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Dedicated to the memory of Andrei Bolibrukh

Fourier–Laplace transform of irreducible regular differential systems on the Riemann sphere

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Abstract. It is shown that the Fourier–Laplace transform of an irreducible regular differential system on the Riemann sphere underlies a polarizable regular twistor $\mathcal{D}$-module if one considers only the part at finite distance. The associated holomorphic bundle defined away from the origin of the complex plane is therefore equipped with a natural harmonic metric having a tame behaviour near the origin.

Contents

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Introduction

A positive answer to the Riemann–Hilbert problem for semisimple[1] linear representations of the fundamental group of the complement of a finite point set on the Riemann sphere is an important result of A. A. Bolibrukh ([2], [3]). Similarly, he proved [4] that the Birkhoff problem has a positive answer when the corresponding system of meromorphic linear differential equations is semisimple, thus generalizing previous results of W. Balser.

The two results can be stated in a similar way, using the language of meromorphic bundles on the Riemann sphere. Namely, let $P = \{p_1, \dots, p_r, p_{r+1} = \infty\}$ be a non-empty finite set of points on $\mathbb{P}^1$ and let M be a finite-rank free $\mathcal{O}(*P)$-module[2] equipped with a connection ∇. We assume that there is a basis of M for which the connection ∇ has Poincaré rank $m_i \geq 0$ at p_i for $i = 1, \dots, r$ (that is, the order of the pole of the matrix of ∇ at p_i is equal to $m_i + 1$); moreover, we assume that ∇ has a regular singularity at the point ∞ (that is, the coefficients in the given basis of horizontal sections are multivalued holomorphic functions with at most power-law growth at infinity[3]). In this case, if (M, ∇) is *irreducible* (or semisimple), then there is a basis of M in which the Poincaré rank of ∇ at p_i is m_i $(i = 1, \dots, r)$ and ∇ has at most a logarithmic pole at ∞ (that is, ∇ has zero Poincaré rank at infinity).

After an easy preliminary reduction we can see that the case when $m_i = 0$ for all i corresponds to the Riemann–Hilbert problem and the case when $r = 1$ corresponds to the Birkhoff problem.

Starting from a system of linear meromorphic differential equations having only regular singularities (or, more precisely, from a regular holonomic $\mathcal{D}$-module) on the Riemann sphere, one obtains a new system by using the Fourier–Laplace transform, and the Riemann–Hilbert problem for the original system is transformed into the Birkhoff problem for the new system. These problems (for a given system and its Fourier–Laplace transform) are not directly related to each other;[4] however, one of these systems is semisimple if and only if the other is, and there is a common condition under which both problems have a positive answer simultaneously.

The semisimple linear representations discussed above share another remarkable property: there is a *tame harmonic metric* on the associated flat bundle (see [11]). This property can be expressed by using the language of *polarized twistor* $\mathcal{D}$*-modules* introduced by the author in [10] by extending a notion due to Simpson [13]. Namely, to any representation of this kind one can assign a regular holonomic $\mathcal{D}$-module on the Riemann sphere, and this module is unique up to isomorphism and has neither submodules nor quotient modules supported at a point. The last property can be expressed as follows: this $\mathcal{D}$-module underlies a polarizable regular twistor $\mathcal{D}$-module in the sense defined in the cited papers (see also below).

In the present paper we study the behaviour of polarized regular twistor $\mathcal{D}$-modules on the Riemann sphere under the Fourier–Laplace transform. However,

[1] Semisimple objects are direct sums of irreducible objects.

[2] *Russian Editor's note*: This means a sheaf of modules over the sheaf of rings of meromorphic functions having poles only at points of the set P.

[3] *Russian Editor's note*: This means the growth of analytic functions in sectors of finite aperture with vertex at infinity.

[4] See, however, § V.2c in [9].

we give no information on the behaviour at infinity in the Fourier plane, where an irregular singularity can occur.

We note that, using another apparatus, Szabo [14] has established a perfect Fourier correspondence in a more general situation in which an irregular singularity at infinity is admitted but some other more restrictive assumptions are imposed.

We refer to [10] for diverse results (used below) related to polarizable twistor $\mathcal{D}$-modules.

§ 1. Statement of the results

We regard the projective line $X = \mathbb{P}^1$ as the union of two affine charts $\operatorname{Spec}\mathbb{C}[t]$ and $\operatorname{Spec}\mathbb{C}[t']$ with $t' = 1/t$ on the intersection, and we define ∞ as the point where $t' = 0$. As above, let $P = \{p_1, \dots, p_r, p_\infty\}$ be a finite set of $r+1$ distinct points in $\mathbb{P}^1$. We set $X^* = \mathbb{P}^1 \setminus P$.

Let the pair (H, D_V) be formed by a C^∞ vector bundle H with a flat connection D_V on X^{*an}. The bundle is holomorphic with respect to the $(0,1)$-part D''_V, and we set $(V, \nabla) = (\ker D'', D'_V)$. The associated local system is $\mathcal{L} \overset{\text{def}}{=} \ker[\nabla\colon V \to V \otimes_{\mathcal{O}_{X^{*an}}} \Omega^1_{X^{*an}}]$. We denote by T_j the local monodromy of this local system at each point p_j of P.

1.1. Fourier–Laplace transform of flat bundles. The notion of Fourier–Laplace transform is *a priori* defined for algebraic $\mathcal{D}$-modules on the affine line $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$ and not for holomorphic bundles with connection on X^* nor for holomorphic vector bundles on $\mathbb{P}^1$ equipped with a meromorphic connection.

We denote by $\mathbb{C}[t]\langle\partial_t\rangle$ the Weyl algebra in dimension one, that is, the quotient by the relation $[\partial_t, t] = 1$ of the free algebra generated by $\mathbb{C}[t]$ and $\mathbb{C}[\partial_t]$ (see, for instance, [5] and [7]). Let M be a holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module. The module M is said to be a *minimal extension* if it has neither submodules nor quotient modules supported by some point of $\mathbb{A}^1$. The following assertion is well known.

Lemma 1 (Riemann–Hilbert correspondence). *The functor assigning to any holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module with all its singularities in P the restriction of this module to X^{*an} induces an equivalence between the category of holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-modules that have regular singularities* (*including the point at infinity*) *and are minimal extensions* (*the morphisms are the morphisms of $\mathbb{C}[t]\langle\partial_t\rangle$-modules*), *and the category of all flat holomorphic bundles on X^{*an}.*

For a given holomorphic bundle with connection (V, ∇) on X^{*an} we denote by M the regular holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module associated with (V, ∇) by Lemma 1. The Fourier–Laplace transform $\widehat{M}$ of M is the $\mathbb{C}$-vector space M equipped with an action of the Weyl algebra $\mathbb{C}[\tau]\langle\partial_\tau\rangle$ (with respect to the variable τ) defined by the formula

$$\tau \cdot m = \partial_t m, \qquad \partial_\tau = -tm. \tag{1.1}$$

We denote by $\widehat{X} = \widehat{\mathbb{A}}^1$ the affine line with the coordinate τ and by $\widehat{\mathbb{P}}^1$ the corresponding projective line. As is known, the module $\widehat{M}$ has a single singularity at a finite distance, namely, at $\tau = 0$, and this singularity is regular. However, the module has an irregular singularity at $\tau = \infty$ in general (for general results concerning Fourier transforms of holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-modules, see, for instance, [7]).

We therefore obtain a holomorphic bundle with connection $(\widehat{V}, \widehat{P})$ on the punctured projective line $\widehat{X} = \widehat{\mathbb{P}}^1 \setminus \widehat{P}$, where $\widehat{\mathbb{P}}^1$ is the projective line with the coordinate τ and $\widehat{P} = \{0, \infty\}$. The associated flat bundle of class C^∞ is denoted by $(\widehat{H}, D_{\widehat{V}})$. However, the module $\widehat{M}$ cannot be recovered from a given pair $(\widehat{V}, \widehat{\nabla})$ in general.

