STOKES PHENOMENON IN DIMENSION TWO

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Introduction

A *meromorphic connection* on a complex analytic manifold X, with poles along a reduced divisor $Z \subset X$ is a coherent $\mathcal{O}_X[*Z]$-module $\mathcal{M}$ (where $\mathcal{O}_X[*Z]$ denotes the sheaf of meromorphic functions on X with poles along Z) equipped with a flat connection $\nabla : \mathcal{M} \to \Omega^1_X \otimes \mathcal{M}$.

In this paper we will assume that X is a surface and Z is a curve. Such a connection is then locally free over $\mathcal{O}[*Z]$ and, given a local basis and local coordinates (x_1, x_2), the matrix Θ of ∇ can be written $\Theta = A_1(x_1, x_2)dx_1 + A_2(x_1, x_2)dx_2$ where A_1, A_2 have poles along Z and satisfy the following integrability condition

$$\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \; = \; [A_1, A_2].$$

The case when the connection is regular (equivalently with regular singularities) is well understood in any dimension (see [3], see also [9] and [13]).

In dimension one, it is easy to give examples of regular or irregular meromorphic connections. Consider for instance a holomorphic differential operator and apply the usual criterion to determine wether its singularities are regular or not.

In dimension bigger than one, due to the integrability condition imposed on the connection, explicit examples are not easily described. Many examples of regular connections are obtained by constructions coming from algebraic geometry (Gauss-Manin connection). The (partial or complete) Fourier transform takes a regular connection on $\mathbf{C}^n$ into a (maybe) irregular connection on $\mathbf{C}^n$ (taking into account singularities at infinity).

Meromorphic connections are a particular case of holonomic $\mathcal{D}_X$-modules. One can apply to them operations of this theory (direct image for instance) and localize to obtain new meromorphic connections.

This paper is a survey concerning (irregular) meromorphic connections and the associated Stokes phenomenon in dimension two. We will give the main statements without proof (which will appear elsewhere). The first part consists of formal properties and the second one is concerned with asymptotic properties.

1 Formal structure of meromorphic connections in dimension two

1.1 Preliminaries

Operations on meromorphic connections. Given two $\mathcal{O}_X[*Z]$-connections, the direct sum and the tensor product over $\mathcal{O}_X$ have a natural structure of a meromorphic connection. A morphism of meromorphic connections is a morphism of $\mathcal{O}_X$-modules which commutes with ∇.

Inverse image of meromorphic connections. Let X' be a complex analytic manifold (of dimension one or two) and let $\pi : X' \to X$ be an analytic map. Assume that $Z' = \pi^{-1}(Z)$ has everywhere codimension one in X'. One can then define the inverse image $\pi^+\mathcal{M}$ as a meromorphic connection on X' with poles along Z': as a $\mathcal{O}_{X'}[*Z']$-module, this is $\pi^*\mathcal{M}$; the connection matrix in a local basis is $\pi^*\Theta$. We will mainly consider the case when π is a sequence of blowing-up and ramified coverings, or the case when π is the normalization of the germ of a curve in X not contained in Z.

Formal meromorphic connections. Let $x^0 \in Z$. We can consider two kinds of formalization of $\mathcal{O}_X$ near x^0:

- $\mathcal{O}_{\widehat{X|Z},x^0}$ is the germ at x^0 of the sheaf $\mathcal{O}_{\widehat{X|Z}}$, which is the formal completion of $\mathcal{O}_X$ along Z;
- $\mathcal{O}_{\widehat{X|x^0}} = \widehat{\mathcal{O}}_{x^0}$ is the ring of formal power series at x^0 (and does not depend on Z).

When $Z = D$ is a curve with only normal crossings, we will denote Y a stratum of D: if Y has dimension one, this is a connected component of the smooth part of D and if Y has dimension zero, this is a crossing point of D. In this situation we will mainly use the formal objects $\mathcal{O}_{\widehat{X|Y}}$ for each stratum Y, instead of the sheaf $\mathcal{O}_{\widehat{X|D}}$ (these objects do not glue as a sheaf on D).

The definition of a formal meromorphic connection is then straightforward.

1.2 Good formal structure

We assume here that $Z = D$ is a curve with normal crossings in X.

Let φ be a local section of $\mathcal{O}_X[*D]$. We will denote $\mathcal{E}^\varphi$ the free $\mathcal{O}_X[*D]$-connection of rank 1, equipped with a basis for which the matrix of the connection is $d\varphi$. The isomorphism class of this connection only depends on the class of φ in $\mathcal{O}_X[*D]/\mathcal{O}_X$, that we shall denote with the same symbol.

Let $x^0 \in D$ and choose local coordinates at x^0 such that $D = \{x_1x_2 = 0\}$ if x^0 is a crossing point of D and $D = \{x_1 = 0\}$ otherwise. We will say that a $\mathcal{O}_X[*D]$-connection $\mathcal{R}$ is *regular* if there exists a finite dimensional $\mathbf{C}$-vector space V equipped with an endomorphism δ_1 (and δ_2 if x^0 is a crossing point, with $[\delta_1, \delta_2] = 0$), such that $\mathcal{R}$ is locally isomorphic to the connection $\mathcal{O}_X[*D] \otimes_{\mathbf{C}} V$, where

$$x_i\partial_{x_i}(f \otimes v) \;=\; x_i\partial_{x_i}(f) \otimes v + f \otimes \delta_i(v)$$

with $i = 1, 2$ if x^0 is a crossing point, and $i = 1$ and

$$\partial_{x_2}(f \otimes v) \;=\; \partial_{x_2}(f) \otimes v$$

otherwise. Every regular connection can be locally realized as a successive extension of rank one regular connections and such a connection will be denoted "x^a" with $a \in \mathbf{C}^2$ (and $a_2 = 0$ if x^0 is not crossing point).

