

GOOD LATTICES OF ALGEBRAIC CONNECTIONS

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ABSTRACT. We construct a logarithmic model of connections on smooth quasi-projective n -dimensional geometrically irreducible varieties defined over an algebraically closed field of characteristic 0. It consists of a good compactification of the variety together with $(n+1)$ lattices on it which are stabilized by log differential operators, and compute algebraically de Rham cohomology. The construction is derived from the existence of good Deligne-Malgrange lattices, a theorem of Kedlaya and Mochizuki which consists first in eliminating the turning points. Moreover, we show that a logarithmic model obtained in this way, called a good model, yields a formula predicted by Michael Groechenig, computing the class of the characteristic variety of the underlying \mathcal{D} -module in the K -theory group of the variety.

1. INTRODUCTION

Let U be a smooth quasi-projective geometrically irreducible variety of dimension n defined over a characteristic 0 field k . An open embedding $j : U \hookrightarrow X$ is said to be a *good compactification* if X is smooth projective and $D = X \setminus U$ is a strict normal crossings divisor. Here strict normal crossings divisor means that the irreducible components of $D_{\bar{k}}$, where $\bar{k} \supset k$ is an algebraic closure and the lower index \bar{k} indicates the base change $\otimes_k \bar{k}$, are smooth and intersect transversally.

Definition 1.1. Let (\mathcal{E}, ∇) be vector bundle on U with an integrable connection relative to k . A tuple

$$(X, (E_0, E_1, \dots, E_n))$$

is called a *logarithmic model* of $(U, (\mathcal{E}, \nabla))$ if the following conditions are fulfilled.

- 0) $j : U \hookrightarrow X$ is a good compactification;
- 1) $E_0 \subset E_1 \subset \dots \subset E_n$ is a tower of locally free lattices of $j_*\mathcal{E}$;
- 2) $\nabla(\Omega_X^i(\log D) \otimes_{\mathcal{O}_X} E_i) \subset \Omega_X^{i+1}(\log D) \otimes_{\mathcal{O}_X} E_{i+1}$;
- 3) For any effective divisor Δ with support in D , the embeddings of k -linear complexes

$$\Omega_X^\bullet(\log D) \otimes_{\mathcal{O}_X} E_\bullet \subset \Omega_X^\bullet(\log D) \otimes_{\mathcal{O}_X} E_\bullet(\Delta) \subset j_*(\Omega_U^\bullet \otimes_{\mathcal{O}_U} \mathcal{E})$$

are quasi-isomorphisms.

There is another natural definition.

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Definition 1.2. Let (\mathcal{E}, ∇) be vector bundle on U with an integrable connection relative to k . A tuple

$$(X, (E_0, E_1, \dots, E_n))$$

is called a *good model* of $(U, (\mathcal{E}, \nabla))$ if the following conditions are fulfilled.

- 0) $j : U \hookrightarrow X$ is a good compactification;
- 0') j resolves the turning points of (\mathcal{E}, ∇) ;
- 1') E_0 is the Deligne-Malgrange lattice of \mathcal{E} (see Definition 2.1 and Theorem 2.4) and E_{i+1} is defined inductively as the \mathcal{O}_X -coherent subsheaf of $j_*\mathcal{E}$ spanned by E_i and $\Theta_X(\log D) \cdot E_i$.

Here, $\Theta_X(\log D)$ is the sheaf of vector fields stabilizing D . That good models exist is due to Kedlaya and Mochizuki. Our first main result is the following theorem.

Theorem 1.3. *Good models are logarithmic models.*

Our purpose is to prove that the lattices E_i as described in 1') verify 1) and 3). It is performed in Section 3 by constructing specific filtrations on the (log)-de Rham complexes, the graded of which are \mathcal{O}_X -linear.

The Deligne-Malgrange lattice E_0 and its log derivatives E_i ($i \geq 1$) can also be used to compute the characteristic class of $j_*\mathcal{E}$ in Grothendieck's K -group $K_0(\mathcal{O}_X)$. Recall that, in [Lau83, §6.1], Laumon defines a group homomorphism

$$\text{Car} : K_0(\mathcal{D}_X) \longrightarrow K_0(\mathcal{O}_{T^*X})$$

as follows. Let $F_\bullet \mathcal{D}_X$ be the filtration of \mathcal{D}_X by the order of differential operators. By an $F\mathcal{D}_X$ -filtration of a \mathcal{D}_X -module M , we mean an increasing filtration $F_\bullet M$ such that $F_k \mathcal{D}_X \cdot F_\ell M \subset F_{k+\ell} M$ for all $k, \ell \in \mathbb{Z}$. A *coherent $F\mathcal{D}_X$ -filtration* is an $F\mathcal{D}_X$ -filtration such that $\text{gr}^F M$ is coherent over $\text{gr}^F \mathcal{D}_X$. For any coherent \mathcal{D}_X -module M , let $F_\bullet M$ be any coherent $F\mathcal{D}_X$ -filtration. Then, denoting by $[\cdot]$ the class in K_0 , one has

$$\text{Car } M = [\text{gr}^F M].$$

Composing with the restriction by the zero section $i : X \hookrightarrow T^*X$, we obtain a group homomorphism

$$Li^* \text{Car} : K_0(\mathcal{D}_X) \longrightarrow K_0(\mathcal{O}_X).$$

Computing $Li^* \text{Car } M$ amounts to computing the class of

$$\text{gr}^F M \otimes_{\mathcal{O}_{T^*X}}^L \mathcal{O}_X \in K_0(\mathcal{O}_X).$$

Our second main result confirms an expectation of Michael Groechenig.

Theorem 1.4. *Let (\mathcal{E}, ∇) be vector bundle on U with an integrable connection relative to k . Let $(X, (E_0, E_1, \dots, E_n))$ be a good model of $(U, (\mathcal{E}, \nabla))$. Then*

$$Li^* \text{Car}(j_*\mathcal{E}) = \sum_{i=0}^n (-1)^{n-i} [\omega_X^{-1} \otimes \Omega_X^i(\log D) \otimes E_i] \quad \text{in } K_0(\mathcal{O}_X).$$

The proof rests on the two Appendices by the second named author. Appendix A is classical, see e.g. [MHM]. Appendix B is inspired by [Wei17] and gives a criterion for exactness of the tensor product of the differential operators over the ones with logarithmic poles.

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2. THE THEOREM OF KEDLAYA-MOCHIZUKI

Sabbah's conjecture [Sab00, Conj. 2.5.1] over $k = \mathbb{C}$, stipulating the existence of a good formal structure of (\mathcal{E}, ∇) at each closed point of $(X_0 \setminus U)$ for a given good compactification $U \hookrightarrow X_0$, after blow up and ramification, was solved by Mochizuki and Kedlaya, and for the latter, over any characteristic 0 field. Mochizuki's theorem is summarized in [Moc09] and contains more structure than Sabbah's initial conjecture. It yields a uniquely defined *lattice* with certain properties. See the definitions in *loc. cit.* 2.2.2, Remark 2.3, Conjecture 2.11, which is Sabbah's conjecture *enhanced* with the existence of good Deligne-Malgrange lattices after blow-up of a given good compactification, and ramification at the formal completion of each closed point at infinity, and see Theorem 2.12 in which Conjecture 2.11 is solved.

We use Kedlaya's more algebraic approach [Ked11, §5.3], that we now recall. Let $j : U \hookrightarrow X$ be a good compactification and x be a finite union of closed points in D . We denote by X_x , resp. D_x the spectrum of the semi-local ring $\mathcal{O}_{X,x}$, resp. $\mathcal{O}_{D,x}$ of X , resp. D at x , by \widehat{X}_x , resp. \widehat{D}_x the formal spectrum of the completion $\widehat{\mathcal{O}}_{X,x}$, resp. $\widehat{\mathcal{O}}_{D,x}$ with respect to the ideal of x . We set

$$U_x = X_x \setminus D_x, \quad \widehat{U}_x = \widehat{X}_x \setminus \widehat{D}_x.$$

and

$$(\mathcal{E}_x, \nabla_x) = \mathcal{O}_{X_x} \otimes_{\mathcal{O}_X} (j_* \mathcal{E}, \nabla), \quad (\widehat{\mathcal{E}}_x, \widehat{\nabla}_x) = \mathcal{O}_{\widehat{X}_x} \otimes_{\mathcal{O}_X} (j_* \mathcal{E}, \nabla).$$

Likewise, for any sheaf \mathcal{F} of \mathcal{O}_X -modules on X , we define \mathcal{F}_x and $\widehat{\mathcal{F}}_x$. We also denote by $\widehat{\mathcal{F}}_x$ an $\mathcal{O}_{\widehat{X}_x}$ -module, which is not necessarily defined over \mathcal{O}_{X_x} , and similarly we use the symbol $\widehat{(-)}_x$ for an $\mathcal{O}_{\widehat{X}_x}$ -morphism which does not necessarily descend to \mathcal{O}_{X_x} .

