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FOURIER–LAPLACE TRANSFORM OF IRREDUCIBLE REGULAR DIFFERENTIAL SYSTEMS ON THE RIEMANN SPHERE, II

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ABSTRACT. This article is devoted to the complete proof of the main theorem in the author's paper of 2004 showing that the Fourier–Laplace transform of an irreducible regular differential system on the Riemann sphere underlies, at finite distance, a polarizable regular twistor $\mathcal{D}$-module.

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Let $P' = \{p_1, \ldots, p_r\}$ be a finite set of points in the complex affine line $\mathbb{A}^1$ and set $P = P' \cup \{\infty\}$, which is a subset of the Riemann sphere $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$. Let (V, ∇) be a holomorphic bundle on $\mathbb{P}^1 \setminus P$ with a holomorphic connection, and let (H, D_V) be the associated C^∞ bundle with flat connection. If we assume that (V, ∇) is *irreducible* (or more generally semisimple), then, by using results of Simpson [10], together with [8] (cf. also [4]), one can associate to (V, ∇) a *polarized regular twistor* $\mathcal{D}$*-module* of weight 0 on $\mathbb{P}^1$ that we denote $(\mathcal{M}, \mathcal{M}, C, \mathrm{Id})$.

Let us quickly recall the notation used in [7], after [8]. We denote by Ω_0 the complex line with coordinate z (or an open neighbourhood of the closed disc $|z| \leqslant 1$, whose boundary $|z| = 1$ is denoted by $\boldsymbol{S}$). If X is a complex manifold (here, $X = \mathbb{P}^1$ or X is a disc), then $\mathcal{X} = X \times \Omega_0$ (e.g., $\mathcal{P}^1 = \mathbb{P}^1 \times \Omega_0$) and $\pi \colon \mathcal{X} \to X$ denotes the projection. We will denote by $p \colon \mathcal{X} \to \Omega_0$ the other projection. Denoting by x a local holomorphic coordinate on X, we consider the ring $\mathcal{R}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}}\langle z\partial_x\rangle$ of holomorphic differential operators, and we denote $\eth_x = z\partial_x$. Recall that $\mathcal{M}$ is a left module over $\mathcal{R}_{\mathcal{X}}$ and C is a pairing on $\mathcal{M}|_{X\times\boldsymbol{S}}$ taking values in the sheaf of distributions on $X \times \boldsymbol{S}$ which are continuous with respect to $z \in \boldsymbol{S}$.

The Laplace transform of a holonomic $\mathcal{D}_{\mathbb{A}^1}$-module is a holonomic $\mathcal{D}_{\widehat{\mathbb{A}}^1_{\mathrm{an}}}$-module. A similar transform, called Fourier–Laplace, is defined for objects $(\mathcal{M}, \mathcal{M}, C, \mathrm{Id})$. The main result of [7] states:

Theorem 1 (Th. 1 in [7])**.** *If* $(\mathcal{M}, \mathcal{M}, C, \mathrm{Id})$ *is polarized regular twistor* $\mathcal{D}$*-module of weight* 0 *on* X*, its Fourier–Laplace transform on the complex line* $\widehat{\mathbb{A}}^1_{\mathrm{an}}$ *is of the same kind.*

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In this article, we give a complete proof of this theorem, correcting the following two errors in the proof given in [7]:

(i) In Section 3.1 of [7], point 5, we assert:

> "By simple homogeneity considerations with respect to τ, it suffices to prove the property in the neighbourhood of $\tau = 0$."

It happens that homogeneity does not lead to such a statement. One has to prove the twistor property for the pairing $\widehat{C}$ at any $\tau^o \neq 0$. While the proof given in [7] holds for $|\tau^o|$ small enough by an argument of degeneration, we will give in Section 2 a proof for any fixed τ^o and it is enough for such a proof to give the argument when $\tau^o = 1$.

(ii) In the proof of Lemma 4 of [7] (a main tool for [7, Prop. 1]), the computation of H^1 cannot follow the same lines as Lemma 6.2.13 in [8]. We will instead use the argument indicated in [7, Rem. 1].

The argument for (ii) is given in Section 1 and that for (i) in Section 2. We implicitly refer to [7] for the notation and the objects considered here.

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1. Proof of Proposition 1 in [7]

The corrected proof of [7, Prop. 1] follows the same lines as in [7, Section 3.4] once we have proved the lemma below. Nevertheless, instead of using the isometry [7, (2.5)] as in the original text, we will use (2.4) of loc. cit. In order to simplify the notation, we will set in the following $\widetilde{\mathfrak{D}}_z = \mathfrak{D}_z - dt$ (this is a small change with respect to the notation of [7]) and $h_z = e^{2\operatorname{Re} z\bar{t}}\pi^* h$, in particular, $h_{z_o} = e^{2\operatorname{Re} z_o\bar{t}}h$.

We set $\widetilde{\mathcal{M}} = \mathcal{O}_{\mathcal{P}^1}(*\infty) \otimes_{\mathcal{O}_X} \mathcal{M}$ and ${}^F\mathcal{M} = \widetilde{\mathcal{M}} \otimes \mathcal{E}^{-t/z}$ (i.e., we twist the z-connection on $\widetilde{\mathcal{M}}$ by adding the term $-dt = dt'/t'^2$, where $t' = 1/t$ is the local coordinate at ∞). We will also set ${}^F\mathfrak{M}_{z_o} = {}^F\mathcal{M}/(z - z_o){}^F\mathcal{M}$.

Lemma 1. *For any $z_o \in \Omega_0$, there is an isomorphism in the derived category $D^b(\mathbb{C}_{\mathbb{P}^1})$:*

$$\mathrm{DR}({}^F\mathfrak{M}_{z_o}) \simeq \mathcal{L}^{1+\bullet}_{(2)}(H, h_{z_o}, \widetilde{\mathfrak{D}}_{z_o}).$$

We also set $\mathrm{L}(t) = \big|\log|t|^2\big|$.

Proof of Lemma 1. We will distinguish whether $z_o = 0$ or not. When $z_o = 0$ we continue using [7, Lemma 4] (Dolbeault Lemma) as it stands (with the supplementary assumption that $z_o = 0$), corrected as in Section 1.1 below. It says that the natural inclusion $\mathrm{DR}({}^F\mathfrak{M}_0)_{(2)} \hookrightarrow \mathcal{L}^{1+\bullet}_{(2)}(H, h, \widetilde{\mathfrak{D}}_0)$ is a quasi-isomorphism. We then use [7, Lemma 3] to conclude.

When $z_o \neq 0$, we will change the argument, and use that indicated in [7, Rem. 1]. This will be done in Section 1.2 below. $\square$

1.1. The Dolbeault lemma. We correct the statement given on page 1178, line 1 of [7] when $z_o = 0$. Let $\omega = \psi \dfrac{dt'}{t'} + \varphi \dfrac{d\overline{t'}}{\overline{t'}}$ be in $\ker \widetilde{\mathfrak{D}}_0^{(1)}$. We wish to prove that, modulo the image by $\widetilde{\mathfrak{D}}_0$ of $\mathfrak{L}^0_{(2)}(H,\, h,\, \widetilde{\mathfrak{D}}_0)$, we can reduce φ to be as written in page 1178, line 3 of [7]. Expanding ω on the basis $(e'^{(0)}_{\beta,\ell,k})$, the L^2 condition reads: $|t'|^{\beta'}\,\mathrm{L}(t')^{\ell/2}\,|\psi_{\beta,\ell,k}|$ and $|t'|^{\beta'}\,\mathrm{L}(t')^{\ell/2}\,|\varphi_{\beta,\ell,k}|$ belong to $L^2(d\theta\, dr/r)$. Then $\widetilde{\mathfrak{D}}_0^{(1)}\omega = 0$ reads, setting $\beta = \beta' + i\beta''$ and $\alpha(t') = 1/(1 + i\beta'' t'/2)$,

$$-\overline{t'}\partial_{\overline{t'}}\psi_{\beta,\ell,k} + \frac{1}{\alpha(t')t'}\varphi_{\beta,\ell,k} + \xi_{\beta,\ell,k} = 0,$$

with $\xi = \sum \xi_{\beta,\ell,k} e'^{(0)}_{\beta,\ell,k}$ defined by $\xi \dfrac{dt'}{t'} \wedge \dfrac{d\overline{t'}}{\overline{t'}} = \Theta'_{0,\mathrm{nilp}}\varphi$. It follows that

$$\varphi_{\beta,\ell,k} = \overline{t'}\partial_{\overline{t'}}\big(\alpha(t')t'\psi_{\beta,\ell,k}\big) - \alpha(t')t'\xi_{\beta,\ell,k}.$$