Let d denote the rank of H. We indicate how to compute the rank $\widehat{d}$ of $\widehat{H}$. As is known, for any $\tau_o \neq 0$ the $\mathbb{C}$-linear morphism

$$M \xrightarrow{\partial_t - \tau_o} M$$

is injective and its cokernel is a finite-dimensional vector space, namely, the fibre of the bundle $\widehat{V}$ at the point τ_o. The dimension of the cokernel, which coincides with the total number μ of 'vanishing cycles' of M at the points of the set $P \setminus \infty$ (that is, the sum of multiplicities of the characteristic variety of M along its components $T^*_{p_i} X$; see, for instance, § 4 (p. 66) in [6]), can be readily computed here by the following formulae (see Proposition 1.5 (p. 79) of [7]):

$$\widehat{d} = \mu = rd - d_1, \quad r = \operatorname{card} P - 1, \quad d_1 \overset{\text{def}}{=} \sum_{j=1}^{r} \dim \operatorname{Ker}(T_j - \operatorname{Id}). \tag{1.2}$$

More precisely, the tensor product $\mathbb{C}[\tau, \tau^{-1}] \otimes_{\mathbb{C}[\tau]} \widehat{M}$ is a free $\mathbb{C}[\tau, \tau^{-1}]$-module of rank μ.

1.2. Fourier–Laplace transform of an irreducible bundle with connection. We assume now that (H, D_V) is irreducible, or, equivalently, that no non-trivial subspace of L is invariant under all T_j $(j = 1, \dots, r+1)$. One can readily prove another equivalent condition claiming that the associated regular holonomic $\mathcal{D}$-module M is irreducible as a $\mathbb{C}[\tau]\langle \partial_\tau \rangle$-module. In turn, this is (very easily) equivalent to the irreducibility of the Fourier–Laplace transform $\widehat{M}$. However, since the module $\widehat{M}$ can be irregular at infinity, this does not imply the irreducibility of $(\widehat{H}, D_{\widehat{V}})$ (see below). The rank $\widehat{d}$ of $\widehat{H}$ is positive unless $(V, \nabla) = (\mathcal{O}_X, d)$, because irreducibility is assumed. In the following we implicitly assume that $\widehat{d} > 0$.

The bundle $(\widehat{H}, D_{\widehat{V}})$ is determined by its monodromy $\widehat{T}_0$ around the point $\tau = 0$. The Jordan structure of the monodromy operator $\widehat{T}_0$ is determined by that of the monodromy T_∞ of (H, D_V) around ∞ (below we denote by A_λ the restriction of an endomorphism A to the generalized eigenspace (root subspace) corresponding to the eigenvalue λ). This is the content of the following lemma, which can be derived from Proposition 8.4.20 of [10].

Lemma 2. *Suppose that a pair (H, D_V) is irreducible. Then*:

1) *for any $\lambda \neq 1$ the Jordan structures of the restrictions of the monodromy operators $\widehat{T}_{0,\lambda}$ and $T_{\infty,\lambda}$ coincide*;
2) *any Jordan block of size $k \geq 1$ of $T_{\infty,1}$ induces a Jordan block of size $k+1$ of $\widehat{T}_{0,1}$*;
3) *the remaining Jordan blocks of $\widehat{T}_{0,1}$ are of size one.*

The semisimplicity of the pair (the decomposability as the direct sum of irreducible bundles with connection) is equivalent to that of the monodromy operator $\widehat{T}_0$. This holds (provided that (H, D_V) is irreducible) if and only if the operator T_∞ is semisimple and 1 is not an eigenvalue of T_∞.

1.3. Fourier–Laplace transform of a bundle with connection and Hermitian metrics. We assume first that the monodromy representation associated with the local system $\mathcal{L} = \ker \nabla$ is unitary. In other words, we assume that there is a *D_V-flat Hermitian metric h* on H. In particular,

1) the local system $\mathcal{L}$ is an orthogonal direct sum of irreducible local systems, so in what follows we assume that the local system is *irreducible* (the case of a constant rank-one local system is trivial, and we therefore also assume that $\mathcal{L}$ is not a constant local system);
2) the local monodromy T_j of the local system at each point p_j of P is (unitary and) semisimple and the eigenvalues of T_j are roots of unity.

We can ask whether the space $\widehat{H}$ can be naturally equipped with a Hermitian metric and whether this metric is $D_{\widehat{V}}$-flat. We note that flatness would imply semisimplicity and unitarity of the monodromy $\widehat{T}_0$. By the above assumptions and Lemma 2, this holds if and only if 1 is not an eigenvalue of the monodromy operator T_∞.

If we only assume that (H, D_V) is irreducible but not necessarily unitary, then there is still a unique *tame harmonic Hermitian metric h* on (H, D_V) (by [11]), which is therefore a natural metric to be considered. Such a metric also exists if (H, D_V) is a semisimple pair, but it can fail to be unique.

Tame harmonic metrics. Let us recall the definition of these metrics. We can fix a choice of a metric connection on H, which we denote by D_E, by the following condition: if we introduce a 1-form $\theta_E = D_V - D_E$ and decompose it into the $(1,0)$ and $(0,1)$ parts, $\theta_E = \theta'_E + \theta''_E$, then the *$h$-adjoint* of θ'_E is defined as the form θ''_E. Since X is of dimension one, it follows that the bundle $E = \ker D''_E$ is holomorphic on X^{*an}, and the form θ'_E satisfies the *Higgs condition* $\theta'_E \wedge \theta'_E = 0$.

A triple (H, D_V, h) (where D_V is a flat connection) is said to be *harmonic* if the Higgs field is holomorphic on E, that is, if the 1-form $\theta'_E \colon E \to E \otimes_{\mathcal{O}_{X^{*an}}} \Omega^1_{\mathcal{O}_{X^{*an}}}$ is holomorphic.

Following [11], we say that a triple (H, D_V, h) of this kind is *tame* if the eigenvalues of the Higgs field (which are multivalued holomorphic one-forms) have at most a simple pole at each point of P.

One can ask whether the pair $(\widehat{H}, D_{\widehat{V}})$ also carries a harmonic metric of this kind with tame behaviour at $\tau = 0$. We give a positive answer in Corollary 1.

Twistor $\mathcal{D}$-modules. We speak in the language of twistor $\mathcal{D}$-modules used in the preprint [10], to which the reader is referred. Let us briefly recall some basic definitions.

We still denote by X the Riemann sphere and write $\mathcal{X} = X \times \mathbb{C}$, using the coordinate z on the factor $\mathbb{C}$. We also denote by $\mathbf{S}$ the circle $\{|z| = 1\}$ and by $\mathcal{D}_X$ the sheaf of holomorphic differential operators on X, and we consider the sheaf $\mathcal{R}_\mathcal{X}$ of z-differential operators on $\mathcal{X}$, namely, in any local coordinate x on X the module $\mathcal{R}_\mathcal{X}$ is equal to $\mathcal{O}_\mathcal{X}\langle \eth_x \rangle$, where $\eth_x = z\partial_x$. In particular, $\mathcal{D}_X = \mathcal{R}_\mathcal{X}/(z-1)\mathcal{R}_\mathcal{X}$.

Let us consider the category $\mathcal{R}$-Triples(X) whose objects are triples of the form $(\mathcal{M}', \mathcal{M}'', C)$, where $\mathcal{M}', \mathcal{M}''$ are coherent $\mathcal{R}_{\mathcal{X}}$-modules and C is a sesquilinear pairing between these modules, that is, for any point z_o on the circle $\mathbf{S}$ this is a pairing between the stalk of the sheaf $\mathcal{M}'_{z_o}$ and the conjugate stalk of $\mathcal{M}''_{-z_o}$, and this pairing takes values in the sheaf $\mathfrak{Db}_{X_\mathbb{R}}$ of distributions on X and is linear with respect to the action of holomorphic differential operators on $\mathcal{M}'_{z_o}$ and of antiholomorphic differential operators on the conjugate module of $\mathcal{M}''_{-z_o}$ under the natural action of the differential operators of both types on the distributions on X. Finally, this pairing must be continuous with respect to the point $z \in \mathbf{S}$. We treat C as a sesquilinear pairing of sheaves $\mathcal{M}'_{|S} \otimes_{\mathcal{O}_{\mathcal{X}|\mathbf{S}}} \overline{\mathcal{M}''_{|S}} \to \mathfrak{Db}_{X_{\mathbb{R}\times\mathbf{S}}/\mathbf{S}}$, where $\mathfrak{Db}_{X_{\mathbb{R}\times\mathbf{S}}/\mathbf{S}}$ stands for the sheaf of distributions on $X \times \mathbf{S}$ that are continuous with respect to $z \in \mathbf{S}$ and the conjugation is opposite to the usual conjugation on $\mathbf{S}$ (see § 1.5.a in [10]).