If $K \supset k$ is any field extension, we denote by a lower index K the base change $\otimes_k K$. Let $\bar{k} \supset k$ be the choice of an algebraic closure. Let $\tau : \bar{k}/\mathbb{Z} \rightarrow \bar{k}$ be the admissible section of the projection $\bar{k} \rightarrow \bar{k}/\mathbb{Z}$ (see Kedlaya's definition [Ked11, Def. 5.3.2]), which for $k = \mathbb{C}$ is the one used by Deligne and is characterized by the property that the real part of the image lies in $[0, 1)$.

Definition 2.1. Let $j : U \hookrightarrow X$ be a good compactification of a smooth quasi-projective geometrically irreducible variety defined over k , let (\mathcal{E}, ∇) be a vector bundle with an integrable connection on U .

- 1) An *unramified good Deligne-Malgrange lattice* $E \subset j_* \mathcal{E}$ is a lattice such that for every closed point $x \in D = X \setminus U$,

$$\widehat{E}_x = \bigoplus_{\varphi \in \Phi} (L_\varphi \otimes R_\varphi),$$

where R_φ is Deligne's extension of a regular singular connection associated to τ , $\Phi \subset \mathcal{O}(\widehat{U}_x)$ is a finite set satisfying the properties in [Ked11, Def. 3.4.6 (a) & (b)],

and L_φ is a purely irregular lattice of a connection of rank 1, i.e., L_φ is isomorphic to $\mathcal{O}_{\widehat{X}_x}$ with $\nabla(1) = d\varphi$.

- 2) A lattice $E \subset j_*\mathcal{E}$ is a *good Deligne-Malgrange lattice* if, for every closed point $x \in D$, there exists a finite Galois cover $\widehat{h}_x : \widehat{X}'_{x'} \rightarrow \widehat{X}_x$, étale over \widehat{U}_x , where x' is a finite union of closed points, with $\widehat{X}'_{x'}$ formally smooth, of group G , a G -invariant lattice

$$\widehat{E}'_{x'} \subset \widehat{h}_x^* \widehat{\mathcal{E}}_x$$

such that for all closed points y in x' , the lattice $(\widehat{E}'_{x'})_y$ is an unramified good Deligne-Malgrange lattice and such that

$$\widehat{E}_x = (\widehat{E}'_{x'})^G.$$

Remark 2.2.

- (1) Recall [Ked11, Th. 5.3.4] that if a good Deligne-Malgrange lattice E exists on a good compactification X of U , it is locally free, and the isomorphism classes of E and \widehat{E}_x are unique for any closed point $x \in D$. See also the proof of Lemma 3.1.
- (2) The Galois cover \widehat{h}_x in 2) could simply be a base change $K \supset k$ necessary to make the geometric irreducible components of D rational over K .

Definition 2.3. The formal integrable connection on \widehat{U}_x is said to be *unramified* if it admits a good Deligne-Malgrange lattice where \widehat{h}_x is the base change by a field extension. A connection (\mathcal{E}, ∇) is said to be unramified if $(\widehat{\mathcal{E}}_x, \widehat{\nabla}_x)$ is unramified for all closed points $x \in D$.

Theorem 2.4 ([Moc09], Theorem 2.12 and Conjecture 2.11 for the definition used in Theorem 2.12, [Moc11], Section 2.7 and Theorem 16.2.1, [Ked11], Theorem 8.1.3, Hypotheses 8.1.1, Theorem 8.2.3). *Let $j : U \hookrightarrow X_0$ be a good compactification of a smooth variety of finite type defined over a field k of characteristic 0. Let (\mathcal{E}, ∇) be a vector bundle with a flat connection on U . Then there exists a proper birational map $X \rightarrow X_0$, which is an isomorphism on U , such that $j : U \hookrightarrow X$ is a good compactification and the locally free $j_*\mathcal{O}_U$ -sheaf $j_*\mathcal{E}$ admits a good Deligne-Malgrange lattice E .*

Mochizuki's proof is analytic, and holds for meromorphic connections on complex projective varieties as well, while Kedlaya's proof is algebraic and rational over the field of definition. The latter does not insist on the projectivity of X when U is assumed to be quasi-projective. However, they both construct X starting from a given X_0 as above. Taking X_0 to be projective, then their X , which is proper over X_0 , admits a proper modification $X' \rightarrow X$ such that $U \hookrightarrow X'$ is a good compactification and X' is projective. Then one applies the covariance of Deligne-Malgrange lattices, see e.g. [Ked11, Rem. 5.3.7, 2nd part], where the formula should read $f_*\mathcal{E}'_0 \cap \mathcal{E} = \mathcal{E}_0$. We remark that if (\mathcal{E}, ∇) is regular singular, then E is Deligne's lattice [Del70, Prop. 5.4] and that, over the complex numbers, good Deligne-Malgrange lattices coincide with the canonical lattices constructed in [Mal96].

3. PROOF OF THEOREM 1.3

The purpose of this section is to prove Theorem 1.3. We fix an effective divisor Δ with support in D .

Let $\Theta_X(\log D)$ be the sheaf of D -logarithmic vector fields on X . Define inductively the sequence E_i of \mathcal{O}_X -coherent subsheaves of $j_*\mathcal{E}$ by the following rule:

- 1) $E_0 = E$;
- 2) For any natural number $i \geq 0$, E_{i+1} is the \mathcal{O}_X -coherent subsheaf of $j_*\mathcal{E}$ generated over \mathcal{O}_X by E_i and $\nabla_\xi E_i$ for any local section ξ of $\Theta_X(\log D)$.

We also denote it by $E_i + \Theta_X(\log D)E_i$. Here $\Theta_X(\log D)E_i$ is understood as the \mathcal{O}_X -coherent subsheaf of $j_*\mathcal{E}$ spanned by the $\nabla_\xi E_i$.

By definition one has, for any divisor Δ with support in D ,

- 1) $E_i(\Delta) \subset j_*\mathcal{E}$ and $\mathcal{E} = j^*(E_i(\Delta))$ for each natural number $i \geq 0$;
- 2) The connection $j_*\nabla : j_*\mathcal{E} \rightarrow \Omega_X^1(\log D) \otimes_{\mathcal{O}_X} j_*\mathcal{E}$ restricts for each natural number $i \geq 0$ to $\nabla : E_i(\Delta) \rightarrow \Omega_X^1(\log D) \otimes_{\mathcal{O}_X} E_{i+1}(\Delta)$ defining the complex

$$\mathrm{DR}_{\log D} E_\bullet(\Delta) := \Omega_X^\bullet(\log D) \otimes_{\mathcal{O}_X} E_\bullet(\Delta).$$

If (\mathcal{E}, ∇) is regular singular, then $E_i = E_0$ and is locally free for all $i \geq 0$ by [Del70], *loc. cit.*

For a closed point $x \in D$, we do on \widehat{X}_x the analogous construction:

- 1) $\widehat{E}_{0,x} = (\widehat{E}_0)_x$;
- 2) For any natural number $i \geq 0$, $\widehat{E}_{i+1,x}$ is the $\mathcal{O}_{\widehat{X}_x}$ -coherent subsheaf of $\widehat{\mathcal{E}}_x$ generated over $\mathcal{O}_{\widehat{X}_x}$ by $\widehat{E}_{i,x}$ and $\nabla_\xi \widehat{E}_{i,x}$ for any local section ξ of $\Theta_{\widehat{X}_x}(\log \widehat{D}_x)$.

Lemma 3.1. *For any closed point $x \in X$, one has $\widehat{E}_{i,x} = (\widehat{E}_i)_x$ and the \mathcal{O}_X -coherent sheaves E_i are all locally free.*

Proof. As $\mathcal{O}_X \rightarrow \mathcal{O}_{\widehat{X}_x}$ is flat, the second part of Lemma 3.1 follows from the first part and $\widehat{E}_{i,x}$ being locally free. We argue by induction in $i \geq 0$. We first assume $i = 0$. Then the first part $\widehat{E}_{0,x} = (\widehat{E}_0)_x$ is by definition. By Definition 2.1 2), $(\widehat{E}_0)_x$ is locally free if and only if $(\widehat{E}_0)_x$ is. So we are reduced to the unramified case, in which case this is part of Definition 2.1 1). We now assume $i + 1 \geq 1$ and the statement for i . By Lemma 3.2, Lemma 3.1 is equivalent to $(\widehat{E}_i)_x + (\Theta_X(\log D)E_i)_x$ being locally free. On the other hand, by definition $(\Theta_X(\log D)E_i)_x = \Theta_X(\log D)_x(\widehat{E}_i)_x$. Thus one has $\widehat{E}_{i+1,x} = (\widehat{E}_{i+1})_x$. It remains to prove local freeness. Again by Definition 2.1 2) it follows from local freeness in the case of the existence of an unramified good Deligne-Malgrange lattice. Then if I_φ denotes the effective pole divisor of φ ,

$$\widehat{E}_{i,x} = \bigoplus_{\varphi \in \Phi} (L_\varphi(i \cdot I_\varphi) \otimes R_\varphi).$$

