A REMARK ON THE IRREGULARITY COMPLEX

by

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Abstract. We prove that, for a good meromorphic flat bundle with poles along a divisor with normal crossings, the restriction of the irregularity complex to each natural stratum of this divisor only depends on the formal flat bundle along this stratum. This answers a question raised by J.-B. Teyssier.

Contents

1. Statement of the results

Let X be a complex manifold of dimension n and let $D = \bigcup_{i \in J} D_i$ be a divisor with normal crossings. We assume that each irreducible component D_i of D is smooth. For any subset $I \subset J$ we set $D_I = \bigcap_{i \in I} D_i$ and $D_I^\circ = D_I \smallsetminus \bigcup_{j \notin I} D_j$. We denote the codimension of D_I° by ℓ, that we regard as a locally constant function on D_I° (which can have many connected components), and by $\iota_I : D_I^\circ \hookrightarrow D$ the inclusion. Let $\mathscr{M}$ be an holonomic $\mathscr{D}_X$-module such that

(1) $\mathscr{M} = \mathscr{M}(*D)$,

(2) $\mathscr{M}_{X \smallsetminus D}$ is locally $\mathscr{O}_X$-free of finite rank.

We then say that $\mathscr{M}$ is a meromorphic flat bundle with poles along D. In this note, we assume that $\mathscr{M}$ has a *good formal structure along D* (we simply say that $\mathscr{M}$ is a good D-meromorphic flat bundle, or a good meromorphic flat bundle on (X, D)). This means the following. For any $I \subset J$, and near every $x_o \in D_I^\circ$, the formalized module $\mathscr{M}_{\widehat{D_I^\circ}} := \mathscr{D}_{\widehat{X|D_I^\circ}} \otimes_{\mathscr{D}_X} \mathscr{M}$, when restricted to a suitable open neighbourhood $\mathrm{nb}(x_o)$ of x_o in X, can be decomposed, after a local ramification $\rho_{\boldsymbol{d}_I} : \mathrm{nb}(x_o)_{\boldsymbol{d}_I} \to \mathrm{nb}(x_o)$ around the branches $(D_i)_{i \in I}$ (hence inducing an isomorphism above D_I), as the direct sum of formal elementary D-meromorphic connections $\mathscr{E}^\varphi \otimes \widehat{\mathscr{R}}_\varphi$, where φ varies in a *good* finite subset $\Phi_I(\mathrm{nb}(x_o)) \subset \mathscr{O}_{\mathrm{nb}(x_o)_{\boldsymbol{d}_I}}(*\rho_{\boldsymbol{d}_I}^{-1}(D))$ and $\mathscr{R}_\varphi$ is a locally free $\mathscr{O}_{(\mathrm{nb}(x_o)_{\boldsymbol{d}_I}|D_I^\circ)^\wedge}(*\rho_{\boldsymbol{d}_I}^{-1}(D))$-module with an integrable connection having a regular singularity along $\rho_{\boldsymbol{d}_I}^{-1}(D)$. Goodness means here that for any $x_o \in D_I^\circ$ and any pair $\varphi \neq \psi \in \Phi_{I,x_o} \cup \{0\}$, the difference $\varphi - \psi$ can be written as $x^{-\boldsymbol{m}} \eta(x)$, with $\boldsymbol{m} \in \mathbb{N}^{\#I}$ and $\eta(x)$ holomorphic near x_o and non vanishing at x_o.[1] The sets $\Phi_I(\mathrm{nb}(x_o))$ glue together all along D_I°, producing a finite non-ramified covering $\Sigma_I^\circ \to D_I^\circ$.

Proposition 1.1. *For every $I \subset J$, there exists a unique good D-meromorphic flat bundle $\mathscr{M}_I^\circ$ in the neighbourhood of D_I° which satisfies the following two properties.*

(1) $\mathscr{D}_{\widehat{X|D_I^\circ}} \otimes_{\mathscr{D}_X} \mathscr{M}_I^\circ \simeq \mathscr{M}_{\widehat{D_I^\circ}}$.

(2) *At each point of D_I°, the local formal decomposition of $\mathscr{M}_I^\circ$ (after a local ramification around D) into elementary formal D-meromorphic flat bundles already holds without taking formalization.*

For any holonomic $\mathscr{D}_X$-module $\mathscr{N}$, the irregularity complexes $\mathrm{Irr}_D \mathscr{N}$ and $\mathrm{Irr}_D^* \mathscr{N}$, as defined by Mebkhout [**Meb90**], are constructible complexes supported on D, which only depend on $\mathscr{N}(*D)$. For $\mathscr{M}$ as above, the cohomology of $\mathrm{Irr}_D \mathscr{M}$ and $\mathrm{Irr}_D^* \mathscr{M}$ is locally constant along each stratum D_I°. The complexes $\mathrm{Irr}_D \mathscr{M}[\dim X], \mathrm{Irr}_D^* \mathscr{M}[\dim X]$ are a perverse sheaves (see loc. cit.).

Theorem 1.2. *For every $I \subset J$, we have*
$$\iota_I^{-1} \mathrm{Irr}_D \mathscr{M} \simeq \iota_I^{-1} \mathrm{Irr}_D(\mathscr{M}_I^\circ), \quad \text{and} \quad \iota_I^{-1} \mathrm{Irr}_D^* \mathscr{M} \simeq \iota_I^{-1} \mathrm{Irr}_D^*(\mathscr{M}_I^\circ).$$

In other words, the complexes $\iota_I^{-1} \mathrm{Irr}_D \mathscr{M}, \iota_I^{-1} \mathrm{Irr}_D^* \mathscr{M}$ only depend (up to isomorphism) on the formalization $\mathscr{M}_{\widehat{D_I^\circ}}$ of $\mathscr{M}$ along D_I°.

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[1] Note that, here, the goodness condition is assumed for $\Phi_I \cup \{0\}$ and not only for Φ_I, because of [**Sab13**, Cor. 12.7]. This is unfortunately not made precise in [**Sab13**, Th. 12.16] and should be corrected.

2. The irregularity complex

Our aim in this section is to show that, under the goodness assumption as above, the irregularity complex is determined by its restriction to the smooth part of D (we will only treat the case of $\mathrm{Irr}_D \mathscr{M}$). More precisely, for every $I \subset J$, and for every connected component of D_I°, we show that there exists a component D_k of D $(k \in I)$ such that $\iota_I^{-1} \mathrm{Irr}_D \mathscr{M}$ (on this connected component) is determined from $\iota_k^{-1} \mathrm{Irr}_D \mathscr{M}$.

Let $\varpi : \widetilde{X} := \widetilde{X}(D_{i \in J}) \to X$ be the real blowing-up of the components D_i of D in X, and set $\partial \widetilde{X} := \varpi^{-1}(D)$. The fibre of ϖ over a point in D_I° is diffeomorphic to $(S^1)^\ell$. *From now on, we will restrict the complexes to D or to $\partial \widetilde{X}$ without writing the pullback functor for simplicity. Similarly, by a D-meromorphic flat bundle we will mean a germ along D of such an object, and $\mathscr{O}_{X|D}, \mathscr{D}_{X|D}$ denote the germs along D of $\mathscr{O}_X, \mathscr{D}_X$.* We consider the sheaf $\mathfrak{I}$ on $\partial \widetilde{X}$ as constructed in [**Sab13**, §9.3]. Each Σ_I° lifts as a covering $\widetilde{\Sigma}_I^\circ$ of $\varpi^{-1}(D_I^\circ)$, and $\widetilde{\Sigma}_I^\circ$ is naturally contained in the étalé space $\mathfrak{I}^{\text{ét}}$ of $\mathfrak{I}$. When I varies as a subset of J, we get a *good stratified $\mathfrak{I}$-covering* $\bigsqcup_I \widetilde{\Sigma}_I^\circ =: \widetilde{\Sigma} \to \partial \widetilde{X}$ of $\partial \widetilde{X}$, in the sense of [**Sab13**, Rem. 11.12].

