ON THE DE RHAM COMPLEX OF MIXED TWISTOR $\mathscr{D}$-MODULES

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Abstract. Given a complex manifold S, we introduce for each complex manifold X a t-structure on the bounded derived category of $\mathbb{C}$-constructible complexes of $\mathscr{O}_S$-modules on $X\times S$. We prove that the de Rham complex of a holonomic $\mathscr{D}_{X\times S/S}$-module which is $\mathscr{O}_S$-flat as well as its dual object is perverse relatively to this t-structure. This result applies to mixed twistor $\mathscr{D}$-modules.

1. Introduction

Given a vector bundle V of rank $d\geqslant 1$ with an integrable connection $\nabla: V\to\Omega^1_X\otimes V$ on a complex manifold X of complex dimension n, the sheaf of horizontal sections $V^\nabla=\ker\nabla$ is a locally constant sheaf of d-dimensional $\mathbb{C}$-vector spaces, and is the only nonzero cohomology sheaf of the de Rham complex $\mathrm{DR}_X(V,\nabla)=(\Omega^\bullet_X\otimes V,\nabla)$. Assume moreover that (V,∇) is equipped with a harmonic metric in the sense of [19, p. 16]. The twistor construction of [20] produces then a holomorphic bundle $\mathscr{V}$ on the product space $\mathscr{X}=X\times\mathbb{C}$, where the factor $\mathbb{C}$ has coordinate z, together with a holomorphic flat z-connection. By restricting to $\mathscr{X}^*:=X\times\mathbb{C}^*$, giving such a z-connection on $\mathscr{V}^*:=\mathscr{V}_{|\mathscr{X}^*}$ is equivalent to giving a flat relative connection ∇ with respect to the projection $p:\mathscr{X}^*\to\mathbb{C}^*$. Similarly, the relative de Rham complex $\mathrm{DR}_{\mathscr{X}^*/\mathbb{C}^*}(\mathscr{V}^*,\nabla)$ has cohomology in degree zero at most, and $(\mathscr{V}^*)^\nabla:=\ker\nabla$ is a locally constant sheaf of locally free $p^{-1}\mathscr{O}_{\mathbb{C}^*}$-modules of rank d.

Holonomic $\mathscr{D}_X$-modules generalize the notion of a holomorphic bundle with flat connection to objects having (possibly wild) singularities, and a well-known theorem of Kashiwara [2] shows that the solution complex of such a holonomic $\mathscr{D}_X$-module has $\mathbb{C}$-constructible cohomology, from which one can deduce that the de Rham complex is of the same kind and more precisely that both are $\mathbb{C}$-perverse sheaves on X up to a shift by $\dim X$.

The notion of a holonomic $\mathscr{D}_X$-module with a harmonic metric has been formalized in [14] and [11] under the name of pure twistor $\mathscr{D}$-module (this

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generalizes holonomic $\mathscr{D}_X$-modules with regular singularities), and then in [15] and [12] under the name of wild twistor $\mathscr{D}$-modules (this takes into account arbitrary irregular singularities). More recently, Mochizuki [13] has fully developed the notion of a mixed (possibly wild) twistor $\mathscr{D}$-module. When restricted to $\mathscr{X}^*$, such an object contains in its definition two holonomic $\mathscr{D}_{\mathscr{X}^*/\mathbb{C}^*}$-modules, and we say that both underlie a mixed twistor $\mathscr{D}$-module.

The main result of this article concerns the de Rham complex and the solution complex of such objects.

Theorem 1.1. *The de Rham complex and the solution complex of a $\mathscr{D}_{\mathscr{X}^*/\mathbb{C}^*}$-module underlying a mixed twistor $\mathscr{D}$-module are perverse sheaves of $p^{-1}\mathscr{O}_{\mathbb{C}^*}$-modules (up to a shift by* $\dim X$*).*

In Section 2, we define the notion of relative constructibility and perversity. This applies to the more general setting where $p : \mathscr{X}^* \to \mathbb{C}^*$ is replaced by a projection $p_X : \mathscr{X} = X \times S \to S$, where S is any complex manifold. We usually set $p = p_X$ when X is fixed. On the other hand, we call *holonomic* any coherent $\mathscr{D}_{X\times S/S}$-module whose relative characteristic variety in $T^*(X \times S/S) = (T^*X) \times S$ is contained in a variety $\Lambda \times S$, where Λ is a conic Lagrangian variety in T^*X. We say that a $\mathscr{D}_{X\times S/S}$-module is *strict* if it is $p^{-1}\mathscr{O}_S$-flat.

Theorem 1.2. *The de Rham complex and the solution complex of a strict holonomic $\mathscr{D}_{X\times S/S}$-module whose dual is also strict are perverse sheaves of $p^{-1}\mathscr{O}_S$-modules (up to a shift by* $\dim X$*).*

A $\mathscr{D}_{\mathscr{X}^*/\mathbb{C}^*}$-module $\mathscr{M}$ underlying a mixed twistor $\mathscr{D}$-module is strict and holonomic (see [13]). Moreover, Mochizuki has defined a duality functor on the category of mixed twistor $\mathscr{D}$-modules, proving in particular that the dual of $\mathscr{M}$ as a $\mathscr{D}_{\mathscr{X}^*/\mathbb{C}^*}$-module is also strict holonomic. Therefore, these results together with Theorem 1.2 imply Theorem 1.1.

Note that, while our definition of perverse objects in the bounded derived category $\mathsf{D}^{\mathrm{b}}(p^{-1}\mathscr{O}_S)$ intends to supply a notion of holomorphic family of perverse sheaves, we are not able, in the case of twistor $\mathscr{D}$-modules, to extend this notion to the case when the parameter $z \in \mathbb{C}^* = S$ also achieves the value zero, and to define a perversity property in the Dolbeault setting of [19] for the associated Higgs module.

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2. Relative constructibility in the case of a projection

We keep the setting as above, but X is only assumed to be a real analytic manifold. Given a real analytic map $f : Y \to X$ between real analytic manifolds, we will denote by f_S (or f if the context is clear) the map $f \times \mathrm{id}_S : Y \times S \to X \times S$.

2.1. **Sheaves of $\mathbb{C}$-vector spaces and of $p^{-1}\mathscr{O}_S$-modules.** Let $f : Y \to X$ be such a map. There are functors $f^{-1}, f^!, Rf_*, Rf_!$ between $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X\times S})$ and $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{Y\times S})$, and functors $f_S^{-1}, f_S^!, Rf_{S,*}, Rf_{S,!}$ between $\mathsf{D}^{\mathrm{b}}(p_X^{-1}\mathscr{O}_S)$ and $\mathsf{D}^{\mathrm{b}}(p_Y^{-1}\mathscr{O}_S)$. These functors correspond pairwise through the forgetful functor $\mathsf{D}^{\mathrm{b}}(p_X^{-1}\mathscr{O}_S) \to \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X\times S})$. Indeed, this is clear except for $f_S^!$ and $f^!$. To check it, one decomposes f as a closed immersion and a projection. In the first case, the compatibility follows from the fact that both are equal to $f^{-1}R\Gamma_{f(X)}$ (see [6, Prop. 3.1.12]) and for the case of a projection one uses [6, Prop. 3.1.11 & 3.3.2]. We note also that the Poincaré-Verdier duality theorem [6, Prop. 3.1.10] holds on $\mathsf{D}^{\mathrm{b}}(p^{-1}\mathscr{O}_S)$ (see [6, Rem. 3.1.6(i)]). From now on, we will write f^{-1}, etc. instead of f_S^{-1}, etc.

The ring $p_X^{-1}\mathscr{O}_S$ is Noetherian, hence coherent (see [4, Prop. A.14]). For each $s_o \in S$ let us denote by $\mathfrak{m}_{s_o}$ the ideal of sections of $\mathscr{O}_S$ vanishing at s_o and by $i_{s_o}^\star$ the functor

$$\begin{aligned}\mathrm{Mod}(p_X^{-1}\mathscr{O}_S) &\longmapsto \mathrm{Mod}(\mathbb{C}_X)\\ F &\longmapsto F \otimes_{p_X^{-1}\mathscr{O}_S} p_X^{-1}(\mathscr{O}_S/\mathfrak{m}_{s_o}).\end{aligned}$$

This functor will be useful for getting properties of $\mathsf{D}^{\mathrm{b}}(p_X^{-1}\mathscr{O}_S)$ from well-known properties of $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_X)$.

Proposition 2.1. *Let F and F' belong to $\mathsf{D}^{\mathrm{b}}(p_X^{-1}\mathscr{O}_S)$. Then, for each $s_o \in S$ there is a well-defined natural morphism*

$$Li_{s_o}^*(R\mathscr{H}om_{p^{-1}(\mathscr{O}_S)}(F, F')) \to R\mathscr{H}om_{\mathbb{C}_X}(Li_{s_o}^*(F), Li_{s_o}^*(F'))$$

which is an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_X)$.

Proof. Let us fix $s_o \in S$. The existence of the morphism follows from [4, (A.10)]. Moreover, since $p_X^{-1}\mathscr{O}_S$ is a coherent ring as remarked above and $p_X^{-1}(\mathscr{O}_S/\mathfrak{m}_{s_o})$ is $p_X^{-1}\mathscr{O}_S$-coherent, we can apply the argument given after (A.10) in loc. cit. to show that it is an isomorphism. q.e.d.

Proposition 2.2. *Let F and F' belong to $\mathsf{D}^{\mathrm{b}}(p_X^{-1}\mathscr{O}_S)$ and let $\phi : F \to F'$ be a morphism. Assume the following conditions:*

(1) *for all $j \in \mathbb{Z}$ and $(x, s) \in X \times S$, $\mathscr{H}^j(F)_{(x,s)}$ and $\mathscr{H}^j(F')_{(x,s)}$ are of finite type over $\mathscr{O}_{S,s}$,*
(2) *for all $s_o \in S$, the natural morphism*

$$Li_{s_o}^*(\phi) : Li_{s_o}^*(F) \to Li_{s_o}^*(F')$$

is an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_X)$.

Then ϕ is an isomorphism.