An *elementary local model* $\mathcal{M}_1$ (at $x^0 \in D$) is a $\mathcal{O}_X[*D]$-connection isomorphic to a direct sum

$$\bigoplus_{\alpha \in A} (\mathcal{E}^{\varphi_\alpha} \otimes \mathcal{R}^\alpha)$$

where $(\varphi_\alpha)_{\alpha \in A}$ is a finite family of local sections (at $x^0 \in D$) of $\mathcal{O}_X[*D]/\mathcal{O}_X$ and $(\mathcal{R}^\alpha)_{\alpha \in A}$ a family of regular $\mathcal{O}_X[*D]$-connections.

We will say that the elementary local model is *good* if it admits such a decomposition for which the family $(\varphi_\alpha)_{\alpha \in A}$ satisfies the following

Property (B):

- If $\alpha \neq \beta$, then $\varphi_\alpha \neq \varphi_\beta$ in $\mathcal{O}_X[*D]/\mathcal{O}_X$; hence there exists at most one $\alpha \in A$ such that $\varphi_\alpha = 0$ and it is denoted α_0;
- the divisors $(\varphi_\alpha)_{\alpha \in A - \{\alpha_0\}}$ and $(\varphi_\alpha - \varphi_\beta)_{\alpha \neq \beta \in A}$ are supported on D and are < 0; in particular the divisors $(\varphi_\alpha)_{\alpha \in A}$ are totally ordered.

In other words, when $D = \{x_1 = 0\}$, each nonzero φ_α or $\varphi_\alpha - \varphi_\beta$ can be written $x_1^{-r_1}u(x_1, x_2)$ with $r_1 > 0$ and $u(0, 0) \neq 0$, and when $D = \{x_1x_2 = 0\}$, each such polar part is written $x_1^{-r_1}x_2^{-r_2}u(x_1, x_2)$ with $(r_1, r_2) \in \mathbf{N}^2 - \{0\}$ and $u(0, 0) \neq 0$.

An elementary local model admits a decomposition as a direct sum of elementary connections $\mathcal{E}^{\varphi_\alpha} \otimes \mathcal{R}^\alpha$, and if property (B) is satisfied, one can see that this decomposition is unique. If the elementary local model is not good, it becomes so after a convenient sequence of point blowing-up.

If $\mathcal{M}_1$ is a good elementary local model and if $\varphi_\alpha \neq 0$ is an exponent appearing in the good decomposition, the connection $\mathcal{E}^{-\varphi_\alpha} \otimes \mathcal{M}_1$ is also a good elementary local model.

All these notions can be transposed for connections defined on $\mathcal{O}_{\widehat{X|Y}}[*D]$ when Y is a stratum of D. One has then to take care that the $\widehat{\varphi}_\alpha$ are local sections of $\mathcal{O}_{\widehat{X|Y}}[*D]/\mathcal{O}_{\widehat{X|Y}}$ and that, if $\dim Y = 0$, the connection $\widehat{\mathcal{E}^{\varphi}}$ does not necessarily come from a $\mathcal{O}_X[*D]$-connection, because if $Y = \{x^0\}$ we have $\mathcal{O}_{X,x^0}[*D]/\mathcal{O}_{X,x^0} \neq \widehat{\mathcal{O}}_{x^0}[*D]/\widehat{\mathcal{O}}_{x^0}$.

Let $\mathcal{M}$ be a $\mathcal{O}_X[*D]$-connection. Let Y be a stratum of D. Denote $\widehat{\mathcal{M}} = \mathcal{O}_{\widehat{X|Y}} \otimes_{\mathcal{O}_X} \mathcal{M}$. We will say that a $\mathcal{O}_X[*D]$-connection $\mathcal{M}$ has a *good formal decomposition* along (D, Y) at $0 \in Y$ if there exists a good elementary local model $\mathcal{M}_1$ in a neighbourhood of 0 and an isomorphism

$$\widehat{\mathcal{M}} \;\simeq\; \widehat{\mathcal{M}_1}.$$

Such a local model is then unique up to isomorphism.

We will say that $\mathcal{M}$ has a *good formal structure* along (D, Y) at $0 \in Y$ if after a covering ρ, cyclically ramified around the components of D, defined in a neighbourhood of 0, the inverse image $\rho^+\mathcal{M}$ has a good formal decomposition along (D, Y) at $0 \in Y$.

We will say that $\mathcal{M}$ has a *good formal structure* along D if for every stratum Y of the natural stratification of D it has a good formal structure along (D, Y) at each point of Y.

One can give a stronger definition of a good formal decomposition (for crossing points of D): if the decomposition is valid after tensorisation with $\mathcal{O}_{\widehat{X|D}}$, (and the local model is good) we will say that the decomposition is *very good*. The notion of a very good formal structure is defined in the same way. This is usually too strong.

One can give a weaker definition of a good formal decomposition by requiring only that the elementary model $\mathcal{M}_1$ is formal, *i.e.* that the exponents φ are formal. This is weaker only at crossing points of D. The following shows that for a meromorphic connection defined over $\mathcal{O}_X[*D]$ (and not only over $\widehat{\mathcal{O}}[*D]$) the two properties (existence of a formal structure or of a weak formal structure) are equivalent. More precisely we have

Proposition 1.2.1 *Let $\mathcal{M}$ be a $\mathcal{O}_X[*D]$-connection. Then*

1. *for all $x^0 \in D$, there exists an open neighbourhood Δ of x^0 in D such that $\mathcal{M}$ has a good formal structure along $\Delta - \{x^0\}$;*

2. *if $\mathcal{M}$ has a weak good formal structure (or decomposition) at $x^0 \in D$ then $\mathcal{M}$ has a good formal structure (or decomposition) in a neighbourhood of x^0; moreover, in the case of a decomposition, an elementary local model at x^0 is an elementary local model at each point in a neighbourhood of x^0.*

The fact that the existence of a good formal structure is generic (first point) is analogous to some results in [1].

1.3 Existence of a good formal structure

Let Z be the germ of an analytic curve at $x^0 \in X$ and let $\mathcal{M}$ be a $\mathcal{O}_X[*Z]$-connection near x^0. The following conjecture is not yet proved in general but the result would certainly be central in the theory.