Firstly, $|t'|^{\beta'}\,\mathrm{L}(t')^{-1+\ell/2}\,|t'\psi_{\beta,\ell,k}|$ clearly belongs to $L^2(d\theta\, dr/r)$, hence on the one hand, $t'\psi_{\beta,\ell,k}e'^{(0)}_{\beta,\ell,k}$ is a section of $\mathfrak{L}^0_{(2)}(H,\, h)$. On the other hand,

$$\Theta'_0\big(\alpha(t')t'\psi_{\beta,\ell,k}e'^{(0)}_{\beta,\ell,k}\big) = \psi_{\beta,\ell,k}e'^{(0)}_{\beta,\ell,k}\frac{dt'}{t'} + \Theta'_{0,\mathrm{nilp}}\big(\alpha(t')t'\psi_{\beta,\ell,k}e'^{(0)}_{\beta,\ell,k}\big).$$

As we know that $\Theta'_{0,\mathrm{nilp}}$ is bounded with respect to the L^2 norms, it follows that the left-hand term is a section of $\mathfrak{L}^{(1,0)}_{(2)}(H,\, h)$. We conclude that $\widetilde{\mathfrak{D}}_0(\alpha(t')t'\psi)$ is L^2 and that the $(0,\, 1)$-part of $\omega - \widetilde{\mathfrak{D}}_0(\alpha(t')t'\psi)$ is equal to $-\alpha(t')t'\xi\, d\overline{t'}/\overline{t'}$.

Secondly, by the property of $\Theta'_{0,\mathrm{nilp}}$, we find that $|t'|^{\beta'}\,\mathrm{L}(t')^{1+\ell/2}\,|\xi_{\beta,\ell,k}|$ also belongs to $L^2(d\theta\, dr/r)$. Let us now argue as in [8, Lemma 6.2.11]. We expand $\xi_{\beta,\ell,k}$ as a Fourier series $\sum_n \xi_{\beta,\ell,k,n}(r)e^{in\theta}$ with $r = |t'|$, and set $\xi_{\beta,\ell,k,\neq 0} = \xi_{\beta,\ell,k} - \xi_{\beta,\ell,k,0}$. We then find that it is possible to solve $\overline{t'}\partial_{\overline{t'}}\eta_{\beta,\ell,k,\neq 0} = \xi_{\beta,\ell,k,\neq 0}$ with $\eta_{\beta,\ell,k,\neq 0}$ being a local section of $\mathfrak{L}^0_{(2)}(H,\, h)$. As above, we then show that $\Theta'_0\big(\alpha(t')t'\eta_{\beta,\ell,k,\neq 0}e'^{(0)}_{\beta,\ell,k}\big)$ is a section of $\mathfrak{L}^{(1,0)}_{(2)}(H,\, h)$.

We finally conclude that $\omega - \widetilde{\mathfrak{D}}_0[\alpha(t')t'(\psi - \eta_{\neq 0})]$ satisfies the desired property.

1.2. The Poincaré lemma. We will now give the proof of Lemma 1 when $z_o \neq 0$, a condition that we assume to hold for the remaining of this subsection.

Reduction of the proof of Lemma 1 to local statements when $z_o \neq 0$. We will first work with the metric h (and not h_{z_o}). We denote by ${}^F\mathfrak{M}_{z_o,\mathrm{loc}}$ the localization of ${}^F\mathfrak{M}_{z_o}$ at the singularities P (note that, at infinity, ${}^F\mathfrak{M}_{z_o}$ is already equal to its localized module) and by $\mathrm{DR}({}^F\mathfrak{M}_{z_o,\mathrm{loc}})_{(2),h}$ the meromorphic L^2 de Rham complex, which is a subcomplex of $\mathrm{DR}\,{}^F\mathfrak{M}_{z_o,\mathrm{loc}}$. In fact, it is a subcomplex of $\mathrm{DR}\,{}^F\mathfrak{M}_{z_o}$: at finite distance, this is [8, Prop. 6.2.4] and at infinity this is clear. The argument of [7, Lemma 3] gives:

Lemma 2. *The inclusion of complexes $\mathrm{DR}({}^F\mathfrak{M}_{z_o,\mathrm{loc}})_{(2),h} \hookrightarrow \mathrm{DR}\,{}^F\mathfrak{M}_{z_o}$ is a quasi-isomorphism.* ◻

On the other hand, by definition, $\mathrm{DR}({}^F\mathfrak{M}_{z_o,\mathrm{loc}})_{(2),h}$ is a sub-complex of the L^2 complex $\mathcal{L}^{1+\bullet}_{(2)}(H,\,h,\,\widetilde{\mathfrak{D}}_{z_o})$ and, according to [8, Th. 6.2.5], the inclusion is a quasi-isomorphism at finite distance. Lemma 1 now follows from the following two statements:

The natural inclusion $\mathrm{DR}({}^F\mathfrak{M}_{z_o,\mathrm{loc}})_{(2),h} \hookrightarrow \mathcal{L}^{1+\bullet}_{(2)}(H,\,h,\,\widetilde{\mathfrak{D}}_{z_o})$ is a quasi-isomorphism near ∞. (1.1)

Both inclusions of complexes

$$\mathcal{L}^{1+\bullet}_{(2)}(H,\,h,\,\widetilde{\mathfrak{D}}_{z_o}) \longleftarrow \mathcal{L}^{1+\bullet}_{(2)}(H,\,h+h_{z_o},\,\widetilde{\mathfrak{D}}_{z_o}) \longrightarrow \mathcal{L}^{1+\bullet}_{(2)}(H,\,h_{z_o},\,\widetilde{\mathfrak{D}}_{z_o}).$$

are quasi-isomorphisms. (1.2)

Both questions are now local near ∞, and we will restrict to an open disc at ∞. So, we set $t' = 1/t$ and we denote by X the open disc centered at 0 and of radius $r_0 < 1$ in $\mathbb{C}$, with coordinate t', and we set $X^* = X \setminus \{0\}$. We still keep the notation h_{z_o} for the metric $e^{2\operatorname{Re} z_o/\overline{t'}}h$. We will work with polar coordinates with respect to t'.

The setting. We consider the real blow-up

$$\rho\colon \widetilde{X} := [0,\,r_0) \times S^1 \longrightarrow X, \qquad (r,\,\theta) \longmapsto t' = re^{i\theta}.$$

We will use the sheaf $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}}$ on $\widetilde{X}$, consisting of holomorphic functions on $\widetilde{X}^* = X^*$ which have moderate growth along $r=0$. It is known that $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}}$ is stable by $\eth_{t'}$. We also consider the differential 1-forms on $\widetilde{X}$:

$$\begin{aligned}\omega_r &= \frac{(z_o+1)}{2}\,\frac{dr}{r} + i\frac{(z_o-1)}{2}\,d\theta\\ \omega_\theta &= -i\frac{(z_o-1)}{2}\,\frac{dr}{r} + \frac{(z_o+1)}{2}\,d\theta,\end{aligned}$$

which form a basis of 1-forms and which satisfy

$$\frac{dr}{r} - i\,d\theta = \omega_r - i\omega_\theta, \qquad \frac{dr}{r} + i\,d\theta = \frac{1}{z_o}\,(\omega_r + i\omega_\theta).$$

Let us denote by d the differential. The decomposition $d = d' + d''$ on X can be lifted to $\widetilde{X}$ and, for a C^∞ function $\varphi(r,\,\theta)$ on $\widetilde{X}$, we have

$$(d'' + z_o d')\varphi = r\partial_r(\varphi)\omega_r + \partial_\theta(\varphi)\omega_\theta.$$

Similarly, for a 1-form $\eta = \varphi\omega_r + \psi\omega_\theta$, we have

$$(d'' + z_o d')\eta = \big(r\partial_r(\psi) - \partial_\theta(\varphi)\big)\omega_r \wedge \omega_\theta.$$

The L^2 complexes. Recall that, in this local setting, we denote by $\widetilde{\mathfrak{D}}_{z_o}$ the connection $\mathfrak{D}_{z_o} + dt'/t'^2$. We are interested in computing the cohomology of the complex $\mathcal{L}^{1+\bullet}_{(2)}(H,\,\boldsymbol{h},\,\widetilde{\mathfrak{D}}_{z_o})$, where $\boldsymbol{h}$ denotes one of the metrics h, h_{z_o} or $h + h_{z_o}$, which is defined exactly like in [8, Section 6.2.b].