Proof. It is enough to prove that the mapping cone of ϕ is quasi-isomorphic to 0. So we are led to proving that for $F \in \mathsf{D}^{\mathrm{b}}(p^{-1}\mathscr{O}_S)$, if $\mathscr{H}^j(F)_{(x,s)}$ are of finite type over $\mathscr{O}_{S,s}$ for all $(x, s) \in X \times S$, and $Li_{s_o}^*(F)$ is quasi-isomorphic to 0 for each $s_o \in S$, then F is quasi-isomorphic to 0.

We may assume that S is an open subset of $\mathbb{C}^n$ with coordinates $s^1, \dots, s^n$ and we will argue by induction on n. Assume $n = 1$. For such an F, for each $s_o \in S$ and any $j \in \mathbb{Z}$ the morphism $(s^1 - s^1_o) : \mathscr{H}^j(F) \to \mathscr{H}^j(F)$ is an isomorphism, hence $\mathscr{H}^j(F)/(s^1 - s^1_o)\mathscr{H}^j(F) = 0$ and by Nakayama's Lemma, for any $x \in X$, $\mathscr{H}^j(F)_{(x,s^1_o)} = 0$ and the result follows. For $n \geqslant 2$, the cone F' of the morphism $(s^n - s^n_o) : F \to F$ also satisfies $Li^*_{s'_o} F' = 0$ for any $s'_o = (s^1_o, \dots, s^{n-1}_o)$, hence is zero by induction, so we can argue as in the case $n = 1$. q.e.d.

2.2. **S-locally constant sheaves.** We say that a sheaf F of $\mathbb{C}$-vector spaces (resp. $p_X^{-1}\mathscr{O}_S$-modules) on $X \times S$ is *S-locally constant* if, for each point $(x, s) \in X \times S$, there exists a neighbourhood $U = V_x \times T_s$ of (x, s) and a sheaf $G^{(x,s)}$ of $\mathbb{C}$-vector spaces (resp. $\mathscr{O}_S$-modules) on T_s, such that $F_{|U} \simeq p_{V_x}^{-1} G^{(x,s)}$. The category of S-locally constant sheaves is an abelian full subcategory of that of sheaves of $\mathbb{C}_{X\times S}$-vector spaces (resp. $p^{-1}\mathscr{O}_S$-modules), which is stable by extensions in the respective categories, by $\mathscr{H}om$ and tensor products. Moreover, if $\pi : Y \times X \times S \to Y \times S$ is the projection, with X contractible, then, if F' is S-locally constant on $Y \times X \times S$,

- $\pi_* F'$ is S-locally constant on $Y \times S$,
- $R^k \pi_* F' = 0$ if $k > 0$,
- $F' \simeq \pi^{-1}\pi_* F'$.

Applying this to $Y = \{\mathrm{pt}\}$, we find that, if F is S-locally constant, then for each $x \in X$ there exist a connected neighbourhood V_x of x and a $\mathbb{C}_S$-module (resp. $\mathscr{O}_S$-module) $G^{(x)}$ such that $F = p_{V_x}^{-1} G^{(x)}$, and one has $G^{(x)} = p_{V_x,*} F_{|V_x \times S} = F_{|\{x\}\times S}$. We shall also denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{lc}}(p_X^{-1}\mathbb{C}_S)$ (resp. $\mathsf{D}^{\mathrm{b}}_{\mathrm{lc}}(p_X^{-1}\mathscr{O}_S)$) the bounded triangulated category whose objects are the complexes having S-locally constant cohomology sheaves. Similarly, for such a complex F we have $F_{|V_x\times S} \simeq p_{V_x}^{-1} Rp_{V_x,*} F_{|V_x\times S} \simeq p_{V_x}^{-1} F_{|\{x\}\times S}$.

We conclude from the previous remarks, by using the natural forgetful functor $\mathsf{D}^{\mathrm{b}}(p_X^{-1}\mathscr{O}_S) \to \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X\times S})$:

Lemma 2.3.

(1) *An object F of $\mathsf{D}^{\mathrm{b}}(p_X^{-1}\mathscr{O}_S)$ belongs to $\mathsf{D}^{\mathrm{b}}_{\mathrm{lc}}(p_X^{-1}\mathscr{O}_S)$ if and only if, when regarded as an object of $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X\times S})$, it belongs to $\mathsf{D}^{\mathrm{b}}_{\mathrm{lc}}(p_X^{-1}\mathbb{C}_S)$.*

(2) *For any object F of $\mathsf{D}^{\mathrm{b}}_{\mathrm{lc}}(p_X^{-1}\mathscr{O}_S)$ and for any $s_o \in S$, $Li^*_{s_o} F$ belongs to $\mathsf{D}^{\mathrm{b}}_{\mathrm{lc}}(\mathbb{C}_X)$.*

2.3. **S-weakly $\mathbb{R}$-constructible sheaves.** As long as the manifold X is fixed, we shall write p instead of p_X.

Definition 2.4. Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X\times S})$ (resp. $F \in \mathsf{D}^{\mathrm{b}}(p^{-1}\mathscr{O}_S)$). We shall say that F is S-weakly $\mathbb{R}$-constructible if there exists a subanalytic μ-stratification (X_α) of X (see [6, Def. 8.3.19]) such that, for all $j \in \mathbb{Z}$, $\mathscr{H}^j(F)|_{X_\alpha\times S}$ is S-locally constant.

This condition characterizes a full triangulated subcategory $\mathsf{D}^{\mathrm{b}}_{\text{w-}\mathbb{R}\text{-c}}(p^{-1}\mathbb{C}_S)$ (resp. $\mathsf{D}^{\mathrm{b}}_{\text{w-}\mathbb{R}\text{-c}}(p^{-1}\mathscr{O}_S)$) of $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X\times S})$ (resp. $\mathsf{D}^{\mathrm{b}}(p^{-1}\mathscr{O}_S)$). Due to Lemma 2.3, an object F of $\mathsf{D}^{\mathrm{b}}(p^{-1}\mathscr{O}_S)$ is in $\mathsf{D}^{\mathrm{b}}_{\text{w-}\mathbb{R}\text{-c}}(p^{-1}\mathscr{O}_S)$ if and only if it belongs to

$\mathsf{D}^{\mathrm{b}}_{\text{w-}\mathbb{R}\text{-c}}(p^{-1}\mathbb{C}_S)$ when considered as an object of $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X\times S})$. As a consequence, this condition is independent of the choice of the μ-stratification. By mimicking for $\mathsf{D}^{\mathrm{b}}_{\text{w-}\mathbb{R}\text{-c}}(p^{-1}\mathbb{C}_S)$ the proof of [6, Prop. 8.4.1 & Th. 8.4.2] and according to the previous remark for $\mathsf{D}^{\mathrm{b}}_{\text{w-}\mathbb{R}\text{-c}}(p^{-1}\mathscr{O}_S)$, we obtain:

Proposition 2.5. *Let F belong to $\mathsf{D}^{\mathrm{b}}(p^{-1}\mathscr{O}_S)$.*

(1) *Assume that F is S-weakly $\mathbb{R}$-constructible on X. Then, given a μ-stratification $X=\bigsqcup_\alpha X_\alpha$ of X, (X_α) is adapted to F if and only if $SS(F)\subset(\bigsqcup_\alpha T^*_{X_\alpha}X)\times T^*S$.*
(2) *F is S-weakly $\mathbb{R}$-constructible on X if and only if there exists a closed conic subanalytic Lagrangian subset Λ of T^*X such that $SS(F)\subset \Lambda\times T^*S$.*

Proposition 2.6. *Let $F\in\mathsf{D}^{\mathrm{b}}_{\text{w-}\mathbb{R}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ and let $s_o\in S$. Then $Li^*_{s_o}(F)\in\mathsf{D}^{\mathrm{b}}_{\text{w-}\mathbb{R}\text{-c}}(\mathbb{C}_X)$.*

Proof. Let $i_\alpha: X_\alpha\hookrightarrow X$ denote the locally closed inclusion of a stratum of an adapted stratification (X_α). It is enough to observe that, for each α, we have $i_\alpha^{-1}Li^*_{s_o}(F)\simeq Li^*_{s_o}(i^{-1}_{\alpha,S}F)$, and to apply Lemma 2.3(2). q.e.d.

Let now Y be another real analytic manifold and consider a real analytic map $f:Y\to X$. The following statements for objects of $\mathsf{D}^{\mathrm{b}}_{\text{w-}\mathbb{R}\text{-c}}(p^{-1}\mathbb{C}_S)$ are easily deduced from Proposition 2.5 similarly to the absolute case treated in [6], as consequences of Theorem 8.3.17, Proposition 8.3.11, Corollary 6.4.4 and Proposition 5.4.4 of loc. cit. In order to get the same statements for objects of $\mathsf{D}^{\mathrm{b}}_{\text{w-}\mathbb{R}\text{-c}}(p^{-1}\mathscr{O}_S)$, one uses Lemma 2.3(1) together with §2.1. We will not distinguish between f and f_S.

Proposition 2.7.

(1) *If F is S-weakly $\mathbb{R}$-constructible on X, then so are $f^{-1}(F)$ and $f^!(F)$.*
(2) *Assume that F' is S-weakly $\mathbb{R}$-constructible on Y and that f is proper on* $\operatorname{Supp}(F')$. *Then $Rf_*(F')$ is S-weakly $\mathbb{R}$-constructible on X.*

Given a closed subanalytic subset $Y\subset X$, we will denote by $i:Y\times S\hookrightarrow X\times S$ the closed inclusion and by j the complementary open inclusion.

Corollary 2.8. *Assume that F^* is S-weakly $\mathbb{R}$-constructible on $X\smallsetminus Y$ with respect to a μ-stratification of $X\smallsetminus Y$ induced from one on X. Then the objects $Rj_!F^*$ and Rj_*F^* are also S-weakly $\mathbb{R}$-constructible on X.*

Proof. Since we can refine the μ-stratification on X so that Y is a union of strata, the statement for $Rj_!F^*$ is obvious. Then Proposition 2.7 implies that $i^!Rj_!F^*$ is S-weakly $\mathbb{R}$-constructible. We conclude by using the distinguished triangle

$$Ri_*i^!Rj_!F^*\to Rj_!F^*\to Rj_*F^*\xrightarrow{+1}$$

and the S-weak $\mathbb{R}$-constructibility of the first two terms. q.e.d.

