ERRATA TO "POLARIZABLE TWISTOR $\mathscr{D}$-MODULES"

by

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(1) On page 21, line 5 and page 22, line 5, replace $\boldsymbol{L}^*_{z_o}$ with $\boldsymbol{L}i^*_{z_o}$.

(2) On page 30, Lemma 1.5.3 and in its proof, replace $\mathscr{C}^{\infty,\mathrm{an}}_{X|\mathbf{S}}$ with $\mathscr{C}^{\infty,0}_{\mathscr{X}|\mathbf{S}}$.

(3) On page 32, line 8, the isomorphism $\mathscr{T}^*(-k) \to \mathscr{T}(k)^*$ is not the morphism obtained by adjunction of (1.6.3), but the inverse morphism obtained from (1.6.3) where we replace $\mathscr{T}$ by $\mathscr{T}^*$. The choice of (1.6.3) is universal and holds for any $\mathscr{T}$.

(4) On page 33, 4th line of 1.6.b, replace $\mathscr{C}^{\infty,\mathrm{an}}_{X|\mathbf{S}}$ with $\mathscr{C}^{\infty,0}_{\mathscr{X}|\mathbf{S}}$.

(5) On page 48, the text of Remark 2.2.1 has to be replaced by the following text:

Remark 2.2.1. — We have seen that the sesquilinear pairing C takes values in $\mathscr{C}^{\infty,0}_{\mathscr{X}|\mathbf{S}}$, according to Lemma 1.5.3. So the restriction to x_o of each component of the smooth twistor structure is well defined. Then, according to (2.1.1), C takes values in $\mathscr{C}^{\infty,\mathrm{an}}_{X|\mathbf{S}}$. It is also nondegenerate and gives a gluing of $\mathscr{H}'^*$ with $\overline{\mathscr{H}''}$, defining thus a $\mathscr{C}^{\infty,\mathrm{an}}_{X\times\mathbb{P}^1}$-bundle $\widetilde{\mathscr{H}}$ on $X \times \mathbb{P}^1$.

(6) The statement of Lemma 3.6.33 (which is not used in the text) has to be replaced with

$$\big\langle \phi_{t,0}C([m_0'],\overline{[m_0'']}),\bullet\big\rangle = \operatorname{Res}_{s=0}\frac{-1}{s}\big\langle(|t|^{2s}-s)C(m_0',\overline{m_0''}),\bullet\wedge\chi(t)\tfrac{i}{2\pi}dt\wedge d\bar t\big\rangle.$$

Proof. — We write $m_0'' = \eth_t m''_{-1} + \mu''_{<0}$. By definition,

$$\begin{aligned}\big\langle \phi_{t,0}C([m_0'],\overline{[m_0'']}),\varphi\big\rangle &= \operatorname{Res}_{s=0}\big\langle C(m_0',\overline{\eth_t m''_{-1}}),\varphi\wedge I_{\widehat\chi}\chi\tfrac{i}{2\pi}dt\wedge d\bar t\big\rangle\\ &= \operatorname{Res}_{s=0}\big\langle C(m_0',\overline{m''_{-1}}),\varphi\wedge(\eth_t I_{\widehat\chi})\chi\tfrac{i}{2\pi}dt\wedge d\bar t\big\rangle\\ &= -z^{-1}\operatorname{Res}_{s=-1}\big\langle C(m_0',\overline{m''_{-1}}),\varphi\wedge t|t|^{2s}\chi(t)\tfrac{i}{2\pi}dt\wedge d\bar t\big\rangle,\end{aligned}$$

by (3.6.23). On the other hand,

$$\begin{aligned}
\operatorname{Res}_{s=0} \frac{-1}{s} & \big\langle |t|^{2s} C(m'_0, \overline{\eth_t m''_{-1}}), \varphi \wedge \chi(t) \tfrac{i}{2\pi} dt \wedge d\bar t \big\rangle \\
&= \operatorname{Res}_{s=-1} \frac{-1}{s+1} \big\langle C(m'_0, \overline{\eth_t m''_{-1}}), \varphi \wedge |t|^{2(s+1)} \chi(t) \tfrac{i}{2\pi} dt \wedge d\bar t \big\rangle \\
&= \operatorname{Res}_{s=-1} \frac{1}{s+1} \big\langle C(m'_0, \overline{m''_{-1}}), \varphi \wedge \eth_t (|t|^{2(s+1)} \chi(t)) \tfrac{i}{2\pi} dt \wedge d\bar t \big\rangle \\
&= -z^{-1} \operatorname{Res}_{s=-1} \big\langle C(m'_0, \overline{m''_{-1}}), \varphi \wedge t|t|^{2s} \chi(t) \tfrac{i}{2\pi} dt \wedge d\bar t \big\rangle \\
&\qquad + \operatorname{Res}_{s=-1} \frac{1}{s+1} \big\langle C(m'_0, \overline{m''_{-1}}), \varphi \wedge |t|^{2(s+1)} \eth_t \chi(t) \tfrac{i}{2\pi} dt \wedge d\bar t \big\rangle \\
&= -z^{-1} \operatorname{Res}_{s=-1} \big\langle C(m'_0, \overline{m''_{-1}}), \varphi \wedge t|t|^{2s} \chi(t) \tfrac{i}{2\pi} dt \wedge d\bar t \big\rangle \\
&\qquad - \big\langle C(m'_0, \overline{\eth_t m''_{-1}}), \varphi \wedge \chi \tfrac{i}{2\pi} dt \wedge d\bar t \big\rangle
\end{aligned}$$