The notion of polarized regular twistor $\mathcal{D}$-module of weight $w \in \mathbb{Z}$ on X was defined in [10]. Namely, these are objects of the form $(\mathcal{T}, \mathcal{S})$, where $\mathcal{T} = (\mathcal{M}', \mathcal{M}'', C)$ are the triples discussed above and $\mathcal{S}$ (the so-called *polarization*) consists of two isomorphisms $\mathcal{M}'' \xrightarrow{\sim} \mathcal{M}'$. Some axioms are introduced (which we do not recall here; see Chapter 4 of [10]). Most arguments can be reduced to the case in which the weight w is equal to 0, $\mathcal{M}' = \mathcal{M}''$, and both isomorphisms in $\mathcal{S}$ are equal to Id. Objects of this kind are denoted simply by $(\mathcal{M}, \mathcal{M}, C, \mathrm{Id})$ or just by $(\mathcal{M}, \mathcal{M}, C)$.

Introducing such an object of weight 0, we obtain a $\mathcal{D}_X$-module by considering the quotient $\mathcal{M}/(z-1)\mathcal{M}$. Moreover, by restricting to X^{*an}, we obtain a holomorphic bundle V with connection ∇. Finally, it follows from the axioms that the sesquilinear pairing C enables one to define a metric h associated with V on the C^∞ bundle H and that this metric is *harmonic*. Moreover, this metric is *tame* by the regularity assumption. More precisely, using results of [11] and [1], the author proved in Chapter 5 of [10] that the category of polarized regular twistor $\mathcal{D}_X$-modules is equivalent to the category of tame harmonic bundles[5] (H, D_V, h) on X^{*an} with a certain parabolic structure (this structure is referred to as being 'of Deligne type' in [10]). According to Simpson [11], the latter category is equivalent to the category of semisimple bundles (H, D_V) with flat connection on X^{*an}.

The Fourier–Laplace transform of a given object of the form $(\mathcal{M}', \mathcal{M}'', C, \mathcal{S})$ is introduced in Chapter 8 of [10] (see also below). The transform is defined as a quadruple $(\widehat{\mathcal{M}}', \widehat{\mathcal{M}}'', \widehat{C}, \widehat{\mathcal{S}})$ over $\widehat{\mathbb{A}}^{1an} \times \mathbb{C}$. The main result of the paper is as follows.

Theorem 1. *If* $(\mathcal{M}, \mathcal{M}, C, \mathrm{Id})$ *is a polarized regular twistor* $\mathcal{D}_X$*-module of weight* 0*, then* $(\widehat{\mathcal{M}}, \widehat{\mathcal{M}}, \widehat{C}, \mathrm{Id})$ *is a polarized regular twistor* $\mathcal{D}$*-module of weight* 0 *over* $\widehat{\mathbb{A}}^{1an} \times \mathbb{C}$.

This statement can be directly extended to polarized regular twistor $\mathcal{D}$-modules of weight w. We note that a part of the theorem was already proved in Theorem 8.4.1 of [10], namely, the condition on cycles near $\tau = 0$. We are therefore mainly interested in the behaviour at points $\tau_o \neq 0$. The 'fibre' of $(\widehat{\mathcal{M}}, \widehat{\mathcal{M}}, \widehat{C})$ at $\tau = \tau_o = 0$ is obtained from that at $\tau = 1$ by a rescaling, that is, by a preliminary change of variable $t \to t/\tau_o$, because the kernel of the Fourier–Laplace transform is $e^{-t\tau/z}$.

We obtain the following assertion as a corollary.

[5] *Russian Editor's note*: That is, bundles with a tame harmonic metric.

Corollary 1. *If a flat bundle* (H, D_V) *is semisimple, then the Fourier–Laplace transform* $(\widehat{H}, D_{\widehat{V}})$ *on* $\widehat{X}^*$ *carries a harmonic metric with tame behaviour at* $\tau = 0$.

§ 2. Exponential twist of harmonic bundles and twistor $\mathfrak{D}$-modules

Let us recall the basic correspondences indicated in § 8.1.b of [10]. We keep the notation of § 1, but now we fix the point $\tau_o = 1$.

2.1. Exponential twist of smooth twistor structures. We begin with a triple (H, h, D_V) on X^*, rescale the metric h, and twist the connection D_V by defining

$$\begin{aligned} {}^F D_V &= e^t \circ D_V \circ e^{-t}, \quad \text{that is,} \quad {}^F D'_V = D'_V - dt, \quad {}^F D''_V = d'', \\ {}^F h &= e^{2\operatorname{Re} t} h. \end{aligned}$$

We recall that, in terms of the definitions in [11] and [12], if the triple (H, D_V, h) is harmonic on X^*, then so is the triple $(H, {}^F D_V, {}^F h)$. The Higgs field is defined by the formulae

$${}^F\theta'_E = \theta'_E - dt, \qquad {}^F\theta''_E = \theta''_E - d\overline{t},$$

and the metric connection ${}^F h$ given by ${}^F D_E = {}^F D'_E + {}^F D''_E$ by the formulae

$${}^F D_E = e^{-\overline{t}} \circ D_E \circ e^{\overline{t}}, \quad \text{that is,} \quad {}^F D'_E = D'_E, \quad {}^F D''_E = D''_E + d\overline{t}.$$

The exponential twist exists at the level of smooth twistor structures. As in [10], we denote by $\mathcal{C}^{\infty,an}_{\mathcal{X}^*}$ the sheaf on $\mathcal{X}^*$ of C^∞ functions holomorphic with respect to z. Let us consider the $\mathcal{C}^{\infty,an}_{\mathcal{X}^*}$-module[6] $\mathcal{H}^{an} = \mathcal{C}^{\infty,an}_{\mathcal{X}^*} \otimes_{\pi^{-1}\mathcal{C}^\infty_{X^*}} \pi^{-1} H$ equipped with the d'' operator

$${}^F\mathfrak{D}''_z = {}^F D''_E + z\, {}^F\theta''_E = \mathfrak{D}''_z + (1 - z) d\overline{t}. \tag{2.1}$$

We obtain a holomorphic subbundle ${}^F\mathcal{H}' = \ker {}^F\mathfrak{D}''_z \subset \mathcal{H}^{an}$ equipped with a z-connection given by ${}^F\mathfrak{D}'_z = z\,{}^F D'_E + {}^F\theta'_E = \mathfrak{D}'_z - dt$. We set

$${}^F\mathfrak{D}_z = {}^F\mathfrak{D}'_z + {}^F\mathfrak{D}''_z = \mathfrak{D}_z - dt + (1 - z) d\overline{t}.$$

Moreover, if $\pi\colon \mathcal{X}^* = X^* \times \mathbb{C} \to X^*$ is the natural projection, then the bundle $\mathcal{H}^{an}$ can be equipped with the metric $\pi^* h$ or the metric $\pi^{*F} h$. These metrics are constant with respect to z. We shall also consider the metric $e^{2\operatorname{Re}(z\overline{t})}\pi^* h$, which varies as z varies.

We have an isomorphism of locally free $\mathcal{C}^{\infty,an}_{\mathcal{X}^*}$-modules with metric and z-connection:

$$(\mathcal{H}^{an}, \pi^{*F} h, {}^F\mathfrak{D}_z) \xrightarrow{\cdot e^{(1-z)\overline{t}}} (\mathcal{H}^{an}, e^{2\operatorname{Re}(z\overline{t})}\pi^* h, \mathfrak{D}_z - dt). \tag{2.2}$$

This isomorphism sends the holomorphic subbundle ${}^F\mathcal{H}'$ to $\mathcal{H}' = \ker \mathfrak{D}''_z$.

It will also be useful to have a model related to the metric $\pi^* h$. This model is defined on the sheaf $\mathcal{C}^\infty_{\mathcal{X}^*}$ rather than on $\mathcal{C}^{\infty,an}_{\mathcal{X}^*}$. We set $\mathcal{H} = \mathcal{C}^\infty_{\mathcal{X}^*} \otimes_{\pi^{-1}\mathcal{C}^\infty_{X^*}} \pi^{-1} H$. There is an isomorphism

$$(\mathcal{H}, \pi^{*F} h, {}^F\mathcal{D}_z) \xrightarrow{\cdot e^{\overline{t} - 2i\operatorname{Im}(z\overline{t})}} (\mathcal{H}, \pi^* h, \mathfrak{D}_z - (1 + |z|^2) dt). \tag{2.3}$$

This isomorphism is not defined over $\mathcal{C}^{\infty,an}_{\mathcal{X}^*}$.