Proposition 2.9. *An object $F\in\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X\times S})$ (resp. $F\in\mathsf{D}^{\mathrm{b}}(p^{-1}(\mathscr{O}_S))$) is S-weakly $\mathbb{R}$-constructible with respect to a μ-stratification (X_α) if and only if, for each α, $i^!_\alpha F$ has S-locally constant cohomology on X_α.*

Proof. Assume that F is S-weakly $\mathbb{R}$-constructible with respect to a μ-stratification (X_α) of X. Then $i^!_\alpha F$ has S-locally constant cohomology on X_α. Indeed the estimation of the micro-support of [6, Cor. 6.4.4(ii)] implies that $SS(i^!_\alpha F)$ (like $SS(i^*_\alpha F)$) is contained in $T^*_{X_\alpha}X_\alpha \times T^*S$, so $i^!_\alpha F$ has locally constant cohomology on X_α for each α, according to Proposition 2.5.

Conversely, if $i^!_\alpha F$ is locally constant for each α, then F is S-weakly $\mathbb{R}$-constructible. Indeed, we argue by induction and we denote by X_k the union of strata of codimension $\leqslant k$ in X. Assume we have proved that $F_{|X_{k-1}\times S}$ is S-weakly $\mathbb{R}$-constructible with respect to the stratification (X_α) with $\operatorname{codim} X_\alpha \leqslant k-1$. We denote by $j_k : X_{k-1} \hookrightarrow X_k$ the open inclusion and by i_k the complementary closed inclusion. According to Corollary 2.8, $Rj_{k,*}j_k^{-1}F$ is S-weakly $\mathbb{R}$-constructible with respect to $(X_\alpha)_{|X_k}$. Now, by using the exact triangle $i^!_k F \to i_k^{-1}F \to i_k^{-1}Rj_{k,*}j_k^{-1}F \xrightarrow{+1}$, we conclude that $i_k^{-1}F$ is locally constant, hence $F_{|X_k\times S}$ is S-weakly $\mathbb{R}$-constructible. q.e.d.

Corollary 2.10. *Let $F, F' \in \mathsf{D}^{\mathrm{b}}_{\text{w-}\mathbb{R}\text{-c}}(p_X^{-1}\mathscr{O}_S)$. Then $R\mathscr{H}om_{p_X^{-1}\mathscr{O}_S}(F,F')$ also belongs to $\mathsf{D}^{\mathrm{b}}_{\text{w-}\mathbb{R}\text{-c}}(p_X^{-1}\mathscr{O}_S)$.*

Proof. In view of Proposition 2.9, it is sufficient to prove that for each α, $i^!_\alpha R\mathscr{H}om_{p_X^{-1}\mathscr{O}_S}(F,F')$ belongs to $\mathsf{D}^{\mathrm{b}}_{\mathrm{lc}}(p_X^{-1}\mathscr{O}_S)$. Setting $p_\alpha = p_{X_\alpha}$ for short, we have:

$$i^!_\alpha R\mathscr{H}om_{p^{-1}\mathscr{O}_S}(F,F') \simeq R\mathscr{H}om_{p_\alpha^{-1}\mathscr{O}_S}(i_\alpha^{-1}F, i^!_\alpha F').$$

Since both $i_\alpha^{-1}F$ and $i^!_\alpha F'$ belong to $\mathsf{D}^{\mathrm{b}}_{\mathrm{lc}}(p_X^{-1}\mathscr{O}_S)$, according to Proposition 2.9, we have locally on X_α isomorphisms $i_\alpha^{-1}F = p_\alpha^{-1}G_\alpha$ and $i^!_\alpha F' = p_\alpha^{-1}G'_\alpha = p^!_\alpha G'_\alpha[-\dim_{\mathbb{R}} X_\alpha]$ for some $\mathscr{O}_S$-modules G_α and G'_α. Then

$$\begin{aligned} R\mathscr{H}om_{p_\alpha^{-1}\mathscr{O}_S}(i_\alpha^{-1}F, i^!_\alpha F') &= R\mathscr{H}om_{p_\alpha^{-1}\mathscr{O}_S}(p_\alpha^{-1}G_\alpha, p^!_\alpha G'_\alpha[-\dim_{\mathbb{R}} X_\alpha]) \\ &\simeq p^!_\alpha R\mathscr{H}om_{\mathscr{O}_S}(G_\alpha, G'_\alpha)[-\dim_{\mathbb{R}} X_\alpha] \\ &= p_\alpha^{-1} R\mathscr{H}om_{\mathscr{O}_S}(G_\alpha, G'_\alpha). \end{aligned}$$

q.e.d.

The following lemma will be useful in the next section. Assume that $X = Y \times Z$ and that the μ-stratification (X_α) of X takes the form $X_\alpha = Y \times Z_\alpha$, where (Z_α) is a μ-stratification of Z. We denote by $q : X \to Y$ the projection. Let $z_o \in Z$ that we can assume to be a stratum, let $U \ni z_o$ be a coordinate neighbourhood of z_o in Z and, for each $\varepsilon > 0$ small enough, let $B_\varepsilon \subset U$ be the open ball of radius ε centered at z_o and let $\overline{B}_\varepsilon$ be the closed ball and S_ε its boundary. For the sake of simplicity, we denote by $q_\varepsilon, q_{\overline{\varepsilon}}, q_{\partial\varepsilon}$ the corresponding projections. The sheaf-theoretic restrictions to these sets will be implicit in the notation below.

We set $Z^* = Z \smallsetminus \{z_o\}$ and $X^* = Y \times Z^*$. We denote by $i : Y \times \{z_o\} \hookrightarrow Y \times Z$ and by $j : Y \times Z^* \hookrightarrow Y \times Z$ the complementary closed and open inclusions.

Lemma 2.11. *Let $F^* \in \mathsf{D}^{\mathrm{b}}_{\text{w-}\mathbb{R}\text{-c}}(p_{X^*}^{-1}\mathbb{C}_S)$ (resp. $F^* \in \mathsf{D}^{\mathrm{b}}_{\text{w-}\mathbb{R}\text{-c}}(p_{X^*}^{-1}\mathscr{O}_S)$) be adapted to the previous stratification. Then there exists $\varepsilon_o > 0$ such that, for each $\varepsilon \in (0, \varepsilon_o)$, the natural morphisms*

$$Rq_{\partial\varepsilon,*}F^* \longleftarrow Rq_{\overline{\varepsilon},*}Rj_*F^* \longrightarrow Rq_{\varepsilon,*}Rj_*F^* \longrightarrow i^{-1}Rj_*F^*$$

are isomorphisms.

Proof. We note that, according to Corollary 2.8, $F := Rj_*F^*$ is S-weakly $\mathbb{R}$-constructible, and is adapted to the stratification $(Y \times Z_\alpha)$. On the other hand, according to §2.1, it is enough to consider the case where F^* is an object of $\mathsf{D}^{\mathrm{b}}_{\text{w-}\mathbb{R}\text{-c}}(p_{X^*}^{-1}\mathbb{C}_S)$.

Let us start with the right morphisms. We can argue with any object $F \in \mathsf{D}^{\mathrm{b}}_{\text{w-}\mathbb{R}\text{-c}}(p_X^{-1}\mathbb{C}_S)$, not necessarily of the form Rj_*F^*. Recall that we have an adjunction morphism $q_\varepsilon^{-1}Rq_{\varepsilon,*} \to \mathrm{id}$ and thus $i^{-1}q_\varepsilon^{-1}Rq_{\varepsilon,*} \to i^{-1}$. Since $q_\varepsilon \circ i = \mathrm{id}_{Y\times S}$, we get the second right morphism. The first one is the restriction morphism.

According to [6, Prop. 8.3.12 and 5.4.17], there exists $\varepsilon_o > 0$ such that, for $\varepsilon' < \varepsilon$ in $(0, \varepsilon_o)$, the restriction morphisms $Rq_{\overline{\varepsilon},*}F \to Rq_{\varepsilon,*}F \to Rq_{\overline{\varepsilon}',*}F \to Rq_{\varepsilon',*}F$ are isomorphisms. In particular, the first right morphism is an isomorphism.

Let us take a q-soft representative of F, that we still denote by F. The inductive system $q_{\varepsilon,*}F$ ($\varepsilon \to 0$) has limit $i^{-1}F$ and all morphisms of this system are quasi-isomorphisms. Hence the second right morphism is a quasi-isomorphism.

For the left morphism, we take a q-soft representative of F^* that we still denote by F^*. For $\varepsilon_- < \varepsilon < \varepsilon_+ < \varepsilon_o$, we denote by $B_{\varepsilon_-,\varepsilon_+}$ the open set $B_{\varepsilon_+} \smallsetminus \overline{B}_{\varepsilon_-}$ and by $q_{\varepsilon_-,\varepsilon_+}$ the corresponding projection. We have $q_{\partial\varepsilon,*}F^* = \varinjlim_{|\varepsilon_+-\varepsilon_-|\to 0} q_{\varepsilon_-,\varepsilon_+,*}F^*$. On the other hand, the morphisms of this inductive system are all quasi-isomorphisms, according to [6, Prop. 5.4.17]. Fixing $\varepsilon' \in (\varepsilon, \varepsilon_o)$ we find a quasi-isomorphism $q_{\varepsilon',*}F^* \to q_{\partial\varepsilon,*}F^*$. On the other hand, from the first part we have $q_{\varepsilon',*}F^* \xrightarrow{\sim} q_{\overline{\varepsilon},*}F^*$, hence the result. q.e.d.

Remark 2.12. A argument similar to that used in the first part of the proof gives an isomorphism $i^!F \xrightarrow{\sim} Rq_{\varepsilon,!}F$, by using [6, Prop. 5.4.17(c)].

