boldsymbol{R}\widetilde{\jmath}_{f*}\boldsymbol{R}\rho_*\rho^{-1}\mathscr{F}^*\\
&= \boldsymbol{R}\varpi_{f_0*}\widetilde{\imath}_f^{\,-1}\boldsymbol{R}\widetilde{\jmath}_{f*}\boldsymbol{R}\rho_*\rho^{-1}\mathscr{F}^* \quad (\varpi_f \text{ proper})\\
&\simeq \boldsymbol{R}\varpi_{f_0*}\boldsymbol{R}\widetilde{\rho}_{0*}\widetilde{\rho}_0^{-1}\widetilde{\psi_f}^*\mathscr{F} \quad \text{(Lemma 5.6)}\\
&= \boldsymbol{R}q_{0*}\widetilde{\rho}_0^{-1}\widetilde{\psi_f}^*\mathscr{F}.
\end{aligned}$$

The compatibility with $\widetilde{\mathrm{T}}, \mathrm{T}$ follows from the previous lemma. □

Proof that (5.5 $*$) *is an isomorphism.*

Lemma 5.7. *Let* $\mathscr{G}$ *be a weakly* $\mathbb{R}$*-constructible bounded complex on* $\partial\widetilde{X}(f)$ *(see* [**KS90**, Def. 8.4.3]*) satisfying the following property:*

(5.7 $*$) *For each* $x \in X_0$ *and* $\widetilde{\imath}_x : \varpi_{f_0}^{-1}(x) \simeq S^1 \times \{x\} \hookrightarrow \partial\widetilde{X}(f) \simeq S^1 \times X_0$, *the cohomology sheaves of the restriction* $\widetilde{\imath}_x^{\,-1}\mathscr{G}$ *to* $S^1 \times \{x\}$ *are locally constant with finite rank.*

Then the adjunction morphism $q_0^{-1}\boldsymbol{R}q_{0*}\widetilde{\rho}_0^{-1}\mathscr{G} \to \widetilde{\rho}_0^{-1}\mathscr{G}$ *is an isomorphism.*

Proof. We first reduce to proving the lemma when X_0 is a point. It is enough to prove that, for every $x \in X_0$, the morphism

$$\widehat{\widetilde{\imath}}_x^{\,-1}q_0^{-1}\boldsymbol{R}q_{0*}\widetilde{\rho}_0^{-1}\mathscr{G} \longrightarrow \widehat{\widetilde{\imath}}_x^{\,-1}\widetilde{\rho}_0^{-1}\mathscr{G}$$

is an isomorphism, and this reduces to showing

$$q_0^{-1}\boldsymbol{R}\Gamma(S^1 \times \{x\}, \widetilde{\imath}_x^{\,-1}\boldsymbol{R}\widetilde{\rho}_{0*}\widetilde{\rho}_0^{-1}\mathscr{G}) \longrightarrow \widetilde{\rho}_0^{-1}\widetilde{\imath}_x^{\,-1}\mathscr{G}$$

is an isomorphism. Due to the assumption on $\mathscr{G}$, we can apply Example A.3 to write the right-hand side as $q_0^{-1}\boldsymbol{R}\Gamma(\mathbb{R}\times\{x\},\widetilde{\rho}_0^{-1}\widetilde{\imath}_x^{-1}\mathscr{G})$, so we are reduced to proving the lemma for $\widetilde{\imath}_x^{-1}\mathscr{G}$ on $S^1\times\{x\}$.

Now, if $\mathscr{G}$ is a bounded complex on S^1 whose cohomology is locally constant and of finite rank, the cohomology of $\widetilde{\rho}_0^{-1}\mathscr{G}$ on $\mathbb{R}$ is constant of finite rank, and $H^k(\mathbb{R},\mathscr{H}^j\mathscr{G})=0$ for $k\neq 0$, so it is easy to conclude that $q_0^{-1}\boldsymbol{R}\Gamma(\mathbb{R},\widetilde{\rho}_0^{-1}\mathscr{G})\to\widetilde{\rho}_0^{-1}\mathscr{G}$ is an isomorphism. □

Lemma 5.8. *Let $\pi:Y\to X$ be a proper morphism and set $g=f\circ\pi$. If the morphism (5.5 $*$) for g and a $\mathbb{C}$-constructible bounded complex $\mathscr{G}$ on Y is an isomorphism, then so is the morphism (5.5 $*$) for f and $\mathscr{F}=\boldsymbol{R}\pi_*\mathscr{G}$ on X.*

Proof. Straightforward due to the base change property for a proper morphism. □

By a standard 'dévissage', we can assume that there exists a divisor $D'\subset X$ with normal crossings and smooth components, such that, denoting by $j:U=X\smallsetminus D'\hookrightarrow X$ the inclusion, $\mathscr{F}=j_!\mathscr{L}$, where $\mathscr{L}$ is a local system on U, and $f^{-1}(0)=D\subset D'$, so $X_0=D$ with the previous notation. Let $\varpi_D:\widetilde{X}(D)\to X$ be the real blowing up of the components of D, so that $\widetilde{X}(D)$ is a manifold with corners. Then we have a decomposition $\varpi_D=\varpi_f\circ\varpi_{D,g}$ with $\varpi_{D,g}:\widetilde{X}(D)\to\widetilde{X}(f)$. We will prove that (5.7 $*$) holds for $\widetilde{\psi_f}^*\mathscr{F}$ with these assumptions.

We can choose local coordinates $(x_1,\dots,x_\ell,y_1,\dots,y_m,z_1,\dots,z_p)$ on X such that $f(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})=\prod_{i=1}^\ell x_i^{e_i}=:\boldsymbol{x}^{\boldsymbol{e}}$ ($e_i>0$ for all i) and $D'=\prod x_i\prod y_j=0$. Then $\widetilde{X}(D)$ has partial polar coordinates $(\boldsymbol{\rho},\mathrm{e}^{i\boldsymbol{\theta}},\boldsymbol{y},\boldsymbol{z})$ ($\rho_i\in\mathbb{R}_+$) and $\partial\widetilde{X}(D)=\{\rho_1\cdots\rho_\ell=0\}$. The map $\varpi_{D,g}:\partial\widetilde{X}(D)\to\partial\widetilde{X}(f)$ is induced by the map $(S^1)^\ell\to S^1$ given by $\mathrm{e}^{i\boldsymbol{\theta}}\mapsto\mathrm{e}^{\sum e_i\theta_i}$.

With obvious notation, we have $\widetilde{\psi_f}^*\mathscr{F}=\boldsymbol{R}\varpi_{D,g*}\widetilde{\imath}_D^{-1}\boldsymbol{R}\widetilde{\jmath}_{D*}^{-1}\mathscr{F}^*$. If we restrict to $\prod y_j\neq 0$, we have $\mathscr{F}^*=\mathscr{L}$ and $\widetilde{\mathscr{L}}:=\widetilde{\imath}_D^{-1}\boldsymbol{R}\widetilde{\jmath}_{D*}^{-1}\mathscr{F}^*$ is a local system with the same monodromy as $\mathscr{L}$. On $\partial\widetilde{X}(D)$ we then have $\widetilde{\imath}_D^{-1}\boldsymbol{R}\widetilde{\jmath}_{D*}^{-1}\mathscr{F}^*=j_!\widetilde{\mathscr{L}}$ (extension by zero along $\prod y_j=0$).

Since the map $\varpi_{D,g}:\partial\widetilde{X}(D)\to\partial\widetilde{X}(f)$ is a fibration and $\widetilde{\mathscr{L}}$ is a local system, we conclude that, for $x\in D\smallsetminus\{\prod y_j=0\}$, the cohomology of $\widetilde{\imath}_x^{-1}\widetilde{\psi_f}^*\mathscr{F}$ is locally constant and of finite rank, while if $x\in D\cap\{\prod y_j=0\}$, it is zero, so (5.7 $*$) holds for $\widetilde{\psi_f}^*\mathscr{F}$. □

5.c. The X_0-support condition and duality for ${}^{\mathrm{p}}\widetilde{\psi_f}^*\mathscr{M}$. Since ${}^{\mathrm{p}}\psi_f\,{}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}$ satisfies the support condition on X_0, the isomorphism (5.5 $*$) shows that ${}^{\mathrm{p}}\widetilde{\psi_f}^*\mathscr{M}:={}^{\mathrm{p}}\widetilde{\psi_f}^*\,{}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}$ satisfies the X_0-support condition.