[6] In [10] we simply denoted this module by $\mathcal{H}$; here we stress its analytic dependence on z.

2.2. Exponential twist in $\mathcal{R}$-Triples(X^*). We recall the following definitions (see § 8.1.a in [10]). Let $\mathcal{M}$ be a left $\mathcal{R}_{\mathcal{X}}$-module, that is, an $\mathcal{O}_{\mathcal{X}}$-module with a flat relative meromorphic connection $\nabla_{\mathcal{X}/\mathbb{C}}$ (relative to z, that is, no differentiation with respect to z is carried out). We denote by $\mathcal{M}_{\text{loc}}$ the localized module along P, that is, $\mathcal{M}_{\text{loc}} = \mathcal{O}_{\mathcal{X}}[*(P \times \mathbb{C})] \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{M}$. The twisted $\mathcal{R}_{\mathcal{X}}$-module ${}^F\mathcal{M}_{\text{loc}} = \mathcal{M}_{\text{loc}} \otimes \mathcal{E}^{-t/z}$ is defined as the $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{M}_{\text{loc}}$ equipped with the twisted connection $e^{t/z} \circ \nabla_{\mathcal{X}/\mathbb{C}} \circ e^{-t/z}$.

Let $C\colon \mathcal{M}'_{|\mathbf{S}} \otimes_{\mathcal{O}_{\mathcal{X}|\mathbf{S}}} \overline{\mathcal{M}_{|\mathbf{S}}}'' \to \mathfrak{Db}_{X_\mathbb{R} \times \mathbf{S}/\mathbf{S}}$ be a sesquilinear pairing. If the restriction of $(\mathcal{M}', \mathcal{M}'', C)$ to $\mathcal{X}^*$ is a smooth twistor structure, then the restriction of C to $X^* \times \mathbf{S}$ takes values in $\mathcal{C}^{\infty,an}_{\mathcal{X}^*}$, and the extension C_{loc} of C to $\mathcal{M}'_{\text{loc}|\mathbf{S}} \otimes_{\mathcal{O}_{\mathcal{X}|\mathbf{S}}} \overline{\mathcal{M}''_{\text{loc}|\mathbf{S}}}$ takes values in the extension of $\mathcal{C}^{\infty,an}_{\mathcal{X}^*}$ formed by the functions on $\mathcal{X}^*$ which can be extended as distributions continuous with respect to $z \in \mathbf{S}$. Moreover, if we assume that $(\mathcal{M}', \mathcal{M}'', C)$ underlies a polarized regular twistor $\mathcal{D}$-module, then, using (5.3.3) in [10], we can see that C_{loc} takes values in the extension of $\mathcal{C}^{\infty,an}_{\mathcal{X}^*}$ formed by the functions on $\mathcal{X}^*$ having moderate growth near each puncture in P, locally uniformly with respect to $z \in \mathbf{S}$.

We note that the number $z\bar{t} - t/z$ is purely imaginary for any $z \in \mathbf{S}$. Then under the above assumption the map ${}^F C_{\text{loc}} := \exp(z\bar{t} - t/z) C_{\text{loc}}$ is a sesquilinear pairing on ${}^F\mathcal{M}'_{\text{loc}|\mathbf{S}} \otimes_{\mathcal{O}_{\mathcal{X}|\mathbf{S}}} \overline{{}^F\mathcal{M}''_{\text{loc}|\mathbf{S}}}$ taking values in the same sheaf of functions with moderate growth.

With a harmonic bundle (H, h, D_V) on X^* one can associate a smooth twistor structure $(\mathcal{H}', \mathcal{H}', \pi^* h_{\mathcal{H}'_{|\mathbf{S}} \otimes \overline{\mathcal{H}'_{|\mathbf{S}}}})$, where $\mathcal{H}' \subset \mathcal{H}^{an}$ is the kernel $\ker \mathfrak{D}''_z$ equipped with the $\mathcal{R}_{\mathcal{X}^*}$-structure given by the z-connection $\mathfrak{D}'_z$.

This harmonic bundle can be exponentially twisted as an object of the category $\mathcal{R}$-Triples(X^*), and the result is the triple

$$\big(\mathcal{H}', \mathcal{H}', \exp(z\bar{t} - t/z) \pi^* h_{\mathcal{H}'_{|\mathbf{S}} \otimes \overline{\mathcal{H}'_{|\mathbf{S}}}}\big),$$

where $\mathcal{H}'$ is equipped with the $\mathcal{R}_{\mathcal{X}^*}$-structure defined by the z-connection $\mathfrak{D}'_z - dt$.

The isomorphism (2.2) identifies the twisted harmonic bundle with the smooth twistor structure associated with $(H, {}^F h, {}^F D_V)$ (see Lemma 8.1.2 of [10]).

2.3. Exponential twist in $\mathcal{R}$-Triples(X). Let $\mathcal{M}$ be a left $\mathcal{R}_{\mathcal{X}}$-module. We denote by $\widetilde{\mathcal{M}}$ the localization of $\mathcal{M}$ *only at infinity*. Then ${}^F\mathcal{M}$ is defined as the twisted $\mathcal{R}_{\mathcal{X}}$-module $\widetilde{\mathcal{M}} \otimes \mathcal{E}^{-t/z}$ (the $\mathcal{R}_{\mathcal{X}}$-structure is defined as above). In particular, the module ${}^F\mathcal{M}$ is localized at ∞, and the module ${}^F\mathcal{M}_{\text{loc}}$ is the localization of ${}^F\mathcal{M}$ at $P \setminus \{\infty\}$. We know (see Proposition 8.3.1(i) in [10]) that the $\mathcal{R}_{\mathcal{X}}$-module ${}^F\mathcal{M}$ is coherent under a certain condition on $\mathcal{M}$ near ∞, and this condition is satisfied if $\mathcal{M}$ corresponds to a (polarized) regular twistor $\mathcal{D}$-module on X.

For a given (polarized) regular twistor $\mathcal{D}$-module $(\mathcal{M}', \mathcal{M}'', C)$ on X the definition of the sesquilinear pairing ${}^F C$ on ${}^F\mathcal{M}'_{|\mathbf{S}} \otimes_{\mathcal{O}_{\mathcal{X}|\mathbf{S}}} \overline{{}^F\mathcal{M}''_{|\mathbf{S}}}$ with values in $\mathfrak{Db}_{X_\mathbb{R} \times \mathbf{S}/\mathbf{S}}$ needs some care, because one must define a lifting of the localized distribution (or C^∞ function of moderate growth) ${}^F C_{\text{loc}}$ to distributions on X. In [10] one first defines a pairing ${}^{\mathcal{F}}C$ on the total exponential twist ${}^{\mathcal{F}}\mathcal{M}'_{|\mathbf{S}} \otimes \overline{{}^{\mathcal{F}}\mathcal{M}''_{|\mathbf{S}}}$ (where one must not forget the variable τ); the module ${}^F\mathcal{M}$ is regarded as the specialization of the module $\mathcal{F}\mathcal{M}$ at $\tau = 1$, and then the pairing ${}^F C$ is defined as the specialization (by means of the Mellin transform) of the pairing ${}^{\mathcal{F}}C$.

2.4. Restriction to the submanifold $z = z_o$. Let us analyze the behaviour of the above constructions under restriction to the submanifold $z = z_o$.