Conjecture 1.3.1 *There exists a sequence of point blowing-up $e : X' \to X$ above x^0 such that, above a neighbourhood of x^0:*

1. *$D = e^{-1}(Z)$ is a divisor with normal crossings;*
2. *for each stratum Y of D, the formalized connection $\mathcal{O}_{\widehat{X'|Y}} \otimes_{\mathcal{O}_{X'}} e^{+}\mathcal{M}$ has a weak good formal structure along D at each point of Y.*

Thanks to the previous proposition, the second statement is equivalent to saying that $e^{+}\mathcal{M}$ has a good formal structure along D. However the statement given in the conjecture applies as well to formal meromorphic connections. Notice also that the conjecture would not be true with the second statement replaced with the fact $e^{+}\mathcal{M}$ admits a very good formal structure along D. At the moment we have

Theorem 1.3.2 *The conjecture is true when the rank of $\mathcal{M}$ is ≤ 5.*

Another example when the conjecture is true was mainly found by B. Malgrange [11]: assume that Z is smooth at x^0 and that the matrix of the connection in some local basis has the following form in local coordinates for which $Z = \{x_1 = 0\}$

$$\Theta = x_1^{-r_1}\left[A(x_1, x_2)\frac{dx_1}{x_1} + B(x_1, x_2)dx_2\right]$$

where A, B have holomorphic coefficients, $r_1 > 0$ and $A_0 = A(0,0) \neq 0$ is a constant regular matrix (*i.e.* its minimal polynomial is equal to its characteristic polynomial). Then $\mathcal{M}$ has a formal decomposition, which becomes good after a convenient sequence of blowing-up. This can be used to prove that the conjecture is true if Z has normal crossings at x^0 and the matrix of the connection in some local basis is

$$\Theta = x^{-r}\left[A(x_1, x_2)\frac{dx_1}{x_1} + B(x_1, x_2)\frac{dx_2}{x_2}\right]$$

where $r = (r_1, r_2) \in \mathbf{N}^2 - \{0\}$, A and B have holomorphic coefficients and for each $n_1, n_2 \in \mathbf{N} - \{0\}$, the matrix $n_1 A(0,0) + n_2 B(0,0)$ is regular and nonzero (notice that $A(0,0)$ and $B(0,0)$ commute, thanks to the integrability condition).

Here is another family of examples for which the conjecture is true. Let U be an open neighbourhood of 0 in $\mathbf{C}$ with coordinate x_2; this open set will be chosen as small as necessary. Let M be a holonomic $\mathcal{O}(U)[x_1]\langle\partial_{x_1},\partial_{x_2}\rangle$-module (see *e.g.* [2]), with regular singularities even along $x_1=\infty$.

Put $\xi_1=\partial_{x_1}$ et $\partial_{\xi_1}=-x_1$, so that $\mathcal{O}(U)[x_1]\langle\partial_{x_1},\partial_{x_2}\rangle=\mathcal{O}(U)[\xi_1]\langle\partial_{\xi_1},\partial_{x_2}\rangle$. We denote $\widehat{M}$ the module M considered as a $\mathcal{O}(U)[\xi_1]\langle\partial_{\xi_1},\partial_{x_2}\rangle$-module: this is the Fourier transform of M in the direction x_1. It is also holonomic but not necessarily regular. Moreover its singular set is a union of components of the divisor $\{\xi_1=0,\infty\}\cup\{x_2=0\}$. Hence, after localizing $\widehat{M}$ along its singular set, one gets a meromorphic connection $\widehat{M}'$ on $\mathbf{P}^1\times U$, with poles along the singular set. We then have

Theorem 1.3.3 *With these assumptions, conjecture* 1.3.1 *is true for* $\widehat{M}'$.

1.4 Semi-continuity of irregularity

Let $\mathcal{N}$ be a meromorphic connection on $(\mathbf{C},0)$ with pole at 0. We will denote $\mathrm{ir}_0\mathcal{N}$ the Malgrange-Komatsu irregularity number of $\mathcal{N}$ at zero. It is equal to the height of the Newton polygon of $\mathcal{N}$ when $\mathcal{N}$ is defined by a holomorphic differential operator.

Let now $X=(\mathbf{C}^2,0)$, Z be a germ of reduced analytic curve at 0 and $\mathcal{M}$ be a $\mathcal{O}_X[*Z]$-connection. There exists an open neighbourhood of 0 in Z, that we shall also denote Z, such that for every germ of smooth curve $(\Gamma,z^0):(\mathbf{C},0)\to(Z,z^0)$ transverse to Z at $z^0\in Z-\{0\}$, the irregularity number of the restriction $\Gamma^+\mathcal{M}$ at 0 is locally constant when z^0 moves on $Z-\{0\}$ (this follows for instance from § 2.4 in [12]).

Let Z_i ($i\in I$) be the irreductible components of the germ Z. Let $\gamma:(\mathbf{C},0)\to(\mathbf{C}^2,0)$ be the germ of an analytic curve whose image is not contained in Z and let $\gamma^+\mathcal{M}$ be the inverse image on $(\mathbf{C},0)$ with pole at 0. Denote (γ,Z_i) the intersection multiplicity at 0 of $\gamma(\mathbf{C},0)$ and Z_i, *i.e.* the valuation of the ideal $\gamma^*\mathcal{J}_{Z_i}$.

Theorem 1.4.1 *Assume that conjecture* 1.3.1 *is true for* $\mathcal{M}$. *Then the irregularity number of* $\gamma^+\mathcal{M}$ *at* 0 *satisfies the following inequality*

$$\mathrm{ir}_0(\gamma^+\mathcal{M}) \;\leq\; \sum_{i\in I}(\gamma,Z_i)\cdot\mathrm{ir}_{Z_i}(\mathcal{M})$$

where $\mathrm{ir}_{Z_i}(\mathcal{M})$ *denotes the irregularity number of the restriction of* $\mathcal{M}$ *to an analytic germ of curve transverse to* $Z_i-\{0\}$.