In order to prove the property for the Verdier dual complex $\boldsymbol{D}\,{}^{\mathrm{p}}\widetilde{\psi_f}^*\mathscr{M}$, it is enough to prove that, for a constructible complex $\mathscr{F}$, we can find an isomorphism $\boldsymbol{D}\,{}^{\mathrm{p}}\widetilde{\psi_f}^*\mathscr{F}\xrightarrow{\sim}{}^{\mathrm{p}}\widetilde{\psi_f}\boldsymbol{D}\mathscr{F}[1]$, since ${}^{\mathrm{p}}\mathrm{DR}$ is compatible with duality. Since ${}^{\mathrm{p}}\widetilde{\psi_f}^*\mathscr{F}$ and $\mathscr{F}$ are $\mathbb{R}$-constructible, we have (see [**KS90**, Chap. 3]):

$$\boldsymbol{D}\widetilde{\psi_f}^*\mathscr{F}\simeq\widetilde{\imath}_f^{\,!}\boldsymbol{R}\widetilde{\jmath}_{f\,!}\boldsymbol{D}\mathscr{F}\simeq(\widetilde{\psi_f}^*\boldsymbol{D}\mathscr{F})[-1].$$

We conclude:

$$\boldsymbol{D}\,{}^{\mathrm{p}}\widetilde{\psi}_f{}^*\mathscr{F} = \boldsymbol{D}(\widetilde{\psi}_f{}^*\mathscr{F}[-1]) = (\boldsymbol{D}\widetilde{\psi}_f{}^*\mathscr{F})[1] \simeq \widetilde{\psi}_f{}^*\boldsymbol{D}\mathscr{F} \simeq ({}^{\mathrm{p}}\widetilde{\psi}_f{}^*\boldsymbol{D}\mathscr{F})[1]. \qquad \square$$

6. The moderate and rapid-decay nearby cycles

6.a. The functors ${}^{\mathrm{p}}\widetilde{\psi}_f{}^\star$. We keep Notation 3.1 and we set

$$\text{(6.1)} \qquad {}^{\mathrm{p}}\widetilde{\psi}_f{}^\star\mathscr{M} = \widetilde{\imath}_f{}^{-1}\,{}^{\mathrm{p}}\mathrm{DR}^\star_{\widetilde{X}(f)}\mathscr{M}[-1].$$

By the faithful flatness of $i_f^{-1}\mathscr{O}_{\widehat{X|X_0}}$ over $i_f^{-1}\mathscr{O}_X$, we have for every $\mathscr{O}_X$-module $\mathscr{M}$ the equality (see §3.a(6) for the notation $\mathscr{Q}_{X_0}$)

$$\text{(6.2)} \qquad \mathscr{Q}_{X_0}\overset{\boldsymbol{L}}{\otimes}_{i_f^{-1}\mathscr{O}_X} i_f^{-1}\mathscr{M} = \mathscr{Q}_{X_0}\otimes_{i_f^{-1}\mathscr{O}_X} i_f^{-1}\mathscr{M}.$$

Lemma 6.3. *We have*

$$\boldsymbol{R}\varpi_{f*}{}^{\mathrm{p}}\widetilde{\psi}_f{}^{\mathrm{mod}}\mathscr{M} \simeq i_f{}^{-1}\,{}^{\mathrm{p}}\mathrm{DR}_X(\mathscr{M}(*X_0))[-1]$$
$$\boldsymbol{R}\varpi_{f*}{}^{\mathrm{p}}\widetilde{\psi}_f{}^{\mathrm{rd}}\mathscr{M} \simeq {}^{\mathrm{p}}\mathrm{DR}(\mathscr{Q}_{X_0}\otimes i_f^{-1}\mathscr{M})[-2]$$

(here we regard $\mathscr{Q}_{X_0}\otimes i_f^{-1}\mathscr{M}$ as an $i_f^{-1}\mathscr{D}_X$-module, and ${}^{\mathrm{p}}\mathrm{DR}(\mathscr{Q}_{X_0}\otimes i_f^{-1}\mathscr{M}) :=$ $\mathrm{DR}(\mathscr{Q}_{X_0}\otimes i_f^{-1}\mathscr{M})[\dim X]$). $\square$

Recall that ${}^{\mathrm{p}}\mathrm{DR}^{>\mathrm{mod}}_{\widetilde{X}(f)}\mathscr{M}$ and ${}^{\mathrm{p}}\mathrm{DR}^{>\mathrm{rd}}_{\widetilde{X}(f)}\mathscr{M}$ are supported on $\partial\widetilde{X}(f)$. According to (4.5) we also have:

$$\text{(6.4)} \qquad \begin{aligned} {}^{\mathrm{p}}\widetilde{\psi}_f{}^{>\mathrm{mod}}\mathscr{M} &:= \widetilde{\imath}_f{}^{-1}\,{}^{\mathrm{p}}\mathrm{DR}^{>\mathrm{mod}}_{\widetilde{X}(f)}\mathscr{M}[-1] \simeq \widetilde{\imath}_f{}^!\,{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}(f)}\mathscr{M},\\ {}^{\mathrm{p}}\widetilde{\psi}_f{}^{>\mathrm{rd}}\mathscr{M} &:= \widetilde{\imath}_f{}^{-1}\,{}^{\mathrm{p}}\mathrm{DR}^{>\mathrm{rd}}_{\widetilde{X}(f)}\mathscr{M}[-1] \simeq \widetilde{\imath}_f{}^!\,{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}(f)}\mathscr{M}.\end{aligned}$$

Applying the functor $\widetilde{\imath}_f{}^{-1}[-1]$ to the distinguished triangles (4.2) or (4.5), we obtain two distinguished triangles

$$\text{(6.5)} \qquad \begin{aligned} &{}^{\mathrm{p}}\widetilde{\psi}_f{}^{\mathrm{mod}}\mathscr{M} \longrightarrow {}^{\mathrm{p}}\widetilde{\psi}_f{}^*\mathscr{M} \longrightarrow {}^{\mathrm{p}}\widetilde{\psi}_f{}^{>\mathrm{mod}}\mathscr{M} \xrightarrow{+1}\\ &{}^{\mathrm{p}}\widetilde{\psi}_f{}^{\mathrm{rd}}\mathscr{M} \longrightarrow {}^{\mathrm{p}}\widetilde{\psi}_f{}^*\mathscr{M} \longrightarrow {}^{\mathrm{p}}\widetilde{\psi}_f{}^{>\mathrm{rd}}\mathscr{M} \xrightarrow{+1}.\end{aligned}$$

Let $\pi : Y \to X$ be a morphism of complex manifold and set $g = f\circ\pi$. There is a natural morphism $\widetilde{\pi} = \widetilde{Y}(g) \to \widetilde{X}(f)$ extending $\pi : Y^* \to X^*$.

***Proposition 6.6* (Compatibility with projective push-forward)**

Assume that π is projective. Let $\mathscr{M}$ be an object of $\mathsf{D}^{\mathrm{b}}_{\pi\text{-good}}(\mathscr{D}_Y)$. We have a functorial isomorphism of distinguished triangles

$$\begin{tikzcd} {}^{\mathrm{p}}\widetilde{\psi}_f{}^{\mathrm{mod}}\pi_+\mathscr{M} \arrow[r] \arrow[d, "\wr"] & {}^{\mathrm{p}}\widetilde{\psi}_f{}^*\pi_+\mathscr{M} \arrow[r] \arrow[d, "\wr"] & {}^{\mathrm{p}}\widetilde{\psi}_f{}^{>\mathrm{mod}}\pi_+\mathscr{M} \arrow[r, "+1"] \arrow[d, "\wr"] & {} \\ \boldsymbol{R}\widetilde{\pi}_*{}^{\mathrm{p}}\widetilde{\psi}_g{}^{\mathrm{mod}}\mathscr{M} \arrow[r] & \boldsymbol{R}\widetilde{\pi}_*{}^{\mathrm{p}}\widetilde{\psi}_g{}^*\mathscr{M} \arrow[r] & \boldsymbol{R}\widetilde{\pi}_*{}^{\mathrm{p}}\widetilde{\psi}_g{}^{>\mathrm{mod}}\mathscr{M} \arrow[r, "+1"] & {} \end{tikzcd}$$

and a similar one with rapid decay.

Proof. This is a direct consequence of Corollary B.7. $\square$

Corollary 6.7. *Assume that $f : X \to \mathbb{C}$ is projective and let t be a coordinate on $\mathbb{C}$. Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Then the long exact sequence*

$$\cdots \longrightarrow \mathscr{H}^k \boldsymbol{R}f_* \widetilde{{}^{\mathrm{p}}\psi_f}^{\mathrm{mod}} \mathscr{M} \longrightarrow \mathscr{H}^k \boldsymbol{R}f_* \widetilde{{}^{\mathrm{p}}\psi_f}^{*} \mathscr{M} \longrightarrow \mathscr{H}^k \boldsymbol{R}f_* \widetilde{{}^{\mathrm{p}}\psi_f}^{>\mathrm{mod}} \mathscr{M} \\ \longrightarrow \mathscr{H}^{k+1} \boldsymbol{R}f_* \widetilde{{}^{\mathrm{p}}\psi_f}^{\mathrm{mod}} \mathscr{M} \longrightarrow \cdots$$

splits into short exact sequences, and the short exact sequence

$$0 \longrightarrow \mathscr{H}^k \boldsymbol{R}f_* \widetilde{{}^{\mathrm{p}}\psi_f}^{\mathrm{mod}} \mathscr{M} \longrightarrow \mathscr{H}^k \boldsymbol{R}f_* \widetilde{{}^{\mathrm{p}}\psi_f}^{*} \mathscr{M} \longrightarrow \mathscr{H}^k \boldsymbol{R}f_* \widetilde{{}^{\mathrm{p}}\psi_f}^{>\mathrm{mod}} \mathscr{M} \longrightarrow 0$$

is identified with the short exact sequence

$$0 \longrightarrow {}^{\mathrm{p}}\widetilde{\psi}_t^{\mathrm{mod}} \mathscr{H}^k f_+ \mathscr{M} \longrightarrow {}^{\mathrm{p}}\widetilde{\psi}_t^{*} \mathscr{H}^k f_+ \mathscr{M} \longrightarrow {}^{\mathrm{p}}\widetilde{\psi}_t^{>\mathrm{mod}} \mathscr{H}^k f_+ \mathscr{M} \longrightarrow 0.$$