In this setting, the Riemann-Hilbert correspondence ([**Moc11a, Moc11b**], [**Sab13**]) is an equivalence between the category of (germs along D of) good meromorphic connections with poles along D and stratified $\mathfrak{I}$-covering contained in $\widetilde{\Sigma}$, and that of good Stokes-filtered local systems (see e.g. [**Sab13**, §9.5]) on $\partial \widetilde{X}$ with stratified $\mathfrak{I}$-covering contained in $\widetilde{\Sigma}$ (see [**Moc11b**, Th. 4.11] and [**Sab13**, Th. 12.16]).

Let $(\mathscr{L}, \mathscr{L}_\bullet)$ be such a Stokes-filtered local system corresponding to a (germ of) good D-meromorphic flat bundle $\mathscr{M}$. We have $\mathscr{L} = \widetilde{\imath}^{-1} \boldsymbol{R}\widetilde{\jmath}_* \mathrm{DR}\, \mathscr{M}_{|X \smallsetminus D}$, where

$$\widetilde{\imath} : \partial \widetilde{X} \hookrightarrow \widetilde{X} \quad \text{and} \quad \widetilde{\jmath} : X \smallsetminus D \hookrightarrow \widetilde{X}$$

are the natural closed and open inclusions. Let us denote by $\mathscr{A}_{\widetilde{X}}^{\mathrm{mod}\, D}$ (resp. $\mathscr{A}_{\widetilde{X}}^{\mathrm{rd}\, D}$) the sheaf on $\widetilde{X}$ of holomorphic functions on $X \smallsetminus D$ having moderate growth (resp. rapid decay) along $\partial \widetilde{X}$. One can then define the moderate (resp. rapidly decaying) de Rham complex $\mathrm{DR}^{\mathrm{mod}\, D} \mathscr{M}$ (resp. $\mathrm{DR}^{\mathrm{rd}\, D} \mathscr{M}$) on $\partial \widetilde{X}$. With the goodness assumption, it is known that both have cohomology in degree zero at most. More precisely, the Riemann-Hilbert correspondence mentioned above gives

$$\mathscr{L}_{\leqslant 0} = \mathscr{H}^0 \mathrm{DR}^{\mathrm{mod}\, D} \mathscr{M} \quad \text{and} \quad \mathscr{H}^j \mathrm{DR}^{\mathrm{mod}\, D} \mathscr{M} = 0 \text{ for } j \neq 0.$$

We define $\mathscr{L}^{>0} := \mathscr{L}/\mathscr{L}_{\leqslant 0}$, and similarly $\mathrm{DR}^{>\mathrm{mod}\, D} \mathscr{M}$ is the cone of

$$\mathrm{DR}^{\mathrm{mod}\, D} \mathscr{M} \longrightarrow \widetilde{\imath}^{-1} \boldsymbol{R}\widetilde{\jmath}_* \mathrm{DR}\, \mathscr{M}_{|X \smallsetminus D},$$

so that $\mathscr{L}^{>0} = \mathscr{H}^0 \mathrm{DR}^{>\mathrm{mod}\, D} \mathscr{M}$ (and $\mathscr{H}^k \mathrm{DR}^{>\mathrm{mod}\, D} \mathscr{M} = 0$ for $k \neq 0$).

Proposition 2.1. *We have* $\mathrm{Irr}_D \mathscr{M} = \boldsymbol{R}\varpi_* \mathscr{L}^{>0}$.

Proof. We have

$$\boldsymbol{R}\varpi_* \mathrm{DR}^{\mathrm{mod}\, D} \mathscr{M} = \mathrm{DR}\, \mathscr{M}(*D) \quad \text{and} \quad \boldsymbol{R}\varpi_* \boldsymbol{R}\widetilde{\jmath}_* \mathrm{DR}\, \mathscr{M}_{|X \smallsetminus D} = \boldsymbol{R}j_* \mathrm{DR}\, \mathscr{M}_{|X \smallsetminus D},$$

where $j : X \smallsetminus D \hookrightarrow X$ is the inclusion. We then apply [**Meb04**, Def. 3.4-1]. $\square$

Remark 2.2 (The irregularity complex $\mathrm{Irr}_D^* \mathscr{M}$**).** Recall that Mebkhout also defined the irregularity complex $\mathrm{Irr}_D^* \mathscr{M}$ in [**Meb90**] (see also [**Meb04**, Def. 3.4-2]), which is non-canonically isomorphic to the complex $\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_{X|D}}(\mathscr{M}, \mathscr{Q}_D)[-1]$, where $\mathscr{Q}_D = \mathscr{O}_{\widehat{X|D}}/\mathscr{O}_{X|D}$ (see [**Meb04**, Cor. 3.4-4]). Let us set $\mathscr{L}_{\prec 0} := \mathscr{H}^0 \mathrm{DR}^{\mathrm{rd}\,D} \mathscr{M}$. We then have

$$(2.2*) \qquad\qquad \boldsymbol{R}\varpi_* \mathscr{L}_{\prec 0} \simeq \mathrm{Irr}_D^* \mathscr{M}^\vee,$$

where $\mathscr{M}^\vee$ is the holonomic $\mathscr{D}_X$-module dual to $\mathscr{M}$. Indeed, According to [**Kas03**, (3.13)] we have

$$\mathrm{DR}(\mathscr{Q}_D \overset{\boldsymbol{L}}{\otimes} \mathscr{M})[-1] \simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_{X|D}}(\mathscr{M}^\vee, \mathscr{Q}_D)[-1] \simeq \mathrm{Irr}_D^* \mathscr{M}^\vee.$$

On the other hand, as $\mathscr{Q}_D$ is flat over $\mathscr{O}_{X|D}$ (because $\mathscr{O}_{\widehat{X|D}}$ is faithfully flat over $\mathscr{O}_{X|D}$) and as $\boldsymbol{R}\varpi_* \mathscr{A}_{\widetilde{X}}^{\mathrm{rd}\,D} \simeq \mathscr{Q}_D[-1]$, we have

$$\mathrm{DR}(\mathscr{Q}_D \otimes \mathscr{M})[-1] \simeq \mathrm{DR}(\mathscr{Q}_D \overset{\boldsymbol{L}}{\otimes} \mathscr{M})[-1] \simeq \boldsymbol{R}\varpi_* \mathrm{DR}^{\mathrm{rd}\,D} \mathscr{M}.$$

We also notice that $\mathrm{Irr}_D^* \mathscr{M}^\vee = \mathrm{Irr}_D^* \mathscr{M}^\vee(*D)$ and $\mathscr{M}^\vee(*D)$ is also a good D-meromorphic flat bundle, which is identified with the dual D-meromorphic flat bundle $\mathscr{H}om_{\mathscr{O}_X(*D)}(\mathscr{M}, \mathscr{O}_X(*D))$.

Let us fix $I \subset J$. Near each point x_o of D_I°, there exists a local ramification $\rho : \mathrm{nb}(x_o)_{\boldsymbol{d}_I} \to \mathrm{nb}(x_o)$ along D such that the pullback of $\mathscr{M}$ has a good formal decomposition at each point in $\mathrm{nb}(x_o)_{\boldsymbol{d}_I}$. The set $\Phi_{x_o} \cup \{0\}$ is good, so in particular the pole divisors of each of its nonzero elements are totally ordered. The smallest such divisor is nonzero, and we denote by $k(x_o) \in I$ the index of a component $D_{k(x_o)}$ of this divisor. In other words, we choose $k(x_o)$ such that each nonzero $\varphi \in \Phi_{x_o}$ has a pole along $D_{k(x_o)}$. One can choose this index constant along any connected component of D_I°. For simplicity, we denote by $k(I)$ the locally constant function $x_o \mapsto k(x_o)$ on D_I°.