and

$$\operatorname{Res}_{s=0} \frac{-1}{s} \big\langle |t|^{2s} C(m'_0, \overline{\mu''_{<0}}), \varphi \wedge \chi(t) \tfrac{i}{2\pi} dt \wedge d\bar t \big\rangle = -\big\langle C(m'_0, \overline{\mu''_{<0}}), \varphi \wedge \chi \tfrac{i}{2\pi} dt \wedge d\bar t \big\rangle. \qquad \square$$

(7) On page 119, in the statement of Corollary 4.2.9, replace $w+1$ with w.

(8) On page 121, the argument given on lines 10–14 is not correct, as the inverse image by the projection is not known to be a polarizable twistor $\mathscr{D}$-module. One can argue as follows.

Choose a finite morphism $\pi : Z \to Z'$ with Z' smooth and projective (a projective line, for instance) and consider the composed morphism $\nu \circ \pi : \widetilde{Z} \to Z'$. On $Z^o \subset \widetilde{Z}$, the object $(\mathscr{T}, \mathscr{S})$ defines a harmonic bundle (H, D''_E, θ_E, h) in the sense of C. Simpson [**1**], according to the correspondence of Lemma 2.2.2 on Z^o. We can restrict Z^o so that $\pi : Z^o \to Z'^o$ is a finite covering. We wish to show that the eigenvalues of the Higgs field are (multivalued) meromorphic one-forms, with a pole of order at most one at each puncture, and a purely imaginary residue at any such punctures. Indeed, this will imply that the harmonic bundle (H, D''_E, θ_E, h) on Z^o is tame on $\widetilde{Z}$, and that its parabolic filtration at the punctures is the trivial one, so, by [**1**], the corresponding local system is semisimple.

It is then enough to prove that such a property is satisfied for the direct image $\pi_*(H, D''_E, \theta_E, h)$ on Z'^o, as locally the covering is trivial (in a local coordinate t on $\widetilde{Z}$ and t' on Z' for which $\pi(t) = t' = t^q$, we have $dt'/t' = q\,dt/t$, and, if the eigenvalues of θ'_E are written as $\alpha(t)dt/t$, the eigenvalues of $\pi_*\theta'_E$ are of the form $\frac{1}{q}\alpha(\zeta t)\frac{dt'}{t'}$, with $\zeta^q = 1$; hence the condition on eigenvalues is satisfied for θ'_E if and only if it is satisfied for $\pi_*\theta'_E = \theta'_{\pi_*E}$).

Now, a particular case of Theorem 6.1.1 (the case when π is finite) implies that $\pi_+(\mathscr{T}, \mathscr{S})$ is an object of $\mathrm{MT}^{(\mathrm{r})}(Z, 0)^{(\mathrm{p})}$, and we apply the correspondence of Theorem 5.0.1.

(9) On page 135, the line after (5.3.5), read $\mathscr{O}_{\mathscr{X}}$ instead of $\mathscr{O}_{\mathscr{Q}}$.

(10) On page 156, line -1 and page 157, line 1, replace $n_j+\beta_j=-1$ by $n_j+\beta_j=0$, and $\ell_z(n_j+\beta_j)=-1$ by $\ell_z(n_j+\beta_j)=0$. This does not affect thre reasoning.

(11) On page 167, line 2: it is implicitly understood that $\omega_{\beta,\ell,k}$ is holomorphic even at $t=0$, although the previous reasoning only gives the holomorphy away from $t=0$. The argument that $(\mathcal{D}'_z\eta_{\neq(0,0)})_{\neq(0,0)}$ is L^2 has to be corrected. I thank T. Mochizuki for pointing out the mistake and providing the following proof.