A similar result holds for the rapid-decay complexes. Moreover, $\mathscr{H}^k \boldsymbol{R}f_* \widetilde{{}^{\mathrm{p}}\psi_f}^{*} \mathscr{M}$ is a local system on S^1 for each k. □

6.b. Proof of Proposition 1.1. Assume $\mathscr{M}$ is holonomic. We have, after [**Kas03**, (3.13)] and (6.2):

$${}^{\mathrm{p}}\mathrm{DR}(\mathscr{Q}_{X_0} \otimes i_f^{-1}\mathscr{M}) \simeq \boldsymbol{R}\mathscr{H}om_{i_f^{-1}\mathscr{D}_X}(\mathscr{M}^\vee, \mathscr{Q}_{X_0})[\dim X] \simeq i_f^{-1}\, {}^{\mathrm{p}}\mathrm{Irr}^*_{X_0} \mathscr{M}^\vee[1],$$

after [**Meb04**, Cor. 3.4-4]. Therefore,

$$\boldsymbol{R}\varpi_{f*} \widetilde{{}^{\mathrm{p}}\psi_f}^{\mathrm{rd}} \mathscr{M} = i_f^{-1}\, {}^{\mathrm{p}}\mathrm{DR}(\mathscr{Q}_{X_0} \otimes i_f^{-1}\mathscr{M})[-2] \simeq i_f^{-1}\, {}^{\mathrm{p}}\mathrm{Irr}^*_{X_0} \mathscr{M}^\vee[-1].$$

Similarly, $\boldsymbol{R}\varpi_{f*} \widetilde{{}^{\mathrm{p}}\psi_f}^{>\mathrm{mod}} \mathscr{M}$ is isomorphic to the cone of

$$i_f^{-1}\, {}^{\mathrm{p}}\mathrm{DR}\, \mathscr{M}(*X_0) \longrightarrow i_f^{-1} \boldsymbol{R}j_{f*} j_f^{-1}\, {}^{\mathrm{p}}\mathrm{DR}\, \mathscr{M},$$

hence is isomorphic to $i_f^{-1}\, {}^{\mathrm{p}}\mathrm{Irr}_{X_0} \mathscr{M}$ (see [**Meb04**, Def. 3.4-1]). □

6.c. Proof of Theorem 1.2. The $\mathbb{R}$-constructibility property follows from Theorem 4.7 and the case of $\widetilde{{}^{\mathrm{p}}\psi_f}^{*} \mathscr{M}$ has been treated in §5.c, hence we are left with proving the X_0-support condition for $\widetilde{{}^{\mathrm{p}}\psi_f}^{\mathrm{mod}} \mathscr{M}[1]$, $\widetilde{{}^{\mathrm{p}}\psi_f}^{\mathrm{rd}} \mathscr{M}[1]$, $\widetilde{{}^{\mathrm{p}}\psi_f}^{>\mathrm{mod}} \mathscr{M}$ and $\widetilde{{}^{\mathrm{p}}\psi_f}^{>\mathrm{rd}} \mathscr{M}$. We will argue for the moderate-growth case, the rapid-decay one being done similarly. We first notice that, obviously,

$$\mathscr{H}^j \widetilde{{}^{\mathrm{p}}\psi_f}^{\mathrm{mod}} \mathscr{M}[1] = \widetilde{i}_f^{-1} \mathscr{H}^{j+\dim X}\, {}^{\mathrm{p}}\mathrm{DR}_{\widetilde{X}(f)} \mathscr{M} = 0 \quad \text{for } j > 0,$$

and the equality $\mathscr{H}^j \widetilde{{}^{\mathrm{p}}\psi_f}^{>\mathrm{mod}} \mathscr{M} = 0$ for $j > 0$ follows then from $\mathscr{H}^j \widetilde{{}^{\mathrm{p}}\psi_f}^{*} \mathscr{M} = 0$ for $j > 0$. On the other hand, Remark 4.19 shows that $\mathscr{H}^0 \widetilde{{}^{\mathrm{p}}\psi_f}^{\mathrm{mod}} \mathscr{M}[1]$ is supported on the pull-back by ϖ_f of a discrete set of points. The same holds for $\mathscr{H}^0 \widetilde{{}^{\mathrm{p}}\psi_f}^{*} \mathscr{M}$, due to Theorem 1.7(1) and the perversity of ${}^{\mathrm{p}}\psi_f\, {}^{\mathrm{p}}\mathrm{DR}\, \mathscr{M}$. Therefore, the same property holds for $\mathscr{H}^0 \widetilde{{}^{\mathrm{p}}\psi_f}^{>\mathrm{mod}} \mathscr{M}$.

Remark 6.8. At this point, we note that if we could prove $\mathscr{H}^0 \widetilde{{}^{\mathrm{p}}\psi_f}^{\mathrm{mod}} \mathscr{M}[1] = 0$, then the proof by induction done below would lead to the X_0-support condition for $\widetilde{{}^{\mathrm{p}}\psi_f}^{\mathrm{mod}} \mathscr{M}$.

Using a standard "dévissage" as in the proof of Theorem 4.7, we can assume that $\mathscr{M}$ is a meromorphic flat bundle with pole divisor P containing X_0. The question is local on X_0, so we fix $x_o \in X_0$, we replace X with a sufficiently small neighbourhood of x_o, that we still denote by X, so that there exists, according to **[Ked11]**, a projective modification $e : Y \to X$ such that $D = e^{-1}(P)$ is a divisor with normal crossings having smooth components and $e^+\mathscr{M}$ has a good formal structure along D. Since the map e is proper, we can stratify X and Y by complex analytic strata such that X_0, P, D are union of strata and the map e is a stratified map, smooth on each stratum of Y to the corresponding stratum of X. We will show by induction on $\dim X$ that, given a meromorphic flat bundle $\mathscr{M}$ with such data, then for $j \geqslant 0$, $\mathscr{H}^{-j}\,{}^{\mathrm{p}}\widetilde{\psi}_f^{\,\mathrm{mod}}\mathscr{M}[1] = 0$ on each stratum S_k of dimension $k > j$.

Up to shrinking X, we can find a local coordinate system centered at x_o such that each coordinate defines a morphism $p : X \to S$ which is smooth on each stratum of dimension $\geqslant 1$. In such a way, Corollary 4.18 reads

$$i_{X_s}^{-1}\,{}^{\mathrm{p}}\widetilde{\psi}_f^{\,\mathrm{mod}}\mathscr{M}[1] \simeq {}^{\mathrm{p}}\widetilde{\psi}_{f_s}^{\,\mathrm{mod}}\mathscr{H}^0 i_{X_s}^{+}\mathscr{M}[2].$$

We also notice that, given $s \neq 0$, $\mathscr{H}^0 i_{X_s}^{+}\mathscr{M}$ is a meromorphic flat bundle, and the restriction to X_s of the data attached to $\mathscr{M}$ are data attached to $\mathscr{H}^0 i_{X_s}^{+}\mathscr{M}$. By the inductive assumption, we have $\mathscr{H}^{-j}\,{}^{\mathrm{p}}\widetilde{\psi}_{f_s}^{\,\mathrm{mod}}\mathscr{H}^0 i_{X_s}^{+}\mathscr{M}_{|S_k\cap X_s}[1] = 0$ if $k-1 > j \geqslant 0$, and thus

$$\mathscr{H}^{-j-1} i_{X_s}^{-1}\,{}^{\mathrm{p}}\widetilde{\psi}_f^{\,\mathrm{mod}}\mathscr{M}_{|S_k\cap X_s}[1] = 0 \quad \text{if } k > j+1 \text{ and } j \geqslant 0.$$

This holds for each $s \neq 0$. By changing the coordinate projection p, we finally obtain

$$\mathscr{H}^{-j} i_{X_s}^{-1}\,{}^{\mathrm{p}}\widetilde{\psi}_f^{\,\mathrm{mod}}\mathscr{M}_{|S_k}[1] = 0 \quad \text{if } k > j \text{ and } j \geqslant 1.$$

The case where $j = 0$ has been treated in the first part of the proof. Since the case where $\dim X = 1$ is obvious, the proof of Theorem 1.2 is complete. $\square$

6.d. An improvement of Theorem 1.2 in the good case. Let (Y, D) be a smooth complex manifold with a normal crossing divisor having smooth components, and let $\mathscr{M}$ be a good meromorphic flat bundle on Y with poles along D. Let $g : Y \to \mathbb{C}$ be a holomorphic function such that $D_g := g^{-1}(0)$ is contained in D. We will use the notation as in the proof of Theorem 4.7.