For every subset $I \subset J$, we have a natural inclusion $\widetilde{\iota}_I : \varpi^{-1}(D_I^\circ) \subset \varpi^{-1}(D)$ lifting ι_I.

Proposition 2.3. *Let us fix $I \subset J$ and let us set $k = k(I)$ for simplicity. Then the natural morphism $\widetilde{\iota}_I^{-1}\mathscr{L}^{>0} \to \widetilde{\iota}_I^{-1}\boldsymbol{R}\widetilde{\iota}_{k*}\widetilde{\iota}_k^{-1}\mathscr{L}^{>0}$ is an isomorphism.*

By applying $\boldsymbol{R}\varpi_*$ and using Proposition 2.1, we obtain:

Corollary 2.4. *With the notation as in Proposition 2.3, the natural morphism $\iota_I^{-1}\mathrm{Irr}_D(\mathscr{M}) \to \iota_I^{-1}\boldsymbol{R}\iota_{k*}\iota_k^{-1}\mathrm{Irr}_D(\mathscr{M})$ is an isomorphism.* $\qquad\square$

Proof of Proposition 2.3. Since the morphism is globally defined, the proof that it is an isomorphism is a local question on $\varpi^{-1}(D_I^\circ)$ near a point $\widetilde{x}_o \in \varpi^{-1}(x_o)$, and one can also reduce to the non-ramified case. One can then apply the higher dimensional Hukuhara-Turrittin theorem (see e.g. [**Sab13**, Th. 12.5]). Let $\mathscr{A}_{\widetilde{X}}$ denote the sheaf

of C^∞ functions on $\widetilde{X}$ which are holomorphic on X^*. We can thus assume that $\mathscr{A}_{\widetilde{X}} \otimes \varpi^{-1} \mathscr{M}$ decomposes as the direct sum of terms $\mathscr{A}_{\widetilde{X}} \otimes \varpi^{-1} (\mathscr{E}^\varphi \otimes \mathscr{R}_\varphi)$. By induction on the rank, we can also assume that $\mathscr{R}_\varphi$ has rank one, and locally on $\varpi^{-1}(D_I^\circ)$ the corresponding local system is trivial, so we can finally assume that $\mathscr{M} = \mathscr{E}^\varphi$.

If $\varphi = 0$ in $\mathscr{O}_{X,x_o}(*D)/\mathscr{O}_{X,x_o}$, then $\mathscr{L}^{>0} = 0$ and there is nothing to prove.

If $\varphi \neq 0$ in $\mathscr{O}_{X,x_o}(*D)/\mathscr{O}_{X,x_o}$, we set $\varphi(x) = u(x)/x^m$, where $u \in \mathscr{O}_{X,x_o}$ satisfies $u(x_o) \neq 0$, and $m_i \in \mathbb{N}$ for $i \in I$. In particular, $m_{k(I)} \neq 0$. We choose local coordinates at $\widetilde{x}_o$ of the form $(\rho_1, \ldots, \rho_\ell, \theta_1, \ldots, \theta_\ell, (x_j)_{j \notin I})$ with $\rho_i \in \mathbb{R}_+$. We can assume that $m_i \neq 0$ for $i = 1, \ldots, p$ with $1 \leqslant p \leqslant \ell$. Then, in these coordinates, $\varpi^{-1}(D) = \prod_{i=1}^\ell \rho_i = 0$ and $\mathscr{L}^{>0}$ is the constant sheaf of rank one on the closed subset of $\varpi^{-1}(D)$ defined by

$$\begin{cases} \sum_{i=1}^p m_i \theta_i \in \arg u(x_0) + [-\pi/2, \pi/2], \\ \prod_{i=1}^p \rho_i = 0, \end{cases}$$

and it is zero outside this closed subset. Each connected component of this set is homeomorphic to a product

$$\partial \mathbb{R}_+^p \times [a,b] \times (S^1)^{p-1} \times (\mathbb{R}_+)^{\ell-p} \times (S^1)^{\ell-p} \times \mathbb{C}^{n-\ell}.$$

Assume $k(I) = 1$ in these coordinates, for simplicity. Then the trace of this set on $\varpi^{-1}(D_{k(I)}^\circ)$ is the set defined by $\prod_{j=2}^\ell \rho_j \neq 0$. This is the subset

$$\{\rho_1 = 0\} \times (\mathbb{R}_+^*)^{p-1} \times [a,b] \times (S^1)^{p-1} \times (\mathbb{R}_+^*)^{\ell-p} \times (S^1)^{\ell-p} \times \mathbb{C}^{n-\ell}.$$

Its closure is the subset

$$\{\rho_1 = 0\} \times \mathbb{R}_+^{p-1} \times [a,b] \times (S^1)^{p-1} \times \mathbb{R}_+^{\ell-p} \times (S^1)^{\ell-p} \times \mathbb{C}^{n-\ell}.$$

The open inclusion is acyclic for the constant sheaf, because so is $(\mathbb{R}_+^*)^{\ell-1} \hookrightarrow \mathbb{R}_+^{\ell-1}$, and the pushforward by this inclusion of the constant sheaf on the open subset is the constant sheaf on the closed subset. Since $\varpi^{-1}(D_I)$ is the subset of the closed subset above defined by $\rho_i = 0$ for $i = 2, \ldots, \ell$, the restriction of the latter sheaf to $\varpi^{-1}(D_I)$ is the constant sheaf on $\varpi^{-1}(D_I)$, as wanted. $\square$

3. An equivalence of categories

Let A be a category and let G be a group. The category G-A is the category whose objects are G-objects of A, that is, pairs (M, ρ) where M is an object of A and ρ is a morphism $G \to \operatorname{Aut}(M)$, and for which $\operatorname{Hom}_{G\text{-}\mathsf{A}}((M, \rho_M), (N, \rho_N)) \subset \operatorname{Hom}_{\mathsf{A}}(M, N)$ is the subset consisting of morphisms $\varphi : M \to N$ such that, for every $g \in G$, $\varphi \circ \rho_M(g) = \rho_N(g)$.

Let $\widetilde{\Sigma} \to D$ be a good stratified $\mathcal{I}$-covering and let $\mathsf{Mod}_{\mathrm{hol}}(X, D, \widetilde{\Sigma})$ denote the full subcategory of that of holonomic $\mathscr{D}_X$-modules whose objects consist in good meromorphic flat bundles on (X, D) with associated stratified $\mathcal{I}$-covering contained in $\widetilde{\Sigma}$.

Let us fix a nonempty subset $I \subset J$, let D_I° the corresponding stratum of D, let $x_o \in D_I^\circ$ and let $D_I^\circ(x_o)$ the connected component of D_I° containing x_o. Let us fix a local holomorphic decomposition

$$\big(\mathrm{nb}(x_o, X), \mathrm{nb}(x_o, D)\big) = (\Omega, D_\Omega) \times \mathrm{nb}(x_o, D_I^\circ),$$

where Ω is an open neighbourhood of 0 in $\mathbb{C}^\ell$ and D_Ω is the union of the coordinate hyperplanes in $\mathbb{C}^\ell$. We consider the category $\mathsf{Mod}_{\mathrm{hol}}((X, D_I^\circ(x_o)), D, \widetilde{\Sigma})$ of germs along $D_I^\circ(x_o)$ of good meromorphic flat bundles on (X, D) with associated stratified $\mathcal{I}$-covering contained in $\widetilde{\Sigma}$ and the similar category $\mathsf{Mod}_{\mathrm{hol}}((\Omega, 0), D_\Omega, \widetilde{\Sigma}_{x_o})$, where $\widetilde{\Sigma}_{x_o}$ is the restriction of $\widetilde{\Sigma}$ above $\partial\widetilde{\Omega} := \varpi^{-1}(D_\Omega)$.