The inequality can be strict, and even the LHS can be zero (*i.e.* the restriction under γ can be regular) without the RHS being so. For instance consider the rank one connection equal to $\mathbf{C}[x,x^{-1},t]\cdot e^{t/x}$ with poles along $x=0$ and take for γ the curve $t=0$.

This theorem also shows (if the conjecture is true) that confluence phenomena from regular singularities to an irregular one cannot occur in an *integrable* family of meromorphic connections.

2 Analytic structure of meromorphic connections in dimension two

2.1 Preliminaries

We recall some notations of [14]. It will be convenient here to work in any dimension. Hence X will denote a complex analytic manifold, D a divisor with normal crossings in X all irreducible components of which are assumed to be smooth. When we consider a local situation, we will use the following notations: $X = V \times Y$ where V is a neighbourhood of 0 in $\mathbf{C}^n$ and Y is a neighbourhood of 0 in $\mathbf{C}^p$; the manifold X has coordinates $x_1, \dots, x_n, y_1, \dots, y_p$ and we will denote $D = \{x_1 \cdots x_n = 0\}$. We will identify Y with $\{0\} \times Y \subset X$. In the application we will have $\dim X = 2$ and $n = 1$ or $n = 2$.

Let $\mathcal{D}_X$ be the sheaf of analytic differential operators on X (see for instance [2]).

Denote $\pi : \widetilde{X}(D) \to X$ the map obtained by composing the real oriented blowing-up of the components of D. Locally we have $\widetilde{X}(D) \simeq (\mathbf{R}_+)^n \times (S^1)^n \times Y$ and we have polar coordinates on $\widetilde{X}(D)$.

Let $\mathcal{C}^\infty_{\widetilde{X}(D)}$ be the sheaf of C^∞ functions on $\widetilde{X}(D)$. It comes equipped with a left action of logarithmic differential operators $\pi^{-1}\mathcal{D}_X\langle\log D\rangle$ and their conjugates $\pi^{-1}\mathcal{D}_{\overline{X}}\langle\log \overline{D}\rangle$ (locally generated by the logarithmic vector fields). Put

$$\mathcal{A}_{\widetilde{X}(D)} = \bigcap_{i=1}^{n} \ker \overline{x}_i \overline{\partial}_{x_i} \cap \bigcap_{j=1}^{p} \ker \overline{\partial}_{y_j} \subset \mathcal{C}^\infty_{\widetilde{X}(D)}.$$

This is a sheaf of left $\pi^{-1}\mathcal{D}_X$-modules (see for instance § 2 in [14]).

Local sections of $\mathcal{A}_{\widetilde{X}(D)}$ on $\widetilde{X}(D)$ are holomorphic functions in multi-sectors which admit asymptotic expansions, which are uniform with respect to Y, when the variables x go to 0.

If Z is a closed subspace of D, we will consider the sheaf $\mathcal{A}^{\leq Z}_{\widetilde{X}(D)}$ which is the intersection of $\mathcal{A}_{\widetilde{X}(D)}$ and $\mathcal{P}^{<\pi^{-1}(Z)}_{\widetilde{X}(D)}$ (C^∞ functions on $\widetilde{X}(D)$, flat on $\pi^{-1}(Z)$). We will mainly consider the case when Z is a smooth submanifold of D or the case when $Z = D$. We denote

$$\mathcal{A}_{\widehat{\widetilde{X}|Z}} = \varprojlim_k \mathcal{A}_{\widetilde{X}} / \mathcal{J}_Z^k \mathcal{A}_{\widetilde{X}}$$

the formal completion of $\mathcal{A}_{\widetilde{X}}$ along (the inverse image of) Z and we denote $T_Z : \mathcal{A}_{\widetilde{X}} \to \mathcal{A}_{\widehat{\widetilde{X}|Z}}$ the natural map.

In the local setting, when Z is defined by $x_1 = \cdots = x_r = 0$, the restriction to Z of the sheaf $\mathcal{O}_{\widehat{X|Z}}$ (which is zero outside Z) is the sheaf defined by

$$U \longmapsto \mathcal{O}_Z(U) [\![x_1, \dots, x_r]\!]$$

(formal power series in $x_1, \dots, x_r$ with coefficients in $\mathcal{O}_Z(U)$). In the same way, let $\widetilde{Z}$ be the oriented real blowing-up Z along $x_{r+1} = 0, \dots, x_n = 0$. We have a map $p : \pi^{-1}(Z) \to \widetilde{Z}$ with fibre $(S^1)^r$: in polar coordinates, the projection $\pi : \widetilde{X}(D) \to X$ is given by $x_j = \rho_j e^{i\theta_j}$ for all $j = 1, \dots, n$, $\pi^{-1}(Z)$ is defined by $\rho_j = 0$ for all $j = 1, \dots, r$ and p forgets the arguments θ_j for $j = 1, \dots, r$. Let $\mathcal{A}_{\widetilde{Z}} [\![x_1, \dots, x_r]\!]$ the sheaf on $\widetilde{Z}$ defined by

$$\widetilde{U} \longmapsto \mathcal{A}_{\widetilde{Z}}(\widetilde{U}) [\![x_1, \dots, x_r]\!]$$

(formal power series in $x_1, \dots, x_r$ with coefficients in $\mathcal{A}_{\widetilde{Z}}(\widetilde{U})$). Then we have $\mathcal{A}_{\widehat{\widetilde{X}|Z}} = p^{-1}\mathcal{A}_{\widetilde{Z}} [\![x_1, \dots, x_r]\!]$ and $\pi_* \mathcal{A}_{\widehat{\widetilde{X}|Z}} = \mathcal{O}_{\widehat{X|Z}}$. One can see in an analogous way that there is an exact sequence

$$0 \longrightarrow \mathcal{A}^{<Z}_{\widetilde{X}} \longrightarrow \mathcal{A}_{\widetilde{X}} \xrightarrow{T_Z} \mathcal{A}_{\widehat{\widetilde{X}|Z}}.$$