Proposition 6.9. *The complexes* ${}^{\mathrm{p}}\widetilde{\psi}_g^{\,\mathrm{mod}D_g}\mathscr{M}$ *and* ${}^{\mathrm{p}}\widetilde{\psi}_g^{\,\mathrm{rd}D_g}\mathscr{M}$ *satisfy the X_0-support condition.*

Proof. The statement is local, so we can use local coordinates $x_1, \dots, x_n$ adapted to D, i.e., such that $D = \{x_1 \cdots x_\ell = 0\}$ and $g(x_1, \dots, x_n) = x_1^{e_1} \cdots x_\ell^{e_\ell} = \boldsymbol{x}^{\boldsymbol{e}}$, with $e_j \geqslant 0$. We have

$$\widetilde{Y}(D) = (S^1)^\ell \times (\mathbb{R}_+)^\ell \times \mathbb{C}^{n-\ell}, \quad \partial\widetilde{Y}(D) = (S^1)^\ell \times \partial(\mathbb{R}_+)^\ell \times \mathbb{C}^{n-\ell},$$

with coordinates $(\theta_1, \dots, \theta_\ell; \rho_1, \dots, \rho_\ell; x_{\ell+1}, \dots, x_n)$, where $\partial(\mathbb{R}_+)^\ell$ is defined by $\rho_1 \cdots \rho_\ell = 0$. The map $\varpi_{D,g}$ is given by the formula

$$(\theta_1, \dots, \theta_\ell; \rho_1, \dots, \rho_\ell; x_{\ell+1}, \dots, x_n) \longmapsto \Big(\sum e_i\theta_i, \prod \rho_i^{e_i}, x_{\ell+1}, \dots, x_n\Big).$$

More precisely, if $e_j > 0$ for $j = 1, \dots, k$ and $e_j = 0$ for $j = k+1, \dots, \ell$, then we have

$$\begin{aligned} \widetilde{Y}(D) &\longrightarrow \widetilde{Y}(D_g) \\ (\theta_1, \dots, \theta_\ell; \rho_1, \dots, \rho_\ell; \boldsymbol{x}_{>\ell}) &\longmapsto (\theta_1, \dots, \theta_k; \rho_1, \dots, \rho_k; \rho_{>k} e^{i\theta_{>k}}, \boldsymbol{x}_{>\ell}) \end{aligned}$$

and

$$\begin{aligned} \widetilde{Y}(D_g) &\longrightarrow \widetilde{Y}(g) \\ (\theta_1, \dots, \theta_k; \rho_1, \dots, \rho_k; \rho_{>k} e^{i\theta_{>k}}, \boldsymbol{x}_{>\ell}) &\longmapsto \Big(\textstyle\sum_{i=1}^k e_i\theta_i, \prod_{i=1}^k \rho_i^{e_i}, \rho_{>k} e^{i\theta_{>k}}, \boldsymbol{x}_{>\ell}\Big). \end{aligned}$$

Let S be the stratum defined by $x_1 = \cdots = x_\ell = 0$ and $x_j \neq 0$ for $j > \ell$. Above this stratum, $\partial\widetilde{Y}(D)$ is defined by $\rho_1 = \cdots = \rho_\ell = 0$ and the map $\varpi_{D,g}$ is given by $(\theta_1, \dots, \theta_\ell, \boldsymbol{x}_{>\ell}) \mapsto (\sum_{i=1}^k e_i\theta_i, , \boldsymbol{x}_{>\ell})$, hence its fibre is a compact manifold of real dimension $\ell - 1$. As a consequence, for every sheaf $\mathscr{G}$ on $\partial\widetilde{Y}(D)_{|S}$, we have $R^j\varpi_{D,g*}\mathscr{G} = 0$ for $j > \ell - 1$. The X_0-support (i.e., D_g-support) condition follows then from (4.10) and the generalized Hukuhara-Turrittin theorem 4.11. $\square$

Remark 6.10. In such a case, the support condition for ${}^{\mathrm{p}}\mathrm{Irr}_{D_g}\mathscr{M}$ and ${}^{\mathrm{p}}\mathrm{Irr}^*_{D_g}\mathscr{M}$ (see [**Meb04**, Th. 3.5-2]) can be obtained as in dimension one, e.g. as in [**Sab13**, Cor. 3.16], by using [**Sab13**, Prop. 9.23].

Appendix A. Base change for a covering map

Let us consider a cartesian square of topological spaces:

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & \square & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

Lemma A.1. *There is a canonical morphism of functors $g^{-1} \circ f_* \to f'_* \circ g'^{-1}$.*

Proof. The adjunction morphism $\mathrm{Id} \to g'_* \circ g'^{-1}$ induces a morphism

$$f_* \longrightarrow f_* \circ g'_* \circ g'^{-1} = g_* \circ f'_* \circ g'^{-1}.$$

We deduce a morphism $g^{-1} \circ f_* \to g^{-1} \circ g_* \circ f'_* \circ g'^{-1}$, and by using now the adjunction $g^{-1} \circ g_* \to \mathrm{Id}$, we obtain the desired morphism. $\square$

For a sheaf $\mathscr{G}$ on Y, we consider the following property:

(P) Each point $y \in Y$ admits a fundamental system $\mathfrak{V}_y$ of open neighbourhoods such that, for each $V \in \mathfrak{V}_y$, the natural morphism $\Gamma(V, \mathscr{G}) \to \mathscr{G}_y$ is an isomorphism.

We say that a bounded complex $\mathscr{G}^\bullet$ satisfies Property (P) if all its cohomology sheaves do so.

Proposition A.2. *Assume moreover that f is a covering map. Let $\mathscr{G}^\bullet$ be a complex of sheaves on Y satisfying Property* (P)*, as well as $g^{-1}\mathscr{G}^\bullet$ on Y'. Then the natural morphism*

$$g^{-1} \circ \boldsymbol{R}f_*(f^{-1}\mathscr{G}^\bullet) \longrightarrow \boldsymbol{R}f'_* \circ g'^{-1}(f^{-1}\mathscr{G}^\bullet) \tag{A.2 $*$}$$

is an isomorphism.

Proof. Assume first that $\mathscr{G}^\bullet$ is a sheaf $\mathscr{G}$. Since the question is local on Y, we can assume that f is the projection $X = Y\times F \to Y$, where F is a discrete set. Then $f^{-1}\mathscr{G}$ also satisfies property (P) at each point $x \in X$ with $\mathfrak{V}_x = \mathfrak{V}_{f(x)}$. Set $\mathscr{F} = f^{-1}\mathscr{G}$ and denote by $\mu : \mathscr{F}^{\text{ét}} \to X$ the étalé space attached to $\mathscr{F}$. Then Property (P) implies that $f\circ\mu : \mathscr{F}^{\text{ét}} \to Y$ is the étalé space attached to $f_*\mathscr{F}$.

On the other hand, the étalé space of $g'^{-1}\mathscr{F} = f'^{-1}(g^{-1}\mathscr{G})$ is by definition $\mu' : (g'^{-1}\mathscr{F})^{\text{ét}} = X' \times_X \mathscr{F}^{\text{ét}} \to X'$, and by applying Property (P) to $g^{-1}\mathscr{G}$ we find that the étalé space of $f'_*g^{-1}\mathscr{G}$ is $f'\circ\mu' : X'\times_X \mathscr{F}^{\text{ét}} \to Y'$. Since we have a cartesian square, we identify the latter with $Y'\times_Y \mathscr{F}^{\text{ét}} \to Y'$, as wanted.

For an arbitrary bounded complex $\mathscr{G}^\bullet$ as in the statement, we note that, since f and f' are covering maps, we have

$$\begin{aligned}\mathscr{H}^j g^{-1}\circ \boldsymbol{R}f_*(f^{-1}\mathscr{G}^\bullet) &= g^{-1}\circ f_*(f^{-1}\mathscr{H}^j\mathscr{G}^\bullet),\\ \mathscr{H}^j \boldsymbol{R}f'_*\circ g'^{-1}(f^{-1}\mathscr{G}^\bullet) &= f'_*\circ g'^{-1}(f^{-1}\mathscr{H}^j\mathscr{G}^\bullet),\end{aligned}$$

so we can apply the first part of the proof. □

Example A.3. Assume that $g : Y' \to Y$ is a morphism of real analytic manifolds and that $\mathscr{G}^\bullet$ is weakly $\mathbb{R}$-constructible (see [**KS90**, Def. 8.4.3]). Then so is $g^{-1}\mathscr{G}^\bullet$, and both satisfy Property (P), according to [**KS90**, Prop. 8.1.4]. Therefore, if f is a covering map, the morphism (A.2 $*$) is an isomorphism.

Appendix B. Proof of the results in §3.a

For the sake of completeness, we indicate how to use the results of [**Moc14**, Chap. 4 & 5] to obtain those stated in §3.

Let $f : X \to \mathbb{C}$ be a holomorphic function. Set $D_X = f^{-1}(0)$. Let $\widetilde{\mathbb{C}}$ denote the real blow up of $\mathbb{C}$ at the origin. The product $X\times\mathbb{C}$ is denoted by $\mathfrak{X}$, the real blowing up map along $X\times\{0\}$ by $\varpi : \widetilde{\mathfrak{X}} \to \mathfrak{X}$ and the open subset $X\times\mathbb{C}^*$ by $\mathfrak{X}^*$. For a complex manifold Y with a normal crossing divisor D_Y with smooth components, we denote by $\widetilde{Y}(D_Y)$ the real blow up of each component of D_Y.