We can similarly define the corresponding L^2 complex $\widetilde{\mathcal{L}}^{1+\bullet}_{(2)}$ by working on $\widetilde{X}$. Let us notice that the use of polar coordinates is convenient to express the L^2 condition.

The local basis $\boldsymbol{e}'^{(z_o)} := (e'^{(z_o)}_{\beta,\ell,k})$ which was introduced in [8] for the bundle $(H, h, \mathfrak{D}_{z_o})$ remains holomorphic with respect to $\widetilde{\mathfrak{D}}''_{z_o}$, and also L^2-adapted for the metric $\boldsymbol{h}$ (in loc. cit., we used the notation $\boldsymbol{e}'^{(z_o)}$ for a frame defined when z varies; in this paragraph, we reduce it modulo $z - z_o$ but keep the same notation).

Let us recall the L^2 condition. We denote by $\widetilde{\jmath}$ the inclusion $X^* \hookrightarrow \widetilde{X}$. Then $\widetilde{\mathfrak{L}}^0_{(2)}(H, \boldsymbol{h})$ is the subsheaf of $\widetilde{\jmath}_* L^1_{\mathrm{loc}}(H)$ consisting of sections which are holomorphic with respect to z_o and L^2 with respect to the metric $\boldsymbol{h}$ on each compact set of the open set on which they are defined.

Given a local section u of $\widetilde{\jmath}_* L^1_{\mathrm{loc}}(H)$ on $\widetilde{X}$, written as $\sum u_{\beta,\ell,k}(r, \theta) e'^{(z_o)}_{\beta,\ell,k}$, it is a local section of $\widetilde{\mathfrak{L}}^0_{(2)}(H, \boldsymbol{h})$ if and only if

$$\big[(r, \theta) \mapsto u_{\beta,\ell,k}(r, \theta) \cdot r^{\ell_{z_o}(q_{\beta,\zeta_o}+\beta)}\, \mathrm{L}(r)^{\ell/2-1} e_{\boldsymbol{h}}\big] \in L^2(d\theta\, dr/r), \tag{1.3}$$

with $e_{\boldsymbol{h}} = 1,\ e^{\mathrm{Re}(z_o/\overline{t'})},\ 1 + e^{\mathrm{Re}(z_o/\overline{t'})}$ if $\boldsymbol{h} = h,\ h_{z_o},\ h + h_{z_o}$ (cf. [8, p. 135] for the notation).

We define similarly $\widetilde{\mathfrak{L}}^1_{(2)}(H, \boldsymbol{h})$ and $\widetilde{\mathfrak{L}}^2_{(2)}(H, \boldsymbol{h})$ by asking moreover that ω_r, ω_θ have norm $\mathrm{L}(r)$ and $\omega_r \wedge \omega_\theta$ has norm $\mathrm{L}(r)^2$ (up to some constant depending on z_o). Therefore, a local section v of $\widetilde{\mathfrak{L}}^1_{(2)}(H, \boldsymbol{h})$ has coefficients $v^{(r)}_{\beta,\ell,k}$ and $v^{(\theta)}_{\beta,\ell,k}$ on $e'^{(z_o)}_{\beta,\ell,k}\omega_r$ and $e'^{(z_o)}_{\beta,\ell,k}\omega_\theta$ respectively, which satisfy (1.3) with $\mathrm{L}(r)^{\ell/2}$ instead of $\mathrm{L}(r)^{\ell/2-1}$. Similarly, a local section w of $\widetilde{\mathfrak{L}}^2_{(2)}(H, \boldsymbol{h})$ has coefficients $w_{\beta,\ell,k}$ on $e'^{(z_o)}_{\beta,\ell,k}\omega_r \wedge \omega_\theta$ which satisfy (1.3) with $\mathrm{L}(r)^{\ell/2+1}$ instead of $\mathrm{L}(r)^{\ell/2-1}$.

Lemma 3 ($\widetilde{L}^2$ Poincaré Lemma)**.** *The complexes $\widetilde{\mathfrak{L}}^{1+\bullet}_{(2)}(H, \boldsymbol{h}, \widetilde{\mathfrak{D}}_{z_o})$ ($\boldsymbol{h} = h,\ h_{z_o}$ or $h + h_{z_o}$) have cohomology in degree -1 at most.*

Keeping the notation of [8, (5.3.7)], the matrix $\widetilde{\Theta}'_{z_o}$ of $\widetilde{\mathfrak{D}}_{z_o}$ in the basis $\boldsymbol{e}'^{(z_o)}$ can be decomposed as

$$\widetilde{\Theta}'_{z_o} = \widetilde{\Theta}'_{z_o,\mathrm{diag}} + \Theta'_{z_o,\mathrm{nilp}} + \Theta'_{z_o,\mathrm{pert}},$$
$$\widetilde{\Theta}'_{z_o,\mathrm{diag}} = \bigoplus_\beta \big[(q_{\beta,\zeta_o} + \beta) \star z_o + 1/t'\big]\,\mathrm{Id}\,\frac{dt'}{t'}$$

with

$$\Theta'_{z_o,\mathrm{nilp}} = \big[\mathrm{Y} + P(0, z_o)\big]\frac{dt'}{t'}, \quad \Theta'_{z_o,\mathrm{pert}} = \big[P(t', z_o) - P(0, z_o)\big]\frac{dt'}{t'},$$

and with $\mathrm{Y} = (\bigoplus_\beta \mathrm{Y}_\beta)$ (cf. [8, Proof of Theorem 6.2.5]). We set $\mathrm{N}_{z_o} = \mathrm{Y} + P(0, z_o)$.

Using [8, Formula (6.2.7)], we see as in loc. cit. that the L^2 condition on derivatives under $\widetilde{\mathfrak{D}}_{z_o}$ can be replaced with the L^2 condition on derivatives under $\widetilde{\mathfrak{D}}_{z_o,\mathrm{diag}}$ (having matrix $\widetilde{\Theta}'_{z_o,\mathrm{diag}}$): indeed, $\Theta'_{z_o,\mathrm{nilp}} + \Theta'_{z_o,\mathrm{pert}}$ sends L^2 sections to L^2 sections, when using the metric $\boldsymbol{h}$.

Let $\theta_o \in S^1$, $r_1 \in (0, r_0)$ and let $U = (0, r_1) \times (\theta_o - \varepsilon, \theta_o + \varepsilon)$ be an open sector in X^* with $\varepsilon > 0$ small enough so that $[\theta_o - \varepsilon, \theta_o + \varepsilon]$ contains at most one zero of $\cos(\theta + \arg z_o) \cdot \sin(\theta + \arg z_o)$ and this zero belongs to the interior of the interval. We denote by $\overline{U}$ its (compact) closure.

If $\mathcal{E}^k$ denotes the sheaf of C^∞ k-forms,

$$\Gamma(\overline{U},\, \mathcal{L}^k_{(2)}(H,\, \boldsymbol{h},\, \widetilde{\mathfrak{D}}_{z_o})) = L^2(\overline{U},\, \mathcal{E}^k_{\overline{U}} \otimes H,\, \boldsymbol{h},\, \widetilde{\mathfrak{D}}_{z_o,\mathrm{diag}})$$

and the right-hand term is a Hilbert space, the norm being given by $\|\cdot\|_{2,\boldsymbol{h}} + \|\widetilde{\mathfrak{D}}_{z_o,\mathrm{diag}}\cdot\|_{2,\boldsymbol{h}}$.

The proof will decompose in 3 steps:

- We first prove the lemma for the L^2 complex
$$\big(L^2(\overline{U},\, \mathcal{E}^{1+\bullet}_{\overline{U}} \otimes H,\, \boldsymbol{h},\, \widetilde{\mathfrak{D}}_{z_o,\mathrm{diag}}),\, \widetilde{\mathfrak{D}}_{z_o,\mathrm{diag}}\big),$$
- without changing the terms of the complex, we change the differential to $\widetilde{\mathfrak{D}}_{z_o,\mathrm{diag}} + \Theta'_{z_o,\mathrm{nilp}}$ and prove the lemma by an extension argument,
- last, we change the differential to $\widetilde{\mathfrak{D}}_{z_o,\mathrm{diag}} + \Theta'_{z_o,\mathrm{nilp}} + \Theta'_{z_o,\mathrm{pert}}$, that we regard as a small perturbation of the previous one.