The restriction to the submanifold $z = z_o$ of the triple $(\mathcal{H}^{an}, \pi^{*F}h, {}^F\mathfrak{D}_z)$ is the bundle H equipped with the metric ${}^Fh$ and the z_o-connection ${}^F\mathfrak{D}_{z_o}$. The isomorphism (2.2) specializes to an isomorphism

$$(H, {}^Fh, {}^F\mathfrak{D}_{z_o}) \xrightarrow{\cdot e^{(1-z_o)\overline{t}}} (H, e^{2\,\mathrm{Re}(z_o\overline{t})}h, \mathfrak{D}_{z_o} - dt), \tag{2.4}$$

and the isomorphism (2.3) specializes to an isomorphism

$$(H, {}^Fh, {}^F\mathfrak{D}_{z_o}) \xrightarrow{\cdot e^{\overline{t}-2i\,\mathrm{Im}(z_o\overline{t})}} (H, h, \mathfrak{D}_{z_o} - (1+|z_o|^2)dt). \tag{2.5}$$

On the other hand, since the module $\mathcal{O}_{\mathcal{X}}[*(P\times\mathbb{C})]$ (or $\mathcal{O}_{\mathcal{X}}[*(\{\infty\}\times\mathbb{C})]$) is flat over the ring $\mathcal{O}_{\mathcal{X}}$, it follows that if $\mathcal{M}$ is a *strict* $\mathcal{R}_{\mathcal{X}}$-module (that is, if it is $\mathcal{O}_{\mathbb{C}}$-torsion-free), then so is its localization $\mathcal{M}_{\mathrm{loc}}$ or $\widetilde{\mathcal{M}}$. If we set $\mathfrak{M}_{z_o} = \mathcal{M}/(z-z_o)\mathcal{M}$, then the localization $_{\mathrm{loc}}$ or $\widetilde{\ }$ of the module $\mathfrak{M}_{z_o}$ is the restriction to $z = z_o$ of the corresponding localization of $\mathcal{M}$.

We introduce the twisted module ${}^F\mathfrak{M}_{z_o}$ for $z_o \neq 0$ as the module $\widetilde{\mathfrak{M}}_{z_o} \otimes \mathcal{E}^{-t/z_o}$ (that is, we twist the z_o-connection by adding $-dt$) and for $z_o = 0$ as the module $\widetilde{\mathfrak{M}}_0$ with the Higgs field obtained by adding $-dt$. In this case if $\mathcal{M}$ is strict, then ${}^F\mathfrak{M}_{z_o} = {}^F\mathcal{M}/(z-z_o){}^F\mathcal{M}$ and ${}^F\mathfrak{M}_{z_o,\mathrm{loc}} = {}^F\mathcal{M}_{\mathrm{loc}}/(z-z_o){}^F\mathcal{M}_{\mathrm{loc}}$.

3. Proof of Theorem 1

Let $(\mathcal{T}, \mathcal{S})$ be a polarized regular twistor $\mathcal{D}$-module of weight 0 on $\mathbb{P}^1$ (that is, an object of $\mathrm{MT}^r(\mathbb{P}^1, w)^{(p)}$; see [10]). We can assume that it is of the form $\mathcal{T} = (\mathcal{M}, \mathcal{M}, C)$ and $\mathcal{S} = (\mathrm{Id}, \mathrm{Id})$. The restriction of this module to X^* corresponds to a harmonic bundle (H, h, D_V). Using the notation of (1.2), we prove the following assertion in this section.

Proposition 1. *The complex* $\mathbf{R}\Gamma(X, \mathrm{DR}\,{}^F\mathcal{M})$ *has non-trivial cohomology only of degree* 0, *and its non-zero cohomology is a locally free* $\mathcal{O}_{\mathbb{C}}$*-module of finite rank* $\widehat{d}$.

3.1. Proof of Theorem 1. We recall (see Chapter 8 of [10]) that we set $\widetilde{\mathcal{M}} = \mathcal{O}_{\mathcal{X}}(*\infty)\otimes_{\mathcal{O}_{\mathcal{X}}}\mathcal{M}$, and if $p\colon X\times\widehat{X}\times\mathbb{C}\to X\times\mathbb{C} = \mathcal{X}$ and $\widehat{p}\colon X\times\widehat{X}\times\mathbb{C}\to\widehat{X}\times\mathbb{C} = \widehat{\mathcal{X}}$ denote the projections and $\otimes\,\mathcal{E}^{-t\tau/z}$ denotes the exponential twist of the $\mathcal{R}$-structure, then we write

$$\widehat{\mathcal{M}} := \widehat{p}_+p^+(\widetilde{\mathcal{M}}\otimes\mathcal{E}^{-t\tau/z}) = \widehat{p}^{\,0}_+p^+(\widetilde{\mathcal{M}}\otimes\mathcal{E}^{-t\tau/z}) := \widehat{p}^{\,0}_+\,{}^{\mathcal{F}}\mathcal{M}.$$

The sesquilinear pairing ${}^{\mathcal{F}}C$ on ${}^{\mathcal{F}}\mathcal{M}_{|\mathbf{S}}\otimes\overline{{}^{\mathcal{F}}\mathcal{M}_{|\mathbf{S}}}$ is defined in Chapter 8 of [10], and we can write $\widehat{C} = \widehat{p}^{\,0}_+\,{}^{\mathcal{F}}C$.

1. It follows from Theorem 8.4.1 of [10] that, along the submanifold $\tau = 0$, all necessary conditions for the existence of a polarized regular twistor $\mathcal{D}$-module (see Definition 4.1.2 in [10]) are satisfied.

2. The main question is now concerned with the behaviour of $\widehat{\mathcal{M}}$ away from the point $\tau = 0$. Let us fix some $\tau_o \neq 0$ in $\widehat{X}$. We recall Proposition 8.3.1(i) of [10]

claiming that the module $\widetilde{\mathcal{M}}\otimes\mathcal{E}^{-t\tau_o/z}$ is $\mathcal{R}_{\mathcal{X}}$-good. In fact, it suffices to take $\tau_o=1$ by the obvious homogeneity considerations. We denote by ${}^F\mathcal{M}$ the $\mathcal{R}_{\mathcal{X}}$-module $\widetilde{\mathcal{M}}\otimes\mathcal{E}^{-t/z}$.

3. Since the module ${}^{\mathcal{F}}\mathcal{M}$ is regular and strictly specializable along $\tau=\tau_o$ and since, according to [10] (Proposition 8.3.1(ii) and (iii), Theorem 3.1.8, and § 3.1.d), the following assertion holds by virtue of Proposition 1 (which holds for any $\tau_o\neq 0$).

Corollary 2. *For any $\tau_o\neq 0$ the module $\widehat{\mathcal{M}}$ is strictly specializable and regular along $\tau=\tau_o$, and for any $\alpha\in\mathbb{C}$*

$$\psi_{\tau-\tau_o,\alpha}\widehat{\mathcal{M}}=\begin{cases}0 & \text{if } \alpha\notin-\mathbb{N}^*,\\ \mathbf{R}^0\Gamma(X,\mathrm{DR}\,\widetilde{\mathcal{M}}\otimes\mathcal{E}^{-t\tau_o/z}) & \text{if } \alpha\in-\mathbb{N}^*.\end{cases}$$

4. This corollary implies that, near any $\tau_o\neq 0$, the module $\widehat{\mathcal{M}}$ is equal to the level -1 of its V-filtration along $\tau=\tau_o$. By the regularity, the module $\widehat{\mathcal{M}}$ is $\mathcal{O}_{\widehat{\mathcal{X}}}$-coherent, and since $\dim\widehat{\mathcal{M}}/(\tau-\tau_o)\widehat{\mathcal{M}}=\dim\psi_{\tau-\tau_o,\alpha}\widehat{\mathcal{M}}=\widehat{d}$ does not depend on $\tau_o\neq 0$, it follows that $\widehat{\mathcal{M}}$ is $\mathcal{O}_{\widehat{\mathcal{X}}}$-locally free of rank $\widehat{d}$ away from the point $\tau=0$. The characteristic variety of this module in $T^*(\widehat{X}\setminus\{0\})\times\mathbb{C}$ is equal to $\{\text{the zero section}\}\times\mathbb{C}$ and the characteristic variety in $T^*(\widehat{X})\times\mathbb{C}$ is contained in

$$\big(\text{the zero section of}\cup T_0^*\widehat{X}\big)\times\mathbb{C},$$

and thus the module $\widehat{\mathcal{M}}$ is holonomic (see Definition 1.2.4 in [10]). It also follows from the corollary that the S-decomposability (see Definition 3.5.1 in [10]) is trivially satisfied near $\tau_o\neq 0$. We have therefore obtained the condition (HSD) of (see Definition 4.1.2 in [10]).