This finishes the proof. □

Lemma 3.2. *Let $R \subset R'$ be a flat extension of commutative rings. Let A, B, C be R -modules with $A, B \subset C$. Let $A' = A \otimes_R R'$ and similarly for B, C be the base changed modules. Then $A', B' \subset C'$, $A' \cap B' = (A \cap B)' \subset C'$ and $A' + B' = (A + B)' \subset C'$.*

Notations 3.3. We define $\mathcal{D}_X(\log D) \subset \mathcal{D}_X$ to be the sheaf of subalgebras spanned by \mathcal{O}_X and $\Theta_X(\log D)$ and set

$$V^0\mathcal{E} = \bigcup_{i \in \mathbb{N}} E_i \subset (V^0\mathcal{E})(\Delta) = \bigcup_{i \in \mathbb{N}} E_i(\Delta) \subset j_*\mathcal{E}.$$

Facts 3.4. By definition, one has

$$\mathcal{D}_X(\log D) \cdot E_0(\Delta) = (V^0\mathcal{E})(\Delta), \quad \mathcal{D}_X(\log D) \cdot V^0\mathcal{E}(\Delta) = V^0\mathcal{E}(\Delta),$$

defining

$$\mathrm{DR}_{\log D} V^0\mathcal{E}(\Delta) = \Omega_X^\bullet(\log D) \otimes V^0\mathcal{E}(\Delta)$$

and the embedding of complexes

$$\mathrm{DR}_{\log D} E_\bullet(\Delta) \xrightarrow{\alpha} \mathrm{DR}_{\log D} V^0\mathcal{E}(\Delta) \xrightarrow{\beta} \mathrm{DR}(j_*\mathcal{E}),$$

where $\mathrm{DR}(j_*\mathcal{E}) = \Omega_X^\bullet \otimes_{\mathcal{O}_X} j_*\mathcal{E} = \mathrm{DR}_{\log D}(j_*\mathcal{E}) = \Omega_X^\bullet(\log D) \otimes j_*\mathcal{E}$. The $\mathcal{D}_X(\log D)$ -module $V^0\mathcal{E}(\Delta)$ is coherent. Moreover, (\mathcal{E}, ∇) is regular singular if and only if $V^0\mathcal{E}(\Delta) = E_0(\Delta)$, if and only if $V^0\mathcal{E}(\Delta)$ is a \mathcal{O}_X -coherent subsheaf of $j_*\mathcal{E}$.

The rest of the section is devoted to prove

Theorem 3.5. *Both α and β are quasi-isomorphisms.*

Clearly this immediately implies Theorem 1.3.

We first treat α . To this aim we regard the left-hand side of α as the zeroth term of the increasing filtration of $\mathrm{DR}_{\log D} V^0\mathcal{E}$ defined by

$$F_q \mathrm{DR}_{\log D} V^0\mathcal{E}(\Delta) = \begin{cases} \{0 \rightarrow E_q(\Delta) \rightarrow \Omega_X^1(\log D) \otimes E_{q+1}(\Delta) \rightarrow \cdots\} & \text{if } q \geq 0, \\ 0 & \text{if } q \leq -1. \end{cases}$$

So Theorem 3.5 for α is equivalent to the following proposition.

Proposition 3.6. *For every $q \geq 1$, the graded complex $\mathrm{gr}_q^F \mathrm{DR}_{\log D} V^0\mathcal{E}(\Delta)$ has \mathcal{O}_X -linear differentials and is quasi-isomorphic to zero.*

Proof. The \mathcal{O}_X -linearity is trivial. Faithful flatness of $\mathcal{O}_X \rightarrow \mathcal{O}_{\widehat{X}_x}$ again implies that the proposition is true if and only if the analogous proposition on \widehat{X}_x is true for all closed points $x \in D$. Again by Definition 2.1 one reduces the problem to the case of the existence of an unramified good Deligne-Malgrange lattice: setting $\widehat{\Delta}'_x = \widehat{h}_x^* \widehat{\Delta}'_x$ and defining $\widehat{E}'_{i,x'}$ from $\widehat{E}'_{0,x'}$ as in 2) before Lemma 3.1, we have $\widehat{E}'_i(\Delta)_x = (\widehat{E}'_{i,x'}(\widehat{\Delta}'_x))^G$ (same argument as in the proof of Proposition 3.9 below). Let us consider the decomposition of Definition 2.1 1). For the component of \widehat{E}_x corresponding to $\varphi = 0$, the filtration F_q is constant and the assertion is obvious. For a component $L_\varphi \otimes R_\varphi$ with $\varphi \neq 0$, the statement is equivalent to the \mathcal{O}_X -linear complex

$$(\Omega_{\widehat{X}_x}^\bullet(\log D) \otimes_{\mathcal{O}_{\widehat{X}_x}} \mathcal{O}_{I_\varphi}(\widehat{\Delta}'_x + (q + \bullet)I_\varphi) \otimes_{\mathcal{O}_{\widehat{X}_x}} R_\varphi, \wedge d\varphi)$$

being quasi-isomorphic to zero. Moreover, it is enough to show this assertion after a finite extension of the ground field, so we can assume that Assumption B.4 holds. We set $\varphi = u(x)x^{-m}$ with $u \in \mathcal{O}_x^\times$ (i.e., $u(0) \neq 0$) and $m \in \mathbb{N}^\ell \setminus \{0\}$. If $m_i \neq 0$, then for any $k \geq 0$,

$$x_i \partial_{x_i} : \mathcal{O}_{I_\varphi}(\widehat{\Delta}'_x + (q + k)I_\varphi) \otimes_{\mathcal{O}_{\widehat{X}_x}} R_\varphi \longrightarrow \mathcal{O}_{I_\varphi}(\widehat{\Delta}'_x + (q + k + 1)I_\varphi) \otimes_{\mathcal{O}_{\widehat{X}_x}} R_\varphi$$

is an isomorphism, being nothing but the multiplication by $-m_i$. The assertion follows. \square

We now treat β . The assertion is local so we fix a closed point $x \in D$. Theorem 2.4 yields for a closed point $x \in D$ natural numbers m depending on i, x and the choice of \widehat{h}_x such that \widehat{h}_x is a Kummer cover along $\widehat{D}_{i,x}$ which ramifies with ramification indices m .

Definition 3.7. Fixing (i, x) , the minimal m is called the ramification index of (\mathcal{E}, ∇) at x along D_i and is denoted by $m_{i,x}$.

Lemma 3.8. *Fixing a closed point $x \in D$, the Galois cover \widehat{h}_x from Definition 2.1 2) is algebraizable in the following sense. There exists a smooth projective variety Z defined over k , a finite union $z \in Z$ of closed points together with a flat finite morphism $g : Z \rightarrow X$, finite étale Galois of group H outside of a strict normal crossings divisor \mathcal{D} containing D , such that $Z \setminus g^{-1}(\mathcal{D}) \hookrightarrow Z$ is a good compactification, and there is a factorization*

$$\widehat{g}_x : \widehat{Z}_z \xrightarrow{\lambda_x} \widehat{X}'_{x'} \xrightarrow{\widehat{h}_x} \widehat{X}_x,$$

where g_x is étale on $X_x \setminus D_x$. In particular, G is a quotient of H .

Proof. We apply [Kaw81, Th. 17] (together with [Mat02, Lem. 5.2.4] for the Galois property) to construct g . We just have to check the last property. To say that g_x is étale on $X_x \setminus D_x$ is to say that x does not lie on the closure of $\mathcal{D} \setminus D$. By the proof of [Kaw81, Th. 17], this divisor is generic in a very ample linear system, thus in particular can be chosen to avoid any 0-dimensional subscheme. This finishes the proof. \square

We set $j' : Z \setminus g^{-1}(\mathcal{D}) \rightarrow Z$ for the closed embedding and $\Delta'' = g^*(\Delta)$. We denote by β' the corresponding embedding of de Rham complexes

$$\mathrm{DR}_{\log g^{-1}(\mathcal{D})}[(V^0 g^* \mathcal{E})(\Delta'')] \xrightarrow{\beta'} \mathrm{DR}(j'_* j'^* g^* \mathcal{E}).$$

We use the notations

$$\beta_x = \beta \otimes_{\mathcal{O}_X} \mathcal{O}_{X_x}, \quad \beta'_z = \beta' \otimes_{\mathcal{O}_Z} \mathcal{O}_{Z_z},$$

and similarly $(\mathrm{DR}_{\log D}(V^0 \mathcal{E})(\Delta))_x$ etc.