Proof of Lemma 3, first step. It is permissible to rescale the basis $\boldsymbol{e}'^{(z_o)}$, which therefore remains L^2-adapted (cf. [8, Section 6.2.b]), by multiplying each term $e'^{(z_o)}_{\beta,\ell,k}$ by the function $e^{1/z_o t'} t'^{-(q_{\beta,\varsigma_o}+\beta)\star z_o/z_o}$ to obtain a basis $\widetilde{\boldsymbol{e}'^{(z_o)}}$, which is $\widetilde{\mathfrak{D}}_{z_o,\mathrm{diag}}$-flat. On the other hand, the h-norm of $e'^{(z_o)}_{\beta,\ell,k}$ is equivalent, when $t' \to 0$, to $|t'|^{\ell_{z_o}(q_{\beta,\varsigma_o}+\beta)}\,\mathrm{L}(t')^{\ell/2}$ up to a multiplicative constant depending on z_o (cf. [8, Formula (5.3.6)]).

Therefore, the h_{z_o}-norm of $\widetilde{e'^{(z_o)}}_{\beta,\ell,k}$ is equivalent (up to a constant) to

$$e^{\frac{1+|z_o|^2}{|z_o|r}\cos(\theta+\arg z_o)} \cdot r^{-\frac{\beta''}{2}(|z_o|+1/|z_o|)\sin\arg z_o} \cdot \mathrm{L}(r)^{\ell/2-1}.$$

On the other hand, the h-norm is given by the same formula, where we replace $e^{\frac{1+|z_o|^2}{|z_o|r}\cos(\theta+\arg z_o)}$ with $e^{\frac{1}{|z_o|r}\cos(\theta+\arg z_o)}$.

The proof of the vanishing of the higher cohomology sheaves in all three cases is then completely similar to that of [5, Lemma 4.1]. □

Proof of Lemma 3, second step. Consider the monodromy filtration of N_{z_o} and apply the first step to each graded piece. Use then an easy extension argument. □

Proof of Lemma 3, third step. We then apply a standard perturbation argument to the complex of Hilbert spaces considered in the second step, as the L^2-norm of $\Theta'_{z_o,\mathrm{pert}}$ can be made small if r_1 is small (see, e.g., [4, Lemma 2.68, p. 53]). □

The complex $\widetilde{\mathrm{DR}}({}^F\mathfrak{M}_{z_o})$. We extend the coefficients of ${}^F\mathfrak{M}_{z_o}$ to $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}}$ and we consider the corresponding de Rham complex, that we denote by $\widetilde{\mathrm{DR}}({}^F\mathfrak{M}_{z_o})$. This is a complex on $\widetilde{X}$. Let us note that, as $\boldsymbol{R}\rho_*\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}} = \mathcal{O}_X[t'^{-1}]$ and as ${}^F\mathfrak{M}_{z_o}$ is $\mathcal{O}_X[t'^{-1}]$-flat (being locally free as such), we have $\boldsymbol{R}\rho_*\,\widetilde{\mathrm{DR}}({}^F\mathfrak{M}_{z_o}) = \mathrm{DR}\,{}^F\mathfrak{M}_{z_o}$.

Lemma 4 ($\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}}$-Poincaré lemma)**.** *The complex* $\widetilde{\mathrm{DR}}({}^F\mathfrak{M}_{z_o})$ *has cohomology in degree* -1 *at most.*

Proof. This is a particular case of a general result on irregular meromorphic connections, see, e.g., [3, App. 1]. □

Lemma 5 (Comparison)**.** *The sheaves* $\mathcal{H}^{-1}\widetilde{\mathrm{DR}}({}^F\mathfrak{M}_{z_o})$ *and* $\mathcal{H}^{-1}\widetilde{\mathcal{L}}^{1+\bullet}_{(2)}(H,\boldsymbol{h},\widetilde{\mathfrak{D}}_{z_o})$ $(\boldsymbol{h}=h,\,h_{z_o},$ *or* $h+h_{z_o})$ *coincide as subsheaves of* $\widetilde{\jmath}_*j^{-1}\mathcal{H}^{-1}\mathrm{DR}({}^F\mathfrak{M}_{z_o})$.

Proof. A $\widetilde{\mathfrak{D}}_{z_o}$-flat local section u of H takes the form $e^{1/z_ot'}v$, where v is a $\mathfrak{D}_{z_o}$-flat local section of H. Using for instance (5.3.6) and Remark 5.3.8(4) in [8], one knows that the h-norm of v grows exactly like $|t'|^b\,\mathrm{L}(t')^\nu$ for some $b\in\mathbb{R}$ and some $\nu\in\frac{1}{2}\mathbb{Z}$ when $t'\to 0$. As the h-norm of u is equal to $e^{\frac{1}{|z_o|r}\cos(\theta+\arg z_o)}\|v\|_h$, this norm is L^2 near (θ_o,z_o) if and only if $\cos(\theta_o+\arg z_o)<0$. The germ of $\mathcal{H}^{-1}\widetilde{\mathcal{L}}^{1+\bullet}_{(2)}(H,h,\widetilde{\mathfrak{D}}_{z_o})$ at θ_o is therefore 0 if $\cos(\theta_0+\arg z_o)\geqslant 0$, and consists of all flat local sections if $\cos(\theta_0+\arg z_o)<0$.

Considering the metric h_{z_o} instead of h will only replace $e^{\frac{1}{|z_o|r}\cos(\theta+\arg z_o)}$ with $e^{\frac{(1+|z_o|^2)}{|z_o|r}\cos(\theta+\arg z_o)}$, so the argument is the same. The argument for $h+h_{z_o}$ is also the same.

A similar argument shows that a $\widetilde{\mathfrak{D}}_{z_o}$-flat section has coefficients with moderate growth in the basis $\boldsymbol{e}'^{(z_o)}$ if and only if $\cos(\theta_o+\arg z_o)<0$ and, in such a case, any flat local section is a section of $\mathcal{H}^{-1}\widetilde{\mathrm{DR}}({}^F\mathfrak{M}_{z_o})$. □

Proof of (1.2). The assertion follows from Lemmas 3 and 5 by taking $\boldsymbol{R}\rho_*$. Let us note indeed that the complexes $\widetilde{\mathcal{L}}^{1+\bullet}_{(2)}$ are c-soft and that $\boldsymbol{R}\rho_*\widetilde{\mathcal{L}}^{1+\bullet}_{(2)}=\mathcal{L}^{1+\bullet}_{(2)}$. □

Proof of (1.1). In order to prove (1.1), we have to compare the complexes $\mathrm{DR}({}^F\mathfrak{M}_{z_o})$ and $\mathcal{L}^{1+\bullet}_{(2)}(H,h,\widetilde{\mathfrak{D}}_{z_o})$. We will compare them with a third complex that we introduce now. We denote by $\mathfrak{Db}^{\mathrm{mod}}_{\widetilde{X}}$ (resp. $\mathfrak{Db}^{\mathrm{mod}}_X$) the sheaf on $\widetilde{X}$ (resp. X) of distributions on X^* which can be lifted as distributions on $\widetilde{X}$ (resp. X). We have $\rho_*\mathfrak{Db}^{\mathrm{mod}}_{\widetilde{X}}=\mathfrak{Db}^{\mathrm{mod}}_X$. If $\mathfrak{Db}_X$ is the sheaf of distributions on X, it is known that $\mathfrak{Db}^{\mathrm{mod}}_X=\mathfrak{Db}_X[t'^{-1}]$. We can define the complex on $\widetilde{X}$ of currents with moderate growth with values in ${}^F\mathfrak{M}_{z_o}$, that we denote by $\mathfrak{Db}^{\mathrm{mod},1+\bullet}_{\widetilde{X}}\otimes\rho^{-1}({}^F\mathfrak{M}_{z_o})$ and we have an inclusion $\widetilde{\mathrm{DR}}({}^F\mathfrak{M}_{z_o})\hookrightarrow\mathfrak{Db}^{\mathrm{mod},1+\bullet}_{\widetilde{X}}\otimes\rho^{-1}({}^F\mathfrak{M}_{z_o})$. By an adaptation of the Dolbeault-Grothendieck theorem (cf. [6, Prop. II.1.1.7]), the complex of moderate currents of type $(0,\bullet)$ with differential d'' is a resolution of $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}}$, hence the previous morphism is a quasi-isomorphism which becomes, after taking $\boldsymbol{R}\rho_*$, the quasi-isomorphism $\mathrm{DR}({}^F\mathfrak{M}_{z_o})\to\mathfrak{Db}^{\mathrm{mod},1+\bullet}_X\otimes{}^F\mathfrak{M}_{z_o}$ (cf. [5, Section 2.c]).