5. At this step we know that the module $\widehat{\mathcal{M}}$ is $\mathcal{O}_{\widehat{\mathcal{X}}}$-locally free of finite rank away from $\tau=0$. By Lemma 1.5.3 in [10], this implies that $(\widehat{\mathcal{M}},\widehat{\mathcal{M}},\widehat{C})$ is a smooth object of the category $\mathcal{R}$-Triples$(\widehat{X}^*)$ on this domain. We claim that the pairing $\widehat{C}$ defines, by gluing, a family of trivial vector bundles on $\mathbb{P}^1$ parametrized by the punctured complex line $\widehat{X}^*$. This family is obtained from a C^∞ vector bundle $\widehat{H}$ on $\widehat{X}^*$ equipped with a Hermitian metric $\widehat{h}$ by using the correspondence described in Lemma 2.2.2 of [10]. By this construction, the metric turns out to be *harmonic*. By simple homogeneity considerations with respect to τ, it suffices to prove this property in some neighbourhood of $\tau=0$, which we still denote by $\widehat{X}^*$.

As is known by Theorem 8.4.1 of [10], the twistor properties are satisfied by the triple $\widehat{\mathcal{T}}=(\widehat{\mathcal{M}},\widehat{\mathcal{M}},\widehat{C})$ equipped with the polarization $\widehat{\mathcal{S}}=(\mathrm{Id},\mathrm{Id})$ along $\tau=0$, and we can apply the argument used in §§ 5.4.c–5.4.e of [10] to obtain the twistor property and the polarizability in some neighbourhood of $\tau=0$. This completes the proof of Theorem 1.

Let us now prove Proposition 1. Since ${}^F\mathcal{M}$ is a good $\mathcal{R}_{\mathcal{X}}$-module, we know *a priori* that the cohomology of the complex $\mathbf{R}\Gamma(X,\mathrm{DR}^F\mathcal{M})$ is $\mathcal{O}_{\mathbb{C}}$-coherent. Therefore, it suffices to prove that for any $z_o\in\mathbb{C}$ the complex $\mathbf{R}\Gamma(X,\mathrm{DR}^F\mathfrak{M}_{z_o})$ has cohomology only of degree 0 and that the dimension of the space $\mathbf{H}^0(X,\mathrm{DR}^F\mathfrak{M}_{z_o})$ is equal to $\widehat{d}$ (we recall that ${}^F\mathfrak{M}_{z_o}={}^F\mathcal{M}/(z-z_o){}^F\mathcal{M}$).

As in [15], we identify the complex $\mathrm{DR}^F\,\mathfrak{M}_{z_o}$ with an L^2 complex. This identification is *local* on X. The L^2 cohomology on X can then be obtained by the L^2-Hodge theory. The independence of the dimension of $\mathbf{H}^*(X, \mathrm{DR}^F\,\mathfrak{M}_{z_o})$ with respect to z_o will follow from the independence of the corresponding Laplacian with respect to z_o (one can extract this argument from [12]).

3.2. The meromorphic L^2 de Rham and Dolbeault complexes. In order to give a common proof which holds both if z_o is zero and if it is non-zero, it is convenient to consider the twisted module ${}^F\mathfrak{M}_{z_o} \otimes \mathcal{E}^{-c(z_o)t}$, where $c(z_o)$ stands for the usual conjugate of z_o, and thus we can write $|z_o|^2 = z_o c(z_o)$ (we keep the more traditional notation $\overline{z}_o$ for the 'geometric conjugate' $-1/z_o$). In other words, ${}^F\mathfrak{M}_{z_o} \otimes \mathcal{E}^{-c(z_o)t}$ is simply the $\mathcal{O}_{\mathcal{X}}$-module $\widetilde{\mathfrak{M}}_{z_o}$ equipped with the twisted z_o-connection $\mathfrak{D}'_{z_o} - (1+|z_o|^2)dt$.

We recall that the symbol ${}^F\mathfrak{M}_{\mathrm{loc},z_o}$ means the localized module of ${}^F\mathfrak{M}_{z_o}$ at all points of P (but localization at ∞ is unnecessary because ${}^F\mathfrak{M}_{z_o}$ is already localized at ∞). We consider the meromorphic L^2 complex of the form $\mathrm{DR}({}^F\mathfrak{M}_{\mathrm{loc},z_o} \otimes \mathcal{E}^{-c(z_o)t})_{(2)}$ obtained by taking sections of the sheaf $\mathfrak{M}_{\mathrm{loc},z_o}$ or the sheaf $\mathfrak{M}_{\mathrm{loc},z_o} \otimes \Omega^1_X$. These are locally L^2 sections, as well as their images under the connection $\mathfrak{D}'_{z_o} - (1+|z_o|^2)dt$ if one takes the metric h on the restriction V_{z_o} of the sheaf $\mathfrak{M}_{\mathrm{loc},z_o}$ to X^{*an} (V_{z_o} stands for the holomorphic subbundle of H determined by the d'' operator $\mathfrak{D}''_{z_o} = D''_V + (z_o - 1)\theta''_E$) and a metric locally equivalent to the Poincaré metric near each puncture in P on X^*. We have a natural morphism

$$\mathrm{DR}({}^F\mathfrak{M}_{\mathrm{loc},z_o} \otimes \mathcal{E}^{-c(z_o)t})_{(2)} \to \mathrm{DR}({}^F\mathfrak{M}_{z_o} \otimes \mathcal{E}^{-c(z_o)t}).$$

Indeed, this holds away from ∞, as was explained in § 6.2.a of [10] (this needs explanation, because it is unclear that the terms of the left-hand complex are contained in the corresponding terms of the right-hand complex). The inclusion is clear near the point at infinity, because the module ${}^F\mathfrak{M}_{z_o}$ is equal there to the module ${}^F\mathfrak{M}_{\mathrm{loc},z_o}$.

Lemma 3. *The natural morphism* $\mathrm{DR}({}^F\mathfrak{M}_{\mathrm{loc},z_o} \otimes \mathcal{E}^{-c(z_o)t})_{(2)} \to \mathrm{DR}({}^F\mathfrak{M}_{z_o} \otimes \mathcal{E}^{-c(z_o)t})$ *is a quasi-isomorphism.*

Proof. Away from the point at infinity, this was proved in Proposition 6.2.4 of [10]. We therefore consider the situation near ∞ with a local coordinate t' and omit the index 'loc', because the sheaf ${}^F\mathfrak{M}_{z_o}$ is equal to its localized module near $t' = 0$.

By the regularity assumption of the module $\mathcal{M}$ near ∞, we know that there is a local meromorphic basis $\mathbf{e}^{(z_o)}$ of $\widetilde{\mathfrak{M}}_{z_o}$ in which the connection matrix of $\mathfrak{D}'_{z_o}$ has a simple pole at $t' = 0$ (see (5.3.7) in [10]). By considering the maximal order of the poles of the coefficients in the basis $\mathbf{e}^{(z_o)}$ for a section of the sheaf $\widetilde{\mathfrak{M}}_{z_o}$ and using the term $(1+|z_o|^2)dt'/t'^2$ in the z_o-connection, we see that $\mathcal{H}^{-1}(\mathrm{DR}({}^F\mathfrak{M}_{z_o} \otimes \mathcal{E}^{-c(z_o)t})) = 0$, and hence $\mathcal{H}^{-1}(\mathrm{DR}({}^F\mathfrak{M}_{z_o} \otimes \mathcal{E}^{-c(z_o)t})_{(2)}) = 0$.

On the other hand, the same argument shows that any local section at $t' = 0$ of $\widetilde{\mathfrak{M}}_{z_o} \otimes \Omega^1_X$ with maximum order of a pole equal to k is equivalent, modulo the image of the operator $\mathfrak{D}'_{z_o} + (1+|e|{z_o}^2)dt'/t'^2$, to a section having a pole of maximum order $\leqslant k-1$. Iterating this process and using the moderate behaviour of the h-norm of each element in the basis $\mathbf{e}^{(z_o)}$, we see that such a section is equivalent to a section of $\widetilde{\mathfrak{M}}_{z_o} \otimes \Omega^1_X$ which is an L^2 section with respect to the metric h, or, equivalently, that the morphism

$$\mathcal{H}^0\big(\mathrm{DR}({}^F\mathfrak{M}_{z_o} \otimes \mathcal{E}^{-c(z_o)t})_{(2)}\big) \to \mathcal{H}^0\big(\mathrm{DR}({}^F\mathfrak{M}_{z_o} \otimes \mathcal{E}^{-c(z_o)t})\big)$$

is onto.