When $Z = Y$, we have $\mathcal{A}_{\widehat{\widetilde{X}|Z}} = \pi^{-1}\mathcal{O}_{\widehat{X|Y}}$. When $Z = D$, a section of the sheaf $\mathcal{A}^{<D}_{\widetilde{X}(D)}$ on an open set Ω of $\widetilde{X}(D)$ is a holomorphic function on $\Omega - \pi^{-1}(D)$ which, for every compact set K of Ω and all $N \in \mathbf{N}^p$, satisfies estimates of the following type on $K - \pi^{-1}(D)$

$$|\varphi(x)| \le C_{K,N} \left| x_1^{N_1} \cdots x_n^{N_n} \right|.$$

This follows from Cauchy integral formulas. The sheaf $\mathcal{A}_{\widehat{\widetilde{X}|D}}$ can be understood using the Mayer-Vietoris complex:

Lemma 2.1.1 *The Mayer-Vietoris complexes*

$$0 \to \mathcal{O}_{\widehat{X|D}} \to \bigoplus_{i=1}^{n} \mathcal{O}_{\widehat{X|D_i}} \to \bigoplus_{i,j} \mathcal{O}_{\widehat{X|D_i \cap D_j}} \to \cdots \to \mathcal{O}_{\widehat{X|Y}} \to 0$$

and

$$0 \to \mathcal{A}_{\widehat{\widetilde{X}|D}} \to \bigoplus_{i=1}^{n} \mathcal{A}_{\widehat{\widetilde{X}|D_i}} \to \bigoplus_{i,j} \mathcal{A}_{\widehat{\widetilde{X}|D_i \cap D_j}} \to \cdots \to \mathcal{A}_{\widehat{\widetilde{X}|Y}} \to 0$$

are exact.

It follows from this lemma that $\pi_* \mathcal{A}_{\widehat{\widetilde{X}|D}} = \mathcal{O}_{\widehat{X|D}}$ and that the sequence

$$0 \longrightarrow \mathcal{A}^{<D}_{\widetilde{X}} \longrightarrow \mathcal{A}_{\widetilde{X}} \xrightarrow{T_D} \mathcal{A}_{\widehat{\widetilde{X}|D}}$$

is exact. The following proposition (Borel-Ritt lemma) is important:

Proposition 2.1.2 (Majima [6]) *If Z is an intersection of components of D the sequence*

$$0 \longrightarrow \mathcal{A}_{\widetilde{X}}^{<Z} \longrightarrow \mathcal{A}_{\widetilde{X}} \xrightarrow{T_Z} \mathcal{A}_{\widehat{\widetilde{X}|Z}} \longrightarrow 0$$

is exact, as well as the sequence

$$0 \longrightarrow \mathcal{A}_{\widetilde{X}}^{<D} \longrightarrow \mathcal{A}_{\widetilde{X}} \xrightarrow{T_D} \mathcal{A}_{\widehat{\widetilde{X}|D}} \longrightarrow 0.$$

2.2 Theorems of Malgrange-Sibuya type

We consider here the local setting introduced above. Denote $\mathrm{GL}_d^{<D}(\mathcal{A}_{\widetilde{X}})$ the subsheaf of $\mathrm{GL}_d(\mathcal{A}_{\widetilde{X}})$ whose local sections are matrices for which the image in $\mathrm{GL}_d(\mathcal{A}_{\widehat{\widetilde{X}|D}})$ is equal to the identity matrix. These are the matrices which can be written $\mathrm{Id} + M$ with $M \in \mathrm{End}((\mathcal{A}_{\widetilde{X}}^{<D})^d)$. Let us give independent variables $z = (z_1, \ldots, z_r)$. One defines in the same way $\mathrm{GL}_d^{<D}(\mathcal{A}_{\widetilde{X}}[\![z]\!])$ (where $\mathcal{A}_{\widetilde{X}}[\![z]\!]$ has been defined in § 2.1). The following result is a variant of a theorem due to Majima ([6], [5]) which is a generalization of a theorem of Malgrange-Sibuya.

Theorem 2.2.1 *The image of*

$$H^1\left(\pi^{-1}(0), \mathrm{GL}_d^{<D}(\mathcal{A}_{\widetilde{X}}[\![z]\!])\right) \longrightarrow H^1\left(\pi^{-1}(0), \mathrm{GL}_d(\mathcal{A}_{\widetilde{X}}[\![z]\!])\right)$$

is (the class of) the identity.

The following corollary is proved by induction on n using Mayer-Vietoris type arguments.

Corollary 2.2.2 *The image of*

$$H^1\left(\pi^{-1}(0), \mathrm{GL}_d^{<Y}(\mathcal{A}_{\widetilde{X}})\right) \longrightarrow H^1\left(\pi^{-1}(0), \mathrm{GL}_d(\mathcal{A}_{\widetilde{X}})\right)$$

is the identity.

One also deduces

Corollary 2.2.3 *If $T \supset Y \ni 0$ is an intersection of components of D and D^T denotes the union of components of D which do not contain T, the image of the map*

$$H^1\left(\pi^{-1}(0), \mathrm{GL}_d^{<(D^T\cap T)}(\mathcal{A}_{\widehat{\widetilde{X}|T}})\right) \longrightarrow H^1\left(\pi^{-1}(0), \mathrm{GL}_d(\mathcal{A}_{\widehat{\widetilde{X}|T}})\right)$$

is the identity, as well as the one of

$$H^1\left(\pi^{-1}(0), \mathrm{GL}_d^{<Y}(\mathcal{A}_{\widehat{\widetilde{X}|T}})\right) \longrightarrow H^1\left(\pi^{-1}(0), \mathrm{GL}_d(\mathcal{A}_{\widehat{\widetilde{X}|T}})\right).$$

2.3 Good $\mathcal{A}$-structure

We assume here that X has dimension two, so we have $n = 1$ or $n = 2$ in the previous notations.

Let $\mathcal{M}$ be a $\mathcal{O}_X[*D]$-connection. Denote $\mathcal{M}_{\widetilde{X}} = \mathcal{A}_{\widetilde{X}(D)} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{M}$, which is a $\mathcal{A}_{\widetilde{X}(D)}[*D]$-module, locallly of finite type, equipped with a compatible action of $\mathcal{D}_{\widetilde{X}} \stackrel{\text{def}}{=} \mathcal{A}_{\widetilde{X}(D)} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{D}_X$. Let Y be a stratum of D (hence of dimension one or zero).