As the basis $\boldsymbol{e}'^{(z_o)}$ is L^2 adapted and as the h-norm of each element of this basis has moderate growth, we have a natural morphism from the L^2 complex to the complex of currents, that is, we have morphisms

$$\widetilde{\mathcal{L}}^{1+\bullet}_{(2)}(H,h,\widetilde{\mathfrak{D}}_{z_o})\hookrightarrow\mathfrak{Db}^{\mathrm{mod},1+\bullet}_{\widetilde{X}}\otimes\rho^{-1}({}^F\mathfrak{M}_{z_o})\xleftarrow{\sim}\widetilde{\mathrm{DR}}({}^F\mathfrak{M}_{z_o}).$$

From Lemma 5 we conclude that the left morphism is a quasi-isomorphism, and finally, taking $\boldsymbol{R}\rho_*$, we find quasi-isomorphisms

$$\mathcal{L}^{1+\bullet}_{(2)}(H,h,\widetilde{\mathfrak{D}}_{z_o})\xrightarrow{\sim}\mathfrak{Db}^{\mathrm{mod},1+\bullet}_X\otimes{}^F\mathfrak{M}_{z_o}\xleftarrow{\sim}\mathrm{DR}({}^F\mathfrak{M}_{z_o}).$$

Using now Lemma 2, we find that the natural morphism

$$\mathrm{DR}({}^F\mathfrak{M}_{z_o,\mathrm{loc}})_{(2)} \longrightarrow \mathcal{L}^{1+\bullet}_{(2)}(H,\,h,\,\widetilde{\mathfrak{D}}_{z_o}) \tag{1.4}$$

is a quasi-isomorphism. □

2. Proof of the Twistor Property

In this section, it will be simpler to replace isometrically $(H,\,{}^Fh,\,{}^F\mathfrak{D}_{z_o})$, as defined in [7, Section 2.1], with $(H,\,h,\,{}^L\mathfrak{D}_z)$, where ${}^L\mathfrak{D}_z = e^{\bar t}\,{}^F\mathfrak{D}_z e^{-\bar t} = \mathfrak{D}_z - dt - z\,d\bar t$. We denote by Harm the space of harmonic sections in $\Gamma(\mathbb{P}^1,\,\mathcal{L}^1_{(2)}(H,\,h,\,{}^L\mathfrak{D}_{z_o}))$. From the proof of [7, Prop. 1] (as corrected above), we know that Harm does not depend on z_o when regarded as a subspace of $\Gamma(\mathbb{P}^1,\,\mathcal{L}^1_{(2)}(H,\,h))$.

Recall that we denote by $\mathcal{P}^1$ (resp. $\widetilde{\mathcal{P}}^1$) the product $\mathbb{P}^1\times\Omega_0$ (resp. $\widetilde{\mathbb{P}}^1\times\Omega_0$), by ρ the projection $\widetilde{\mathcal{P}}^1\to\mathcal{P}^1$ and by $p\colon\mathcal{P}^1\to\Omega_0$ (resp. $\widetilde p=\rho\circ p\colon\widetilde{\mathcal{P}}^1\to\Omega_0$) the natural projection. We define the L^2 sheaves on $\mathcal{P}^1$ (resp. $\widetilde{\mathcal{P}}^1$) in the same way as we did in [8, Section 6.2.b]. These sheaves are p-soft (resp. $\widetilde p$-soft) (cf. [2, Def. 3.1.1]). We thus have a natural morphism $\mathrm{Harm}\otimes_{\mathbb{C}}\mathcal{O}_{\Omega_0}\to p_*\mathcal{L}^1_{(2)}(\mathcal{H},\,h)$ constructed as in [8, Section 2.2.b], and harmonic sections are in the kernel of ${}^L\mathfrak{D}_z$ for any z, so the morphism takes values in $p_*\mathcal{L}^1_{(2)}(\mathcal{H},\,h,\,{}^L\mathfrak{D}_z)$. Using the isometry given by the multiplication by $e^{-z\bar t}$, we find a natural morphism

$$\mathrm{Harm}\otimes_{\mathbb{C}}\mathcal{O}_{\Omega_0}\xrightarrow{\ \cdot\, e^{-z\bar t}\ } p_*\mathcal{L}^1_{(2)}(\mathcal{H},\,h_z,\,\widetilde{\mathfrak{D}}_z). \tag{2.1}$$

We want to show that Harm is a lattice in $\boldsymbol{R}^0p_*\,\mathrm{DR}\,{}^F\mathcal{M}$, and we will first find a morphism $\mathrm{Harm}\otimes_{\mathbb{C}}\mathcal{O}_{\Omega_0}\to\boldsymbol{R}^0p_*\,\mathrm{DR}\,{}^F\mathcal{M}$.

The meromorphic L^2 de Rham complex. Let us first state an analogue of [7, Lemma 3]. We denote by ${}^F\mathcal{M}_{\mathrm{loc}}$ the $\mathcal{R}_{\mathcal{P}^1}[*P]$-module obtained by localizing ${}^F\mathcal{M}$ at its singularities P. Note that, ${}^F\mathcal{M}_{\mathrm{loc}}$ coincides with ${}^F\mathcal{M}$ near ∞. The meromorphic L^2 de Rham complex, with respect to the metric h, is denoted by $\mathrm{DR}({}^F\mathcal{M}_{\mathrm{loc}})_{(2),h}$. It is the sub-complex of $\mathrm{DR}\,{}^F\mathcal{M}_{\mathrm{loc}}$ defined by L^2 conditions with respect to h for the sections and their derivatives. We have a natural morphism $\mathrm{DR}({}^F\mathcal{M}_{\mathrm{loc}})_{(2),h}\to\mathrm{DR}\,{}^F\mathcal{M}$: this is shown in [8, Section 6.2.a] at finite distance, and is clear near ∞.

Lemma 6. *The natural morphism $\mathrm{DR}({}^F\mathcal{M}_{\mathrm{loc}})_{(2),h}\to\mathrm{DR}\,{}^F\mathcal{M}$ is a quasi-isomorphism.*

Proof. This is [8, Prop. 6.2.4] at finite distance and is proved as in [7, Lemma 3] near ∞. □

The complex $\mathcal{F}^\bullet$. As in the proof of Lemma 1, we wish to work with moderate distributions near ∞, while keeping L^2 complexes at finite distance. We will denote by X an open disc near ∞ in $\mathbb{P}^1$ which contains no other singularity of ${}^F\mathcal{M}$ than ∞ and by Y the complement of ∞ in $\mathbb{P}^1$. Lastly, we set $Z=X\cap Y$, which is a punctured disc. We will denote by $j_X: X\hookrightarrow\mathbb{P}^1$ the inclusion, and similarly for j_Y and j_Z. We denote by the same letters the inclusion $\mathcal{X}\hookrightarrow\mathcal{P}^1$, with $\mathcal{X}=X\times\Omega_0$, etc.

We denote by $\mathfrak{Db}_{\mathcal{X}}$ the sheaf of distributions on X and by $\mathfrak{Db}_{\mathcal{X}}^{\mathrm{an}}$ the sub-sheaf of distributions which are holomorphic with respect to z, i.e., the kernel of $\overline{\partial}_z$. We denote by $(\mathfrak{Db}_{\mathcal{X}}^{\mathrm{an},1+\bullet}, zd' + d'')$ the sheaf of z-holomorphic currents on $\mathcal{X}$ (we use the same rescaling on forms and currents as in [8, Section 0.3]). The Dolbeault–Grothendieck theorem implies that the complex of currents $(\mathfrak{Db}_{\mathcal{X}}^{\mathrm{an},(k,0)}, d'')$ is a resolution of $\Omega^k_{\mathcal{X}}$. As ${}^F\mathcal{M}|_{\mathcal{X}}$ is $\mathcal{O}_{\mathcal{X}}[*\infty]$-locally free (this follows from [8, Lemma 5.4.1 and Lemma 3.4.1]) it is $\mathcal{O}_{\mathcal{X}}$-flat and $(\mathfrak{Db}_{\mathcal{X}}^{\mathrm{an},(k,0)} \otimes_{\mathcal{O}_{\mathcal{X}}} {}^F\mathcal{M}|_{\mathcal{X}}, d'')$ is a resolution of $\Omega^k_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}} {}^F\mathcal{M}|_{\mathcal{X}}$. Finally, we find that the natural morphism $\mathrm{DR}\, {}^F\mathcal{M}|_{\mathcal{X}} \to \mathfrak{Db}_{\mathcal{X}}^{\mathrm{an},1+\bullet} \otimes_{\mathcal{O}_{\mathcal{X}}} {}^F\mathcal{M}|_{\mathcal{X}}$ is a quasi-isomorphism.