Proposition 3.9. *If β'_z is a quasi-isomorphism, then so is β_x .*

Proof. The morphism $\beta_x : (\mathrm{DR}_{\log D} V^0 \mathcal{E})_x \rightarrow (\mathrm{DR} j_* \mathcal{E})_x$ factors as

$$(\mathrm{DR}_{\log D} V^0 \mathcal{E})_x \longrightarrow (\mathrm{DR}_{\log D} \mathcal{E})_x \longrightarrow (\mathrm{DR} j_* \mathcal{E})_x,$$

and, since the second morphism is clearly an isomorphism, we are reduced to showing that $(\mathrm{DR}_{\log D}(V^0 \mathcal{E})(\Delta))_x \rightarrow (\mathrm{DR}_{\log D} j_* \mathcal{E})_x$ is a quasi-isomorphism. By Lemma 3.8, and uniqueness in Theorem 2.4, the sheaf F_0^K of $K = \ker(H \rightarrow G)$ -invariants of the Deligne-Malgrange lattice F_0 of $(j'_* j'^* g^* \mathcal{E})_z$ has formal germ $(\widehat{F_0})_z^K = (\widehat{F_0}^K)_{x'}$ equal to the Deligne-Malgrange lattice of $\widehat{h}_x^* \widehat{\mathcal{E}}_x$ on $\widehat{X}'_{x'}$. This implies that $(\widehat{F_0}^H)_x = (\widehat{F_0}^H)_z = ((\widehat{F_0})_z^K)^G$ is equal to $(\widehat{E_0})_x$. We deduce from this that the two lattices F_0^H and $E_{0,x}$ of $(j_* \mathcal{E})_x$ coincide.

We now prove inductively on i that $F_i^H = E_{i,x}$. We assume it is true for some $i \geq 0$. Recall that g_x is étale away from D_x . As $g_x^* \Theta_{X_x}(\log D_x) = \Theta_{Z_z}(\log g_x^{-1} D_x)$, one has

$$(\Theta_{Z_z}(\log g_x^{-1} D_x) F_i)^H = \Theta_{X_x}(\log D_x) F_i^H = \Theta_{X_x}(\log D_x) E_{i,x}.$$

On the other hand,

$E_{i+1,x} = F_i^H + (\Theta_{Z_z}(\log g_x^{-1} D_x) F_i)^H = (F_i + (\Theta_{Z_z}(\log g_x^{-1} D_x) F_i))^H = F_{i+1}^H$
as $E_{i,x}$ and $\Theta_{X_x}(\log D_x) E_{i,x}$ are sheaves of \mathbb{Q} -vector spaces, thus any local section

$$(v + w) \in (F_i + (\Theta_{Z_z}(\log g_x^{-1} D_x) F_i))^H$$

with $v \in F_i$, $w \in (\Theta_{Z_z}(\log g_x^{-1} D_x) F_i)$ can be written as

$$\frac{1}{|H|} \sum_{h \in H} h \cdot v + \frac{1}{|H|} \sum_{h \in H} h \cdot w \in F_i^H + (\Theta_{Z_z}(\log g_x^{-1} D_x) F_i)^H.$$

We conclude that

$$((V^0 g^* \mathcal{E})_z)^H = (V^0 \mathcal{E})_x,$$

and similarly $((V^0 g^* \mathcal{E})(\Delta'')_z)^H = V^0 \mathcal{E}(\Delta)_x$, from which again by the compatibility of the differential forms we conclude

$$(\mathrm{DR}_{\log \mathcal{D}}(V^0 g^* \mathcal{E})(\Delta''))_z^H = (\mathrm{DR}_{\log \mathcal{D}}(V^0 \mathcal{E})(\Delta))_x = \mathrm{DR}_{\log D_x}(V^0 \mathcal{E})(\Delta)_x.$$

On the other hand, one trivially has

$$\mathrm{DR}(g^* j_* \mathcal{E})_z^H = \mathrm{DR}(j_* \mathcal{E})_x.$$

This finishes the proof. \square

We may now assume that \mathcal{E}_x has an unramified good Deligne-Malgrange lattice. The sheaf \mathcal{D}_X of differential operators on X is canonically endowed with the filtration $F_\bullet \mathcal{D}_X$ by the order of differential operators. In particular, $F_p \mathcal{D}_X = 0$ for $p \leq -1$, $F_0 \mathcal{D}_X = \mathcal{O}_X$ and $F_1 \mathcal{D}_X$ is generated by \mathcal{O}_X and Θ_X . We first mimic the definition of the *filtration by the order of the poles* (shifted by Δ) introduced by Deligne [Del70, Chap. 6], and its relation with the stupid filtration of the logarithmic de Rham complex. We set, for $p \in \mathbb{Z}$,

$$P^p j_* \mathcal{E} = (F_{-p} \mathcal{D}_X) \cdot (V^0 \mathcal{E})(D + \Delta) \subset j_* \mathcal{E},$$

so that, in particular,

$$P^p j_* \mathcal{E} = 0 \ (p \geq 1), \quad P^0 j_* \mathcal{E} = (V^0 \mathcal{E})(D + \Delta), \quad P^{-1} j_* \mathcal{E} = P^0 j_* \mathcal{E} + \Theta_X \cdot P^0 j_* \mathcal{E}.$$

The de Rham complex is then filtered as usual, for $p \in \mathbb{Z}$,

$$P^p \mathrm{DR}(j_* \mathcal{E}) = \{0 \rightarrow P^p j_* \mathcal{E} \rightarrow \Omega_X^1 \otimes P^{p-1} j_* \mathcal{E} \rightarrow \dots \rightarrow \Omega_X^{\dim X} \otimes P^{p-\dim X} j_* \mathcal{E} \rightarrow 0\}.$$

By definition, the differentials on the graded complex $P^p \mathrm{DR} j_* \mathcal{E} / P^{p+1} \mathrm{DR} j_* \mathcal{E}$ are \mathcal{O}_X -linear. On the other hand, one has the stupid filtration $\sigma^{\geq p}$ on the logarithmic de Rham complex of $V^0 := (V^0 \mathcal{E})(\Delta)$:

$$\sigma^{\geq p} \mathrm{DR}_{\log D} V^0 = \{0 \rightarrow \dots \rightarrow 0 \rightarrow \Omega_X^p(\log D) \otimes V^0 \rightarrow \dots \rightarrow \Omega_X^{\dim X}(\log D) \otimes V^0 \rightarrow 0\},$$

for which the graded complexes are just sheaves in various degrees. We use the same notations for the localization at a closed point x . Theorem 3.5 for β is then a consequence of the following more precise theorem.

Theorem 3.10. *If \mathcal{E}_x has an unramified good Deligne-Malgrange lattice, the inclusion of filtered complexes*

$$(\mathrm{DR}_{\log D} V_x^0, \sigma^{\geq \bullet}) \hookrightarrow (\mathrm{DR} \mathcal{E}_x, P^\bullet)$$

is a filtered quasi-isomorphism.

Proof. The differentials of the graded complexes $\mathrm{gr}_\sigma^p \mathrm{DR}_{\log D} V_x^0$ and $\mathrm{gr}_P^p \mathrm{DR} \mathcal{E}_x$ ($p \in \mathbb{Z}$) being \mathcal{O}_X -linear, again by Definition 2.1 we are reduced to the formal case. In the regular case, the statement follows from the formal version of [Del70, Prop. II.3.13] as τ is admissible thus the condition on the residues is fulfilled. In the case where $\varphi \neq 0$, if $|I_\varphi| = D$, we have $V^0 = \mathcal{E}$ and $P^p \mathcal{E} = \mathcal{E}$ for $p \leq 0$. The assertion amounts to proving that the natural morphism

$$(\mathrm{DR}_{\log D} \mathcal{E}, \sigma^{\geq \bullet}) \hookrightarrow (\mathrm{DR} \mathcal{E}, \sigma^{\geq \bullet})$$

is a filtered quasi-isomorphism, which is obvious since $\Omega_X^p(\log D) \otimes \mathcal{E} = \Omega_X^p \otimes \mathcal{E}$ for every $p \geq 0$. On the other hand, if $|I_\varphi| \subsetneq D$, \mathcal{E} is a successive extension of rank-one connections and it is enough to prove the assertion for such connections. Each such connection can be written as an external product of two terms, one term satisfying the assumption above, the other one being regular. For such a rank-one term, the assertion follows by using the regular case and the case $D_\varphi = D$, both proved above, by arguing as in [Del70, p. 81]. \square

4. REMARKS

4.1. Dimension one and Deligne's theorem. If $n = 1$, in which case X is necessarily the normal compactification of U , this concept has been developed by Deligne [Del70, §6]. He shows over $k = \mathbb{C}$ the existence of pairs $E_0 \subset E_1$ with 1), 2), 3). He proves that although those pairs are not unique, $\dim_{\mathbb{C}}(E_1/E_0)_x$ is independent of the choice for all closed points x on D and defines the irregularity divisor $\sum_{x \in D} \dim_{\mathbb{C}}(E_1/E_0)_x \cdot x$ of (\mathcal{E}, ∇) [Del70, Lem. 6.21]. It has been then deemed ‘pairs of good lattices’ in [BE04, §3] in order to define local Fourier transforms of connections, then in [BBDE04, §3.1] to compute the periods of the local Fourier transforms. While Deligne in *loc. cit.* constructed the lattices using the existence of a cyclic vector, the construction in [BE04], [BBDE04] *loc. cit.* and [BBE02, §5] uses the Turrittin-Levelt decomposition.

4.2. Non-negative shifts. The proof of Theorem 1.3 yields that for any natural number a , the embeddings of complexes

$$\Omega_X^\bullet \otimes_{\mathcal{O}_X} E_\bullet \hookrightarrow \Omega_X^\bullet \otimes_{\mathcal{O}_X} E_{a+\bullet} \hookrightarrow j_*(\Omega_U^\bullet \otimes_{\mathcal{O}_U} \mathcal{E})$$

are quasi-isomorphisms.