Theorem 3.1. *Set* $G = \pi_1\big(D_I^\circ(x_o), x_o\big)$. *There is a natural equivalence of categories:*

$$\mathsf{Mod}_{\mathrm{hol}}\big((X, D_I^\circ(x_o)), D, \widetilde{\Sigma}\big) \simeq G\text{-}\mathsf{Mod}_{\mathrm{hol}}\big((\Omega, 0), D_\Omega, \widetilde{\Sigma}_{x_o}\big).$$

Proof. We set $\partial\widetilde{X}_I^\circ(x_o) := \varpi^{-1}(D_I^\circ(x_o))$ and we denote similarly by $\widetilde{\Sigma}_I^\circ(x_o)$ the restriction of $\widetilde{\Sigma}$ above this set.

(1) By the Riemann-Hilbert correspondence (see [**Moc11b**, Th. 4.11], [**Sab13**, Th. 12.16]), the category $\mathsf{Mod}_{\mathrm{hol}}((X, D_I^\circ(x_o)), D, \widetilde{\Sigma})$ is equivalent to the category of Stokes-filtered local systems $(\mathscr{L}, \mathscr{L}_\bullet)$ on $(\partial\widetilde{X}, \partial\widetilde{X}_I^\circ(x_o))$ with associated $\mathcal{I}$-covering contained in $\widetilde{\Sigma}$.

(2) The restriction functor to $\partial\widetilde{X}_I^\circ(x_o)$ induces an equivalence between the latter category and the category of Stokes-filtered local systems $(\mathscr{L}, \mathscr{L}_\bullet)$ on $\partial\widetilde{X}_I^\circ(x_o)$ with associated $\mathcal{I}$-covering contained in $\widetilde{\Sigma}_I^\circ(x_o)$. This is explained below. This reduction is helpful as $\widetilde{\Sigma}_I^\circ$ is Hausdorff (while $\widetilde{\Sigma}$ may be non Hausdorff).

(3) Let $\pi : (E_I^\circ(x_o), y_o) \to (D_I^\circ(x_o), x_o)$ be a universal covering of $D_I^\circ(x_o)$ with base point y_o above x_o and let $G = \mathrm{Gal}(\pi)$ be the corresponding Galois group. Let $\partial\widetilde{Y}_I^\circ(x_o)$ be the pre-image of $\partial\widetilde{X}_I^\circ(x_o)$ by π. Then the latter category is equivalent to the category of G-equivariant Stokes-filtered local systems $(\mathscr{L}, \mathscr{L}_\bullet)$ on $\partial\widetilde{Y}_I^\circ(x_o)$ with associated $\pi^{-1}\mathcal{I}$-covering contained in $\pi^{-1}\widetilde{\Sigma}_I^\circ(x_o)$. This is a standard argument.

(4) (see [**Moc11b**, Th. 4.13]) The sheaf-theoretic restriction functor is an equivalence from the latter category to the category of $\pi_1\big(D_I^\circ(x_o), x_o\big)$-Stokes-filtered local systems $(\mathscr{L}, \mathscr{L}_\bullet)$ on $\partial\widetilde{\Omega}$ with associated $\mathcal{I}_{x_o}$-covering contained in $\widetilde{\Sigma}_{x_o}$. This proof will be reviewed in the appendix.

(5) Applying now (2) and then (1) to $((\Omega, 0), D_\Omega, \widetilde{\Sigma}_{x_o})$ ends the proof of the theorem. □

Proof of (2). One can find a neighbourhood $\mathrm{nb}(D_I^\circ)$ in X together with a C^∞ retraction $\mathrm{nb}(D_I^\circ) \to D_I^\circ$ (by using a tubular neighbourhood of D_I° in X). This retraction can be lifted as a retraction

$$\partial\widetilde{X}_{|\mathrm{nb}(D_I^\circ)} := \varpi^{-1}(\mathrm{nb}(D_I^\circ) \cap D) \longrightarrow \varpi^{-1}(D_I^\circ) =: \partial\widetilde{X}_{|D_I^\circ}.$$

Locally near $x_o \in D_I^\circ$, we have

$$\partial \widetilde{X}_{|\mathrm{nb}(D_I^\circ)} \simeq D_I^\circ \times (S^1)^\ell \times \partial(\mathbb{R}_+)^\ell \quad \text{and} \quad \partial \widetilde{X}_{|D_I^\circ} \simeq D_I^\circ \times (S^1)^\ell \times \{0\}.$$

This can be extended to the $\mathfrak{I}$-stratified covering $\widetilde{\Sigma}$ as follows. There exists a map

$$(3.2) \qquad\qquad \widetilde{\Sigma}_I^\circ \times_{\partial \widetilde{X}_{|D_I^\circ}} \partial \widetilde{X}_{|\mathrm{nb}(D_I^\circ)} \longrightarrow \widetilde{\Sigma}_{|\mathrm{nb}(D_I^\circ)}$$

which is a local homeomorphism, and moreover is a finite covering when restricted to each stratum in the source and target spaces, and this covering is trivial when the map is restricted to $\mathrm{nb}(x_o)$ for any $x_o \in D_I^\circ$.

Let $(\mathscr{L}, \mathscr{L}_\bullet)$ be a Stokes-filtered local system on $\partial \widetilde{X}_{|\mathrm{nb}(D_I^\circ)}$ with stratified $\mathfrak{I}$-covering contained in $\widetilde{\Sigma}_{|\mathrm{nb}(D_I^\circ)}$, assumed to be good. Then $(\mathscr{L}, \mathscr{L}_\bullet)$ is isomorphic to the Stokes-filtered local system on $\partial \widetilde{X}_{|\mathrm{nb}(D_I^\circ)}$ obtained from $(\mathscr{L}, \mathscr{L}_\bullet)_{|\partial \widetilde{X}_{|D_I^\circ}}$ by the following operations:

- Pullback of $(\mathscr{L}, \mathscr{L}_\bullet)_{|\partial \widetilde{X}_{|D_I^\circ}}$ on $\widetilde{\Sigma}_I^\circ \times_{\partial \widetilde{X}_{|D_I^\circ}} \partial \widetilde{X}_{|\mathrm{nb}(D_I^\circ)}$.
- Trace of the latter sheaf by the étale map (3.2) (see [**Sab13**, §1.5] for the trace by an étale map).