On the other hand, we have a morphism of complexes

$$\mathcal{L}^{1+\bullet}_{(2)}(\mathcal{H}, h, \widetilde{\mathfrak{D}}_z)|_{\mathcal{X}} \xrightarrow{\iota} \mathfrak{Db}_{\mathcal{X}}^{\mathrm{an},1+\bullet} \otimes_{\mathcal{O}_{\mathcal{X}}} {}^F\mathcal{M}|_{\mathcal{X}} \tag{2.2}$$

which, when restricted to Z, is a quasi-isomorphism. Indeed, on Z this is clear. Near ∞, this can bee seen by using the local $\mathcal{O}_{\mathcal{X}}[*\infty]$-basis $\boldsymbol{e}'^{(z_o)}$ of $\mathcal{M}_{\mathrm{loc}}$ near ∞: this is a L^2-adapted basis and the h-norm of its elements has moderate growth near ∞, locally uniformly with respect to z; this implies that a section of $\mathcal{L}_{(2)}(\mathcal{H}, h)$ belongs to $\mathfrak{Db}_{\mathcal{X}}^{\mathrm{an}} \otimes_{\mathcal{O}_{\mathcal{X}}} {}^F\mathcal{M}|_{\mathcal{X}}$. Let us check the compatibility of the differentials of the complexes. On $\mathcal{L}_{(2)}$, the derivative is not taken in the distributional sense on $\mathcal{X}$, but only on $\mathcal{X}^* = (X \setminus \{\infty\}) \times \Omega_0$. In other words, it is obtained by taking the derivative in the distributional sense on $\mathcal{X}$ and then restricting to $\mathcal{X}^*$. But the morphism ι is clearly compatible with this way of taking derivatives, as $|t'|$ acts in an invertible way on the right-hand side of (2.2), hence any distribution supported on $\{\infty\} \times \Omega_0$ is annihilated by ι. (Let us notice that this point is exactly what prevents us from using distributions near singularities at finite distance, as ${}^F\mathcal{M} \neq {}^F\mathcal{M}_{\mathrm{loc}}$ near such a singular point.)

The complex $\mathcal{F}^\bullet$ is defined by the exact sequence of complexes

$$\begin{aligned} 0 \longrightarrow\; & j_{Z,!}\mathcal{L}^{1+\bullet}_{(2)}(\mathcal{H}, h, \widetilde{\mathfrak{D}}_z)|_Z \\ & \xrightarrow{(\mathrm{Id},-\iota)} j_{Y,!}\mathcal{L}^{1+\bullet}_{(2)}(\mathcal{H}, h, \widetilde{\mathfrak{D}}_z)|_Y \oplus j_{X,!}(\mathfrak{Db}_{\mathcal{X}}^{\mathrm{an},1+\bullet} \otimes_{\mathcal{O}_{\mathcal{X}}} {}^F\mathcal{M}|_{\mathcal{X}}) \\ & \longrightarrow \mathcal{F}^\bullet \longrightarrow 0. \end{aligned}$$

Let us note that each term in $\mathcal{F}^\bullet$ is p-soft (cf. [2, Prop. 2.5.7(ii) and Cor. 2.5.9]).

Lemma 7. *We have a natural morphism of complexes* $\mathrm{DR}({}^F\mathcal{M}_{\mathrm{loc}})_{(2),h} \to \mathcal{F}^\bullet$ *which is a quasi-isomorphism.*

Proof. We use the exact sequence

$$\begin{aligned} 0 \longrightarrow\; & j_{Z,!}j_Z^{-1}\,\mathrm{DR}({}^F\mathcal{M}_{\mathrm{loc}})_{(2),h} \\ & \longrightarrow j_{Y,!}j_Y^{-1}\,\mathrm{DR}({}^F\mathcal{M}_{\mathrm{loc}})_{(2),h} \oplus j_{X,!}j_X^{-1}\,\mathrm{DR}({}^F\mathcal{M}_{\mathrm{loc}})_{(2),h} \\ & \longrightarrow \mathrm{DR}({}^F\mathcal{M}_{\mathrm{loc}})_{(2),h} \longrightarrow 0 \end{aligned}$$

to reduce the question to each of the open sets X, Y, Z. On Y, this is [8, Th. 6.2.5]. On Z, this is easy, and on X, this follows from Lemma 6. The compatibility with the arrows in the previous exact sequences is easy. □

Lemma 8. *We have a natural morphism* $\mathrm{Harm} \otimes_{\mathbb{C}} \mathcal{O}_{\Omega_0} \to \boldsymbol{R}^0 p_* \mathcal{F}^\bullet = \mathcal{H}^0(p_*\mathcal{F}^\bullet)$.

Proof. Let us first note that the second equality comes from the p-softness of the terms in $\mathcal{F}^\bullet$. Using (2.2), we have a natural morphism $\mathcal{L}^{1+\bullet}_{(2)}(\mathcal{H}, h, \widetilde{\mathfrak{D}}_z) \to \mathcal{F}^\bullet$. Therefore, it is enough to find a morphism

$$\mathrm{Harm}\otimes_{\mathbb{C}}\mathcal{O}_{\Omega_0} \longrightarrow \boldsymbol{R}^0 p_*\mathcal{L}^{1+\bullet}_{(2)}(\mathcal{H}, h, \widetilde{\mathfrak{D}}_z) = \mathcal{H}^0\big(p_*\mathcal{L}^{1+\bullet}_{(2)}(\mathcal{H}, h, \widetilde{\mathfrak{D}}_z)\big). \tag{2.3}$$

We have inclusions of L^2 complexes

$$\mathcal{L}^{1+\bullet}_{(2)}(\mathcal{H}, h, \widetilde{\mathfrak{D}}_z) \xleftarrow{\iota_h} \mathcal{L}^{1+\bullet}_{(2)}(\mathcal{H}, h+h_z, \widetilde{\mathfrak{D}}_z) \xrightarrow{\iota_{h_z}} \mathcal{L}^{1+\bullet}_{(2)}(\mathcal{H}, h_z, \widetilde{\mathfrak{D}}_z).$$

We will prove:

On some open neighbourhood $\mathrm{nb}(0)$ of 0 in Ω_0, the morphism (2.1) factorizes through $p_*\iota_{h_z}$. (2.4)

On $\Omega_0\setminus\{0\}$, the morphism ι_{h_z} is a quasi-isomorphism. (2.5)

This will be enough to conclude that we have a natural morphism

$$\mathrm{Harm}\otimes_{\mathbb{C}}\mathcal{O}_{\Omega_0} \longrightarrow \boldsymbol{R}^0 p_*\mathcal{L}^{1+\bullet}_{(2)}(\mathcal{H}, h+h_z, \widetilde{\mathfrak{D}}_z) = \mathcal{H}^0\big(p_*\mathcal{L}^{1+\bullet}_{(2)}(\mathcal{H}, h+h_z, \widetilde{\mathfrak{D}}_z)\big),$$

giving thus (2.3) by composing with $\boldsymbol{R}^0 p_*\iota_h$. □

Proof of (2.4). By construction, Harm is a subspace of $\Gamma(\mathbb{P}^1, \mathcal{L}^1_{(2)}(H, h, {}^L\mathfrak{D}_z))$. We will use the following lemma, whose proof is due to the the referee of [9] (note that S. Szabo proves a similar result in [11, Lemma 2.32], with different methods however). If f is a section of H (resp. ω is a section of H with values in 1-forms), we will denote by $|f|_{\boldsymbol{h}}$ (resp. $|\omega|_{\boldsymbol{h}}$) the $\boldsymbol{h}$-norm of f (resp. the norm of ω with respect to $\boldsymbol{h}$ and the norm induced by the Poincaré metric on 1-forms, that we call the P-norm).