2.4. ***S*-coherent local systems and *S*-ℝ-constructible sheaves.**

Notation 2.13. We shall denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{lc\,coh}}(p_X^{-1}\mathscr{O}_S)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}_{\mathrm{lc}}(p_X^{-1}\mathscr{O}_S)$ whose objects satisfy, locally on X, $F \simeq p_X^{-1}G$ with $G \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_S)$). Equivalently, for each $x \in X$, $F_{|\{x\}\times S} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_S)$ (see the remarks before Lemma 2.3).

Definition 2.14. Given $F \in \mathsf{D}^{\mathrm{b}}_{\text{w-}\mathbb{R}\text{-c}}(p_X^{-1}\mathscr{O}_S)$, we say that F is $\mathbb{R}$-constructible if, for some μ-stratification of X, $X = \bigsqcup_\alpha X_\alpha$, for all $j \in \mathbb{Z}$, $\mathscr{H}^j(F)|_{X_\alpha\times S} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{lc\,coh}}(p_{X_\alpha}^{-1}\mathscr{O}_S)$. This condition characterizes a full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}_{\text{w-}\mathbb{R}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ which we denote by $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p_X^{-1}\mathscr{O}_S)$.

Similarly to Proposition 2.6 we have:

Proposition 2.15. *Let $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ and let $s_o \in S$. Then $Li^*_{s_o}(F) \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$.*

Remark 2.16. An object of $\mathsf{D}^{\mathrm{b}}_{\text{w-}\mathbb{R}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ is in $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ if and only if, for any $x \in X$, $F_{|\{x\}\times S}$ belongs to $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_S)$.

A straightforward adaptation of [6, Prop. 8.4.8] gives:

Proposition 2.17. *Let $f : Y \to X$ be a a morphism of manifolds and let $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p_Y^{-1}\mathscr{O}_S)$. Assume that f_S is proper on* $\mathrm{Supp}(F)$. *Then*

$$Rf_{S,*}F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p_X^{-1}\mathscr{O}_S).$$

We can also characterize $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ as in Corollary 2.9.

Corollary 2.18. *An object $F \in \mathsf{D}^{\mathrm{b}}(p_X^{-1}\mathscr{O}_S)$ is in $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ if and only if, for some subanalytic Whitney stratification (X_α) of X, the complexes $i_\alpha^!F$ belong to $\mathsf{D}^{\mathrm{b}}_{\mathrm{lc\,coh}}(p_\alpha^{-1}\mathscr{O}_S)$.*

Proof. Assume F is in $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p_X^{-1}\mathscr{O}_S)$. We need to prove the coherence of $i_\alpha^!F$. We argue by induction as in Corollary 2.9, with the same notation. Since the question is local on X_k, by the Whitney property of the stratification (X_α) we can assume that $X_{k-1} = Z \times Y_k$ and that there exists a Whitney stratification (Z_α) of Z such that $X_\alpha = Z_\alpha \times Y_k$ for each α such that $X_\alpha \subset X_{k-1}$ (see e.g. [1, §1.4]). Proving that $i_k^!F$ is $p^{-1}\mathscr{O}_S$-coherent is equivalent to proving that $i_k^{-1}Rj_{k,*}j_k^{-1}F$ is so, since we already know that $i_k^{-1}F$ is so. According to Lemma 2.11, $i_k^{-1}Rj_{k,*}j_k^{-1}F$ is computed as $Rq_{\partial\varepsilon,*}j_k^{-1}F$, and since $q_{\partial\varepsilon}$ is proper, we can apply Proposition 2.17 to get the coherence.

Conversely, Corollary 2.9 already implies that F is an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{w}\text{-}\mathbb{R}\text{-c}}(p_X^{-1}\mathscr{O}_S)$. We argue then as above: since we know by assumption that $i_k^!F$ is coherent, it suffices to prove that $i_k^{-1}Rj_{k,*}j_k^{-1}F$ is so, and the previous argument applies. q.e.d.

2.5. **S-weakly $\mathbb{C}$-constructible sheaves and S-$\mathbb{C}$-constructible sheaves.** Let now assume that X is a complex analytic manifold.

Definition 2.19.

(1) Let $F \in \mathsf{D}^{\mathrm{b}}_{\mathrm{w}\text{-}\mathbb{R}\text{-c}}(p_X^{-1}\mathbb{C}_S)$ (resp. $F \in \mathsf{D}^{\mathrm{b}}_{\mathrm{w}\text{-}\mathbb{R}\text{-c}}(p_X^{-1}\mathscr{O}_S)$). We shall say that F is S-weakly $\mathbb{C}$-constructible if $SS(F)$ is $\mathbb{C}^*$-conic. The corresponding categories are denoted by $\mathsf{D}^{\mathrm{b}}_{\mathrm{w}\text{-}\mathbb{C}\text{-c}}(p_X^{-1}\mathbb{C}_S)$ (resp. $F \in \mathsf{D}^{\mathrm{b}}_{\mathrm{w}\text{-}\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$).

(2) If F belongs to $\mathsf{D}^{\mathrm{b}}_{\mathrm{w}\text{-}\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$, we say that F is S-$\mathbb{C}$-constructible if $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p_X^{-1}\mathscr{O}_S)$, and we denote by $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ the corresponding category, which is full triangulated sub-category of $\mathsf{D}^{\mathrm{b}}(p_X^{-1}\mathscr{O}_S)$.

The following properties are obtained in a straightforward way, by using [6, Th. 8.5.5] in a way similar to [6, Prop. 8.5.7].

Properties 2.20.

(1) An object F of $\mathsf{D}^{\mathrm{b}}(p_X^{-1}\mathscr{O}_S)$ belongs to $\mathsf{D}^{\mathrm{b}}_{\mathrm{w}\text{-}\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ if and only if it belongs to $\mathsf{D}^{\mathrm{b}}_{\mathrm{w}\text{-}\mathbb{C}\text{-c}}(p_X^{-1}\mathbb{C}_S)$.

(2) Proposition 2.5 applies to S-weakly $\mathbb{C}$-constructible complexes provided that one assumes the stratification to be $\mathbb{C}$-analytic and the Lagrangian varieties to be $\mathbb{C}^*$-conic. We will implicitly make this assumption in such a case.

(3) Remark 2.16 applies to $\mathsf{D}^{\mathrm{b}}_{\mathrm{w}\text{-}\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ and $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$.

(4) Proposition 2.7 applies to $\mathsf{D}^{\mathrm{b}}_{\text{w-}\mathbb{C}\text{-c}}$.
(5) Propositions 2.15, 2.17, and Corollary 2.18 apply to $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$.
(6) Corollary 2.10 applies to $\mathsf{D}^{\mathrm{b}}_{\text{w-}\mathbb{C}\text{-c}}$, $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}$ and $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}$.

2.6. **Duality.** According to the syzygy theorem for the regular local ring $\mathscr{O}_{S,s}$ (for any $s \in S$) and e.g. [7, Prop. 13.2.2(ii)] (for the opposite category), any object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_S)$ is locally quasi-isomorphic to a bounded complex of locally free $\mathscr{O}_S$-modules of finite rank $L^\bullet$. As a consequence, the local duality functor

$$\boldsymbol{D} : \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_S) \to \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_S), \quad \boldsymbol{D}(\mathscr{F}) := R\mathscr{H}om_{\mathscr{O}_S}(\mathscr{F}, \mathscr{O}_S)$$

is seen to be an involution, i.e., the natural morphism $\mathrm{id} \to \boldsymbol{D} \circ \boldsymbol{D}$ is an isomorphism. However, the standard t-structure

$$\big(\mathsf{D}^{\mathrm{b},\leqslant 0}_{\mathrm{coh}}(\mathscr{O}_S), \mathsf{D}^{\mathrm{b},\geqslant 0}_{\mathrm{coh}}(\mathscr{O}_S)\big)$$

defined by $\mathscr{H}^j G = 0$ for $j > 0$ (resp. for $j < 0$) is not interchanged by duality when $\dim S \geqslant 1$ (see e.g., [5, Prop. 4.3] in the algebraic setting). Nevertheless, we have:

Lemma 2.21. *Let G be an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_S)$. Assume that $\boldsymbol{D}G$ belongs to $\mathsf{D}^{\mathrm{b},\leqslant 0}_{\mathrm{coh}}(\mathscr{O}_S)$. Then G belongs to $\mathsf{D}^{\mathrm{b},\geqslant 0}_{\mathrm{coh}}(\mathscr{O}_S)$.*

Proof. Setting $G' = \boldsymbol{D}G$, the biduality isomorphism makes it equivalent to proving that $\boldsymbol{D}G'$ belongs to $\mathsf{D}^{\mathrm{b},\geqslant 0}_{\mathrm{coh}}(\mathscr{O}_S)$. The question is local on S and we may therefore replace G' with a bounded complex $L^\bullet$ as above. Moreover, $L^\bullet$ is quasi-isomorphic to such a bounded complex, still denoted by $L^\bullet$, such that $L^k = 0$ for $k > 0$. Indeed, note first that the kernel K of a surjective morphism of locally free $\mathscr{O}_S$-modules of finite rank is also locally free of finite rank (being $\mathscr{O}_S$-coherent and having all its germs K_s free over $\mathscr{O}_{S,s}$, because they are projective and $\mathscr{O}_{S,s}$ is a regular local ring). By assumption, we have $\mathscr{H}^j(L^\bullet) = 0$ for $j > 0$. Let $k > 0$ be such that $L^k \neq 0$ and $L^\ell = 0$ for $\ell > k$, and let $L'^{k-1} = \ker[L^{k-1} \to L^k]$. Then $L^\bullet$ is quasi-isomorphic to $L'^\bullet$ defined by $L'^j = L^j$ for $j < k-1$ and $L'^j = 0$ for $j \geqslant k$. We conclude by induction on k.

Now it is clear that $\boldsymbol{D}G' \simeq \boldsymbol{D}L^\bullet$ is a bounded complex having terms in nonnegative degrees at most, and thus is an object of $\mathsf{D}^{\mathrm{b},\geqslant 0}_{\mathrm{coh}}(\mathscr{O}_S)$. q.e.d.