This correspondence gives the desired equivalence of categories. $\qquad \square$

4. Proof of the main results

4.a. Proof of Proposition 1.1. It is enough to prove existence and uniqueness on each connected component of D_I°. It is enough to describe $\mathscr{M}_I^\circ$ by means of its corresponding representation given by Theorem 3.1. Let us denote by $^\perp\!\mathscr{M}_{I,x_o}$ the restriction of $\mathscr{M}$ to the transversal slice Ω at x_o, and by $\widehat{^\perp\!\mathscr{M}}_{I,x_o}$ its formalization at $0 \in \Omega$. From the representation $\pi_1(D_I^\circ(x_o), x_o) \to \mathrm{Aut}_{\mathscr{D}_{\Omega,0}}(^\perp\!\mathscr{M}_{I,x_o})$ defined by $\mathscr{M}$ we get the representation $\pi_1(D_I^\circ(x_o), x_o) \to \mathrm{Aut}_{\mathscr{D}_{\Omega,0}}(\widehat{^\perp\!\mathscr{M}}_{I,x_o})$ obtained by formalization, corresponding to $\mathscr{M}_{\widehat{D_I^\circ}}$. It is therefore enough

(1) to show the existence and uniqueness of a germ $\mathscr{N}$ on $(\Omega, 0)$ such that $\widehat{\mathscr{N}} \simeq \widehat{^\perp\!\mathscr{M}}_{I,x_o}$ and which decomposes holomorphically, after a local ramification, in the same way as $\widehat{\mathscr{N}} := \widehat{^\perp\!\mathscr{M}}_{I,x_o}$ does (in particular with the goodness condition);

(2) to prove that the natural formalization morphism $\mathrm{Aut}(\mathscr{N}) \to \mathrm{Aut}(\widehat{\mathscr{N}})$ is an isomorphism.

We first prove (1) and (2) in the non-ramified case. In such a case, the existence of $\mathscr{N}$ is clear since the elementary formal models come from meromorphic connections. On the other hand, one checks that there is no non-zero morphism $\mathscr{E}^\varphi \otimes \mathscr{R}_\varphi \to \mathscr{E}^\psi \otimes \mathscr{R}_\psi$ if $\varphi - \psi \neq 0$, and similarly at the formal level. On the other hand, if $\varphi = \psi$, any formal automorphism $\widehat{\mathscr{R}}_\varphi \to \widehat{\mathscr{R}}_\varphi$ is the formalization of a unique automorphism of $\mathscr{R}_\varphi$. We then have $\mathrm{Aut}(\mathscr{N}) = \mathrm{Aut}(\widehat{\mathscr{N}})$ and we also get in this way the uniqueness of $\mathscr{N}$ up to isomorphism.

We can now treat the ramified case. Let ρ be a local ramification of (X, x_o) around D such that $\rho^+ \widehat{\mathscr{N}}$ is non-ramified, and let G denote the corresponding Galois group. The category of germs at x_o of formal meromorphic connections with poles along D which decompose into elementary formal connections after the ramification ρ is equivalent to the category of G-equivariant decomposable formal connections, and we have a similar equivalence for meromorphic connections. Now, $\widehat{\mathscr{N}}$ corresponds to $\widehat{\mathscr{N}'}$ as above in the non-ramified case, together with a G-action $\widehat{\sigma}_g : \widehat{\mathscr{N}'} \simeq g^* \widehat{\mathscr{N}'}$ for $g \in G$. From the non-ramified case treated above, $\widehat{\sigma}_g$ comes from a unique $\sigma_g : \mathscr{N}' \simeq g^* \mathscr{N}'$, which defines thus, by uniqueness, a G-equivariant structure on $\mathscr{N}'$. Applying the equivalence of categories as in the formal case, we conclude that $\mathscr{N}' = \rho^* \mathscr{N}$ for a unique $\mathscr{N}$, whose formalization is thus $\widehat{\mathscr{N}}$. Now, an automorphism of $\mathscr{N}$ is a G-equivariant automorphism of $\mathscr{N}'$. From (2) above, we easily deduce an isomorphism $\mathrm{Aut}_G(\mathscr{N}') \xrightarrow{\sim} \mathrm{Aut}_G(\widehat{\mathscr{N}'})$, concluding the proof. $\qquad\square$

4.b. Proof of Theorem 1.2. Recall that $\mathrm{Irr}_D \mathscr{M}$ is a complex whose cohomology is locally constant on each D_I°. Similarly, we consider for every k the vector space $\mathscr{H}^k \mathrm{Irr}_{D_\Omega}(^\perp\mathscr{M}_{I,x_o})_0$, which is the germ at $0 \in \Omega$ of the constructible sheaf $\mathscr{H}^k \mathrm{Irr}_{D_\Omega}(^\perp\mathscr{M}_{I,x_o})$. The representation $\pi_1(D_I^\circ(x_o), x_o) \to \mathrm{Aut}_{\mathscr{D}_{\Omega,0}}(^\perp\mathscr{M}_{I,x_o})$ defined by the germ of $\mathscr{M}$ along $D_I^\circ(x_o)$ induces a linear representation

$$\pi_1(D_I^\circ(x_o), x_o) \longrightarrow \mathrm{Aut}\big(\mathscr{H}^k \mathrm{Irr}_{D_\Omega}(^\perp\mathscr{M}_{I,x_o})_0\big)$$

for every k, which is isomorphic to the representation defined by the local system $\iota_I^{-1} \mathscr{H}^k \mathrm{Irr}_D \mathscr{M}$ on $D_I^\circ(x_o)$. Similarly, for $\mathscr{N}$ as in the proof of Proposition 1.1, the representation $\pi_1(D_I^\circ(x_o), x_o) \to \mathrm{Aut}_{\mathscr{D}_{\Omega,0}}(\mathscr{N})$ induces the linear representation $\pi_1(D_I^\circ(x_o), x_o) \to \mathrm{Aut}\big(\mathscr{H}^k \mathrm{Irr}_{D_\Omega}(\mathscr{N})_0\big)$ which corresponds to $\iota_I^{-1} \mathscr{H}^k \mathrm{Irr}_D \mathscr{M}_I^\circ$ on $D_I^\circ(x_o)$.

The case $\ell = 1$. We first assume that $I = \{i\}$. Then, because $\dim \Omega = 1$ and $D_\Omega = \{0\}$, it is well-known that $\mathscr{H}^k \mathrm{Irr}_{D_\Omega}(^\perp\mathscr{M}_{i,x_o})$ and $\mathscr{H}^k \mathrm{Irr}_{D_\Omega}(\mathscr{N})$ have the same rank for any k, and vanish except for $k = 0$. We conclude that both local systems in Theorem 1.2 have the same rank.

Lemma 4.1. *There exists an isomorphism between the vector spaces $\mathscr{H}^0 \mathrm{Irr}_{D_\Omega}(^\perp\mathscr{M}_{i,x_o})$ and $\mathscr{H}^0 \mathrm{Irr}_{D_\Omega}(\mathscr{N})$ such that, for any automorphism λ of $^\perp\mathscr{M}_{i,x_o}$, the induced automorphism of $\mathscr{H}^0 \mathrm{Irr}_{D_\Omega}(^\perp\mathscr{M}_{i,x_o})$ corresponds to the automorphism of $\mathscr{H}^0 \mathrm{Irr}_{D_\Omega}(\mathscr{N})$ induced by the formalized automorphism $\widehat{\lambda}$, by means of (1) and (2) in Section 4.a.*

Proof. We consider the Stokes-filtered local system $(\mathscr{L}, \mathscr{L}_\bullet)$ corresponding to $^\perp\mathscr{M}_{i,x_o}$ on $S^1 = \partial\widetilde{\Omega}$. We will indicate the proof for $\mathrm{Irr}_D^* \mathscr{N}^\vee$, that is, we will consider $\mathscr{L}_{\prec 0}$ instead of $\mathscr{L}^{>0}$. Let us cover S^1 by open intervals $(U_\alpha)_{\alpha=1,\dots,N}$ such that

- every open interval which contains at most one Stokes direction for every pair of distinct exponential factors (see e.g. Example 1.4 in [**Sab13**]),

- the intersection of two intervals of the covering is an interval not containing any Stokes direction,
- there are no triple intersections of intervals of the covering.