Lemma 9 (Exponential decay of harmonic sections)**.** *For any $\omega\in\mathrm{Harm}$, there exists $C>0$ and a neighbourhood of ∞ in X on which the h-norm of ω is bounded by $e^{-C|t|}$.*

Once this lemma is proved, we obtain that $|e^{-z\bar t}\omega|_{h_z} = |\omega|_h \leqslant e^{-C|t|}$ for any $\omega\in\mathrm{Harm}$ on a suitable neighbourhood of ∞, hence $|\omega|_{h_z}\leqslant e^{-C|t|+\operatorname{Re} z\bar t}$. If $|z|$ is small enough, we thus get $|\omega|_{h_z}\leqslant e^{-C'|t|}$, and therefore ω is L^2 with respect to h_z, as wanted. □

Proof of Lemma 9. Let $\omega\in\mathrm{Harm}$. Then ${}^L\mathfrak{D}_z\omega=0$ for any $z\in\Omega_0$, hence, if we set ${}^L\theta'_E=\theta'_E-dt$ and ${}^L\theta''_E=\theta''_E-d\bar t$, we have $(D''_E+{}^L\theta'_E)\omega=0$ and $(D'_E+{}^L\theta''_E)\omega=0$. We will now restrict the question near ∞ and we will work with the coordinate t'.

By the Dolbeault lemma for $z_o=0$ ([7, Lemma 4] corrected as in Section 1.1), the complex $\mathcal{L}^{1+\bullet}_{(2)}(H, h, \widetilde{\mathfrak{D}}_0) = \mathcal{L}^{1+\bullet}_{(2)}(H, h, (D''_E+{}^L\theta'_E))$ is quasi-isomorphic to $\mathrm{DR}\,{}^F\mathfrak{M}_0$. Let us note that the germ of $\mathrm{DR}\,{}^F\mathfrak{M}_0$ at ∞ is quasi-isomorphic to 0, as ${}^L\theta'_E = t'^{-2}(\mathrm{Id}+\cdots)$ is invertible on the germ ${}^F\mathfrak{M}_0$ at ∞. Therefore, the germ $\mathcal{L}^{1+\bullet}_{(2)}(H, h, (D''_E+{}^L\theta'_E))_\infty$ is quasi-isomorphic to 0 and there exists a neighbourhood X of ∞ and a section $f\in L^2(X, H, h)$ such that $(D''_E+{}^L\theta'_E)f=\omega$. Assume we prove $|f|_h\leqslant e^{-C'/|t'|}$ for some constant $C'>0$. Then, according to the moderate growth of ${}^L\theta'_E$, we will also have $|{}^L\theta'_E f|_h\leqslant e^{-C''/|t'|}$ for some $C''>0$ on some neighbourhood of ∞, and thus the desired inequality for the (1, 0) part of ω. Arguing with a conjugate argument, we get the same kind of inequality for the (0, 1) part, hence the lemma.

Let us note that $(D'_E + {}^L\theta''_E)(D''_E + {}^L\theta'_E)f = (D'_E + {}^L\theta''_E)\omega = 0$, hence $D'_E D''_E f = -{}^L\theta''_E {}^L\theta'_E f$. Since $D'_E D''_E + D''_E D'_E = -({}^L\theta'_E {}^L\theta''_E + {}^L\theta''_E {}^L\theta'_E)$, we also get $D''_E D'_E f = -{}^L\theta'_E {}^L\theta''_E f$ (all these equalities are taken on X^* in the distributional sense).

In particular, as $D'_E D''_E + {}^L\theta''_E {}^L\theta'_E$ is elliptic on X^*, f is C^∞ on X^*. If we set ${}^L\theta'_E = {}^L\Theta'_E\, dt'$ and ${}^L\theta''_E = {}^L\Theta''_E\, d\overline{t'}$, ${}^L\Theta''_E$ is the h-adjoint of ${}^L\Theta'_E$. We then have on X^*

$$d'd''|f|_h^2 = h(D'_E D''_E f, \overline{f}) - h(D''_E f, \overline{D''_E f}) + h(D'_E f, \overline{D'_E f}) + h(f, \overline{D''_E D'_E f}),$$

so that, dividing by $dt' \wedge d\overline{t'}$ and using the previous relations, we find

$$\partial_{t'}\partial_{\overline{t'}}|f|_h^2 \geqslant |{}^L\Theta'_E f|_h^2 + |{}^L\Theta''_E f|_h^2 \geqslant C|t'|^{-4}|f|_h^2. \tag{2.6}$$

This relation holds on X^*.

Assertion. *The inequality (2.6) holds on X in the weak sense, that is, for any nonnegative test function χ on X, and denoting by $d\,\mathrm{vol_E}$ the Euclidean volume $\frac{i}{2\pi}dt' \wedge d\overline{t'}$,*

$$\int_X |f|_h^2(\partial_{t'}\partial_{\overline{t'}}\chi)\, d\,\mathrm{vol_E} \geqslant C\int_X |t'|^{-4}|f|_h^2\, \chi\, d\,\mathrm{vol_E}\,.$$

Proof of the assertion. Let us first note that $|t'|^{-2}|f|_h$ (hence also $|f|_h$) is in $L^2(d\,\mathrm{vol_E})$, as $|{}^L\Theta'_E f|_h$ is in $L^2(d\,\mathrm{vol_P})$, where $d\,\mathrm{vol_P} = |t'|^{-2}\,\mathrm{L}(t')^{-2} d\,\mathrm{vol_E}$ is the Poincaré volume, and $|{}^L\Theta'_E f|_h \sim |t'|^{-2}|f|_h |dt'|_{\mathrm{P}}$, with $|dt'|_{\mathrm{P}} \sim |t'|\,\mathrm{L}(t')$. In particular, $\|f\|_{h,\mathrm{P}} < +\infty$. Similarly, if $\psi\, d\overline{t'}$ is the (0, 1) component of ω, we have $\left|\int_X h(D''_E f, \overline{D''_E f})\right| = 2\pi\int_X |\psi|_h^2\, d\,\mathrm{vol_E} = 2\pi\int_X |\psi|_h^2 |d\overline{t'}|_{\mathrm{P}}^2\, d\,\mathrm{vol_P} < \infty$, hence $|\psi|_h \in L^2(d\,\mathrm{vol_E})$.

We now claim that $\left|\int_X h(D'_E f, \overline{D'_E f})\right| < \infty$, that is, $\|D'_E f\|_{h,\mathrm{P}} < +\infty$. This follows from the acceptability (in the sense of [10]) of the Hermitian bundle (H, D''_E, h) (as the Higgs field θ_E is tame). Indeed, the P-norm of the curvature $R(h)$ of h is bounded near ∞. For any test function η on X^*, we have (cf. [4, (2.23)])

$$\begin{aligned}\left|\int_X h(D'_E\eta, \overline{D'_E\eta})\right| &\leqslant \left|\int_X h(D''_E\eta, \overline{D''_E\eta})\right| + \left|\int_X (\eta R(h), \eta\, d\,\mathrm{vol_P})_{h,\mathrm{P}}\, d\,\mathrm{vol_P}\right| \\ &\leqslant \left|\int_X h(D''_E\eta, \overline{D''_E\eta})\right| + \left|\int_X |\eta|_h^2 |R(h)|_{\mathrm{P}}\, |d\,\mathrm{vol_P}|_{\mathrm{P}}\, d\,\mathrm{vol_P}\right| \\ &\leqslant \left|\int_X h(D''_E\eta, \overline{D''_E\eta})\right| + C\left|\int_X |\eta|_h^2\, d\,\mathrm{vol_P}\right|,\end{aligned}$$

hence, $\|D'_E\eta\|_{h,\mathrm{P}} \leqslant \|D''_E\eta\|_{h,\mathrm{P}} + C\|\eta\|_{h,\mathrm{P}}$. Since the Poincaré metric is complete near ∞, we can find a sequence of nonnegative test functions η_n on X^* which tend pointwise to 1 in some punctured neighbourhood of ∞, such that $\eta_n \leqslant 1$ and $|d\eta_n|_{\mathrm{P}} \leqslant 2^{-n}$ (see, e.g., [1, Lemme 12.1]). Applying the previous result to $\eta_n f$, we find $\|\eta_n D'_E f\|_{h,\mathrm{P}} \leqslant \|\eta_n D''_E f\|_{h,\mathrm{P}} + (C + 2^{-n+1})\|f\|_{h,\mathrm{P}}$, hence the claim.