Remark 2.22. Let G be an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_S)$. Assume that G and $\boldsymbol{D}G$ belong to $\mathsf{D}^{\mathrm{b},\leqslant 0}_{\mathrm{coh}}(\mathscr{O}_S)$. Then G and $\boldsymbol{D}G$ are $\mathscr{O}_S$-coherent sheaves, hence G and $\boldsymbol{D}G$ are $\mathscr{O}_S$-locally free.

We now set $\omega_{X,S} = p_X^{-1}\mathscr{O}_S[2\dim X] = p_X^!\mathscr{O}_S$.

Proposition 2.23. *The functor $\boldsymbol{D} : \mathsf{D}^{\mathrm{b}}(p_X^{-1}\mathscr{O}_S) \to \mathsf{D}^{+}(p_X^{-1}\mathscr{O}_S)$ defined by $\boldsymbol{D}F = R\mathscr{H}om_{p_X^{-1}\mathscr{O}_S}(F, \omega_{X,S})$ induces an involution $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p_X^{-1}\mathscr{O}_S) \to \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ and $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S) \to \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$.*

We will also set $\boldsymbol{D}'F = R\mathscr{H}om_{p_X^{-1}\mathscr{O}_S}(F, p_X^{-1}\mathscr{O}_S)$.

Proof. Let us first show that, for F in $\mathsf{D}^{\mathrm{b}}_{\text{w-}\mathbb{R}\text{-c}}(p_X^{-1}\mathscr{O}_S)$, the dual $\boldsymbol{D}F$ also belongs to $\mathsf{D}^{\mathrm{b}}_{\text{w-}\mathbb{R}\text{-c}}(p_X^{-1}\mathscr{O}_S)$. Let (X_α) be a μ-stratification adapted to F. According to Corollary 2.9, it is enough to show that $i_\alpha^!\boldsymbol{D}F$ has locally constant cohomology for each α. One can use [6, Prop. 3.1.13] in our setting and get

$$i_\alpha^!\boldsymbol{D}F = R\mathscr{H}om_{p_\alpha^{-1}\mathscr{O}_S}(i_\alpha^{-1}F, \omega_{X_\alpha,S}).$$

Locally on X_α, $i_\alpha^{-1}F = p_\alpha^{-1}G$ for some G in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_S)$ or $\mathsf{D}^{\mathrm{b}}(\mathscr{O}_S)$. Then, locally on X_α,

$$\begin{aligned} i_\alpha^!\boldsymbol{D}F \simeq R\mathscr{H}om_{p_\alpha^{-1}\mathscr{O}_S}(p_\alpha^{-1}G, p_\alpha^!\mathscr{O}_S) &= p_\alpha^! R\mathscr{H}om_{\mathscr{O}_S}(G,\mathscr{O}_S)\\ &= p_\alpha^{-1}(\boldsymbol{D}G)[2\dim X_\alpha]. \end{aligned}$$

The proof for F in $\mathsf{D}^{\mathrm{b}}_{\text{w-}\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ is similar. Moreover, by using Corollary 2.18 instead of Corollary 2.9 one shows that $\boldsymbol{D}$ sends $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ to itself and, according to Properties 2.20(5), $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ to itself.

Let us prove the involution property. We have a natural morphism of functors $\mathrm{id} \to \boldsymbol{D}\boldsymbol{D}$. It is enough to prove the isomorphism property after applying $Li_{s_o}^*$ for each $s_o \in S$, according to Proposition 2.2. On the other hand, Proposition 2.1 implies that $Li_{s_o}^*$ commutes with $\boldsymbol{D}$, so we are reduced to applying the involution property on $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$, according to the $\mathbb{C}$-c-analogue of Proposition 2.15, which is known to be true (see e.g. [6]). q.e.d.

Remark 2.24. By using the biduality isomorphism and the isomorphism $i_x^!\boldsymbol{D}F \simeq \boldsymbol{D}i_x^{-1}F$ for F in $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ or $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$, where $i_x : \{x\}\times S \hookrightarrow X \times S$ denotes the inclusion, we find a functorial isomorphism $i_x^{-1}\boldsymbol{D}F \simeq \boldsymbol{D}i_x^!F$.

2.7. **Perversity.** We will now restrict to the case of S-$\mathbb{C}$-constructible complexes, which is the only case which will be of interest for us, although one could consider the case of S-$\mathbb{R}$-constructible complexes as in [6, §10.2].

We define the category ${}^{\mathrm{p}}\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ as the full subcategory of $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ whose objects are the S-$\mathbb{C}$-constructible bounded complexes F such that, for some adapted μ-stratification (X_α) (i_x is as above),

(Supp) $\quad \forall\,\alpha,\ \forall\, x\in X_\alpha,\ \forall\, j > -\dim X_\alpha, \quad \mathscr{H}^j i_x^{-1}F = 0.$

Similarly, ${}^{\mathrm{p}}\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ consists of objects F such that

(Cosupp) $\quad \forall\,\alpha,\ \forall\, x\in X_\alpha,\ \forall\, j < \dim X_\alpha, \quad \mathscr{H}^j i_x^!F = 0.$

In the preceding situation in view of Corollary 2.18 we have, similarly to [6, Prop.10.2.4]:

Lemma 2.25.

(1) $F \in {}^{\mathrm{p}}\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ *if and only if for any* α *and* $j > -\dim(X_\alpha)$,
$$\mathscr{H}^j(i_\alpha^{-1}F) = 0.$$

(2) $F \in {}^{\mathrm{p}}\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ *if and only if for any* α *and* $j < -\dim(X_\alpha)$,
$$\mathscr{H}^j(i_\alpha^!F) = 0.$$

Namely, if $F \in {}^{\mathrm{p}}\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ and Z is a pure k-dimensional stratum of a μ-stratification adapted to F, then $i_{Z\times S}^{-1}F$ is concentrated in degrees $\leqslant -k$, and if $F' \in {}^{\mathrm{p}}\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$, then $i_{Z\times S}^{!}F'$ is concentrated in degrees $\geqslant -k$. We have the following variant of [6, Prop.10.2.7]:

Proposition 2.26. *Let F be an object of ${}^{\mathrm{p}}\mathsf{D}^{\leqslant 0}_{\text{w-}\mathbb{R}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ and F' an object of ${}^{\mathrm{p}}\mathsf{D}^{\geqslant 0}_{\text{w-}\mathbb{R}\text{-c}}(p_X^{-1}\mathscr{O}_S)$. Then*

$$\mathscr{H}^j R\mathscr{H}om_{p_X^{-1}\mathscr{O}_S}(F,F') = 0, \quad \text{for } j<0.$$

Proof. Let (X_α) be a μ-stratification of X adapted to F and F'. By assumption, for each α, $i_\alpha^{-1}\mathscr{H}^jF = \mathscr{H}^j i_\alpha^{-1}F = 0$ for $j > -\dim X_\alpha$. Similarly, $\mathscr{H}^j i_\alpha^! F' = 0$ for $j < -\dim X_\alpha$.

Let X_α be a stratum of maximal dimension such that

$$i_\alpha^{-1}\mathscr{H}^j R\mathscr{H}om_{p_X^{-1}\mathscr{O}_S}(F,F') \neq 0 \quad \text{for some } j<0.$$

Let V be an open neighbourhood of X_α in X such that $V \smallsetminus X_\alpha$ intersects only strata of dimension $> \dim X_\alpha$, and let $j_\alpha : (V \smallsetminus X_\alpha)\times S \hookrightarrow V\times S$ be the inclusion. Then the complex $i_\alpha^{-1}Rj_{\alpha,*}j_\alpha^{-1}R\mathscr{H}om_{p_X^{-1}\mathscr{O}_S}(F,F')$ has nonzero cohomology in nonnegative degrees only: indeed, by the definition of X_α, this property holds for $j_\alpha^{-1}R\mathscr{H}om_{p_X^{-1}\mathscr{O}_S}(F,F')$, hence it holds for $Rj_{\alpha,*}j_\alpha^{-1}R\mathscr{H}om_{p_X^{-1}\mathscr{O}_S}(F,F')$, and then clearly for the complex $i_\alpha^{-1}Rj_{\alpha,*}j_\alpha^{-1}R\mathscr{H}om_{p_X^{-1}\mathscr{O}_S}(F,F')$. From the distinguished triangle

$$\begin{aligned} i_\alpha^! R\mathscr{H}om_{p_X^{-1}\mathscr{O}_S}(F,F') &\to i_\alpha^{-1}R\mathscr{H}om_{p_X^{-1}\mathscr{O}_S}(F,F') \\ &\to i_\alpha^{-1}Rj_{\alpha,*}j_\alpha^{-1}R\mathscr{H}om_{p_X^{-1}\mathscr{O}_S}(F,F') \xrightarrow{+1} \end{aligned}$$

we conclude that $\mathscr{H}^j i_\alpha^! R\mathscr{H}om_{p_X^{-1}\mathscr{O}_S}(F,F') \to \mathscr{H}^j i_\alpha^{-1}R\mathscr{H}om_{p_X^{-1}\mathscr{O}_S}(F,F') = i_\alpha^{-1}\mathscr{H}^j R\mathscr{H}om_{p_X^{-1}\mathscr{O}_S}(F,F')$ is an isomorphism for all $j<0$. Therefore, we obtain, for this stratum X_α and for any $j<0$,

$$\begin{aligned} i_\alpha^{-1}\mathscr{H}^j R\mathscr{H}om_{p_X^{-1}\mathscr{O}_S}(F,F') &\simeq \mathscr{H}^j i_\alpha^! R\mathscr{H}om_{p_X^{-1}\mathscr{O}_S}(F,F') \\ &\simeq \mathscr{H}^j R\mathscr{H}om_{p_X^{-1}\mathscr{O}_S}(i_\alpha^{-1}F, i_\alpha^! F'). \end{aligned}$$

Since $i_\alpha^{-1}F$ has nonzero cohomology in degrees $\leqslant -\dim X_\alpha$ at most and $i_\alpha^! F'$ in degrees $\geqslant -\dim X_\alpha$ at most, $\mathscr{H}^j R\mathscr{H}om_{p_X^{-1}\mathscr{O}_S}(i_\alpha^{-1}F, i_\alpha^! F') = 0$ for $j<0$, a contradiction with the definition of X_α. q.e.d.