Then this covering is a Leray covering for $\mathscr{L}_{\prec 0}$ (see e.g. the proof of Lemma 3.12 in loc. cit.), and moreover the only nonzero term of the associated Čech complex is the term in degree one. It follows that

$$H^1(S^1, \mathscr{L}_{\prec 0}) = \bigoplus_{\alpha = 1, \ldots, N} H^0(U_\alpha \cap U_{\alpha+1}, \mathscr{L}_{\prec 0}),$$

if we set $U_{N+1} = U_1$.

Recall that, on each interval U_α, the Stokes-filtered local system $(\mathscr{L}, \mathscr{L}_\bullet)$ is graded, i.e., the Stokes filtration splits (see e.g. Lemma 3.12 in loc. cit.). Let us then choose a splitting on $U_\alpha \cap U_{\alpha+1}$. Then Theorem 3.5 (and its proof) in loc. cit. shows that any automorphism λ is graded with respect to the chosen splitting on $U_\alpha \cap U_{\alpha+1}$. It follows that the action of the automorphism on $H^0(U_\alpha \cap U_{\alpha+1}, \mathscr{L}_{\prec 0})$ is the same as the action of the associated graded automorphism on $H^0(U_\alpha \cap U_{\alpha+1}, (\mathrm{gr}\,\mathscr{L})_{\prec 0})$, so we have found a model where both actions are equal. $\qquad\square$

The case $\ell \geqslant 2$. We set $k = k(I)$ as defined after Proposition 2.1. Let $\mathrm{nb}(D_I^\circ)$ be an open neighbourhood of D_I° in X on which $\mathscr{M}_I^\circ$ is defined. We claim that

$$\iota_k^{-1} \mathscr{M}_I^\circ = \mathscr{M}_{k\,|\mathrm{nb}(D_I^\circ)}^\circ.$$

Indeed, this follows from the uniqueness of $\mathscr{M}_k^\circ$, and from the fact that $\mathscr{M}_I^\circ$ also decomposes after ramification along D at each point of $\mathrm{nb}(D_I^\circ) \cap D_i^\circ$ if this neighbourhood is chosen small enough. Since $\iota_k^{-1} \mathrm{Irr}_D(\bullet) = \iota_k^{-1} \mathscr{H}^0 \mathrm{Irr}_D(\bullet)$, we then have

$$\begin{aligned}
\mathrm{Irr}_D(\iota_k^{-1} \mathscr{M}_I^\circ) &= \mathrm{Irr}_D(\mathscr{M}_k^\circ)_{|\mathrm{nb}(D_I^\circ)} \\
&= \mathrm{Irr}_D(\iota_k^{-1} \mathscr{M})_{|\mathrm{nb}(D_I^\circ)} \quad (\text{case } \ell = 1),
\end{aligned}$$

and therefore, by applying $\iota_I^{-1} \boldsymbol{R}\iota_{k*}$,

$$\iota_I^{-1} \boldsymbol{R}\iota_{k*} \iota_k^{-1} \mathrm{Irr}_D(\mathscr{M}_I^\circ) \simeq \iota_I^{-1} \boldsymbol{R}\iota_{k*} \iota_k^{-1} \mathrm{Irr}_D(\mathscr{M}).$$

The assertion of Theorem 1.2 now follows from Corollary 2.4, applied both to $\mathscr{M}$ and $\mathscr{M}_I^\circ$. $\qquad\square$

Appendix. Some properties of Stokes-filtered local systems

In this appendix we keep the setting of Section 2. We review the proof of [**Moc11b**, Th. 4.13] together with some essential results which are proved in loc. cit.

A.a. Grading of a Stokes-filtered local system. The result in this subsection is local with respect to D, hence we allow a ramification around the components of D. We fix a nonempty subset $I \subset J$. We fix a simply connected open set $U_I^\circ \subset D_I^\circ$.

We assume that $(\mathscr{L}, \mathscr{L}_\bullet)$ is non-ramified in the neighbourhood of U_I°. The covering $\widetilde{\Sigma}_I^\circ$ can then be trivialized on $U_I^\circ \times (S^1)^\ell = \varpi^{-1}(U_I^\circ)$, and we set

$$\widetilde{\Sigma}_I^\circ = \Phi \times U_I^\circ \times (S^1)^\ell,$$

where Φ is a finite subset of $\Gamma\big(U_I^\circ, (\mathscr{O}_X(*D)/\mathscr{O}_X)_{|U_I^\circ}\big)$. Moreover, by the goodness assumption on $\widetilde{\Sigma}$, Φ is a good set, namely, for every pair $\varphi \neq \psi$, the polar divisor of $\varphi - \psi$ is negative. The set $\mathrm{St}(\varphi, \psi) \subset U_I^\circ \times (S^1)^\ell$ of Stokes directions is smooth over U_I° with fibers equal to a union of translated codimension-one subtori

$$(\mathrm{A.1}) \qquad \mathrm{St}(\varphi, \psi)_x = \Big\{ (\theta_1, \ldots, \theta_\ell) \in (S^1)^\ell \mid \textstyle\sum_j m_j \theta_j - \arg c(x) = \pm\pi/2 \bmod 2\pi \Big\},$$

where $c(x)$ is an invertible holomorphic function on U_I° and $(m_1, \ldots, m_\ell) \in \mathbb{N}^\ell \smallsetminus \{0\}$. We denote by $\mathrm{St}(\Phi)$ the union of the subsets $\mathrm{St}(\varphi, \psi)$ for all pairs $\varphi \neq \psi \in \Phi$.

Let us fix $\theta_o = (\theta_{o,1}, \ldots, \theta_{o,\ell}) \in (S^1)^\ell$ and $\alpha_1, \ldots, \alpha_\ell \in \mathbb{N}^*$ such that $\gcd(\alpha_1, \ldots, \alpha_\ell) = 1$. The map $\theta \mapsto (\alpha_1 \theta + \theta_{o,1}, \ldots, \alpha_\ell \theta + \theta_{o,\ell})$ embeds S^1 in $(S^1)^\ell$. In the following, $S^1_{\boldsymbol{\alpha}, \theta_o}$ denotes this circle.

Proposition A.2. *Let A° be an open interval of length $< 2\pi$ in $S^1_{\boldsymbol{\alpha}, \theta_o}$ and let A be its closure. Assume that A satisfies the following property.*

- *For every $x \in U_I^\circ$ and every pair $\varphi \neq \psi \in \Phi$,*

$$\#\big(A \cap \mathrm{St}(\varphi, \psi)\big) = \#\big(A^\circ \cap \mathrm{St}(\varphi, \psi)\big) \leqslant 1.$$

If moreover U_I° is contractible, then $(\mathscr{L}, \mathscr{L}_\bullet)$ is graded when restricted to a sufficiently small neighbourhood $U_I^\circ \times A$ in $U_I^\circ \times (S^1)^\ell$.

Proof. We first prove that, for every $\varphi \in \Phi$, we have $H^k(U_I^\circ \times A, \mathscr{L}_{<\varphi}) = 0$ for $k \geqslant 1$. Note that, since $\varpi : U_I^\circ \times A \to U_I^\circ$ is proper, $R^k \varpi_* \mathscr{L}_{<\varphi|U_I^\circ \times A}$ is compatible with base change, hence its germ at x is equal to $H^k(A, \mathscr{L}_{<\varphi|\{x\} \times A})$. By our assumption on A, this is also equal to $H^k(A^\circ, \mathscr{L}_{<\varphi|\{x\} \times A^\circ})$, and by the proof of [**Sab13**, Lem. 9.26], this is zero for $k \geqslant 1$.