The closure of $\gamma_f(X^*)$ in $\widetilde{\mathfrak{X}}$ is the real blow up $\widetilde{X}(f)$ of X along f. We have a commutative diagram

(B.1)
$$\begin{tikzcd} & \widetilde{X}(f) \arrow[r, hook, "\widetilde{\gamma}_f"] \arrow[d, "\varpi_f"] & \widetilde{\mathfrak{X}} \arrow[d, "\varpi"] & \\ X^* \arrow[ur, hook, "\widetilde{\jmath}_f"] \arrow[r, hook, "j_f"] \arrow[rrr, hook, bend right, "{}_*\gamma_f := \gamma_{f|X^*}"'] & X \arrow[r, hook, "\gamma_f"] & \mathfrak{X} & \mathfrak{X}^* \arrow[l, hook', "j"'] \arrow[ul, hook', "\widetilde{\jmath}"'] \end{tikzcd}$$

Proposition B.2 **([Moc14, Th. 4.5.1]).** *For all values of $\star$, the following holds:*

(1) *The derived tensor product $\varpi^{-1}\mathscr{O}_{\gamma_f(X)} \otimes^L_{\varpi^{-1}\mathscr{O}_{\mathfrak{X}}} \mathscr{A}^\star_{\widetilde{\mathfrak{X}}}$ has cohomology in degree zero only, and is supported on $\widetilde{X}(f)$.*

(2) *Let $\rho : Y \to X$ be a birational morphism which induces an isomorphism $Y \smallsetminus \rho^{-1}(D_X) \xrightarrow{\sim} X \smallsetminus D_X$ and such that $D_Y := \rho^{-1}(D_X)$ has normal crossings with smooth components, and let $\widetilde{\rho} : \widetilde{Y}(D_Y) \to \widetilde{\mathfrak{X}}$ be the induced morphism. Then $R^k\widetilde{\rho}_*\mathscr{A}^\star_{\widetilde{Y}(D_Y)} = 0$ for $k > 0$ and the natural morphism*

$$\varpi^{-1}\mathscr{O}_{\gamma_f(X)} \otimes_{\varpi^{-1}\mathscr{O}_{\mathfrak{X}}} \mathscr{A}^\star_{\widetilde{\mathfrak{X}}} \longrightarrow \widetilde{\rho}_*\mathscr{A}^\star_{\widetilde{Y}(D_Y)}$$

induced by ρ^* is an isomorphism.

Proof. The values $\star = \mathrm{mod}, \mathrm{rd}$ are treated in loc. cit. (case $\ell = 1$ there), and the value $\star = *$ is obtained in a very similar way. Proving (1) for $\star = {>}\mathrm{mod}, {>}\mathrm{rd}, \mathrm{mod}/\mathrm{rd}$ amounts to proving injectivity of

$$\varpi^{-1}\mathscr{O}_{\gamma_f(X)} \otimes_{\varpi^{-1}\mathscr{O}_{\mathfrak{X}}} \mathscr{A}^{\mathrm{mod}}_{\widetilde{\mathfrak{X}}} \longrightarrow \varpi^{-1}\mathscr{O}_{\gamma_f(X)} \otimes_{\varpi^{-1}\mathscr{O}_{\mathfrak{X}}} \mathscr{A}^{*}_{\widetilde{\mathfrak{X}}}$$

and similarly for the pairs $(\mathrm{rd}, *)$ and $(\mathrm{rd}, \mathrm{mod})$. This follows from (2) for these pairs. On the other hand, (2) for $\star = {>}\mathrm{mod}, {>}\mathrm{rd}, \mathrm{mod}/\mathrm{rd}$ is obtained by a similar argument. □

Corollary B.3 **([Moc14, Th. 4.5.3]).** *For all values of $\star$, the natural morphism*

$$\varpi^{-1}\mathscr{O}_{\gamma_f(X)} \otimes_{\varpi^{-1}\mathscr{O}_{\mathfrak{X}}} \mathscr{A}^\star_{\widetilde{\mathfrak{X}}} \longrightarrow \widetilde{\gamma}_{f*}\mathscr{A}^\star_{\widetilde{X}(f)}$$

is an isomorphism.

Proof. For $\star = *, \mathrm{mod}, \mathrm{rd}$, we have a natural identification $\widetilde{\rho}_*\mathscr{A}^\star_{\widetilde{Y}(D_Y)} \simeq \widetilde{\gamma}_{f*}\mathscr{A}^\star_{\widetilde{X}(f)}$. For the remaining cases, e.g. for $\star = {>}\mathrm{mod}$, $\widetilde{\rho}_*\mathscr{A}^{>\mathrm{mod}}_{\widetilde{Y}(D_Y)}$ is identified with the cokernel of $\widetilde{\rho}_*\mathscr{A}^{\mathrm{mod}}_{\widetilde{Y}(D_Y)} \to \widetilde{\rho}_*\mathscr{A}^{*}_{\widetilde{Y}(D_Y)}$, according to B.2(2), and by definition $\widetilde{\gamma}_{f*}\mathscr{A}^{>\mathrm{mod}}_{\widetilde{X}(f)}$ is the cokernel of $\widetilde{\gamma}_{f*}\mathscr{A}^{\mathrm{mod}}_{\widetilde{X}(f)} \to \widetilde{\gamma}_{f*}\mathscr{A}^{*}_{\widetilde{X}(f)}$, hence the assertion. □

Let $\pi : Y \to X$ be a morphism of complex manifold and set $g = f \circ \pi$. It can be extended in a unique way as a morphism $\widetilde{\pi} : \widetilde{Y}(g) \to \widetilde{X}(f)$.

Corollary B.4 **([Moc14, Th. 4.4.3 & Th. 4.5.4]).** *Let $\pi : Y \to X$ be a projective morphism and let $\mathscr{N}$ be an inductive limit of coherent $\mathscr{O}_Y$-modules. Then, for any values of $\star$, the natural morphism*

$$\mathscr{A}^\star_{\widetilde{X}(f)} \otimes^L_{\varpi_f^{-1}\mathscr{O}_X} \varpi_f^{-1}\boldsymbol{R}\pi_*\mathscr{N} \longrightarrow \boldsymbol{R}\widetilde{\pi}_*\big(\mathscr{A}^\star_{\widetilde{Y}(g)} \otimes^L_{\varpi_g^{-1}\mathscr{O}_Y} \varpi_g^{-1}\mathscr{N}\big)$$

is an isomorphism. □

Theorem B.5 **(Flatness, [Moc14, Th. 4.6.1]).** *For all values of $\star$, the sheaves $\mathscr{A}^\star_{\widetilde{X}(f)}$ are $\varpi_f^{-1}\mathscr{O}_X$-flat. Furthermore, for any coherent $\mathscr{O}_X$-module $\mathscr{M}$, and for $\star = \mathrm{mod}, \mathrm{rd}$, the natural morphism*

$$\mathscr{A}^\star_{\widetilde{X}(f)} \otimes_{\varpi_f^{-1}\mathscr{O}_X} \varpi_f^{-1}\mathscr{M} \longrightarrow \widetilde{\jmath}_{f*}\mathscr{O}_{X^*} \otimes_{\varpi_f^{-1}\mathscr{O}_X} \varpi_f^{-1}\mathscr{M} = \widetilde{\jmath}_{f*}\mathscr{M}_{|X^*}$$

is injective.

Proof. The theorem is proved in loc. cit. for $\star = \mathrm{mod}, \mathrm{rd}$. Flatness in the case $\star = *$ is easy: it amounts to proving that $R\widetilde{\jmath}_{f*}\widetilde{\jmath}_f^{-1}\mathscr{M} = \widetilde{\jmath}_{f*}\widetilde{\jmath}_f^{-1}\mathscr{M}$ for any coherent $\mathscr{O}_X$-module $\mathscr{M}$, and this follows from the argument used to prove (b) in Section 3.a. Last, flatness in the remaining cases, expressed as the vanishing the cohomology of $\mathscr{A}^\star_{\widetilde{X}(f)} \otimes^L_{\varpi_f^{-1}\mathscr{O}_X} \varpi_f^{-1}\mathscr{M}$ in negative degrees for any coherent $\mathscr{O}_X$-module $\mathscr{M}$, follows from both statements in the cases $\star = *, \mathrm{mod}, \mathrm{rd}$ by the snake lemma. $\square$

For the following results, we consider the case of a right $\mathscr{D}_Y$-module $\mathscr{M}$ for simplicity and we use the Spencer complex Sp.

Corollary B.6 **([Moc14, Prop. 4.7.1]).** *Setting as in Corollary* B.4. *For any coherent $\mathscr{D}_Y$-module $\mathscr{M}$ having a good filtration (locally with respect to X), there is a natural isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}^\star_{\widetilde{X}(f)})$:*

$$\text{(B.6}\,*) \quad \mathscr{A}^\star_{\widetilde{X}(f)} \otimes^L_{\varpi_f^{-1}\mathscr{O}_X} \pi_+\mathscr{M} \longrightarrow \boldsymbol{R}\widetilde{\pi}_*\big((\mathscr{A}^\star_{\widetilde{Y}(g)} \otimes_{\varpi_g^{-1}\mathscr{O}_Y} \varpi_g^{-1}\mathscr{M}) \otimes_{\varpi_g^{-1}\mathscr{D}_Y} \varpi_g^{-1}\,\mathrm{Sp}_{Y\to X}\big).$$

Proof. We can replace $\mathscr{M}$ by its canonical resolution by induced $\mathscr{D}_Y$-modules $\mathscr{M} \otimes_{\mathscr{O}_Y} \mathrm{Sp}_Y$, so that we can assume that $\mathscr{M} = \mathscr{N} \otimes_{\mathscr{O}_Y} \mathscr{D}_Y$, where $\mathscr{N}$ is an inductive limit of coherent $\mathscr{D}_Y$-modules, due to the assumption of existence of a good filtration. Then $\pi_+\mathscr{M} = \boldsymbol{R}\pi_*\mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ and the natural morphism

$$\mathscr{A}^\star_{\widetilde{X}(f)} \otimes^L_{\varpi_f^{-1}\mathscr{O}_X} \varpi_f^{-1}\pi_+\mathscr{M} \longrightarrow \boldsymbol{R}\widetilde{\pi}_*\big(\mathscr{A}^\star_{\widetilde{Y}(g)} \otimes^L_{\varpi_g^{-1}\mathscr{O}_Y} \varpi_g^{-1}\mathscr{M}\big)$$

is an isomorphism by Corollary B.4. Due to the $\varpi_g^{-1}\mathscr{O}_Y$-flatness of $\mathscr{A}^\star_{\widetilde{Y}(g)}$, the latter term is equal to the right-hand side of (B.6 $*$) with $\mathscr{M} = \mathscr{N} \otimes_{\mathscr{O}_Y} \mathscr{D}_Y$. $\square$