Theorem 2.27. *${}^{\mathrm{p}}\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ and ${}^{\mathrm{p}}\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ form a t-structure of $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$, whose heart is denoted by* $\mathrm{Perv}(p_X^{-1}\mathscr{O}_S)$.

Sketch of proof. We have to prove:

(1) ${}^{\mathrm{p}}\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}} \subset {}^{\mathrm{p}}\mathsf{D}^{\leqslant 1}_{\mathbb{C}\text{-c}}$ and ${}^{\mathrm{p}}\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}} \supset {}^{\mathrm{p}}\mathsf{D}^{\geqslant 1}_{\mathbb{C}\text{-c}}$.

(2) For $F \in {}^{\mathrm{p}}\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ and $F' \in {}^{\mathrm{p}}\mathsf{D}^{\geqslant 1}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$,

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(p_X^{-1}\mathscr{O}_S)}(F,F') = 0.$$

(3) For any $F\in\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ there exist $F'\in{}^{\mathrm{p}}\mathsf{D}^{\leqslant0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ and $F''\in{}^{\mathrm{p}}\mathsf{D}^{\geqslant1}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$, giving rise to a distinguished triangle $F'\to F\to F''\xrightarrow{+1}$.

Then, following the line of the proof of [6, Theorem 10.2.8], we observe that (1) is obvious and (2) follows from Proposition 2.26. Now, (3) is deduced by mimicking stepwise the proof of (c) in [6, Theorem 10.2.8]. q.e.d.

According to the preliminary remarks before Lemma 2.21, one cannot expect that the previous t-structure is interchanged by duality when $\dim S\geqslant 1$. However we have:

Proposition 2.28. *Let F be an object of ${}^{\mathrm{p}}\mathsf{D}^{\leqslant0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$ such that $\boldsymbol{D}F$ also belongs to ${}^{\mathrm{p}}\mathsf{D}^{\leqslant0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$. Then F and $\boldsymbol{D}F$ are objects of $\mathrm{Perv}(p_X^{-1}\mathscr{O}_S)$.*

Proof. Let us fix $x\in X_\alpha$. We have $i_x^!F\simeq\boldsymbol{D}(i_x^{-1}\boldsymbol{D}F)$, as already observed in Remark 2.24. By assumption $G:=i_x^{-1}\boldsymbol{D}F$ belongs to $\mathsf{D}^{\mathrm{b},\leqslant-\dim X_\alpha}_{\mathrm{coh}}(\mathscr{O}_S)$, and Lemma 2.21 suitably shifted and applied to $\boldsymbol{D}G$ implies that $\boldsymbol{D}G$ belongs to $\mathsf{D}^{\mathrm{b},\geqslant\dim X_\alpha}_{\mathrm{coh}}(\mathscr{O}_S)$, which is the cosupport condition (Cosupp) for F. q.e.d.

Assume $F\in\mathrm{Perv}(p_X^{-1}\mathscr{O}_S)$. The description of the dual standard t-structure on $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_S)$ given in [5, §4] supplies the following refinement to (Supp) and (Cosupp) when $\boldsymbol{D}F$ is also perverse.

Corollary 2.29. *Let $F\in\mathrm{Perv}(p_X^{-1}\mathscr{O}_S)$ and assume that $\boldsymbol{D}F\in\mathrm{Perv}(p_X^{-1}\mathscr{O}_S)$. Let (X_α) be a stratification adapted to F. Then for each α, each $x\in X_\alpha$ and each closed analytic subset $\Sigma\subset S$, we have*

$$\text{(Cosupp+)}\qquad \mathscr{H}^k(i^!_{\{x\}\times\Sigma}F)=0,\quad \forall\,k<\mathrm{codim}_S\Sigma+\dim X_\alpha.$$

(The perversity of F only gives the previous property when $\Sigma=S$.)

3. The de Rham complex of a holonomic $\mathscr{D}_{X\times S/S}$-module

In what follows X and S denote complex manifolds and we set $n=\dim X$, $\ell=\dim S$. We shall keep the notation of the preceding section. Let $\pi: T^*(X\times S)\to T^*X\times S$ denote the projection and let $\mathscr{D}_{X\times S/S}$ denote the subsheaf of $\mathscr{D}_{X\times S}$ of relative differential operators with respect to p_X (see [18, §2.1 & 2.2]).

Recall that $p_X^{-1}\mathscr{O}_S$ is contained in the center of $\mathscr{D}_{X\times S/S}$. With the same proof as for Proposition 2.1 we obtain:

Proposition 3.1. *Let $s_o\in S$ be given. Let $\mathscr{M}$ and $\mathscr{N}$ be objects of $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{X\times S/S})$. Then, there is a well-defined natural morphism*

$$Li^*_{s_o}(R\mathscr{H}om_{\mathscr{D}_{X\times S/S}}(\mathscr{M},\mathscr{N}))\to R\mathscr{H}om_{i^*_{s_o}(\mathscr{D}_{X\times S/S})}(Li^*_{s_o}(\mathscr{M}),Li^*_{s_o}(\mathscr{N}))$$

which is an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_X)$.

3.1. **Duality for coherent $\mathscr{D}_{X\times S/S}$-modules.** We refer for instance to [4, Appendix] for the coherence properties of the ring $\mathscr{D}_{X\times S/S}$. The classical methods used in the absolute case, i.e, for coherent $\mathscr{D}_X$-objects (see for instance [9, Prop. 2.1.16], [3, Lem. 3.1.1] or [10, Prop. 2.7-3]) apply here:

Proposition 3.2. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_{X\times S/S}$-module. Then $\mathscr{M}$ locally admits a resolution of length at most $2n+\ell$ by free $\mathscr{D}_{X\times S/S}$-modules of finite rank.*

Proposition 3.2 and [7, Prop. 13.2.2(ii)] (for the opposite category) imply:

Corollary 3.3. *Let $\mathscr{M}\in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_{X\times S/S})$. Let us assume that $\mathscr{M}$ is concentrated in degrees $[a,b]$. Then, in a neighborhhod of each $(x,z)\in X\times S$, there exist a complex $\mathscr{L}^\bullet$ of free $\mathscr{D}_{X\times S/S}$-modules of finite rank concentrated in degrees $[a-2n-\ell,b]$ and a quasi-isomorphism $\mathscr{L}^\bullet\to\mathscr{M}$.*

We set $\Omega_{X\times S/S}=\Omega^n_{X\times S/S}$, where $\Omega^n_{X\times S/S}$ denotes the sheaf of relative differential forms of degree $n=\dim X$.

Definition 3.4. The duality functor $\boldsymbol{D}(\cdot):\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{X\times S/S})\to\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{X\times S/S})$ is defined as:

$$\mathscr{M}\mapsto\boldsymbol{D}\mathscr{M}=R\mathscr{H}om_{\mathscr{D}_{X\times S/S}}(\mathscr{M},\mathscr{D}_{X\times S/S}\otimes_{\mathscr{O}_{X\times S}}\Omega^{\otimes -1}_{X\times S/S})[n].$$

We also set $\boldsymbol{D}'\mathscr{M}:=R\mathscr{H}om_{\mathscr{D}_{X\times S/S}}(\mathscr{M},\mathscr{D}_{X\times S/S})\in\mathsf{D}^{\mathrm{b}}(\mathscr{D}^{\mathrm{opp}}_{X\times S/S})$.

By Proposition 3.2, $\mathscr{D}_{X\times S/S}$ has finite cohomological dimension, so [4, (A.11)] gives a natural morphism in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{X\times S/S})$:

$$\mathscr{M}\to\boldsymbol{D}'\boldsymbol{D}'\mathscr{M}\simeq\boldsymbol{D}\boldsymbol{D}\mathscr{M}. \tag{1}$$

Moreover, in view of Corollary 3.3, if $\mathscr{M}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_{X\times S/S})$, then $\boldsymbol{D}'\mathscr{M}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}^{\mathrm{opp}}_{X\times S/S})$. Indeed, we may choose a local free finite resolution $\mathscr{L}^\bullet$ of $\mathscr{M}$, so that $\boldsymbol{D}'\mathscr{M}$ is quasi isomorphic to the transposed complex $(\mathscr{L}^\bullet)^t$ whose entries are free.

By the same argument we deduce that (1) is an isomorphism whenever $\mathscr{M}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_{X\times S/S})$.

Again by Proposition 3.2, $\mathscr{D}_{X\times S/S}$ has finite flat dimension so we are in conditions to apply [4, (A.10)]: given $\mathscr{M},\mathscr{N}\in\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{X\times S/S})$ there is a natural morphism:

$$\boldsymbol{D}'\mathscr{M}\overset{L}{\otimes}_{\mathscr{D}_{X\times S/S}}\mathscr{N}\to R\mathscr{H}om_{\mathscr{D}_{X\times S/S}}(\mathscr{M},\mathscr{N}) \tag{2}$$

which is an isomorphism provided that $\mathscr{M}$ or $\mathscr{N}$ belong to $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_{X\times S/S})$. When $\mathscr{M},\mathscr{N}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_{X\times S/S})$, composing (2) with the biduality isomorphism (1) gives a natural isomorphism

$$R\mathscr{H}om_{\mathscr{D}_{X\times S/S}}(\mathscr{M},\mathscr{N})\simeq R\mathscr{H}om_{\mathscr{D}_{X\times S/S}}(\boldsymbol{D}\mathscr{N},\boldsymbol{D}\mathscr{M}). \tag{3}$$

3.2. **Characteristic variety.** Recall (see [17, §III.1.3]) that the characteristic variety $\operatorname{Char}\mathscr{M}$ of a coherent $\mathscr{D}_{X\times S/S}$-module $\mathscr{M}$ is the support in $T^*X\times S$ of its graded module with respect to any (local) good filtration. One has (see [17, Prop. III.1.3.2])

$$\begin{aligned}\operatorname{Char}(\mathscr{D}_{X\times S}\otimes_{\mathscr{D}_{X\times S/S}}\mathscr{M}) &= \pi^{-1}\operatorname{Char}\mathscr{M},\\ \operatorname{Char}\mathscr{M} &= \pi\big(\operatorname{Char}(\mathscr{D}_{X\times S}\otimes_{\mathscr{D}_{X\times S/S}}\mathscr{M})\big).\end{aligned}\tag{4}$$

One may as well define the characteristic variety of an object $\mathscr{M}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_{X\times S/S})$ as the union of the characteristic varieties of its cohomology modules. By the flatness of $\mathscr{D}_{X\times S}$ over $\mathscr{D}_{X\times S/S}$, (4) holds for any object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_{X\times S/S})$.