We argue as in loc. cit. to obtain that $(\mathscr{L}, \mathscr{L}_\bullet)$ is graded in the neighbourhood of $\{x\} \times A$ for every $x \in U_I^\circ$. In particular, it is easy to check that $\varpi_* \mathscr{L}_{<\varphi|U_I^\circ \times A}$ is locally constant, hence constant, on U_I°. Since U_I° is assumed contractible, we obtain the vanishing of $H^k(U_I^\circ \times A, \mathscr{L}_{<\varphi})$ $(k \geqslant 1)$. Using once more the argument of loc. cit., we obtain the grading property all over $U_I^\circ \times A$, hence in some open neighbourhood of it. $\qquad\square$

A.b. Closedness. Let U_I° be an open subset of D_I° with closure $\overline{U_I^\circ}$ in D_I° and boundary ∂U_I°, and let $j : U_I^\circ \hookrightarrow \overline{U_I^\circ}$ and $\widetilde{j} : \varpi^{-1}(U_I^\circ) \to \varpi^{-1}(\overline{U_I^\circ})$ be the open

inclusions. Let $(\mathscr{L}, \mathscr{L}_\bullet)$ be a Stokes-filtered local system on $\varpi^{-1}(U_I^\circ)$ with associated covering contained in $\widetilde{\Sigma}_{I|U_I^\circ}^\circ$. Assume that

(∗) for any $x \in \partial U_I^\circ$, x has a fundamental system of open neighbourhoods V in D_I° such that $V \cap U_I^\circ$ and $V \cap \overline{U_I^\circ}$ are contractible.

Proposition A.3. *Under this assumption, the functor $\widetilde{\jmath}_*$ induces an equivalence between the category of Stokes-filtered local systems $(\mathscr{L}, \mathscr{L}_\bullet)$ on $\varpi^{-1}(U_I^\circ)$ with associated covering contained in $\widetilde{\Sigma}_{I|U_I^\circ}^\circ$, and that on $\varpi^{-1}(\overline{U_I^\circ})$ with associated covering contained in $\widetilde{\Sigma}_{I|\overline{U_I^\circ}}^\circ$, a quasi-inverse functor being the restriction $\widetilde{\jmath}^{-1}$.*

Proof. Since the functor is globally defined, the question is local near a point $x_o \in \partial U_I^\circ$. Moreover, as in Section A.a, we can assume that $\widetilde{\Sigma}_I^\circ$ is a trivial covering on some neighbourhood of x_o. It is enough to prove the statement in the non-ramified case since, by uniqueness the construction, it will descend by means of the Galois action of the ramification. We will work with the corresponding set Φ of exponential factors.

Firstly, we note that Assumption (∗) also holds for $\varpi^{-1}(\overline{U_I^\circ})$, since any point in $\varpi^{-1}(x)$ has a fundamental systems of neighbourhoods of the form of the product of neighbourhoods V with a product of ℓ open intervals. It follows that the local system $\mathscr{L}$ extends in a unique way as a local system on $\varpi^{-1}(\overline{U_I^\circ})$, and the latter is $\widetilde{\jmath}_*\mathscr{L}$. Similarly, a morphism between local systems extends in a unique way by the functor $\widetilde{\jmath}_*$. The same property holds for the local systems $\mathrm{gr}_\psi \mathscr{L}$ for $\psi \in \Phi$.

Let us first show that the functor $\widetilde{\jmath}_*$ takes values in the category of Stokes-filtered local systems. The point is to check that every $\widetilde{\jmath}_*\mathscr{L}_{\leqslant\varphi}$ decomposes as $\bigoplus_{\psi\in\Phi}\beta_{\psi\leqslant\varphi}\widetilde{\jmath}_* \mathrm{gr}_\psi \mathscr{L}$ in the neighbourhood of every point (x_o, θ_o) of $\varpi^{-1}(x_o)$. If we fix a small interval A° containing this point as in Proposition A.2, we find that, according to this proposition and Assumption (∗),

$$(A.4) \qquad \mathscr{L}_{\leqslant\varphi|(V\cap U_I^\circ)\times\mathrm{nb}(A^\circ)} \simeq \bigoplus_{\psi\in\Phi} \beta_{\psi\leqslant\varphi}(\mathrm{gr}_\psi \mathscr{L})_{|(V\cap U_I^\circ)\times\mathrm{nb}(A^\circ)}.$$

We are thus reduced to checking that $\widetilde{\jmath}_*$ commutes with $\beta_{\psi\leqslant\varphi}$ on local systems, that is, that the natural morphism $\beta_{\psi\leqslant\varphi} \to \widetilde{\jmath}_*\widetilde{\jmath}^{-1}\beta_{\psi\leqslant\varphi}$ is an isomorphism when applied to a local system, since $\mathrm{Id} \to \widetilde{\jmath}_*\widetilde{\jmath}^{-1}$ is an isomorphism when applied to a local system. The question is then local, and we can work in the neighbourhood of (x_o, θ_o), with the constant sheaf of rank one as the given local system.

If $(x_o, \theta_o) \notin \mathrm{St}(\varphi, \psi)_{x_o}$, the result is easy. We will thus focus on the case where $(x_o, \theta_o) \in \mathrm{St}(\varphi, \psi)_{x_o}$. We need to check that the germ at (x_o, θ_o) of $\widetilde{\jmath}_*\widetilde{\jmath}^{-1}\beta_{\psi\leqslant\varphi}\mathbb{C}$ is zero for any such (x_o, θ_o). This amounts to proving that the cohomology of the constant sheaf on the set

$$(A.5) \qquad (V \times \mathrm{nb}(\theta_o)) \cap \left\{\sum m_j\theta_j - \arg c(x) \in (\pi/2 - \varepsilon, \pi/2)\right\}$$

extended by zero on the set

$$(V \times \mathrm{nb}(\theta_o)) \cap \left\{\sum m_j\theta_j - \arg c(x) = \pi/2\right\}$$

is zero for $0 < \varepsilon \ll 1$ and V small enough (and similarly with $(-\pi/2, -\pi/2 + \varepsilon)$ and $-\pi/2$). For V small enough, we obtain the same result by considering the semi-closed interval $[\pi/2 - \varepsilon, \pi/2)$ and extending by zero at $\pi/2$. The set corresponding to (A.5) is a topological fibration above V, and the fibers are homeomorphic to a product of $\ell - 1$ open intervals and a closed interval, the sheaf being zero on the product of $\ell - 1$ open intervals and one boundary point of the closed interval, and constant elsewhere. Since the projection to V is proper, the base change formula shows that the pushforward to V of this sheaf is identically zero, as the cohomology with compact support of a semi-closed interval is zero, hence its global cohomology on

$$\text{(A.6)} \qquad (V \times \mathrm{nb}(\theta_o)) \cap \left\{ \sum m_j \theta_j - \arg c(x) \in [\pi/2 - \varepsilon, \pi/2] \right\}$$

is also zero.

Once the functor $\widetilde{\jmath}_*$ is defined, that it is essentially surjective is proven similarly, since in the neighbourhood of any point (x_o, θ_o) the sheaves $\mathscr{L}_{\leqslant \varphi}$ are given by a formula like (A.4).

The full faithfulness follows from the full faithfulness for the underlying local systems. $\qquad \square$

A.c. Openness. We keep the notation as above.