4.3. Boundedness. We remark the following.

Claim 4.1. The ramification indices $m_{i,x}$ of Definition 3.7 are bounded. Equivalently there is a natural number M such that $m_{i,x}$ divides M for all i, x .

Proof. As $(U, (\mathcal{E}, \nabla))$ and X are defined over a field k_0 of finite type over k , Definition 3.7 over k_0 yields ramification indices $m_{i,u}(0)$ say for all closed points $u \in D_i$ over k_0 . Then $D_i \otimes_{k_0} k$ and $u \otimes_{k_0} k$ might further split, with x, D_i being one component. Then \widehat{h}_x over k is the pull-back of \widehat{h}_u over k_0 localized at x and D_i . Thus the $m_{i,x}$ over k are the same as $m_{u,i}$ over k_0 for k . We now choose a complex embedding $k_0 \hookrightarrow \mathbb{C}$. Again the $D_i \otimes_{k_0} \mathbb{C}$ and $u \otimes_{k_0} \mathbb{C}$

might further split and with the same argument, we just have to show boundedness for the $m_{a,z}$ where z, D_a is one component of $(u, D_i) \otimes_{k_0} \mathbb{C}$. By [Moc09, p. 2827, bullet point], for each complex point z (denoted by P in *loc. cit.*) there exists an analytic neighbourhood X_z (denoted by X_P in *loc. cit.*) and a Kummer cover $\varphi : X'_z \rightarrow X_z$ such that $\varphi^*(\mathcal{E}, \nabla)$ is unramified. Thus for all points $z' \in D \cap X_z$, $m_{z',a}$ divides $m_{\varphi,a}$, the ramification index of φ along $D_a \cap X_z$. As D is compact in the analytic topology, it is covered by finitely many such analytic open sets X_z . This finishes the proof. \square

5. THE LOGARITHMIC CHARACTERISTIC VARIETY

In this section, we give the proof of Theorem 1.4 by reducing it to Theorem 5.1 below. The sheaf $\mathcal{D}_X(\log D) \subset \mathcal{D}_X$ is endowed with the induced filtration $F_\bullet \mathcal{D}_X(\log D)$ by the order of differential operators. Definitions and results similar to those recalled in Sections A.1 and A.2 hold when D is a normal crossings divisor in X and upon replacing \mathcal{D}_X with $\mathcal{D}_X(\log D)$ and correspondingly T^*X with $T^*X(\log D)$, ω_X with $\omega_X(\log D) = \omega_X(D)$. For a coherent $\mathcal{D}_X(\log D)$ -module M , one obtains

$$\text{Car}_{\log} M \text{ in } K_0(\mathcal{O}_{T^*X(\log D)}).$$

The relation between the approach of Section A.3 and the logarithmic approach will be obtained by factoring the log zero-section embedding $i_{\log} : X \hookrightarrow T^*X(\log D)$ as

$$X \xrightarrow{i} T^*X \xrightarrow{p} T^*X(\log D).$$

We now take $(X, (E_0, \dots, E_n))$ to be a good model of $(U, (\mathcal{E}, \nabla))$, see Definition 1.2. In particular, $j_*\mathcal{E}$ has no turning point along D . By definition, $E_i = F_i \mathcal{D}_X(\log D) \cdot E_0$ ($i \geq 0$).

It is convenient to set $E_{-1} = 0$. On the other hand, we set $V^0\mathcal{E} = \bigcup_i E_i$ and $V^{-1}\mathcal{E} = (V^0\mathcal{E})(D)$. The formation of $E_0, E_i, V^0\mathcal{E}$ and $V^{-1}\mathcal{E}$ is compatible with the restriction to the formal neighbourhood \widehat{X}_x of a closed point $x \in X$.

Let us endow the $\mathcal{D}_X(\log D)$ -module $V^{-1}\mathcal{E} := (V^0\mathcal{E})(D)$ with the filtration $0 = E_{-1} \subset E_0(D) \subset E_1(D) \subset \dots$, so that its log de Rham complex $\text{DR}_{\log D}(V^{-1}\mathcal{E})$ is filtered by the formula

$$F_q \text{DR}_{\log D}(V^{-1}\mathcal{E}) = \left\{ 0 \longrightarrow E_q(D) \longrightarrow \Omega_X^1(\log D) \otimes E_{q+1}(D) \longrightarrow \dots \right\}$$

for every $q \in \mathbb{Z}$. By Proposition 3.6, $\text{gr}_q^F \text{DR}_{\log D}(V^{-1}\mathcal{E})$ is acyclic for $q \geq 1$ hence, by mimicking in the logarithmic case the argument leading to (A.20), we find

$$\begin{aligned} (5.1) \quad Li_{\log}^* \text{Car}_{\log}(V^{-1}\mathcal{E}) &= \sum_{i=0}^n (-1)^{n-i} \left[\omega_X(\log D)^{-1} \otimes \Omega_X^i(\log D) \otimes E_i(D) \right] \\ &= \sum_{i=0}^n (-1)^{n-i} \left[\omega_X^{-1} \otimes \Omega_X^i(\log D) \otimes E_i \right]. \end{aligned}$$

Theorem 1.4 immediately follows from the comparison between the characteristic class and the pullback of the log-characteristic class in the Grothendieck group $K_0(\mathcal{O}_{T^*X})$.

Theorem 5.1. *We have the equality in $K_0(\mathcal{O}_{T^*X})$:*

$$\text{Car}(j_*\mathcal{E}) = Lp^* \text{Car}_{\log}(V^{-1}\mathcal{E}).$$

In order to prove the theorem, one is led to compare $j_*\mathcal{E}$ with $\mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} V^{-1}\mathcal{E}$ and to extend this comparison to the graded modules with respect to a coherent F -filtration.

5.1. Base change of the $\mathcal{D}_X(\log D)$ -module $V^{-1}\mathcal{E}$. The following proposition is proved in [Moc15, §§5.3.2–5.3.3] in a more general context over the field \mathbb{C} .

Proposition 5.2. *The natural morphism $\mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} V^{-1}\mathcal{E} \rightarrow \mathcal{E}$ is an isomorphism.*

Proof. The assertion is local formal, and is compatible with base change after a finite extension of the ground field, so we can assume that Assumption B.4 holds. We can also assume that the unramified good Deligne-Malgrange lattice E_0 comes from (see Definition 2.1 2)) has only one component $L_\varphi \otimes R_\varphi$. We keep the notation D for \widehat{D}_x . We use the following notation:

- Let $\widehat{h}_x : \widehat{X}'_x \rightarrow \widehat{X}_x$ be a finite morphism ramified along D with Galois group G , such that $L_\varphi \otimes R_\varphi = \widehat{\mathcal{E}}'_x := \widehat{h}_x^* \widehat{\mathcal{E}}_x$. Let (x_1, \dots, x_n) and (x'_1, \dots, x'_n) be local coordinates adapted to \widehat{h}_x and D , so that $D = (x_1 \cdots x_\ell)$ and $\widehat{h}_x^*(x_i) = x_i^{\rho_i}$ ($i = 1, \dots, \ell$). Denoting by E'_0 the Deligne-Malgrange lattice of $\widehat{\mathcal{E}}'_x$, we have $E_0 = (E'_0)^G$. It follows that $E_p = (E'_p)^G$ and thus $V^0 \widehat{\mathcal{E}}_x = (V^0 \widehat{\mathcal{E}}'_x)^G$. We then have $V^{-1} \widehat{\mathcal{E}}_x = (V^0 \widehat{\mathcal{E}}'_x (\widehat{h}_x^* D))^G$. Moreover, with this identification, the action of $x_i \partial_{x_i}$ is induced by that of $(1/\rho_i) x'_i \partial_{x'_i}$ ($i = 1, \dots, \ell$).
- We identify D with the support of its pull-back by \widehat{h}_x and we decompose it as $D = D_1 \cup D_2$, where D_2 supports the pole divisor of φ .
- We set $x'^\rho = x_1^{-\rho_1} \cdots x_\ell^{-\rho_\ell}$ and assume that $D_1 = (x_1 \cdots x_k)$ with $k \leq \ell$.
- We set $\widehat{\mathcal{D}}_x = \mathcal{O}_{\widehat{X}_x} \otimes_{\mathcal{O}_X} \mathcal{D}_X$.

We will prove the following two assertions.

- (a) $V^{-1} \widehat{\mathcal{E}}_x = (V^{-1} \widehat{\mathcal{E}}_x)(*D_2)$, so that $\widehat{\mathcal{E}}_x = (V^{-1} \widehat{\mathcal{E}}_x)(*D_1)$,
- (b) $\widehat{\mathcal{D}}_x \otimes_{\widehat{\mathcal{D}}_x(\log D_1)} V^{-1} \widehat{\mathcal{E}}_x \simeq (V^{-1} \widehat{\mathcal{E}}_x)(*D_1)$.