Proposition 3.5 ([18, Prop. 2.5])**.** *For $\mathscr{M}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_{X\times S/S})$ we have*

$$\operatorname{Char}(\mathscr{M})=\operatorname{Char}(\boldsymbol{D}\mathscr{M}).$$

3.3. **The de Rham and solution complexes.** For an object $\mathscr{M}$ of $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{X\times S/S})$ we define the functors

$$\begin{aligned}\operatorname{DR}\mathscr{M} &:= R\mathscr{H}om_{\mathscr{D}_{X\times S/S}}(\mathscr{O}_{X\times S},\mathscr{M}),\\ \operatorname{Sol}\mathscr{M} &:= R\mathscr{H}om_{\mathscr{D}_{X\times S/S}}(\mathscr{M},\mathscr{O}_{X\times S})\end{aligned}$$

which take values in $\mathsf{D}^{\mathrm{b}}(p_X^{-1}\mathscr{O}_S)$. If $\mathscr{M}$ is a $\mathscr{D}_{X\times S/S}$-module, that is, an $\mathscr{O}_{X\times S}$-module equipped with an integrable relative connection $\nabla:\mathscr{M}\to\Omega^1_{X\times S/S}\otimes\mathscr{M}$, the object $\operatorname{DR}\mathscr{M}$ is represented by the complex $(\Omega^{\bullet}_{X\times S/S}\otimes_{\mathscr{O}_{X\times S}}\mathscr{M},\nabla)$.

Noting that $R\mathscr{H}om_{\mathscr{D}_{X\times S/S}}(\mathscr{O}_{X\times S},\mathscr{D}_{X\times S/S})\simeq\Omega_{X\times S/S}[-\dim X]$ we get

$$\boldsymbol{D}\mathscr{O}_{X\times S}\simeq\mathscr{O}_{X\times S}.$$

For $\mathscr{N}=\mathscr{O}_{X\times S}$, (3) implies a natural isomorphism, for $\mathscr{M}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_{X\times S/S})$:

$$\operatorname{Sol}\mathscr{M}\simeq\operatorname{DR}\boldsymbol{D}\mathscr{M}.\tag{5}$$

3.4. **Holonomic $\mathscr{D}_{X\times S/S}$-modules.** Let $\mathscr{M}$ be a coherent $\mathscr{D}_{X\times S/S}$-module. We say that it is *holonomic* if its characteristic variety $\operatorname{Char}\mathscr{M}\subset T^*X\times S$ is contained in $\Lambda\times S$ for some closed conic Lagrangian complex analytic subset of T^*X. We will say that a complex μ-stratification (X_α) is adapted to $\mathscr{M}$ if $\Lambda\subset\bigcup_\alpha T^*_{X_\alpha}X$.

An object $\mathscr{M}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_{X\times S/S})$ is said to be holonomic if its cohomology modules are holonomic. We denote the full triangulated category of holonomic complexes by $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{X\times S/S})$. A complex μ-stratification (X_α) is said to be adapted to $\mathscr{M}$ if it is adapted to each cohomology module.

Corollary 3.6 (of Prop. 3.5)**.** *If $\mathscr{M}$ is an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{X\times S/S})$, then so is $\boldsymbol{D}\mathscr{M}$.*

Theorem 3.7. *Let $\mathscr{M}$ be an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{X\times S/S})$. Then $\operatorname{DR}(\mathscr{M})$ and $\operatorname{Sol}\mathscr{M}$ belong to $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$.*

Proof. Firstly, it follows [6, Prop. 11.3.3], that $\mathrm{Sol}(\mathscr{M})$ and $\mathrm{DR}(\mathscr{M})$ have their micro-support contained in $\Lambda \times T^*S$ (see [18, p. 11 & Th. 2.13]) and, according to Proposition 2.5, these complexes are objects of $\mathsf{D}^{\mathrm{b}}_{\text{w-}\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$.

Let $x \in X$. In order to prove that $i_x^{-1}\,\mathrm{DR}\,\mathscr{M}$ has $\mathscr{O}_S$-coherent cohomology, we can assume that x is a stratum of a stratification adapted to $\mathrm{DR}\,\mathscr{M}$ and we use Lemma 2.11 to get $i_x^{-1}\,\mathrm{DR}\,\mathscr{M} \simeq Rp_{\overline{\varepsilon},*}(\mathbb{C}_{\overline{B}_\varepsilon\times S}\otimes_{\mathbb{C}}\mathrm{DR}\,\mathscr{M})$ for ε small enough, where $\overline{B}_\varepsilon$ is a closed ball of radius ε centered at x. One then remarks that $(\mathbb{C}_{\overline{B}_\varepsilon\times S},\mathscr{M})$ forms a relative elliptic pair in the sense of [18], and Proposition 4.1 of loc. cit. gives the desired coherence.

The statement for $\mathrm{Sol}\,\mathscr{M}$ is proved similarly. q.e.d.

Lemma 3.8 (see [14, Prop. 1.2.5])**.** *For $\mathscr{M}$ in $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{X\times S/S})$ with adapted stratification (X_α) and for any $s_o \in S$, $Li^*_{s_o}\mathscr{M}$ is $\mathscr{D}_X$-holonomic and (X_α) is adapted to it.*

Corollary 3.9. *For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{X\times S/S})$, there is a natural isomorphism $\boldsymbol{D}'\,\mathrm{Sol}\,\mathscr{M} \simeq \mathrm{DR}\,\mathscr{M}$.*

Proof. We consider the canonical pairing

$$\mathrm{DR}\,\mathscr{M} \overset{L}{\otimes}_{p_X^{-1}\mathscr{O}_S} \mathrm{Sol}\,\mathscr{M} \to p_X^{-1}\mathscr{O}_S$$

which gives a natural morphism

$$\mathrm{DR}\,\mathscr{M} \to \boldsymbol{D}'\,\mathrm{Sol}\,\mathscr{M}$$

in $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathscr{O}_S)$. We have for each $s_o \in S$, by Proposition 3.1

$$Li^*_{s_o}(\mathrm{DR}\,\mathscr{M}) \simeq \mathrm{DR}\,Li^*_{s_o}(\mathscr{M}),$$
$$Li^*_{s_o}(\mathrm{Sol}\,\mathscr{M}) \simeq \mathrm{Sol}\,Li^*_{s_o}(\mathscr{M}).$$

Since $Li^*_{s_o}(\mathscr{M}) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$ by Lemma 3.8, we have

$$\mathrm{DR}\,Li^*_{s_o}(\mathscr{M}) \simeq \boldsymbol{D}'\,\mathrm{Sol}\,Li^*_{s_o}(\mathscr{M}),$$

so by Proposition 3.1 and Proposition 2.1

$$\boldsymbol{D}'\,\mathrm{Sol}\,Li^*_{s_o}(\mathscr{M}) \simeq \boldsymbol{D}' Li^*_{s_o}(\mathrm{Sol}\,\mathscr{M}) \simeq Li^*_{s_o}(\boldsymbol{D}'\,\mathrm{Sol}\,\mathscr{M}).$$

The assertion then follows by Proposition 2.2. q.e.d.

In the following proposition, the main argument is that of strictness, which is essential. We will set ${}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M} := \mathrm{DR}\,\mathscr{M}[\dim X]$ and ${}^{\mathrm{p}}\mathrm{Sol}\,\mathscr{M} = \mathrm{Sol}\,\mathscr{M}[\dim X]$.

Proposition 3.10. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_{X\times S/S}$-module which is strict, i.e., which is $p^{-1}\mathscr{O}_S$-flat. Then ${}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}$ satisfies the support condition* (Supp) *with respect to a μ-stratification adapted to $\mathscr{M}$.*

Proof. We prove the result by induction on $\dim S$. Since it is local on S, we consider a local coordinate s on S and we set $S' = \{s = 0\}$. The strictness property implies that we have an exact sequence

$$0 \to \mathscr{M} \xrightarrow{\;s\;} \mathscr{M} \to i^*_{S'}\mathscr{M} \to 0,$$

and $i_{S'}^*\mathscr{M}$ is $\mathscr{D}_{X\times S'/S'}$-holonomic and $p^{-1}\mathscr{O}_{S'}$-flat. We deduce an exact sequence of complexes $0\to {}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}\xrightarrow{s}{}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}\to{}^{\mathrm{p}}\mathrm{DR}\,i_{S'}^*\mathscr{M}\to 0$.

Let X_α be a stratum of a μ-stratification of X adapted to $\mathscr{M}$ (hence to $i_{S'}^*\mathscr{M}$, after Lemma 3.8). For $x\in X_\alpha$, let k be the maximum of the indices j such that $\mathscr{H}^j i_x^{-1}\,{}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}\neq 0$. For any S' as above, we have a long exact sequence

$$\cdots\to\mathscr{H}^k i_x^{-1}\,{}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}\xrightarrow{\;s\;}\mathscr{H}^k i_x^{-1}\,{}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}\to\mathscr{H}^k i_x^{-1}\,{}^{\mathrm{p}}\mathrm{DR}\,i_{S'}^*\mathscr{M}\to 0.$$

If $k>-\dim X_\alpha$, we have $\mathscr{H}^k i_x^{-1}\,{}^{\mathrm{p}}\mathrm{DR}\,i_{S'}^*\mathscr{M}=0$, according to the support condition for $i_{S'}^*\mathscr{M}$ (inductive assumption), since (X_α) is adapted to it. Therefore, $s:\mathscr{H}^k i_x^{-1}\,{}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}\to\mathscr{H}^k i_x^{-1}\,{}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}$ is onto. On the other hand, by Theorem 3.7, $\mathscr{H}^k i_x^{-1}\,{}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}$ is $\mathscr{O}_S$-coherent. Then Nakayama's lemma implies that $\mathscr{H}^k i_x^{-1}\,{}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}=0$ in some neighbourhood of S'. Since S' was arbitrary, this holds all over S, hence the assertion. q.e.d.