Proposition A.7. *Let $x_o \in D_I^\circ$ and let $(\mathscr{L}, \mathscr{L}_\bullet)_{x_o}$ be a Stokes-filtered local system on $\varpi^{-1}(x_o) \simeq (S^1)^\ell$ with associated $\mathfrak{I}$-covering contained in $\widetilde{\Sigma}^\circ_{I, x_o}$. Then there exists an open neighbourhood $\mathrm{nb}(x_o)$ in D_I° such that $(\mathscr{L}, \mathscr{L}_\bullet)_{x_o}$ extends in a unique way as a Stokes-filtered local system on $\varpi^{-1}(\mathrm{nb}(x_o)) \simeq \mathrm{nb}(x_o) \times (S^1)^\ell$ with associated $\mathfrak{I}$-covering contained in $\widetilde{\Sigma}^\circ_{I | \mathrm{nb}(x_o)}$. Any morphism $(\mathscr{L}, \mathscr{L}_\bullet)_{x_o} \to (\mathscr{L}', \mathscr{L}'_\bullet)_{x_o}$ between such objects also extends locally in a unique way.*

Proof. The problem is local on D_I° and, by the uniqueness of the extension of morphisms, one can reduce the proof to the non-ramified case. We can therefore assume that $\Sigma_I^\circ = \Phi \times \mathrm{nb}(x_o)$. Moreover, the unique extension of local systems and morphisms between them is clear, so the question reduces to checking that Stokes filtrations extend as well, and that the extended morphism between the extended local systems is compatible with the extended Stokes filtrations.

By Proposition A.2, we can cover $(S^1)^\ell = \varpi^{-1}(x_o)$ by simply connected open sets U_α such that, for every α, there exists a neighbourhood V_α of the compact subset $\overline{U}_\alpha$ and an isomorphism

$$\text{(A.8)} \qquad \mathscr{L}_{x_o | V_\alpha} \simeq \bigoplus_{\varphi \in \Phi} \mathrm{gr}_\varphi \mathscr{L}_{x_o | V_\alpha},$$

and the Stokes filtration on V_α is given by

$$\text{(A.9)} \qquad \mathscr{L}_{x_o, \leqslant \varphi | V_\alpha} \simeq \bigoplus_{\psi \in \Phi} \beta_{\psi \leqslant \varphi} \, \mathrm{gr}_\psi \mathscr{L}_{x_o | V_\alpha}.$$

The transition maps $\lambda_{\alpha\beta}$ on $V_{\alpha\beta} := V_\alpha \cap V_\beta$ satisfy the cocycle condition and are compatible with the Stokes filtration, that is, $\lambda_{\alpha\beta}^{\psi,\varphi} : \mathrm{gr}_\psi \mathscr{L}_{x_o|V_{\alpha\beta}} \to \mathrm{gr}_\varphi \mathscr{L}_{x_o|V_{\alpha\beta}}$ is zero unless $\psi \leqslant \varphi$ on $V_{\alpha\beta}$.

Let us shrink $\mathrm{nb}(x_o)$ to a contractible open neighbourhood such that, for all $\psi \neq \varphi \in \Phi$, $\psi < \varphi$ on $V_{\alpha\beta}$ implies $\psi < \varphi$ on $\mathrm{nb}(x_o) \times U_{\alpha\beta}$. The local system $\mathrm{gr}_\varphi \mathscr{L}_{x_o|U_\alpha}$ extends in a unique way to a local system $\mathrm{gr}_\varphi \mathscr{L}_{|\mathrm{nb}(x_o)\times U_\alpha}$ on $\mathrm{nb}(x_o) \times U_\alpha$, and so do the morphisms $\lambda_{\alpha\beta}^{\psi\varphi}$, which satisfy thus the cocycle condition. In particular, if such an extension $\lambda_{\alpha\beta}^{\psi\varphi}$ is non-zero at one point of $\mathrm{nb}(x_o) \times U_{\alpha\beta}$, it is nonzero everywhere on this open set and we have $\psi < \varphi$ on this open set. Let us set $\mathscr{L}_{|\mathrm{nb}(x_o)\times U_\alpha} := \bigoplus_{\varphi\in\Phi} \mathrm{gr}_\varphi \mathscr{L}_{|\mathrm{nb}(x_o)\times U_\alpha}$, that we equip with the Stokes filtration given by a formula similar to (A.9). It follows that $\lambda_{\alpha\beta}$ is compatible with the Stokes filtrations. We regard now $\lambda_{\alpha\beta}$ as gluing data. The cocycle condition shows that they define a local system $\mathscr{L}$ on $\varpi^{-1}(\mathrm{nb}(x_o))$ whose restriction to $\varpi^{-1}(x_o)$ is isomorphic to $\mathscr{L}$. It is thus uniquely isomorphic to the unique extension of $\mathscr{L}_{x_o}$. Moreover, due to the compatibility with the Stokes filtrations, the latter also glue correspondingly as a Stokes filtration $(\mathscr{L}, \mathscr{L}_\bullet)$ of this local system, and its restriction to $\varpi^{-1}(x_o)$ is equal to $(\mathscr{L}, \mathscr{L}_\bullet)_{x_o}$.

Let $\mu_{x_o} : (\mathscr{L}, \mathscr{L}_\bullet)_{x_o} \to (\mathscr{L}', \mathscr{L}'_\bullet)_{x_o}$ be a morphism. We can choose the covering (U_α) and the decomposition (A.8) so that each $\mu_{x_o,\alpha}$ is graded (see [**Sab13**, Prop. 9.21]). It extends uniquely as a morphism $\mu : \mathscr{L}_{|\mathrm{nb}(x_o)\times U_\alpha} \to \mathscr{L}'_{|\mathrm{nb}(x_o)\times U_\alpha}$, and it is graded with respect to the corresponding decomposition. It follows that μ is strictly compatible with the Stokes filtrations $\mathscr{L}_\bullet$ and $\mathscr{L}'_\bullet$, where these Stokes-filtered local systems $(\mathscr{L}, \mathscr{L}_\bullet)$ and $(\mathscr{L}', \mathscr{L}'_\bullet)$ are obtained as in the first part.

We can now prove the uniqueness (i.e., up to unique isomorphism) of $(\mathscr{L}, \mathscr{L}_\bullet)$ constructed in the first part: the identity automorphism $(\mathscr{L}, \mathscr{L}_\bullet)_{x_o}$ extends in a unique way as an isomorphism between two such extensions. $\square$

A.d. An equivalence of categories. We will use the notation as in Section 3. Let $\pi : (E_I^\circ(x_o), y_o) \to (D_I^\circ(x_o), x_o)$ be a universal covering of $D_I^\circ(x_o)$ with base point y_o above x_o, and let $\partial \widetilde{Y}_I^\circ(x_o)$ be the pre-image of $\partial \widetilde{X}_I^\circ(x_o)$ by π.

Proposition A.10. *The restriction functor*

- *from the category of Stokes-filtered local systems on $\partial \widetilde{Y}_I^\circ(x_o)$ with associated $\pi^{-1}\mathfrak{I}$-covering contained in $\pi^{-1}\widetilde{\Sigma}_I^\circ(x_o)$*
- *to the category of Stokes-filtered local systems on $\partial \widetilde{\Omega}$ with associated $\mathfrak{I}_{x_o}$-covering contained in $\widetilde{\Sigma}_{x_o}$*

is an equivalence.

Proof. Let $\Gamma : [0,1]^2 \to E_I^\circ(x_o)$ be a continuous map sending $(0,0)$ to y_o. We pullback by Γ the data from the first item of the proposition. Let us consider the subset of $[0,1]$ consisting of ε such that the equivalence of the proposition holds with respect to the

restriction corresponding to the inclusion $(0,0) \in [0,\varepsilon]^2$. Propositions A.3 and A.7 imply that this set is open and closed, and contains 0, hence it is equal to $[0,1]$. This shows that one can uniquely extend an object in the second category to an object in the first category along paths starting from y_o and that this extension does not depend on the choice of the path. A similar assertion holds for morphisms. $\square$

Remark A.11. The uniqueness of the extension of morphisms enables one to obtain the equivalence between the corresponding G-equivariant categories, and this gives the implication $(3) \Rightarrow (4)$ in the proof of Theorem 3.1.

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