We will say that $\mathcal{M}$ has a *good $\mathcal{A}$-decomposition* along (D, Y) at $0 \in Y$ if there exists a good elementary local model $\mathcal{M}_1$ in a neighbourhood of 0 and for all $\theta^0 \in \pi^{-1}(0)$ an isomorphism

$$\mathcal{M}_{\widetilde{X},\theta^0} \;\simeq\; \mathcal{M}_{1\,\widetilde{X},\theta^0}$$

inducing a formal isomorphism $\widehat{\mathcal{M}} \simeq \widehat{\mathcal{M}}_1$ *independent of* $\theta^0 \in \pi^{-1}(0)$, where $\widehat{\mathcal{M}} = \mathcal{A}_{\widehat{\widetilde{X}|Y}} \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{M}$ (recall that on $\pi^{-1}(Y)$ one has $\mathcal{A}_{\widehat{\widetilde{X}|Y}} = \pi^{-1}\mathcal{O}_{\widehat{X|Y}}$). In particular $\mathcal{M}$ has a good formal decomposition along (D, Y) at 0.

We will say that $\mathcal{M}$ has a *good $\mathcal{A}$-structure* along (D, Y) at $0 \in Y$ if after a convenient local ramification ρ around the components of D, the inverse image $\rho^+\mathcal{M}$ has a good $\mathcal{A}$-decomposition along (D', Y) at 0.

We will say that $\mathcal{M}$ has a *good $\mathcal{A}$-structure* along D if for every stratum Y of D it has a good $\mathcal{A}$-structure along (D, Y) at each point of Y. In particular $\mathcal{M}$ has a good formal structure along D.

Theorem 2.3.1 *If $\mathcal{M}$ has a good formal decomposition (resp. a good formal structure) along (D, Y) at $0 \in Y$, then $\mathcal{M}$ has a good $\mathcal{A}$-decomposition (resp. a good $\mathcal{A}$-structure) there.*

From this follows

Corollary 2.3.2 *Let X be a complex analytic surface, Z be a divisor of X and $\mathcal{M}$ be a $\mathcal{O}_X[*Z]$-connection. If conjecture* 1.3.1 *is true for $\mathcal{M}$, there exists a sequence of point blowing-up $e : X' \to X$ such that all irreducible components of $e^{-1}(Z)$ are smooth and intersect each other normally, and such that the inverse image $e^+\mathcal{M}$ has a good $\mathcal{A}$-structure along $e^{-1}(Z)$.*

When $\dim X = 1$, the map e is equal to identity and the theorem is a result of Hukuhara-Turrittin (see for instance [15,10]).

The proof of theorem 2.3.1 is done in two steps. The first one is a consequence of the work of H. Majima [6]:

Theorem 2.3.3 *Let $x^0 \in D$ and $\theta^0 \in \pi^{-1}(x^0) \subset \widetilde{X}(D)$. Let $\mathcal{M}$ be a $\mathcal{O}_X[*D]$-connection in a neighbourhood of x^0 and assume that we are given an isomorphism*

$$\mathcal{A}_{\widehat{\widetilde{X}|D},\theta^0} \underset{\mathcal{O}_{X,x^0}}{\otimes} \mathcal{M} \;\xrightarrow{\sim}\; \mathcal{A}_{\widehat{\widetilde{X}|D},\theta^0} \underset{\mathcal{O}_{X,x^0}}{\otimes} \mathcal{M}_1$$

where $\mathcal{M}_1$ *is a* good *elementary model. Then this isomorphism can be lifted as an isomorphism*

$$\mathcal{A}_{\widetilde{X},\theta^0} \mathop{\otimes}_{\mathcal{O}_{X,x^0}} \mathcal{M} \xrightarrow{\sim} \mathcal{A}_{\widetilde{X},\theta^0} \mathop{\otimes}_{\mathcal{O}_{X,x^0}} \mathcal{M}_1.$$

When x^0 is not a crossing point of D, this theorem is due to Sibuya and implies theorem 2.3.1. Indeed by assumption there exists an elementary model $\mathcal{M}_1$ such that

$$\mathcal{O}_{\widehat{X|D},x^0} \mathop{\otimes}_{\mathcal{O}_{X,x^0}} \mathcal{M} \simeq \mathcal{O}_{\widehat{X|D},x^0} \mathop{\otimes}_{\mathcal{O}_{X,x^0}} \mathcal{M}_1 \tag{1}$$

and one gets immediately for all $\theta^0 \in \pi^{-1}(x^0)$ an isomorphism (independent of θ^0)

$$\mathcal{A}_{\widehat{\widetilde{X}|D},\theta^0} \mathop{\otimes}_{\mathcal{O}_{X,x^0}} \mathcal{M} \simeq \mathcal{A}_{\widehat{\widetilde{X}|D},\theta^0} \mathop{\otimes}_{\mathcal{O}_{X,x^0}} \mathcal{M}_1 \tag{2}$$

hence (from the theorem) an isomorphism (depending on θ^0)

$$\mathcal{A}_{\widetilde{X},\theta^0} \mathop{\otimes}_{\mathcal{O}_{X,x^0}} \mathcal{M} \simeq \mathcal{A}_{\widetilde{X},\theta^0} \mathop{\otimes}_{\mathcal{O}_{X,x^0}} \mathcal{M}_1 \tag{3}$$

which lifts (2) hence (1).