Corollary B.7 **([Moc14, Cor. 4.7.3]).** *Setting as in Corollary* B.4. *For any coherent $\mathscr{D}_Y$-module $\mathscr{M}$ there is a functorial isomorphism*

$$^{\mathrm{p}}\mathrm{DR}^\star_{\widetilde{X}(f)}\,\pi_+\mathscr{M} \simeq \boldsymbol{R}\widetilde{\pi}_*\,{}^{\mathrm{p}}\mathrm{DR}^\star_{\widetilde{Y}(g)}\,\mathscr{M}.$$

Proof. We tensor (B.6 $*$) on the right by $\varpi_f^{-1}\,\mathrm{Sp}_X$ and we apply the projection formula to the right-hand side:

$$\begin{aligned}
\boldsymbol{R}\widetilde{\pi}_*\Big[\mathscr{A}^\star_{\widetilde{Y}(g)} \otimes_{\varpi_g^{-1}\mathscr{O}_Y} \big(\mathscr{M} \otimes_{\mathscr{D}_Y} \mathrm{Sp}_{Y\to X}\big)\Big] &\otimes_{\varpi_f^{-1}\mathscr{D}_X} \varpi_f^{-1}\,\mathrm{Sp}_X \\
&\xrightarrow{\sim} \boldsymbol{R}\widetilde{\pi}_*\Big[\mathscr{A}^\star_{\widetilde{Y}(g)} \otimes_{\varpi_g^{-1}\mathscr{O}_Y} \big(\mathscr{M} \otimes_{\mathscr{D}_Y} \mathrm{Sp}_{Y\to X} \otimes_{\pi^{-1}\mathscr{D}_X} \pi^{-1}\,\mathrm{Sp}_X\big)\Big] \\
&\simeq \boldsymbol{R}\widetilde{\pi}_*\Big[\mathscr{A}^\star_{\widetilde{Y}(g)} \otimes_{\varpi_g^{-1}\mathscr{O}_Y} \big(\mathscr{M} \otimes_{\mathscr{D}_Y} \mathrm{Sp}_Y\big)\Big] = \boldsymbol{R}\widetilde{\pi}_*\,{}^{\mathrm{p}}\mathrm{DR}^\star_{\widetilde{Y}(g)}\,\mathscr{M}. \quad \square
\end{aligned}$$

Appendix C. Results about duality

We prove a special case of Conjecture 4.13. Let $f, g : X \to \mathbb{C}$ be holomorphic functions and let V be a meromorphic flat bundle on X with poles along $f^{-1}(0) \cup g^{-1}(0)$. By definition, we have $V = V(*f, *g)$. We denote by $V^\vee$ the dual meromorphic flat bundle.

Proposition C.1. *Locally on X, there exist isomorphisms*

$$\begin{aligned}\boldsymbol{D}\,{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}(f)}\,V &\simeq {}^{\mathrm{p}}\mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}(f)}\,V^\vee(!g),\\ \boldsymbol{D}\,{}^{\mathrm{p}}\mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}(f)}\,V &\simeq {}^{\mathrm{p}}\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}(f)}\,V^\vee(!g)\end{aligned}$$

compatible with the natural morphisms from rapid decay to moderate de Rham complexes and inducing the natural isomorphisms existing on X^*.

C.a. The case of normal crossing divisors and good meromorphic flat bundle. Let (Y,D) be a complex manifold with a normal crossing divisor D with a partition of its components in two disjoint sets, giving rise to the decomposition $D=D_1\cup D_2$. Let V be a meromorphic flat bundle with poles along D and let $V^\vee$ be the dual meromorphic flat bundle. The dual localization $V(!D)$ is defined as $\boldsymbol{D}\big(V^\vee\big)$. We take up the notation of [**Moc14**, §4.1.4] where $<D$ means rapid decay along D and $\leqslant D$ means of Nilsson's class along D. The following de Rham complexes on $\widetilde{Y}(D)$

$$\mathrm{DR}^{\leqslant D}_{\widetilde{Y}(D)}(V),\quad \mathrm{DR}^{<D_1,\leqslant D_2}_{\widetilde{Y}(D)}(V),\quad \mathrm{DR}^{<D_2,\leqslant D_1}_{\widetilde{Y}(D)}(V),\quad \mathrm{DR}^{<D}_{\widetilde{Y}(D)}(V)$$

enter a natural commutative diagram

$$\begin{array}{ccc}\mathrm{DR}^{<D}_{\widetilde{Y}(D)}(M) & \longrightarrow & \mathrm{DR}^{<D_2,\leqslant D_1}_{\widetilde{Y}(D)}(M)\\ \Big\downarrow & & \Big\downarrow\\ \mathrm{DR}^{<D_1,\leqslant D_2}_{\widetilde{Y}(D)}(M) & \longrightarrow & \mathrm{DR}^{\leqslant D}_{\widetilde{Y}(D)}(M)\end{array}\tag{C.2}$$

The dual diagram $\boldsymbol{D}$(C.2) is obtained by taking the Verdier dual at each vertex and by considering the dual arrow, which then point toward the opposite direction. In order to obtain a diagram similar to (C.2), it is thus necessary to consider first a diagram (C.2)$^\perp$ obtained from (C.2) by symmetry with respect to its center. Then $\boldsymbol{D}$(C.2)$^\perp$ is

$$\begin{array}{ccc}\boldsymbol{D}\,\mathrm{DR}^{\leqslant D}_{\widetilde{Y}(D)}(V) & \longrightarrow & \boldsymbol{D}\,\mathrm{DR}^{<D_1,\leqslant D_2}_{\widetilde{Y}(D)}(V)\\ \Big\downarrow & & \Big\downarrow\\ \boldsymbol{D}\,\mathrm{DR}^{<D_2,\leqslant D_1}_{\widetilde{Y}(D)}(V) & \longrightarrow & \boldsymbol{D}\,\mathrm{DR}^{<D}_{\widetilde{Y}(D)}(V)\end{array}\tag{$\boldsymbol{D}$(C.2)$^\perp$}$$

There is a natural morphism of squares

$$\text{(C.2)}(V^\vee)\longrightarrow(\boldsymbol{D}\text{(C.2)}^\perp)(V),\tag{C.3}$$

meaning that there is a natural morphism between the corresponding vertices (in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{\widetilde{Y}(D)})$) and these morphisms are compatible with the arrows in the squares.

Theorem C.4 **([Moc14**, Th. 5.2.2]). *If V is a good meromorphic flat bundle along (Y,D), the morphism* (C.3) *is an isomorphism.*

C.b. De Rham complexes on $\widetilde{Y}(D_1)$. We now consider the real blow-up $\varpi_1 : \widetilde{Y}(D_1) \to Y$ and the commutative diagram of morphisms

$$\begin{tikzcd} \widetilde{Y}(D) \arrow[rr, "\rho"] \arrow[dr, "\varpi"'] & & \widetilde{Y}(D_1) \arrow[dl, "\varpi_1"] \\ & Y & \end{tikzcd}$$

We also consider the following de Rham complexes on $\widetilde{Y}(D_1)$:

$$\mathrm{DR}^{\leqslant D_1}_{\widetilde{Y}(D_1)}(V), \quad \mathrm{DR}^{< D_1}_{\widetilde{Y}(D_1)}(V), \quad \mathrm{DR}^{\leqslant D_1}_{\widetilde{Y}(D_1)}(V(!D_2)), \quad \mathrm{DR}^{< D_1}_{\widetilde{Y}(D_1)}(V(!D_2)).$$

There is a natural commutative diagram

$$\begin{tikzcd} \mathrm{DR}^{< D_1}_{\widetilde{Y}(D_1)}(V(!D_2)) \arrow[r] \arrow[d] & \mathrm{DR}^{\leqslant D_1}_{\widetilde{Y}(D_1)}(V(!D_2)) \arrow[d] \\ \mathrm{DR}^{< D_1}_{\widetilde{Y}(D_1)}(V) \arrow[r] & \mathrm{DR}^{\leqslant D_1}_{\widetilde{Y}(D_1)}(V) \end{tikzcd} \tag{C.5}$$

Proposition C.6. *If V is a good meromorphic flat bundle along (Y, D), there is a functorial isomorphism (in the derived category)* (C.5) $\to \boldsymbol{R}\rho_*$(C.2) *(i.e., a morphism between the corresponding vertices which is compatible with the arrows).*

Proof. We denote by $\square$ one of the symbols $< D, \dots, \leqslant D$ entering the diagram (C.2).