Finally, for a given local section of the sheaf $\widetilde{\mathfrak{M}}_{z_o} \otimes \Omega^1_X$ which is an L^2 section (with respect to the metric h) and belongs to the image of $(\mathfrak{D}'_{z_o} - (1+|z_o|^2))\widetilde{\mathfrak{M}}_{z_o}$, an argument of the same kind shows that this section is in the image of an L^2 section of $\widetilde{\mathfrak{M}}_{z_o}$; equivalently, the morphism

$$\mathcal{H}^0\big(\mathrm{DR}({}^F\mathfrak{M}_{\mathrm{loc},z_o} \otimes \mathcal{E}^{-c(z_o)t})_{(2)}\big) \to \mathcal{H}^0\big(\mathrm{DR}({}^F\mathfrak{M}_{\mathrm{loc},z_o} \otimes \mathcal{E}^{-c(z_o)t})\big)$$

is injective. This completes the proof of Lemma 3.

3.3. L^2 de Rham–Dolbeault lemma. We consider the C^∞ bundle H equipped with the metric h and with the z_o-connection $\mathfrak{D}_{z_o} - (1+|z_o|^2)dt$ (which we denote below by $\widetilde{\mathfrak{D}}_{z_o}$ for simplicity) together with the associated L^2 complex $\mathcal{L}^\bullet_{(2)}(H, h, \widetilde{\mathfrak{D}}_{z_o})$. In particular, we note that the d''-operator is $\mathfrak{D}''_{z_o}$, the corresponding holomorphic subbundle is V_{z_o}, and the extension of this holomorphic subbundle obtained by considering the sections with h-norm of moderate growth is $\mathfrak{M}_{\mathrm{loc},z_o}$ (see Corollary 5.3.1(1) in [10]).

The 'holomorphic' L^2 subcomplex is the following subcomplex of the L^2 complex $\mathcal{L}^\bullet_{(2)}(H, h, \widetilde{\mathfrak{D}}_{z_o})$:

$$0 \longrightarrow \ker \mathfrak{D}''^{(0)}_{z_o} \xrightarrow{\widetilde{\mathfrak{D}'}_{z_o}} \ker \mathfrak{D}''^{(1)}_{z_o} \cap \mathcal{L}^{(1,0)}_{(2)}(H, h, \mathfrak{D}''_{z_o}) \longrightarrow 0, \tag{3.1}$$

where $\mathfrak{D}''^{(k)}_{z_o}$ stands for the action of the operator $\mathfrak{D}''_{z_o}$ on $\mathcal{L}^k_{(2)}(H, h, \mathfrak{D}''_{z_o})$. Our objective in this subsection is to prove the following assertion.

Lemma 4 (L^2 de Rham–Dolbeault lemma). *Suppose that (H, h, D_V) is a tame harmonic bundle on X^{*an}. In this case the inclusion map of the holomorphic L^2 subcomplex* (3.1) *into $\mathcal{L}^\bullet_{(2)}(H, h, \widetilde{\mathfrak{D}}_{z_o})$ is a quasi-isomorphism of complexes.*

The proof is analogous to that of the Dolbeault lemma in [15] and is parallel to the proof of Theorem 6.2.5 in the preprint [10], to which we shall repeatedly refer. As above, we work near ∞ because the result away from the point ∞ is contained in §§ 6.2.d and 6.2.e of [10].

In the definition of the L^2 complex the L^2 condition on sections and the condition concerning the action of the anti-holomorphic part of the connection are the same as in §§ 6.2.d and 6.2.e of [10]. The L^2 condition on the derivative of sections is changed. The new term $(1+|z_o|^2)dt'/t'^2$ in the holomorphic part of the connection simplifies the proofs.

We use polar coordinates: $t' = re^{i\theta}$. Let us first recall some notation used in [10]. Near the point $t' = 0$ the bundle H is equipped with a $\mathfrak{D}''_{z_o}$-holomorphic basis $\mathbf{e}'^{(z_o)}$. The h-norms of the elements of this basis are of moderate growth near $t' = 0$. We denote these elements by $e'^{(z_o)}_{\beta,\ell,k}$, where $\beta = \beta' + i\beta''$ ranges over a finite set of complex numbers whose real parts β' belong to $[0,1[$, ℓ is an integer (the weight of the element), and k is an index used to distinguish different elements having the same data β and ℓ. Let Θ'_{z_o} be the connection matrix of $\mathfrak{D}'_{z_o}$ in the basis. This matrix can be represented as the sum of a diagonal part and a nilpotent part, $\Theta'_{z_o,\mathrm{diag}} + \Theta'_{z_o,\mathrm{nilp}}$, with

$$\begin{aligned}\Theta'_{z_o,\mathrm{diag}} &= \bigoplus_{\beta}(q_{\beta,\zeta_o}+\beta)\star z\,\mathrm{Id}\,\frac{dt'}{t'}\,,\\ \Theta'_{z_o,\mathrm{nilp}} &= \big[\mathrm{Y}+P(t,z)\big]\frac{dt'}{t'},\end{aligned}$$

where $\mathrm{Y} = \big(\bigoplus_\beta \mathrm{Y}_\beta\big)$ and q_{β,ζ_o} stands for an integer chosen in such a way that the number $\ell_{z_o}(q_{\beta,\zeta_o}+\beta) := q_{\beta,\zeta_o}+\beta'-\zeta_o\beta''$ belongs to $[0,1[$, ζ_o being the imaginary part of z_o. Let the basis be indexed so that $\mathrm{Y}(e'^{(z_o)}_{\beta,\ell,k}) = e'^{(z_o)}_{\beta,\ell-2,k}$ for any ℓ and k and let the term $P(t,z_o)$ be given by the formula (6.2.7) in [10].

We recall the notation $\widetilde{\mathfrak{D}}_{z_o} = \mathfrak{D}_{z_o} + (1+|z_o|^2)dt'/t'^2$. Then $\widetilde{\Theta}'_{z_o,\mathrm{nilp}} = \Theta'_{z_o,\mathrm{nilp}}$ and $\widetilde{\Theta}'_{z_o,\mathrm{diag}} = \Theta'_{z_o,\mathrm{diag}} + (1+|z_o|^2)\,\mathrm{Id}\,dt'/t'^2$ in an obvious notation.

Vanishing of H^2. First, we can apply Lemma 6.2.11 of [10] with a fixed value $z=z_o$ without any modification. The entire proof is thus reduced to showing that if the expression $f(r)e'_{\beta,\ell,k}\dfrac{dt'}{t'}\wedge\dfrac{d\overline{t'}}{\overline{t'}}$ defines a local section of the sheaf $\mathcal{L}^2_{(2)}(H)$ for any β with $\ell_{z_o}(q_{\beta,\zeta_o}+\beta)=0$ and any $\ell\leqslant -1$ (in fact, it suffices to use $\ell=-1$, because z is equated to z_o here), then this section belongs to the image of the operator $\widetilde{\mathfrak{D}}_{z_o}$.

We note that

$$\begin{aligned}\widetilde{\mathfrak{D}}_{z_o}\bigg(t'f(r)e'_{\beta,\ell,k}\bigg(z_o\frac{dt'}{t'}+\frac{d\overline{t'}}{\overline{t'}}\bigg)\bigg) &= \big(1+|z_o|^2+z_o+(\beta\star z_o)t'\big)f(r)e'_{\beta,\ell,k}\frac{dt'}{t'}\wedge\frac{d\overline{t'}}{\overline{t'}}\\ &\quad+\Theta'_{z_o,\mathrm{nilp}}\bigg(t'f(r)e'_{\beta,\ell,k}\bigg(z_o\frac{dt'}{t'}+\frac{d\overline{t'}}{\overline{t'}}\bigg)\bigg).\end{aligned}$$

As in [15] and [10], one can readily see that the last term is in L^2. Thanks to the factor t', this term belongs to the image of the operator $\widetilde{\mathfrak{D}}''_{z_o} = \mathfrak{D}''_{z_o}$ (see Lemma 6.2.11 in [10]). For the same reason, the part multiplied by t' in the middle term is in the image of $\mathfrak{D}''_{z_o}$. Hence, both expressions belong to the image of $\widetilde{\mathfrak{D}}_{z_o}$. To complete the proof, it remains to note that the constant $1+|z_o|^2+z_o$ cannot vanish.