When x^0 is a crossing point, this result is not enough to prove theorem 2.3.1. One has to show moreover

Proposition 2.3.4 *Let* $\mathcal{M}$ *be a* $\mathcal{O}_X[*D]$*-connection in a neighbourhood of a crossing point* x^0 *having a good elementary model* $\mathcal{M}_1$ *at* x^0*, i.e. equipped with an isomorphism* $\widehat{\mathcal{M}} \simeq \widehat{\mathcal{M}}_1$*. Then for all* $\theta^0 \in \pi^{-1}(x^0)$*, this isomorphism can be lifted as an isomorphism*

$$\mathcal{A}_{\widehat{\widetilde{X}|D},\theta^0} \mathop{\otimes}_{\mathcal{O}_{X,x^0}} \mathcal{M} \simeq \mathcal{A}_{\widehat{\widetilde{X}|D},\theta^0} \mathop{\otimes}_{\mathcal{O}_{X,x^0}} \mathcal{M}_1.$$

From lemma 2.1.1, it is equivalent to find liftings

$$\mathcal{A}_{\widehat{\widetilde{X}|D_i},\theta^0} \mathop{\otimes}_{\mathcal{O}_{X,x^0}} \mathcal{M} \simeq \mathcal{A}_{\widehat{\widetilde{X}|D_i},\theta^0} \mathop{\otimes}_{\mathcal{O}_{X,x^0}} \mathcal{M}_1 \tag{4}$$

for $i = 1, 2$.

2.4 Local classification of meromorphic connections having a good $\mathcal{A}$*-decomposition*

We consider here a local situation and we choose coordinates x_1, x_2 on X in such a way that $D = \{x_1 = 0\}$ or $D = \{x_1x_2 = 0\}$. Let $\mathcal{M}$ be a $\mathcal{O}_X[*D]$-connection having a good $\mathcal{A}$-decomposition along (D, Y) at $0 \in Y$ and let $\mathcal{M}_1$ be a good elementary local model. One can associate with it

- a formal isomorphism $\widehat{f} : \widehat{\mathcal{M}} \to \widehat{\mathcal{M}_1}$;
- for all $\theta^0 \in \pi^{-1}(0)$ a lifting $f_{\theta^0} : \mathcal{M}_{\widetilde{X},\theta^0} \to \mathcal{M}_{1\,\widetilde{X},\theta^0}$ of $\widehat{f}$.

We will say that two pairs $(\mathcal{M}, \widehat{f})$ and $(\mathcal{M}', \widehat{f}')$ are equivalent if (in a neighbourhood of $0 \in Y$) there exists an isomorphism of $\mathcal{O}_X[*D]$-connections $\mathcal{M} \xrightarrow{\sim} \mathcal{M}'$ such that the composition

$$\widehat{\mathcal{M}} \xrightarrow{\sim} \widehat{\mathcal{M}'} \xrightarrow{\widehat{f}'} \widehat{\mathcal{M}_1}$$

is equal to $\widehat{f}$. Given a sheaf of left modules M over a sheaf of rings A, we denote $\mathrm{Aut}_A M$ the sheaf of local A-isomorphisms of M into itself. Then as in dimension one (see [8]) one can associate with such a pair a class in $H^1\left(\pi^{-1}(0), \mathrm{Aut}_{\mathcal{D}_{\widetilde{X}}[*D]}(\mathcal{M}_{1\,\widetilde{X}})\right)$ whose image in

$$H^1\left(\pi^{-1}(0), \mathrm{Aut}_{\mathcal{D}_{\widehat{\widetilde{X}|Y}}[*D]}(\mathcal{M}_{1\,\widehat{\widetilde{X}|Y}})\right) = H^1\left(\pi^{-1}(0), \pi^{-1}\,\mathrm{Aut}_{\mathcal{D}_{\widehat{X|Y}}[*D]}(\widehat{\mathcal{M}_1})\right)$$

is (the class of) the identity.

Lemma 2.4.1 *The maps*

$$\begin{array}{ccc}\mathrm{Aut}_{\mathcal{D}_{\widetilde{X}}[*D]}(\mathcal{M}_{1\,\widetilde{X}}) & \longrightarrow & \mathrm{Aut}_{\mathcal{D}_{\widehat{\widetilde{X}|Y}}[*D]}(\mathcal{M}_{1\,\widehat{\widetilde{X}|Y}}) \\ & \textit{and} & \\ \mathrm{Aut}_{\mathcal{D}_X[*D]}(\mathcal{M}_1) & \longrightarrow & \mathrm{Aut}_{\mathcal{D}_{\widehat{X|Y}}[*D]}(\widehat{\mathcal{M}_1})\end{array}$$

are onto.

We shall denote the kernels $\mathrm{Aut}^{<Y}_{\mathcal{D}_{\widetilde{X}}[*D]}(\mathcal{M}_{1\,\widetilde{X}})$ and $\mathrm{Aut}^{<Y}_{\mathcal{D}_X[*D]}(\mathcal{M}_1)$.

It follows from this lemma that we have an exact sequence of pointed sets

$$\begin{aligned}&\mathrm{Id} \to H^1\left(\pi^{-1}(0), \mathrm{Aut}^{<Y}_{\mathcal{D}_{\widetilde{X}}[*D]}(\mathcal{M}_{1\,\widetilde{X}})\right) \to \\ &\quad \to H^1\left(\pi^{-1}(0), \mathrm{Aut}_{\mathcal{D}_{\widetilde{X}}[*D]}(\mathcal{M}_{1\,\widetilde{X}})\right) \to H^1\left(\pi^{-1}(0), \mathrm{Aut}_{\mathcal{D}_{\widehat{\widetilde{X}|Y}}[*D]}(\mathcal{M}_{1\,\widehat{\widetilde{X}|Y}})\right)\end{aligned}$$

and one gets a map from the set of classes $(\mathcal{M}, \widehat{f})$ (pointed by the class $(\mathcal{M}_1, \mathrm{Id})$) to $H^1\left(\pi^{-1}(0), \mathrm{Aut}^{<Y}_{\mathcal{D}_{\widetilde{X}}[*D]}(\mathcal{M}_{1\,\widetilde{X}})\right)$.

Theorem 2.4.2 *This map is bijective.*

2.5 Solutions of meromorphic connections with values in the sheaf of distributions

In [4] M. Kashiwara has introduced a conjugation functor for holonomic $\mathcal{D}$-modules. We shall consider here the analogue of this functor for meromorphic connections. Recall first some notations.