Lemma C.7. *We have the following identifications:*

$$\begin{aligned} \boldsymbol{R}\rho_*\mathscr{A}^{\leqslant D}_{\widetilde{Y}(D)} &\simeq \mathscr{A}^{\leqslant D_1}_{\widetilde{Y}(D_1)}(*D_2), \\ \boldsymbol{R}\rho_*\mathscr{A}^{< D_1, \leqslant D_2}_{\widetilde{Y}(D)} &\simeq \mathscr{A}^{< D_1}_{\widetilde{Y}(D_1)}(*D_2), \\ \boldsymbol{R}\rho_*\mathscr{A}^{< D_2, \leqslant D_1}_{\widetilde{Y}(D)} &\simeq \big\{\mathscr{A}^{\leqslant D_1}_{\widetilde{Y}(D_1)} \longrightarrow \mathscr{A}^{\leqslant D_1}_{\widetilde{Y}(D_1)|\widehat{D}_2}\big\}, \\ \boldsymbol{R}\rho_*\mathscr{A}^{< D}_{\widetilde{Y}(D)} &\simeq \big\{\mathscr{A}^{< D_1}_{\widetilde{Y}(D_1)} \longrightarrow \mathscr{A}^{< D_1}_{\widetilde{Y}(D_1)|\widehat{D}_2}\big\}, \end{aligned}$$

where, denoting by f_2 an equation of D_2, we have set

$$\mathscr{A}^{\leqslant D_1}_{\widetilde{Y}(D_1)|\widehat{D}_2} := \varprojlim_k \mathscr{A}^{\leqslant D_1}_{\widetilde{Y}(D_1)} \big/ f_2^k \mathscr{A}^{\leqslant D_1}_{\widetilde{Y}(D_1)}, \quad \mathscr{A}^{< D_1}_{\widetilde{Y}(D_1)|\widehat{D}_2} := \varprojlim_k \mathscr{A}^{< D_1}_{\widetilde{Y}(D_1)} \big/ f_2^k \mathscr{A}^{< D_1}_{\widetilde{Y}(D_1)}.$$

Proof. The first two identifications follow from [**Moc14**, Th. 4.3.2 & Th. 4.3.1] and the last two from the same references together with [**Sab00**, Lem. II.1.1.18]. $\square$

We know from [**Moc14**, Th. 4.3.1] that $\mathscr{A}^{\square}_{\widetilde{Y}(D)}$ is $\varpi^{-1}\mathscr{O}_Y$-flat. For any holonomic $\mathscr{D}_Y$-module M, we have

$$\begin{aligned}\big(\boldsymbol{R}\rho_*\mathscr{A}^{\square}_{\widetilde{Y}(D)} \otimes^L_{\varpi_1^{-1}\mathscr{O}_Y} \varpi_1^{-1}M\big) \otimes_{\varpi_1^{-1}\mathscr{D}_Y} \varpi_1^{-1}\operatorname{Sp}_Y &\\ \xrightarrow{\sim} \boldsymbol{R}\rho_*\big(\mathscr{A}^{\square}_{\widetilde{Y}(D)} \otimes^L_{\varpi^{-1}\mathscr{O}_Y} \varpi^{-1}M\big) \otimes_{\varpi_1^{-1}\mathscr{D}_Y} \varpi_1^{-1}\operatorname{Sp}_Y &\\ = \boldsymbol{R}\rho_*\big(\mathscr{A}^{\square}_{\widetilde{Y}(D)} \otimes_{\varpi^{-1}\mathscr{O}_Y} \varpi^{-1}M\big) \otimes_{\varpi_1^{-1}\mathscr{D}_Y} \varpi_1^{-1}\operatorname{Sp}_Y &\\ \xrightarrow{\sim} \boldsymbol{R}\rho_*\Big[\big(\mathscr{A}^{\square}_{\widetilde{Y}(D)} \otimes_{\varpi^{-1}\mathscr{O}_Y} \varpi^{-1}M\big) \otimes_{\varpi^{-1}\mathscr{D}_Y} \varpi^{-1}\operatorname{Sp}_Y\Big]&.\end{aligned}$$

Let us denote by $\mathscr{A}^{\square}_{\widetilde{Y}(D_1)}$ the object defined by the right-hand sides in Lemma C.7. From the first two lines of Lemma C.7 together with flatness properties (that also hold for $\mathscr{A}^{\leqslant D_1}_{\widetilde{Y}(D_1)|\widehat{D}_2}$ and $\mathscr{A}^{<D_1}_{\widetilde{Y}(D_1)|\widehat{D}_2}$ according to loc. cit.), the first line above reads

$$\big(\mathscr{A}^{\square}_{\widetilde{Y}(D_1)} \otimes_{\varpi_1^{-1}\mathscr{O}_Y} \varpi_1^{-1}M\big) \otimes_{\varpi_1^{-1}\mathscr{D}_Y} \varpi_1^{-1}\operatorname{Sp}_Y .$$

Let us consider the square

$$\text{(C.5}'\text{)}\qquad \begin{array}{ccc} \operatorname{DR}^{\leq D}_{\widetilde{Y}(D_1)}(M) & \longrightarrow & \operatorname{DR}^{<D_2,\leqslant D_1}_{\widetilde{Y}(D_1)}(M) \\ \big\downarrow & & \big\downarrow \\ \operatorname{DR}^{<D_1}_{\widetilde{Y}(D_1)}(M(*D_2)) & \longrightarrow & \operatorname{DR}^{\leqslant D_1}_{\widetilde{Y}(D_1)}(M(*D_2)) \end{array}$$

Then for any holonomic $\mathscr{D}_Y$-module M, we have a functorial isomorphism (C.5′) $\xrightarrow{\sim}$ $\boldsymbol{R}\rho_*$(C.2).

Let us now assume that $M = M(!D_2)$. Then there is a natural morphism (C.5′) $\to$ (C.5) which is the identity on the lower lines. It is induced by the morphism of complexes (for $\star = \,<D_1, \leqslant D_1$)

$$\big\{\mathscr{A}^{\star}_{\widetilde{Y}(D_1)} \otimes \varpi_1^{-1}M \to \mathscr{A}^{\star}_{\widetilde{Y}(D_1)|\widehat{D}_2} \otimes \varpi_1^{-1}M\big\} \longrightarrow \mathscr{A}^{\star}_{\widetilde{Y}(D_1)} \otimes \varpi_1^{-1}M.$$

The proposition follows then from Lemma C.8 below. □

Lemma C.8 **([Moc14]).** *Assume that V is a good meromorphic flat bundle along (Y, D). Then $\operatorname{DR}^{\star}_{\widetilde{Y}(D_1)|\widehat{D}_2}(V(!D_2)) = 0$.*

Proof. This statement is proved in [**Moc14**, Prop. 3.2.2] when one replaces $\widetilde{Y}(D_1)$ with Y. We adapt a first proof of the latter statement which appeared in a preliminary version of [**Moc14**]. A similar proof is given in Lemmas 5.1.6 and 5.1.8 of loc. cit.

Let us set $D_2 = \bigcup_{i\in J} D_i$ ($J \neq \varnothing$) and, for any nonempty subset $I \subset J$, let us set $D_I = \bigcap_{i\in I} D_i$. By a Mayer-Vietoris argument (see [**Sab00**, Lem. II.1.1.13]), it is enough to prove the vanishing $\operatorname{DR}^{\star}_{\widetilde{Y}(D_1)|\widehat{D}_I}(V(!D_2)) = 0$. This is a local question, so that we can use a local coordinate system $(x_1, \dots, x_n)$ adapted to D. It is then enough to prove that, for some $i \in I$, the differential

$$\partial_{x_i} : \mathscr{A}^{\star}_{\widetilde{Y}(D_1)|\widehat{D}_I} \otimes_{\varpi_1^{-1}\mathscr{O}_Y} \varpi_1^{-1}(V(!D_2)) \longrightarrow \mathscr{A}^{\star}_{\widetilde{Y}(D_1)|\widehat{D}_I} \otimes_{\varpi_1^{-1}\mathscr{O}_Y} \varpi_1^{-1}(V(!D_2))$$

is bijective. In the following, we fix such an i. We have an identification

$$\mathscr{A}^{\star}_{\widetilde{Y}(D_1)|\widehat{D}_I}\otimes_{\varpi_1^{-1}\mathscr{O}_Y}\varpi_1^{-1}(V(!D_2))=\mathscr{A}^{\star}_{\widetilde{Y}(D_1)|\widehat{D}_I}\otimes_{\varpi_1^{-1}\mathscr{O}_{Y|\widehat{D}_I}}\varpi_1^{-1}(\mathscr{O}_{Y|\widehat{D}_I}\otimes_{\varpi_1^{-1}\mathscr{O}_Y}V(!D_2)),$$

and by goodness, $\mathscr{O}_{Y|\widehat{D}_I}\otimes_{\varpi_1^{-1}\mathscr{O}_Y}V$ locally decomposes, possibly after a finite ramification around D, into a direct sum of terms obtained from regular holonomic $\mathscr{D}_{Y|\widehat{D}_I}(*D_2)$-modules R by twisting their connection by $\mathrm{d}\varphi$ for some good local section φ of $\mathscr{O}_Y(*D)/\mathscr{O}_Y$ (i.e., φ is a the product of a monomial in $x_1,\dots,x_n$ with negative exponents by a unit in $\mathscr{O}_Y$). Since R is a successive extension of rank-one objects, it is enough to assume that R has rank one by an easy induction. We can thus assume that $V=(\mathscr{O}_Y(*D),\mathrm{d}+\mathrm{d}\varphi+\omega$, where ω is a logarithmic 1-form with constant coefficients. Then the computation is standard (such computations are done in [**Moc14**, Proof of Lem. 5.1.6]). □

Corollary C.9. *There exists an isomorphisms of squares from*

$$\begin{tikzcd}
\mathrm{DR}^{<D_1}_{\widetilde{Y}(D_1)}(V^\vee(!D_2)) \arrow[r]\arrow[d] & \mathrm{DR}^{\leqslant D_1}_{\widetilde{Y}(D_1)}(V^\vee(!D_2))\arrow[d]\\
\mathrm{DR}^{<D_1}_{\widetilde{Y}(D_1)}(V^\vee) \arrow[r] & \mathrm{DR}^{\leqslant D_1}_{\widetilde{Y}(D_1)}(V^\vee)
\end{tikzcd}$$

to

$$\begin{tikzcd}
\boldsymbol{D}\,\mathrm{DR}^{\leqslant D_1}_{\widetilde{Y}(D_1)}(V) \arrow[r]\arrow[d] & \boldsymbol{D}\,\mathrm{DR}^{<D_1}_{\widetilde{Y}(D_1)}(V)\arrow[d]\\
\boldsymbol{D}\,\mathrm{DR}^{\leqslant D_1}_{\widetilde{Y}(D_1)}(V(!D_2)) \arrow[r] & \boldsymbol{D}\,\mathrm{DR}^{<D_1}_{\widetilde{Y}(D_1)}(V(!D_2))
\end{tikzcd}$$

which extend the natural isomorphisms existing on $Y^*:=Y\smallsetminus D_1$.