Computation of H^1. By the previous result, the L^2 complex $\mathcal{L}^\bullet_{(2)}(H,h,\widetilde{\mathfrak{D}}_{z_o})$ is quasi-isomorphic to its subcomplex

$$0\longrightarrow \mathcal{L}^0_{(2)}(H,h,\widetilde{\mathfrak{D}}_{z_o}) \xrightarrow{\widetilde{\mathfrak{D}}_{z_o}} \ker\widetilde{\mathfrak{D}}^{(1)}_{z_o}\longrightarrow 0.$$

Let us now prove an analogue of Lemma 6.2.13 in [10]. That is, we claim that any local section $\psi\,dt/t+\varphi\,d\overline{t}/\overline{t}$ in $\ker\widetilde{\mathfrak{D}}^{(1)}_{z_o}\subset\mathcal{L}^1_{(2)}(H,h,\widetilde{\mathfrak{D}}_{z_o})$ can be represented as the sum of a term in $\mathrm{Image}\,\widetilde{\mathfrak{D}}_{z_o}$ and a term in $\mathcal{L}^{(1,0)}_{(2)}(H,h)\cap\ker\widetilde{\mathfrak{D}}^{(1)}_{z_o}$.

The first part of the proof of Lemma 6.2.13 in [10] can be applied similarly to the present situation, and this reduces the proof to the case in which we start from a local section $\omega = \psi \dfrac{dt'}{t'} + \varphi \dfrac{d\overline{t'}}{\overline{t'}}$ in $\ker \widetilde{\mathfrak{D}}^{(1)}_{z_o}$, where $\varphi = \sum_{\beta,\ell,k} \varphi_{\beta,\ell,k}(r) e'^{(z_o)}_{\beta,\ell,k}$, and ω satisfies the equation $\widetilde{\mathfrak{D}}_{z_o}\omega = 0$.

Further, we consider the coefficient of $e^{-i\theta} e'^{(z_o)}_{\beta,\ell,k} \dfrac{dt'}{t'} \wedge \dfrac{d\overline{t'}}{\overline{t'}}$ in the relation

$$\mathfrak{D}''_{z_o}\left(\psi \frac{dt'}{t'}\right) + \widetilde{\mathfrak{D}}'_{z_o}\left(\varphi \frac{d\overline{t'}}{\overline{t'}}\right) = 0.$$

Denoting by $\psi_{\beta,\ell,k;-1}(r)$ the coefficient of $e^{-i\theta}$ in the Fourier expansion of $\psi_{\beta,\ell,k}$, we see that for any β, ℓ, k

$$\begin{aligned}\varphi_{\beta,\ell,k}(r) e'^{(z_o)}_{\beta,\ell,k} \frac{d\overline{t'}}{\overline{t'}} &= \frac{1}{2} r(r\partial_r - 1)\psi_{\beta,\ell,k;-1}(r) e'^{(z_o)}_{\beta,\ell,k} \frac{d\overline{t'}}{\overline{t'}} \\ &= \mathfrak{D}''_{z_o}\big(re^{-2i\theta}\psi_{\beta,\ell,k;-1}(r) e'^{(z_o)}_{\beta,\ell,k}\big).\end{aligned}$$

Since the local section $\psi dt'/t'$ is an L^2 section, it follows that $r\psi$ is also, and hence $\widetilde{\mathfrak{D}}'_0(re^{-2i\theta}\psi_{\beta,\ell,k;-1} e'^{(z_o)}_{\beta,\ell,k})$ is also an L^2 section.

This computation shows that ω is equivalent modulo $\operatorname{Image} \widetilde{\mathfrak{D}}_{z_o}$ to a $(1,0)$-section which is an L^2 section and belongs to $\ker \widetilde{\mathfrak{D}}^{(1)}_{z_o}$ (because $\widetilde{\mathfrak{D}}_{z_o}\omega = 0$), as was expected.

3.4. End of the proof of Proposition 1. We present the proof in four steps.

1. Arguing exactly as in § 6.2.f of [10], we show that the 'holomorphic' L^2 complex (3.1) is equal to its subcomplex $\mathrm{DR}({}^F\mathfrak{M}_{\mathrm{loc},z_o} \otimes \mathcal{E}^{-c(z_o)t})_{(2)}$. By the coherence, the hypercohomology of the complex $\mathrm{DR}({}^F\mathfrak{M}_{z_o} \otimes \mathcal{E}^{-c(z_o)t})$ is finite-dimensional. By Lemma 3 and the above arguments, so is the hypercohomology of the holomorphic L^2 complex (3.1).

2. It follows from Lemma 4 and the previous result that the cohomology of the complex of sections $\Gamma(X, \mathcal{L}^\bullet_{(2)}(H, h, \widetilde{\mathfrak{D}}_{z_o}))$ is finite-dimensional. According to the isometry (2.5), the cohomology of the complex $\Gamma(X, \mathcal{L}^\bullet_{(2)}(H, {}^F h, {}^F\mathfrak{D}_{z_o}))$ is also finite-dimensional. We can therefore apply Hodge theory to this L^2 cohomology. The corresponding space of harmonic k-forms ($k = 0, 1, 2$) is finite-dimensional, and its dimension does not depend on z_o, because the Laplacian of ${}^F\mathfrak{D}_{z_o}$ with respect to the metric ${}^F h$ is essentially independent of z_o, since the triple $(H, {}^F h, {}^F D_V)$ is harmonic.

3. Arguing in the reverse direction, we see that the dimension of the space $\mathbf{H}^k(X, \mathrm{DR}({}^F\mathfrak{M}_{z_o} \otimes \mathcal{E}^{-c(z_o)t}))$ ($k = -1, 0, 1$) does not depend on z_o. If $z_o = 1$, then the non-trivial cohomology is of degree 0 only (this is well known for a regular holonomic $\mathcal{D}_X$-module twisted by an exponential $e^{\lambda t}$ with $\lambda \in \mathbb{C}^*$). This is therefore true for any z_o; moreover, the dimension of $\mathbf{H}^0$ is independent of z_o.

4. It remains to note that the hypercohomologies of the complexes $\mathrm{DR}({}^F\mathfrak{M}_{z_o} \otimes \mathcal{E}^{-c(z_o)t})$ and $\mathrm{DR}({}^F\mathfrak{M}_{z_o})$ are of the same dimension. This is clear if $z_o = 0$, because the objects are equal in this case. On the other hand, if $z_o \neq 0$, then we reduce the problem to $\mathcal{D}_X$-modules. Working algebraically, we reduce the problem to proving

the following fact: for a given regular holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module the dimension of the cokernel of the operator

$$\partial_t - \lambda\colon M \to M$$

does not depend on $\lambda \in \mathbb{C}^*$. This follows from the regularity of the module M at infinity. ∎

Remark 1. One can give another proof of Proposition 1 if $z_o \neq 0$ by using the z_o-connection $\mathfrak{D}_{z_o} - dt$ on H with the metric $e^{2\operatorname{Re}(z_o\overline{t})}h$. This proof would be analogous to that in [8] and one can use the isometry (2.4) instead of (2.5).[7] Nevertheless, the intermediate steps will be different, because an analogue of Lemma 3 in which the L^2 condition is taken with respect to the metric $e^{2\operatorname{Re}(z_o\overline{t})}h$ fails. As in [8], the lemma works in the space obtained from X by a blowing-up at infinity over the reals. The comparison between various complexes must be made on this space. However, such a proof seems to have no extension to the case $z_o = 0$, and we do not present it here for that reason.

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[7] We use this opportunity to correct a minor mistake in [8]: the inequality on page 1283 should read

$$\int_\rho^{r_1} r^{2\beta}|\log r|^{k-2}\psi(r)\,\frac{dr}{r} \leqslant \rho^{2\beta}|\log\rho|^{k-2}\psi(\rho)(|\log\rho| - |\log r_1|),$$

and the constant C is bounded above by the quantity $4|\log r_1|^{-1} < +\infty$. Similarly, on page 1284, line 5, the constant C is bounded above by $4\kappa(\epsilon)|\log r_1|^{-1} < +\infty$.

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