Let us first check that these assertions imply the proposition. We first claim that

$$(5.2) \quad \widehat{\mathcal{D}}_x(\log D_1) \otimes_{\widehat{\mathcal{D}}_x(\log D)} V^{-1} \widehat{\mathcal{E}}_x \longrightarrow V^{-1} \widehat{\mathcal{E}}_x \quad \text{is an isomorphism.}$$

Indeed, the composed natural morphism

$$(5.3) \quad V^{-1} \widehat{\mathcal{E}}_x \longrightarrow \widehat{\mathcal{D}}_x(\log D_1) \otimes_{\widehat{\mathcal{D}}_x(\log D)} V^{-1} \widehat{\mathcal{E}}_x \longrightarrow V^{-1} \widehat{\mathcal{E}}_x$$

is an isomorphism, hence the first morphism is injective. Let us check it is onto. Set $D_2 = \{x_{k+1} \cdots x_\ell = 0\}$. For any $P \in \widehat{\mathcal{D}}_x(\log D_1)$, there exists a sufficiently large integer N such that $P \cdot (x_{k+1} \cdots x_\ell)^N \in \widehat{\mathcal{D}}_x(\log D)$. Since any section $P \otimes m$ of $\widehat{\mathcal{D}}_x(\log D_1) \otimes_{\widehat{\mathcal{D}}_x(\log D)} V^{-1} \widehat{\mathcal{E}}_x$ can also be written as $P \cdot (x_{k+1} \cdots x_\ell)^N \otimes (x_{k+1} \cdots x_\ell)^{-N} m$ because of (a), it is equal to

$$1 \otimes P \cdot (x_{k+1} \cdots x_\ell)^N [(x_{k+1} \cdots x_\ell)^{-N} m] = 1 \otimes P m \in 1 \otimes V^{-1} \widehat{\mathcal{E}}_x,$$

proving the surjectivity. The first morphism in (5.3) is thus bijective, and so is the second one. We conclude

$$\begin{aligned} \widehat{\mathcal{D}}_x \otimes_{\widehat{\mathcal{D}}_x(\log D)} V^{-1}\widehat{\mathcal{E}}_x &\simeq \widehat{\mathcal{D}}_x \otimes_{\widehat{\mathcal{D}}_x(\log D_1)} (\widehat{\mathcal{D}}_x(\log D_1) \otimes_{\widehat{\mathcal{D}}_x(\log D)} V^{-1}\widehat{\mathcal{E}}_x) \\ &\xrightarrow{\sim} \widehat{\mathcal{D}}_x \otimes_{\widehat{\mathcal{D}}_x(\log D_1)} V^{-1}\widehat{\mathcal{E}}_x \quad \text{by (5.2)} \\ &\simeq (V^{-1}\widehat{\mathcal{E}}_x)(*D_1) \simeq \widehat{\mathcal{E}}_x \quad \text{by (a)}. \end{aligned}$$

Let us now prove (a) and (b). For (b), we apply Proposition B.7 to D_1 and $V^{-1}\widehat{\mathcal{E}}_x$ up to side-changing. In the left setting, the operators to be considered in loc. cit. are (up to sign) $\text{Eu}_i - j$ for $j \geq 0$. The properties B.7(1) and (2) read as follows for $V^0\widehat{\mathcal{E}}_{x'}(\widehat{h}_x^* D)$: For any subset $I \subset \{1, \dots, k\}$, the operators

- x'_i ($i \notin I$),
- $\text{Eu}'_i - \rho_i j$ ($i \in I, j \geq 0$)

act in an invertible way on $(x'^{-\rho})V^0(L_\varphi \otimes R_\varphi)/(x'_i)_{i \in I}(x'^{-\rho})V^0(L_\varphi \otimes R_\varphi)$. These properties are easily checked, showing thereby, after taking G -invariants, that (b) holds.

Similarly, it is enough to prove (a) for $(x'^{-\rho})V^0(L_\varphi \otimes R_\varphi)$, for which the assertion is also easy. \square

5.2. Proof of Theorem 5.1. We use the notation and the terminology explained in the appendix, Section B.1.

Recall that we denote by $F_\bullet \mathcal{D}_X(\log D)$ (resp. $F_\bullet \mathcal{D}_X$) the increasing filtration by the order of differential operators. We have in both cases $F_{-1} = 0$ and $F_0 = \mathcal{O}_X$. Moreover, when regarding \mathcal{D}_X as a (left or right) $\mathcal{D}_X(\log D)$ -module, the filtration $F_\bullet \mathcal{D}_X$ is an $F\mathcal{D}_X(\log D)$ -filtration of \mathcal{D}_X .

Recall that $(E_i(D))_i$ is a coherent filtration of $V^{-1}\mathcal{E}$ with respect to $F_\bullet \mathcal{D}_X(\log D)$. The \mathcal{D}_X -module $\mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} V^{-1}\mathcal{E}$ is then endowed with a natural coherent filtration, namely

$$(5.4) \quad F_i(\mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} V^{-1}\mathcal{E}) = \sum_{j+k \leq i} \text{im} \left[F_j \mathcal{D}_X \otimes_{\mathcal{O}_X} E_k(D) \rightarrow \mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} V^{-1}\mathcal{E} \right].$$

The \mathcal{D}_X -linear isomorphism $\mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} V^{-1}\mathcal{E} \rightarrow \mathcal{E}$ (Proposition 5.2) enables us to transport this filtration as a coherent filtration $(F_i \mathcal{E})$ of the \mathcal{D}_X -module \mathcal{E} , by defining

$$(5.5) \quad F_i \mathcal{E} := \sum_{j+k \leq i} F_j \mathcal{D}_X \cdot E_k(D),$$

where the sum is taken in \mathcal{E} . We simply denote by $\text{gr}(V^{-1}\mathcal{E})$ and $\text{gr}\mathcal{E}$ the graded modules with respect to these coherent filtrations $E_\bullet(D)$ and $F_\bullet \mathcal{E}$ respectively. By definition we have

$$(5.6) \quad \text{gr}\mathcal{E} \simeq \text{gr}^F(\mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} V^{-1}\mathcal{E}).$$

The Rees module $\widetilde{\mathcal{E}} := \bigoplus_i F_i \mathcal{E} \cdot z^i \subset \mathcal{E} \otimes_k k[z]$ with respect to the filtration (5.5) is a left $\widetilde{\mathcal{D}}_X$ -module (see Section B.1). Then

$$Lz^* \widetilde{\mathcal{E}} = z^* \widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}/z \widetilde{\mathcal{E}} = \text{gr}\mathcal{E},$$

where we regard $\text{gr}\mathcal{E}$ as a graded $\widetilde{\mathcal{D}}_X$ -module on which z acts by zero, that is, a $\text{gr}^F\mathcal{D}_X$ -module.

Similarly, by using the filtration $(E_i(D))_i$ of $V^{-1}\mathcal{E}$, we define $\widetilde{V^{-1}\mathcal{E}} = \bigoplus_i E_i(D) \cdot z^i$, which is a left $\widetilde{\mathcal{D}}_X(\log D)$ -module. With this notation, (5.4) reads (see Lemma B.1 up to changing the side):

$$(5.7) \quad (\mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} V^{-1}\mathcal{E})^\sim = \left(\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{D}}_X(\log D)} \widetilde{V^{-1}\mathcal{E}} \right) / z\text{-torsion}.$$

According to Lemma B.3 (up to side-changing), it is enough to prove the following.

Proposition 5.3. *We have*

$$\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{D}}_X(\log D)}^L \widetilde{V^{-1}\mathcal{E}} \simeq \widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{D}}_X(\log D)} \widetilde{V^{-1}\mathcal{E}}.$$

Proof. Since the statement is local, it can be proved after restricting to the formal neighbourhood of any point $x \in D$. Moreover, the formation of V^{-1} and that of $\widetilde{V^{-1}}$ commute with tensoring with the ring $\mathcal{O}_{\widehat{X}_x}$ since $\widehat{E}_{0,x} = \widehat{E}_{0,x}$ (see 1) before Lemma 3.1). Therefore, in the remaining part of the proof, *we will assume that X is the formal neighbourhood of $x \in D$, but we will not change the notation for the sake of simplicity.* As both sides of the equation are compatible with a finite base field extension, we can suppose that Assumption B.4 holds. The proof of the proposition relies on the first part of Proposition B.5 of the appendix, after changing the side. It is enough to prove that any subsequence $(x_i)_{i \in I}$ of (x_1, \dots, x_ℓ) is a regular sequence for $\widetilde{V^{-1}\mathcal{E}}$. Moreover, since $\widetilde{V^{-1}\mathcal{E}} = \widetilde{V^0\mathcal{E}(D)}$, $(x_i)_{i \in I}$ is a regular sequence for $\widetilde{V^{-1}\mathcal{E}}$ if and only if it is so for $\widetilde{V^0\mathcal{E}}$. We thus argue with the latter module.