Proof. We first apply $\boldsymbol{R}\rho_*$ to the isomorphism (C.3). By applying the isomorphism of Proposition C.6 together with the commutation isomorphism $\boldsymbol{R}\rho_*\boldsymbol{D}\simeq\boldsymbol{D}\boldsymbol{R}\rho_*$, we obtain the desired isomorphism. Let us check compatibility with the natural isomorphisms when restricted to Y^*. We note that $\rho:\widetilde{Y}^*(D_2)\to Y^*$ is the real blow-up of the components of D_2. We apply [**Moc14**, Prop. 5.2.1] to this real blow-up (denoted by π in loc. cit.) and with $D=D_2$. □

C.c. End of the proof of Proposition C.1. The question is local on X. Let $\pi:Y\to X$ be a projective modification with Y smooth such that $fg\circ\pi$ defines a divisor with normal crossings D. We denote by D_1 the divisor defined by $f\circ\pi$ and by D_2 the union of the remaining components of D. There exists a unique lift $\widetilde{\pi}:\widetilde{Y}(D_1)\to\widetilde{X}(f)$ of π. Let V be a meromorphic flat bundle on X with poles on $(fg)^{-1}(0)$ and let V_Y denote its pullback on Y, which has poles along D. We have $\pi_+V_Y=V$ (this is seen for example by computing the pushforward by means of induced $\mathscr{D}$-modules). Since duality commutes with pushforward, we deduce that $\pi_+(V_Y(!g\circ\pi))\simeq V(!g)$.

On the other hand, one checks that $V_Y(!g\circ\pi)(*D_1)=V_Y(!D_2)$, so that, when considering the de Rham complexes of Corollary C.9 one can replace $V_Y(!g\circ\pi)$ with $V_Y(!D_2)$. Finally, applying $\boldsymbol{R}\varpi_*$ to the isomorphism of Corollary C.9 together with the isomorphism of Corollary B.7 (stated for $\widetilde{Y}(f\circ\pi)$, but which also applies to $\widetilde{Y}(D_1)$), we obtain the isomorphisms of Proposition C.1.

Last, the compatibility statement is a consequence of that obtained in Corollary C.9. □

Appendix D. A complement on $\psi_f\mathscr{M}$

A shift is missing in [**Sab13**, Cor. 14.5]. We correct the error here. Let $f:X\to\mathbb{C}$ be a holomorphic function on a smooth manifold X and let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. The error comes from the identification $i_{D+}i_D^+\mathscr{M}$ with $R\Gamma_{[D]}\mathscr{M}$. For the hypersurface $D=f^{-1}(0)$, we should make the identification of $i_{D+}i_D^+\mathscr{M}$ with $R\Gamma_{[D]}\mathscr{M}[1]$. We denote $R\Gamma_{[!D]}\mathscr{M}=\boldsymbol{D}R\Gamma_{[D]}\boldsymbol{D}\mathscr{M}$.

Proposition D.1. *There is a natural isomorphism*

$$ {}^{\mathrm{p}}\mathrm{DR}\,\psi_{f,\lambda}\mathscr{M}\simeq\varinjlim_k {}^{\mathrm{p}}\mathrm{DR}(\mathscr{O}_{\widehat{D}}\otimes_{\mathscr{O}_X}\mathscr{M}_{\lambda,k})[-1]. $$

Proof. Let $\gamma_f:X\hookrightarrow X\times\mathbb{C}$ denote the graph inclusion and let $i_0:X\times\{0\}\hookrightarrow X\times\mathbb{C}$ denote the natural inclusion. We first note that $\gamma_{f+}(\mathscr{M}_{\lambda,k})\simeq(\gamma_{f+}\mathscr{M})_{\lambda,k}$ (notation of loc. cit.). By definition,

$$ \psi_{f,\lambda}\mathscr{M}=\varinjlim_k\mathscr{H}^0(i_0^\dagger\gamma_{f+}\mathscr{M}_{\lambda,k}), \tag{D.2} $$

and the limit is achieved on any compact set of $f^{-1}(0)$ for k large enough. Furthermore, the limit of $\mathscr{H}^1$ is zero (see e.g. [**MM04**]).

By definition we have $i_0^\dagger=\boldsymbol{D}i_0^+\boldsymbol{D}$, and so

$$ i_{0+}i_0^\dagger\simeq\boldsymbol{D}i_{0+}i_0^+\boldsymbol{D}=\boldsymbol{D}R\Gamma_{[(X\times\{0\})]}[1]\boldsymbol{D}=R\Gamma_{[!(X\times\{0\})]}[-1]. $$

Furthermore, since $\gamma_{f+}R\Gamma_{[D]}\simeq R\Gamma_{[(X\times\{0\})]}\gamma_{f+}$, and since γ_{f+} commutes with duality, we deduce that

$$ \gamma_{f+}R\Gamma_{[!D]}\simeq R\Gamma_{[!(X\times\{0\})]}\gamma_{f+} $$

On the other hand, since $\psi_{f,\lambda}\mathscr{M}$ is supported on $D=f^{-1}(0)$, we have $i_{0+}\psi_{f,\lambda}\mathscr{M}\simeq\gamma_{f+}\psi_{f,\lambda}\mathscr{M}$ according to the following lemma.

Lemma D.3. *Let $\mathscr{N}$ be a $\mathscr{D}_X$-module supported on $f^{-1}(0)$. Then there exists a natural isomorphism*

$$ \gamma_{f+}\mathscr{N}\simeq i_{0+}\mathscr{N}. $$

Proof. We write $\gamma_{f+}\mathscr{N}=\sum_j\mathscr{N}\partial_t^j\delta(t-f)$, which is supported on $X\times\{0\}$, hence of the form $i_{0+}\mathscr{N}'$. We have $\mathscr{N}'=\ker[t:\gamma_{f+}\mathscr{N}\to\gamma_{f+}\mathscr{N}]$. An element $\sum_j n_j\partial_t^j\delta(t-f)$ belongs to $\ker t$ if and only if the coefficients $n_j\in\mathscr{N}$ satisfy

$$ n_{j+1}=\frac{1}{j+1}fn_j\quad(j\geqslant 0). $$

the isomorphism $\mathscr{N} \xrightarrow{\sim} \mathscr{N}'$ is defined by $n \mapsto \sum_{j\geqslant 0} f^j n/j!$ (the sum is finite since $\mathscr{N}$ is supported on $f^{-1}(0)$). $\square$

Form the lemma and (D.2) we conclude

$$\begin{aligned}\gamma_{f+}\psi_{f,\lambda}\mathscr{M} &\simeq \varinjlim_k \mathscr{H}_0(i_{0+}i_0^\dagger \gamma_{f+}\mathscr{M}_{\lambda,k})\\ &\simeq \varinjlim_k \mathscr{H}_0(R\Gamma_{[!(X\times\{0\})]}\gamma_{f+}\mathscr{M}_{\lambda,k}[-1])\\ &\simeq \varinjlim_k \mathscr{H}_0(\gamma_{f+}R\Gamma_{[!D]}\mathscr{M}_{\lambda,k}[-1])\\ &\simeq \gamma_{f+}\varinjlim_k \mathscr{H}_0(R\Gamma_{[!D]}\mathscr{M}_{\lambda,k}[-1]).\end{aligned}$$

We conclude that $\psi_{f,\lambda}\mathscr{M} \simeq \varinjlim_k \mathscr{H}_0(R\Gamma_{[!D]}\mathscr{M}_{\lambda,k}[-1])$ and we finally obtain

$$^{\mathrm{p}}\mathrm{DR}(\psi_{f,\lambda}\mathscr{M}) \simeq \varinjlim_k {}^{\mathrm{p}}\mathrm{DR}\,\mathscr{H}_0(R\Gamma_{[!D]}\mathscr{M}_{\lambda,k})[-1].$$

We have a natural morphism

$$^{\mathrm{p}}\mathrm{DR}\,\mathscr{H}_0(R\Gamma_{[!D]}\mathscr{M}_{\lambda,k}) \longrightarrow {}^{\mathrm{p}}\mathrm{DR}(R\Gamma_{[!D]}\mathscr{M}_{\lambda,k}),$$

which becomes an isomorphism after passing to the limit (on a compact set of D), and the latter space is identified with $^{\mathrm{p}}\mathrm{DR}(\mathscr{O}_{\widehat{D}} \otimes_{\mathscr{O}_X} \mathscr{M}_{\lambda,k})$. This concludes the proof of the proposition. $\square$

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