Let $Z \subset X$ be a closed analytic subset of a complex analytic manifold X. Let $\mathcal{D}b_X^{\mathrm{mod}\, Z}$ denote the sheaf (on X) of distributions on $X - Z$ which have moderate growth along Z. This is (locally on X) the dual of $\mathcal{P}_X^{<Z}$, the sheaf of C^∞ functions on X which are flat on Z (*i.e.* for which all derivatives vanish identically on Z). Moreover (see chap. VII, § 1 in [7]), it is the image of $\mathcal{D}b_X$ (the sheaf of distributions on X) by the natural morphism $\mathcal{D}b_X \to j_* \mathcal{D}b_{X-Z}$, where $j : X - Z \hookrightarrow X$ denotes the inclusion. We hence have an exact sequence

$$0 \to \mathcal{D}b_{X,Z} \longrightarrow \mathcal{D}b_X \longrightarrow \mathcal{D}b_X^{\mathrm{mod}\, Z} \to 0$$

where $\mathcal{D}b_{X,Z}$ denotes the sheaf of distributions supported on Z.

If Z is a divisor in X given locally by an equation $f = 0$, one can see that f and $\overline{f}$ act in an invertible way on $\mathcal{D}b_X^{\mathrm{mod}\, Z}$. Using the injectivity of $\mathcal{D}b_X$ over $\mathcal{O}_X$, one verifies that

$$\mathcal{D}b_X^{\mathrm{mod}\, Z} = \mathcal{O}_X[*Z] \underset{\mathcal{O}_X}{\otimes} \mathcal{D}b_X = \mathcal{O}_{\overline{X}}[*\overline{Z}] \underset{\mathcal{O}_{\overline{X}}}{\otimes} \mathcal{D}b_X\,.$$

The sheaf $\mathcal{D}b_X^{\mathrm{mod}\, Z}$ is a $\mathcal{O}_X[*Z]$-module equipped with a flat connection (*i.e.* with a structure of a $\mathcal{D}_X$-module), but of course is not $\mathcal{O}_X[*Z]$-coherent. If $\overline{X}$ denotes the complex conjugate manifold (for which the sheaf of holomorphic function is the sheaf of antiholomorphic functions on X), then $\mathcal{D}b_X^{\mathrm{mod}\, Z}$ is also a $\mathcal{O}_{\overline{X}}[*\overline{Z}]$-module equipped with a flat connection, *i.e.* with a structure of a $\mathcal{D}_{\overline{X}}$-module. If $\mathcal{M}$ is a meromorphic connection, then $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_X^{\mathrm{mod}\, Z})$ comes naturally equipped with a structure of a $\mathcal{D}_{\overline{X}}[*\overline{Z}]$-module.

Theorem 2.5.1 *Let Z be a divisor in the surface X and let $\mathcal{M}$ be a $\mathcal{O}_X[*Z]$-connection. Assume that conjecture* 1.3.1 *is true for $\mathcal{M}$. Then*

1. $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_X^{\mathrm{mod}\, Z}) = 0$ *for $i \neq 0$ and*
2. $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_X^{\mathrm{mod}\, Z})$ *is a meromorphic $\mathcal{O}_{\overline{X}}[*\overline{Z}]$-connection.*

This theorem is proved by analysing the effect of complex conjugation on the Stokes matrices which represent the 1-cocycle given by the classification theorem 2.4.2.

1. D. G. Babbit, V. S. Varadarajan, *Deformation of nilpotent matrices over rings and reduction of analytic families of meromorphic differential equations*, Memoirs Amer. Math. Soc. vol. 55 n° 325, American Mathematical Society, Providence RI, 1985.
2. J.-E. Björk, *Rings of Differential Operators*, North Holland, Amsterdam, 1979.
3. P. Deligne, *Equations différentielles à points singuliers réguliers*, Lect. Notes in Math. vol. 163, Springer Verlag, 1970.
4. M. Kashiwara, *Regular holonomic $\mathcal{D}$-modules and distributions on complex manifolds,* in *Complex analytic singularities*, Advanced Studies in Pure Math. **8** (1986) 199-206.
5. H. Majima, *Analogues of Cartan decomposition theorem in asymptotic analysis,* Funkcialaj Ekvacioj **26** (1983) 131-154.
6. H. Majima, *Asymptotic analysis for integrable connections with irregular singular points*, Lect. Notes in Math. vol. 1075, Springer Verlag, 1984.
7. B. Malgrange, *Ideals of differentiable functions*, Oxford University Press, 1966.
8. B. Malgrange, *La classification des connexions irrégulières à une variable,* in *Séminaire E.N.S. Mathématique et Physique*, L. Boutet de Monvel, A. Douady, J.L. Verdier eds., Progress in Math. vol. 37, Birkhaüser, Boston, (1983) 381-399.
9. B. Malgrange, *Regular connexions after Deligne,* in *Algebraic $\mathcal{D}$-modules*, A. Borel ed., Perspectives in Math. vol. 2, Academic Press, Boston, (1987) Chapter IV, 151-172.
10. B. Malgrange, *Equations différentielles à coefficients polynomiaux*, Progress in Math. vol. 96, Birkhaüser, Boston, 1991.
11. B. Malgrange, *Connexions méromorphes* 2*, le réseau canonique,* Prépublication Institut Fourier, Grenoble (1994).
12. Z. Mebkhout, *Sur le théorème de semi-continuité de l'irrégularité des équations différentielles,* in *Systèmes différentiels et singularités*, Astérisque **130** (1985) 365-417.
13. Z. Mebkhout, *Le théorème de comparaison entre cohomologies de de Rham d'une variété algébrique complexe et le théorème d'existence de Riemann,* Publ. Math. I.H.E.S. **69** (1989) 47-89.
14. C. Sabbah, *Equations différentielles à points singuliers irréguliers en dimension* 2*,* Ann. Inst. Fourier **43** (1993) 1619-1688.
15. W. Wasow, *Asymptotic Expansions for Ordinary Differential Equations*, Interscience, New York, 1965.