In order to end the proof of the assertion, it is enough to showing that the difference $\int_{|t'|\geqslant\varepsilon}\big[|f|_h^2(\partial_{t'}\partial_{\overline{t'}}\chi)-(\partial_{t'}\partial_{\overline{t'}}|f|_h^2)\chi\big]\,d\,\mathrm{vol}_{\mathrm{E}}$ tends to 0 with ε. It is then enough to find a sequence $\varepsilon_n\to 0$ such that $\int_{|t'|=\varepsilon_n}|f|_h^2\,d\theta$, $\int_{|t'|=\varepsilon_n}\partial_{t'}|f|_h^2\,d\theta$ and $\int_{|t'|=\varepsilon_n}\partial_{\overline{t'}}|f|_h^2\,d\theta$ tend to 0, and it is enough to checking that the integrals of $|f|_h^2$, $|\partial_{t'}|f|_h^2|$, $|\partial_{\overline{t'}}|f|_h^2|$ with respect to $d\theta\,dr/r$ are finite. For the first one, this follows from $|t'|^{-2}|f|_h\in L^2(d\,\mathrm{vol}_{\mathrm{E}})$. For the second one (and similarly the third one), we use that $|\varphi|_h$, $|\psi|_h$ and $|t'|^{-2}|f|_h$ belong to $L^2(d\,\mathrm{vol}_{\mathrm{E}})$. □

Once the assertion is proved, we can use the same trick (a variant of Ahlfors lemma) as in [10]. Recall that we want to prove $|f|_h\leqslant\exp(-C'|t'|^{-1})$ for some $C'>0$. Let us remark first that, because $\partial_{t'}|f|_h^2$ and $\partial_{\overline{t'}}|f|_h^2$ are $L^1_{\mathrm{loc}}(d\,\mathrm{vol}_{\mathrm{E}})$ at $t'=0$, $|f|_h^2$ is continuous (and C^∞ on X^*). Let us consider the auxiliary function $\exp(-C^{1/2}|t'|^{-1})$. A simple computation shows that $\partial_{t'}\partial_{\overline{t'}}\exp(-C^{1/2}|t'|^{-1})\leqslant C|t'|^{-4}\exp(-C^{1/2}|t'|^{-1})$. Let us then choose $\lambda>0$ such that $|f|_h^2\leqslant\lambda\exp(-C^{1/2}|t'|^{-1})$ in some neighbourhood of ∂X and let $U\subset X$ be the open set where $|f|_h^2>\lambda\exp(-C^{1/2}|t'|^{-1})$. The previous inequalities show that $|f|_h^2-\lambda\exp(-C^{1/2}|t'|^{-1})$ is continuous and subharmonic in U. If U is not empty then, at a boundary point of U in X we have $|f|_h^2=\lambda\exp(-C^{1/2}|t'|^{-1})$ and, by the maximum principle, we have $|f|_h^2-\lambda\exp(-C^{1/2}|t'|^{-1})\leqslant 0$ on U, a contradiction. □

Proof of (2.5). The proof is similar to that of Lemma 3. Let us work near $z_o\in\Omega_0^*$. Using the L^2-adapted basis $\boldsymbol{e}^{(z_o)}$ we trivialize the bundle $\mathcal{H}$ near z_o. Given $\theta_o\in S^1$, we choose an open neighbourhood $\mathrm{nb}(z_o)$ such that the choice of r_1 and ε in the proof of Lemma 3 can be done uniformly with respect to $z\in\overline{\mathrm{nb}(z_o)}$. Let $\mathrm{H}(\mathrm{nb}(z_o))$ denote the Banach space of continuous functions on $\overline{\mathrm{nb}(z_o)}$ which are holomorphic in $\mathrm{nb}(z_o)$. We then consider the complex whose terms are the $\bigoplus_{\beta,\ell,k}L^2\big(\overline{U},\,\mathrm{H}(\mathrm{nb}(z_o)),\,\boldsymbol{h}_{\beta,\ell,k},\,\widetilde{\mathfrak{D}}_{z,\mathrm{diag}}\big)$ twisted by differential forms, where $\boldsymbol{h}_{\beta,\ell,k}$ is $\|e^{(z_o)}_{\beta,\ell,k}\|^2_{\boldsymbol{h},2}\boldsymbol{h}$, and differential as in the three steps of the proof of Lemma 3. We show as in Lemma 3 that this complex has vanishing higher cohomology, and we obtain (2.5). □

Proof that Harm *is a lattice.* From Lemmas 6, 7 and 8 we get a morphism

$$\mathrm{Harm}\otimes_{\mathbb{C}}\mathcal{O}_{\Omega_0}\longrightarrow\boldsymbol{R}^0p_*\,\mathrm{DR}\,{}^F\mathcal{M}.\tag{2.7}$$

As both terms are locally free $\mathcal{O}_{\Omega_0}$-modules of the same rank, it will be an isomorphism as soon as its restriction to each fibre $z=z_o$ is an isomorphism of $\mathbb{C}$-vector spaces. We will shorten the notation and denote by $|_{z=z_o}$ the quotient by the image of $(z-z_o)$.

For any complex $\mathcal{G}^\bullet$ entering in the definition of the morphism (2.7), we have natural morphisms (with obvious notation)

$$(\boldsymbol{R}^0p_*\mathcal{G}^\bullet)|_{z=z_o}\longrightarrow\boldsymbol{R}^0p_*(\mathcal{G}^\bullet|_{z=z_o})\longrightarrow\boldsymbol{R}^0p_*\mathcal{G}^\bullet_{z_o}.$$

According to the exact sequence

$$0\longrightarrow\mathrm{DR}\,{}^F\mathcal{M}\xrightarrow{z-z_o}\mathrm{DR}\,{}^F\mathcal{M}\longrightarrow\mathrm{DR}\,{}^F\mathfrak{M}_{z_o}\longrightarrow 0,$$

and since each of these complexes have hypercohomology in degree 0 at most, the natural morphism $(\boldsymbol{R}^0 p_* \operatorname{DR} {}^F\mathcal{M})|_{z=z_o} \to \boldsymbol{H}^0(\mathbb{P}^1, \operatorname{DR} {}^F\mathfrak{M}_{z_o})$ is an isomorphism.

As a consequence, it is enough to prove that, for any $z_o \in \Omega_0$, the morphism $\mathrm{Harm} \to \boldsymbol{H}^0(\mathbb{P}^1, \operatorname{DR} {}^F\mathfrak{M}_{z_o})$ constructed as (2.7) by fixing $z = z_o$, is an isomorphism. Let us recall how it is constructed, by considering the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Harm} \arrow[r, "\sim"] & H^0(\mathbb{P}^1, \mathcal{L}^{1+\bullet}_{(2)}(H, h_{z_o}, \widetilde{\mathfrak{D}}_{z_o}) & \\
& H^0(\mathbb{P}^1, \mathcal{L}^{1+\bullet}_{(2)}(H, h+h_{z_o}, \widetilde{\mathfrak{D}}_{z_o}) \arrow[u, "\wr"'] \arrow[d, "\wr"] & \\
& H^0(\mathbb{P}^1, \mathcal{L}^{1+\bullet}_{(2)}(H, h, \widetilde{\mathfrak{D}}_{z_o}) \arrow[d, "c"'] & \boldsymbol{H}^0(\mathbb{P}^1, \operatorname{DR}({}^F\mathfrak{M}_{z_o,\mathrm{loc}})_{(2)}) \arrow[l, "a"'] \arrow[d, "\wr"] \\
& H^0(\mathbb{P}^1, \mathcal{F}^\bullet_{z_o}) & \boldsymbol{H}^0(\mathbb{P}^1, \operatorname{DR} {}^F\mathfrak{M}_{z_o}). \arrow[l, "b", "\sim"']
\end{tikzcd}$$

Then $(2.7)_{z_o}$ is obtained by factorizing through $H^0(\mathbb{P}^1, \mathcal{F}^\bullet_{z_o})$ and b^{-1}. On the other hand, we know that a is an isomorphism (this is (1.4) if $z_o \neq 0$ and [7, Lemma 4] as corrected in Section 1.1 if $z_o = 0$). Therefore, c is also an isomorphism. $\square$

End of the proof of the twistor property. The proof is done as in [8, p. 53], where we use the L^2 complex instead of the C^∞ de Rham complex. $\square$

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