Reduction to the unramified case. As in the proof of Proposition 5.2, and with similar notation, we find that $\widetilde{V^0\mathcal{E}} = (\widetilde{V^0\mathcal{E}'})^G$, and we can assume that $\mathcal{E}' \simeq L_\varphi \otimes R_\varphi$. If we know that $(x_i^{\rho_i})_{i \in I}$ is a regular sequence for $\widetilde{V^0\mathcal{E}'}$, then $(\widehat{h}_x^*(x_i))_{i \in I}$ is a regular sequence, and by taking G -invariants we conclude that $(x_i)_{i \in I}$ is a regular sequence for $\widetilde{V^0\mathcal{E}}$, as wanted.

The unramified case. We now assume that \mathcal{E} is unramified. Let $\varphi(x) = u(x)/x^m$, where $m = (m_1, \dots, m_\ell, 0, \dots, 0)$, with $m_i \geq 1$ for $i = k+1, \dots, \ell$ and $u(x) \in \mathcal{O}_X$ with $u(0) \neq 0$. Set $\mathcal{L}_\varphi = (\mathcal{O}_X(*D), d + d\varphi)$ and $D_m = \sum_{i=1}^\ell m_i D_i$. Then $L_{\varphi,p} = \mathcal{O}_X(D_{pm})$ ($p \geq 0$) and $E_p(D) = (V^0\mathcal{R})(D_{pm})$, so that, forgetting the connection, we have

$$(5.8) \quad \widetilde{V^0\mathcal{E}} = \left(\bigoplus_{p \geq 0} (V^0 R_\varphi)(D_{pm}) \cdot z^p \right) \subset R_\varphi[z].$$

Since $\widetilde{V^0\mathcal{E}}$ is graded, $(x_i^{\rho_i})_{i \in I}$ is a regular sequence for $\widetilde{V^0\mathcal{E}}$ if and only if it is so for each graded piece $(V^0 R_\varphi)(D_{pm})$. This holds since $(x_i^{\rho_i})_{i \in I}$ is a regular sequence for $V^0 R_\varphi$. \square

APPENDIX

BY CLAUDE SABBAH

APPENDIX A. A REMINDER ON CHARACTERISTIC VARIETIES

In this section we reproduce [MHM, §A5]. Recall that $n = \dim X$.

A.1. Right \mathcal{D}_X -modules. Recall that if N is a right \mathcal{D}_X -module, the Spencer complex of N is the complex

$$\mathrm{Sp}(N) := \{0 \rightarrow N \otimes_{\mathcal{O}_X} \wedge^n \Theta_X \xrightarrow{\delta} \cdots \xrightarrow{\delta} N \otimes_{\mathcal{O}_X} \Theta_X \xrightarrow{\delta} \bullet \rightarrow 0\},$$

where the \bullet indicates the term in degree zero and where the differential δ is the k -linear map given for $m \in N$ by ($\widehat{\xi}_i$ means omitting ξ_i in the wedge product)

$$(A.1) \quad m \otimes (\xi_1 \wedge \cdots \wedge \xi_k) \xrightarrow{\delta} \sum_{i=1}^k (-1)^{i-1} (m \xi_i) \otimes (\xi_1 \wedge \cdots \wedge \widehat{\xi}_i \wedge \cdots \wedge \xi_k) \\ + \sum_{i < j} (-1)^{i+j} m \otimes ([\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \wedge \widehat{\xi}_i \wedge \cdots \wedge \widehat{\xi}_j \wedge \cdots \wedge \xi_k).$$

Regarding \mathcal{D}_X as a right \mathcal{D}_X -module, the Spencer complex $\mathrm{Sp}(\mathcal{D}_X)$ is a complex in the category of left \mathcal{D}_X -modules, by using the left \mathcal{D}_X -module structure on \mathcal{D}_X , and is a resolution of \mathcal{O}_X as a left \mathcal{D}_X -module, by locally free \mathcal{D}_X -modules. Moreover, there is a natural isomorphism

$$(A.2) \quad \mathrm{Sp}(N) \simeq N \otimes_{\mathcal{D}_X} \mathrm{Sp}(\mathcal{D}_X) = N \otimes_{\mathcal{D}_X}^L \mathrm{Sp}(\mathcal{D}_X).$$

One concludes that

$$(A.3) \quad N \otimes_{\mathcal{D}_X}^L \mathcal{O}_X \simeq N \otimes_{\mathcal{D}_X} \mathrm{Sp}(\mathcal{D}_X) \simeq \mathrm{Sp}(N).$$

If N is endowed with an $F\mathcal{D}_X$ -filtration, the formula for the differential δ shows that $\delta(F_p N \otimes \wedge^k \Theta_X) \subset F_{p+1} N \otimes \wedge^{k-1} \Theta_X$, and the Spencer complex is filtered by the formula

$$(A.4) \quad F_p \mathrm{Sp}(N) = \{0 \rightarrow F_{p-n} N \otimes_{\mathcal{O}_X} \wedge^n \Theta_X \xrightarrow{\delta} \cdots \xrightarrow{\delta} F_{p-1} N \otimes_{\mathcal{O}_X} \Theta_X \xrightarrow{\delta} F_p N \rightarrow 0\}.$$

The graded complex $\mathrm{gr}^F \mathrm{Sp}(N)$ is thus expressed as

$$(A.5) \quad \mathrm{gr}^F \mathrm{Sp}(N) \\ = \{0 \rightarrow \mathrm{gr}^F N \otimes_{\mathcal{O}_X} \wedge^n \Theta_X \xrightarrow{\mathrm{gr}_1 \delta} \cdots \xrightarrow{\mathrm{gr}_1 \delta} \mathrm{gr}^F N \otimes_{\mathcal{O}_X} \Theta_X \xrightarrow{\mathrm{gr}_1 \delta} \mathrm{gr}^F N \rightarrow 0\},$$

where $\mathrm{gr}_1 \delta$ is defined by the first part of (A.1), that is,

$$(A.6) \quad m \otimes (\xi_1 \wedge \cdots \wedge \xi_k) \xrightarrow{\mathrm{gr}_1 \delta} \sum_{i=1}^k (-1)^{i-1} (\xi_i m) \otimes (\xi_1 \wedge \cdots \wedge \widehat{\xi}_i \wedge \cdots \wedge \xi_k),$$

where ξ_i is now regarded as a linear form on T^*X .

We now explain that (A.2) is compatible with taking gr^F . Setting $\mathcal{O}_{T^*X} := \mathrm{Sym} \Theta_X$, one similarly regards $\mathrm{gr}^F \mathrm{Sp}(\mathcal{D}_X)$ as a resolution of \mathcal{O}_X as an \mathcal{O}_{T^*X} -module: one has

$$\mathrm{gr}^F \mathrm{Sp}(\mathcal{D}_X) = \{0 \rightarrow \mathcal{O}_{T^*X} \otimes_{\mathcal{O}_X} \wedge^n \Theta_X \xrightarrow{\mathrm{gr}_1 \delta} \cdots \xrightarrow{\mathrm{gr}_1 \delta} \mathcal{O}_{T^*X} \otimes_{\mathcal{O}_X} \Theta_X \xrightarrow{\mathrm{gr}_1 \delta} \mathcal{O}_{T^*X} \rightarrow 0\}.$$

Grading (A.2) gives

$$(A.7) \quad \mathrm{gr}^F \mathrm{Sp}(N) \simeq \mathrm{gr}^F N \otimes_{\mathcal{O}_{T^*X}} \mathrm{gr}^F \mathrm{Sp}(\mathcal{D}_X) = \mathrm{gr}^F N \otimes_{\mathcal{O}_{T^*X}}^L \mathrm{gr}^F \mathrm{Sp}(\mathcal{D}_X),$$

since each term of the complex $\mathrm{gr}^F \mathrm{Sp}(\mathcal{D}_X)$ is \mathcal{O}_{T^*X} -locally free. The graded analogue of (A.3) is now

$$(A.8) \quad \mathrm{gr}^F N \otimes_{\mathcal{O}_{T^*X}}^L \mathcal{O}_X \simeq \mathrm{Car} N \otimes_{\mathcal{O}_{T^*X}}^L \mathrm{gr}^F \mathrm{Sp}(\mathcal{D}_X) \simeq \mathrm{gr}^F \mathrm{Sp}(N)$$

in the bounded derived category of \mathcal{O}_X -modules. In the previous formulas, one forgets the information given by the grading (e.g. $\mathrm{gr}^F N = \bigoplus_p F_p N / F_{p-1} N$), as it is not to be used. In $K_0(\mathcal{O}_X)$, (A.8) reads

$$(A.9) \quad Li^* \mathrm{Car} N = [\mathrm{gr}^F \mathrm{Sp}(N)].$$

A.2. Left \mathcal{D}_X -modules. If M is a left \mathcal{D}_X -module, its de Rham complex is defined as

$$(A.10) \quad \mathrm{DR} M = \{0 \rightarrow M \xrightarrow{\nabla} \Omega_X^1 \otimes M \rightarrow \cdots \rightarrow \Omega_X^n \otimes M \rightarrow 0\}.$$

Recall the \bullet indicated the degree zero term of the complex. If M is endowed with a coherent filtration $F_\bullet M$, we set

$$(A.11) \quad F_p \mathrm{DR} M = \{0 \rightarrow F_p M \xrightarrow{\nabla} \Omega_X^1 \otimes F_{p+1} M \rightarrow \cdots \rightarrow \Omega_X^n \otimes F_{p+n} M \rightarrow 0\}$$

and we have

$$(A.12) \quad \mathrm{gr}^F \mathrm{DR} M = \{0 \rightarrow \mathrm{gr}^F M \xrightarrow{\mathrm{gr}_1^F \nabla} \Omega_X^1 \otimes \mathrm{gr}^F M \rightarrow \cdots \rightarrow \Omega_X^n \otimes \mathrm{gr}^F M \rightarrow 0\},$$

which is a bounded complex in the category of \mathcal{O}_X -modules.