(a) Let us set $\widetilde{\omega}_{\beta,\ell,k}=t\omega_{\beta,\ell,k}$, which is holomorphic on $D^*\times\mathrm{nb}(z_o)$. Assume first (see (b) below) we have proved that $\widetilde{\omega}_{\beta,\ell,k}e'^{(z_o)}_{\beta,\ell,k}$ is L^2 when we fix z in $\mathrm{nb}(z_o)$. Then, if we expand $\widetilde{\omega}_{\beta,\ell,k}=\sum_{n\in\mathbb{Z}}\widetilde{\omega}_{\beta,\ell,k,n}(z)t^n$, we claim that the coefficients $\widetilde{\omega}_{\beta,\ell,k,n}(z)$ identically vanish when $n\leqslant-1$. In order to prove this, we can argue with z fixed. The L^2 condition we assume is that, for any $n\in\mathbb{Z}$, $|\widetilde{\omega}_{\beta,\ell,k,n}(z)|\,r^{n+\ell_z(q_{\beta,\zeta_o}+\beta)}\mathrm{L}(r)^{\ell/2-1}\in L^2_{\mathrm{loc}}(d\theta\,dr/r)$. But when $n\leqslant-1$ and $a<1$ (as is $\ell_z(q_{\beta,\zeta_o}+\beta)$ for z near z_o), $r^{n+a}\mathrm{L}(r)^{k/2}$ does not belong to $L^2_{\mathrm{loc}}(d\theta,dr/r)$, hence the coefficients $\widetilde{\omega}_{\beta,\ell,k,n}(z)$ have to vanish when $n\leqslant-1$.

In order to conclude, we want to show that $\widetilde{\omega}_{\beta,\ell,k}e'^{(z_o)}_{\beta,\ell,k}dt/t$ is L^2_{loc}, while we have only assumed that $\widetilde{\omega}_{\beta,\ell,k}e'^{(z_o)}_{\beta,\ell,k}$ is so. If $\ell_{z_o}(q_{\beta,\zeta_o}+\beta)\neq0$, multiplying by $\mathrm{L}(r)$ will not cause an escape from the L^2 space, as the L^2 condition is governed by terms like $r^{n+\ell_z(q_{\beta,\zeta_o}+\beta)}$. If $\ell_{z_o}(q_{\beta,\zeta_o}+\beta)=0$, the previous argument is not valid if $n=0$. But we precisely considered the $\neq(0,0)$ parts, so the corresponding coefficient $\widetilde{\omega}_{\beta,\ell,k,0}(z)$ is identically 0 by definition.

(b) Let us now fix $z\in\mathrm{nb}(z_o)$, that we still denote by z_o for simplicity. The operator $D_E+z_o\theta''_E-\overline{z_o}\theta'_E=\mathcal{D}''_{z_o}+\delta'_{z_o}$ is compatible the harmonic metric h on H by definition, and we have $\mathcal{D}'_{z_o}=z_o\delta'_{z_o}+(1+|z_o|^2)\theta'_E$. If we know (*cf.* (c) below) that $(\mathcal{D}''_{z_o}+\delta'_{z_o})(\eta_{\neq(0,0)})$ is a section of $\mathscr{L}^1_{(2)}(H,h)$ then, by the definition of η, the same property holds for $\delta'_{z_o}(\eta_{\neq(0,0)})$. On the other hand, by the expression of Θ'_{z_o} given before (6.2.7), $\mathrm{L}(t)^{-1}\theta'_{z_o}(\eta_{\neq(0,0)})$ is also in $\mathscr{L}^1_{(2)}(H,h)$ (the term $\mathrm{L}(t)^{-1}$ is here to compensate the norm of dt/t). Therefore, we find that $\mathrm{L}(t)^{-1}\mathcal{D}'_{z_o}(\eta_{\neq(0,0)})$ is in $\mathscr{L}^1_{(2)}(H,h)$ and finally, by definition of ω, that $\mathrm{L}(t)^{-1}\omega$ is in $\mathscr{L}^1_{(2)}(H,h)$, so the assumption in (a) above is fulfilled.

(c) As $D_{z_o}\overset{\mathrm{def}}{=}\mathcal{D}''_{z_o}+\delta'_{z_o}$ is compatible with h, we have, for a C^∞_c section e of H on D^*:

$$0=d^2h(e,\overline{e})=2\|D_{z_o}e\|^2_h+h(R_{z_o}e,\overline{e})+h(e,\overline{R_{z_o}e}),$$

where R_{z_o} denotes the curvature operator of D_{z_o}, and where the (fiberwise) norm of $D_{z_o}e$ is computed with the metric h and the Poincaré metric (for the 1-form components). Arguing as in [**1**, page 737], we find the the L^2 norm of the operator R_{z_o} with respect to the metric h and the Poincaré metric is bounded by a constant. It follows that $\|D_{z_o}e\|_h\leqslant C\|e\|_h$ and therefore, if e moreover is a local section of $\mathscr{L}^0_{(2)}(H,h)$, then $D_{z_o}e$ is a local section of $\mathscr{L}^1_{(2)}(H,h)$. By density, we conclude that this holds for any local section of $\mathscr{L}^0_{(2)}(H,h)$. We apply this to $\eta_{\neq(0,0)}$ to get (b).

References

[1] C. Simpson – Harmonic bundles on noncompact curves, *J. Amer. Math. Soc.* **3** (1990), p. 713–770.

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