Proof of Theorem 1.2. It is a direct consequence of the following.

Theorem 3.11. *Let $\mathscr{M}$ be an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{X\times S/S})$ and let $\boldsymbol{D}\mathscr{M}$ be the dual object. Then there is an isomorphism ${}^{\mathrm{p}}\mathrm{DR}\,\boldsymbol{D}\mathscr{M}\simeq\boldsymbol{D}\,{}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}$.*

Indeed, with the assumptions of Theorem 1.2, $\boldsymbol{D}\mathscr{M}$ is holonomic since $\mathscr{M}$ is so (see Corollary 3.6), and both $\mathscr{M}$ and $\boldsymbol{D}\mathscr{M}$ are strict. Then both ${}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}$ and ${}^{\mathrm{p}}\mathrm{DR}\,\boldsymbol{D}\mathscr{M}$ satisfy the support condition, according to Proposition 3.10. Hence, according to Theorem 3.11 and Proposition 2.28, ${}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}$ satisfies the cosupport condition.

Similarly, ${}^{\mathrm{p}}\mathrm{Sol}\,\mathscr{M}\simeq\boldsymbol{D}\,{}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}$ and $\boldsymbol{D}({}^{\mathrm{p}}\mathrm{Sol}\,\mathscr{M})\simeq{}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}$ both satisfy the support condition, hence $\mathrm{Sol}\,\mathscr{M}[\dim X]$ is a perverse object. q.e.d.

Proof of Theorem 3.11. According to (5), we have $\mathrm{DR}\,\boldsymbol{D}\mathscr{M}=\mathrm{Sol}\,\mathscr{M}$, so by Corollary 3.9 we get $\boldsymbol{D}'\,\mathrm{DR}\,\boldsymbol{D}\mathscr{M}\simeq\mathrm{DR}\,\mathscr{M}$, that is, $\boldsymbol{D}\,{}^{\mathrm{p}}\mathrm{DR}\,\boldsymbol{D}\mathscr{M}\simeq{}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}$. Due to Theorem 3.7, we can apply Proposition 2.23 to conclude by biduality. q.e.d.

Example 3.12. Let X be the open unit disc in $\mathbb{C}$ with coordinate x and let S be a connected open set of $\mathbb{C}$ with coordinate s. Let $\varphi:S\to\mathbb{C}$ be a non constant holomorphic function on S and consider the holonomic $\mathscr{D}_{X\times S/S}$-module $\mathscr{M}=\mathscr{D}_{X\times S/S}/\mathscr{D}_{X\times S/S}\cdot P$, with $P=x\partial_x-\varphi(s)$. It is easy to check that $\mathscr{M}$ has no $\mathscr{O}_S$-torsion and admits the resolution $0\to\mathscr{D}_{X\times S/S}\xrightarrow{\cdot P}\mathscr{D}_{X\times S/S}\to\mathscr{M}\to 0$, so that the dual module $\boldsymbol{D}\mathscr{M}$ has a similar presentation and is also $\mathscr{O}_S$-flat. The complex ${}^{\mathrm{p}}\mathrm{Sol}\,\mathscr{M}$ is represented by $0\to\mathscr{O}_{X\times S}\xrightarrow{P\cdot}\mathscr{O}_{X\times S}\to 0$ (terms in degrees -1 and 0). Consider the stratification $X_1=X\smallsetminus\{0\}$ and $X_0=\{0\}$ of X. Then $\mathscr{H}^{-1}\,{}^{\mathrm{p}}\mathrm{Sol}\,\mathscr{M}_{|X_1}$ is a locally constant sheaf of free $p_X^{-1}\mathscr{O}_S$-modules generated by a local determination of $x^{\varphi(s)}$, and $\mathscr{H}^0\,{}^{\mathrm{p}}\mathrm{Sol}\,\mathscr{M}_{|X_1}=0$. On the other hand, $\mathscr{H}^{-1}\,{}^{\mathrm{p}}\mathrm{Sol}\,\mathscr{M}_{|X_0}=0$ and $\mathscr{H}^0\,{}^{\mathrm{p}}\mathrm{Sol}\,\mathscr{M}_{|X_0}$ is a skyscraper sheaf on $X_0\times S$ supported on $\{s\in S\mid\varphi(s)\in\mathbb{Z}\}$.

For each x_0 we have

$$\begin{aligned} &i^!_{x_0}({}^{\mathrm{p}}\mathrm{Sol}\,\mathscr{M}) \\ &\simeq i^{-1}_{\{x_0\}\times S} R\mathscr{H}om_{\mathscr{D}_{X\times S}}(\mathscr{D}_{X\times S}\otimes_{\mathscr{D}_{X\times S/S}}\mathscr{M}, R\Gamma_{\{x_0\}\times S|X\times S}\mathscr{O}_{X\times S})[\dim X] \\ &\qquad\qquad \simeq i^{-1}_{\{x_0\}\times S} R\mathscr{H}om_{\mathscr{D}_{X\times S}}(\mathscr{D}_{X\times S}\otimes_{\mathscr{D}_{X\times S/S}}\mathscr{M}, B_{\{x_0\}\times S|X\times S}) \end{aligned}$$

where $B_{\{x_0\}\times S|X\times S} := \mathscr{H}^1_{[\{x_0\}\times S]}(\mathscr{O}_{X\times S})$ denotes the sheaf of holomorphic hyperfunctions (of finite order) along $x = x_0$ (cf. [16]). The second isomorphism follows from the fact that $\mathscr{D}_{X\times S}\otimes_{\mathscr{D}_{X\times S/S}}\mathscr{M}$ is regular specializable along the submanifold $x = x_0$ (cf. [8]).

Recall that the sheaves $B_{\{x_0\}\times S|X\times S}$ are flat over $p_X^{-1}\mathscr{O}_S$ because locally they are inductive limits of free $p_X^{-1}\mathscr{O}_S$-modules of finite rank.

Since $i^!_{x_0}({}^{\mathrm{p}}\mathrm{Sol}\,\mathscr{M})$ is quasi isomorphic to the complex

$$0 \to B_{\{x_0\}\times S|X\times S}|_{\{x_0\}\times S} \xrightarrow{P\cdot} B_{\{x_0\}\times S|X\times S}|_{\{x_0\}\times S} \to 0$$

it follows that the flat dimension over $\mathscr{O}_S$ of $i^!_{x_0}({}^{\mathrm{p}}\mathrm{Sol}\,\mathscr{M})$ in the sense of [5, §4] is $\leqslant 0$ for any x_0. Moreover, $\mathscr{H}^0 i^!_{x_0}({}^{\mathrm{p}}\mathrm{Sol}\,\mathscr{M}) = 0$ and, if $x_0 \neq 0$, $\mathscr{H}^1 i^!_{x_0}({}^{\mathrm{p}}\mathrm{Sol}\,\mathscr{M})$ is a locally free $\mathscr{O}_S$-module of rank 1. Hence the flat dimension of $i^!_{x_0}({}^{\mathrm{p}}\mathrm{Sol}\,\mathscr{M})$ is $\leqslant 1$. This shows explicitly that ${}^{\mathrm{p}}\mathrm{Sol}\,\mathscr{M}$ satisfies the condition (Cosupp+) of Corollary 2.29.

4. Application to mixed twistor $\mathscr{D}$-modules

Let $\mathscr{R}_{X\times\mathbb{C}}$ be the sheaf on $X\times\mathbb{C}$ of z-differential operators, locally generated by $\mathscr{O}_{X\times\mathbb{C}}$ and the z-vector fields $z\partial_{x_i}$ in local coordinates $(x_1,\dots,x_n)$ on X. When restricted to $X\times\mathbb{C}^*$, the sheaf $\mathscr{R}_{X\times\mathbb{C}^*}$ is isomorphic to $\mathscr{D}_{X\times\mathbb{C}^*/\mathbb{C}^*}$.

A mixed twistor $\mathscr{D}$-module on X (see [13]) is a triple $\mathscr{T} = (\mathscr{M}', \mathscr{M}'', C)$, where $\mathscr{M}', \mathscr{M}''$ are holonomic $\mathscr{R}_{X\times\mathbb{C}}$-modules and C is a certain pairing with values in distributions, that we will not need to make precise here. Such a triple is subject to various conditions. We say that a $\mathscr{D}_{X\times\mathbb{C}^*/\mathbb{C}^*}$-module $\mathscr{M}$ underlies a mixed twistor $\mathscr{D}$-module $\mathscr{T}$ if $\mathscr{M}$ is the restriction to $X\times\mathbb{C}^*$ of $\mathscr{M}'$ or $\mathscr{M}''$.

Theorem 1.1 is now a direct consequence of the following properties of mixed twistor $\mathscr{D}$-modules, since they imply that $\mathscr{M}$ satisfies the assumptions of Theorem 1.2. If $\mathscr{M}$ underlies a mixed twistor $\mathscr{D}$-module, then

- there exists a locally finite filtration $W_\bullet\mathscr{M}$ indexed by $\mathbb{Z}$ by $\mathscr{R}_{X\times\mathbb{C}}$-submodules such that each graded module underlies a pure polarizable twistor $\mathscr{D}$-module; then each $\mathrm{gr}^W_\ell\mathscr{M}$ is strict and holonomic (see [14, Prop. 4.1.3] and [12, §17.1.1]), and thus so is $\mathscr{M}$;
- the dual of $\mathscr{M}$ as a $\mathscr{R}_{X\times\mathbb{C}^*}$-module also underlies a mixed twistor $\mathscr{D}$-module, hence is also strict holonomic (see [13, Th. 12.9]); using the isomorphism $\mathscr{R}_{X\times\mathbb{C}^*}\simeq\mathscr{D}_{X\times\mathbb{C}^*/\mathbb{C}^*}$, we see that the dual $\boldsymbol{D}\mathscr{M}$ as a $\mathscr{D}_{X\times\mathbb{C}^*/\mathbb{C}^*}$-module is strict and holonomic. q.e.d.

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