The side-changing functor is defined by $N = \omega_X \otimes_{\mathcal{O}_X} M$. Then N is a right \mathcal{D}_X -module, and we have natural identifications (see e.g. [MHM, Ex. A.5.9])

$$(A.13) \quad \mathrm{Sp}(N) \simeq \mathrm{DR}(M)[n], \quad \mathrm{gr}^F \mathrm{Sp}(N) \simeq \mathrm{gr}^F \mathrm{DR}(M)[n].$$

Now, (A.8) gives

$$(A.14) \quad \mathrm{gr}^F(\omega_X \otimes_{\mathcal{O}_X} M) \otimes_{\mathcal{O}_{T^*X}}^L \mathcal{O}_X \simeq \mathrm{gr}^F \mathrm{Sp}(\omega_X \otimes_{\mathcal{O}_X} M) \simeq \mathrm{gr}^F \mathrm{DR}(M)[n]$$

It follows that

$$(A.15) \quad \mathrm{gr}^F M \otimes_{\mathcal{O}_{T^*X}}^L \mathcal{O}_X \simeq \omega_X^{-1} \otimes \mathrm{gr}^F \mathrm{DR}(M)[n],$$

and therefore, in $K_0(\mathcal{O}_X)$,

$$(A.16) \quad Li^* \mathrm{Car} M = (-1)^n [\omega_X^{-1} \otimes \mathrm{gr}^F \mathrm{DR}(M)].$$

A.3. Computing $Li^* \mathrm{Car} M$. Let M (resp. N) be a coherent left (resp. right) \mathcal{D}_X -module and let $F_\bullet M$ (resp. $F_\bullet N$) be a coherent filtration. It is well known that there exists p_0 such that

$$(A.17) \quad \mathrm{gr}_p^F \mathrm{DR} M \text{ (resp. } \mathrm{gr}_p^F \mathrm{Sp}(N)) \text{ is acyclic for any } p > p_0.$$

Indeed, one first proves this for $N = \mathcal{D}_X$, for which one knows that $\mathrm{gr}_p^F \mathrm{Sp}(\mathcal{D}_X)$ is acyclic for any $p \geq 1$ (see e.g. [MHM, Ex. A.5.4(3)]) and then for any coherent $(N, F_\bullet N)$ by using a suitable resolution of it by right filtered \mathcal{D}_X -modules of the form $L \otimes_{\mathcal{O}_X} \mathcal{D}_X$, where L is \mathcal{O}_X -coherent. One deduces the lemma for $(M, F_\bullet M)$ by using the side-changing formulas (A.13).

One sets $\mathrm{gr}_{\leq p}^F M := \bigoplus_{q \leq p} \mathrm{gr}_q^F M$ and

$$(A.18) \quad \mathrm{gr}_{\leq p}^F \mathrm{DR} M := \{0 \rightarrow \mathrm{gr}_{\leq p}^F M \xrightarrow{\mathrm{gr}_1^F \nabla} \Omega_X^1 \otimes \mathrm{gr}_{\leq p+1}^F M \rightarrow \cdots \rightarrow \Omega_X^n \otimes \mathrm{gr}_{\leq p+n}^F M \rightarrow 0\}.$$

The acyclicity property for $p > p_0$ implies that the inclusion of complexes

$$\mathrm{gr}_{\leq p_0}^F \mathrm{DR} M \hookrightarrow \mathrm{gr}^F \mathrm{DR} M$$

is a quasi-isomorphism. Then (A.16) reads

$$(A.19) \quad Li^* \mathrm{Car} M = (-1)^n [\omega_X^{-1} \otimes \mathrm{gr}_{\leq p_0}^F \mathrm{DR}(M)],$$

that is

$$(A.20) \quad \begin{aligned} Li^* \mathrm{Car} M &= \sum_{i=1}^n (-1)^{n-i} [\omega_X^{-1} \otimes \Omega_X^i \otimes \mathrm{gr}_{\leq p_0+i}^F M] \\ &= \sum_{i=1}^n (-1)^{n-i} [\omega_X^{-1} \otimes \Omega_X^i \otimes F_{p_0+i} M], \end{aligned}$$

where the latter equality follows from $[\mathrm{gr}_{\leq p}^F M] = [F_p M]$ in $K_0(\mathcal{O}_X)$.

However, it is in general difficult to determine the smallest p_0 . One can nevertheless always assume that $F_{p_0+i} M = F_i \mathcal{D}_X \cdot F_{p_0} M$ by taking p_0 large enough.

APPENDIX B. DEFECT OF FLATNESS OF THE DIFFERENTIAL OPERATORS ON THE DIFFERENTIAL OPERATORS WITH LOGARITHMIC POLES

In this section, we work with right \mathcal{D} -modules.

B.1. Rees modules. The sheaf of rings \mathcal{D}_X is not flat over $\mathcal{D}_X(\log D)$. In this appendix, we make explicit conditions on a $\mathcal{D}_X(\log D)$ -module \mathcal{N} so that, nevertheless, tensoring with \mathcal{N} is exact. We enlarge the point of view to *filtered* $\mathcal{D}_X(\log D)$ -modules by considering the associated Rees modules, in order to control the graded modules.

Recall that, to any object M of an abelian category of (sheaves of) k -vector spaces endowed with an increasing filtration $F_\bullet M$, we associate its Rees module by introducing a new variable z and by defining $R_F M$ as $\bigoplus_p F_p M z^p \subset k[z, z^{-1}] \otimes_k M$. Then, since the filtration is increasing, $R_F M$ is a $k[z]$ -module. We apply this construction to $(\mathcal{D}_X, F_\bullet \mathcal{D}_X)$ and get

$$\tilde{\mathcal{D}}_X := \bigoplus_{p \in \mathbb{N}} F_p \mathcal{D}_X \cdot z^p \subset \mathcal{D}_X[z] := \mathcal{D}_X \otimes_k k[z],$$

that we regard as a *graded ring*, which is locally free over the graded ring $\tilde{\mathcal{O}}_X := \mathcal{O}_X[z]$ and generated as an $\tilde{\mathcal{O}}_X$ -algebra by $\tilde{\mathcal{O}}_X$ and $\tilde{\Theta}_X := \Theta_X \otimes_k z k[z]$. In particular, z is a central element in $\tilde{\mathcal{D}}_X$. Starting with $\mathcal{D}_X(\log D)$, we define similarly $\tilde{\mathcal{D}}_X(\log D)$ and $\tilde{\Theta}_X(\log D)$. We will work in the category $\mathrm{Mod}_{\mathrm{gr}}$ of *graded modules* over these graded rings, and morphisms will be similarly graded of degree zero. We always assume that *the grading is bounded from below*, that is, if $\tilde{\mathcal{N}} = \bigoplus_p (\tilde{\mathcal{N}})_p$, we have $(\tilde{\mathcal{N}})_p = 0$ for $p \ll 0$.

Let us note that

$$k[z, z^{-1}] \otimes_{k[z]} \tilde{\mathcal{D}}_X = k[z, z^{-1}] \otimes_k \mathcal{D}_X$$

and a similar property for $\tilde{\mathcal{D}}_X(\log D)$.

Let $\tilde{\mathcal{N}} = \bigoplus_p (\tilde{\mathcal{N}})_p$ be a graded $\tilde{\mathcal{D}}_X(\log D)$ -module. Let $(\tilde{\mathcal{N}})'_p$ denote the image of $(\tilde{\mathcal{N}})_p$ in $\tilde{\mathcal{N}}[z^{-1}] := k[z, z^{-1}] \otimes_{k[z]} \tilde{\mathcal{N}}$. Then $(\tilde{\mathcal{N}})'_p z^{-p} \subset (\tilde{\mathcal{N}})'_{p+1} z^{-p-1}$ and

$$(B.1) \quad \mathcal{N} := \bigcup_p (\tilde{\mathcal{N}})'_p z^{-p} \subset \tilde{\mathcal{N}}[z^{-1}]$$

is a \mathcal{D}_X -module, that we call the *underlying \mathcal{D}_X -module of $\tilde{\mathcal{N}}$* . We have

$$\tilde{\mathcal{N}}[z^{-1}] = k[z, z^{-1}] \otimes_k \mathcal{N}.$$

We denote by i_z^* the functor $\bullet \otimes_{k[z]} (k[z]/zk[z])$ and by Li_z^* the associated derived functor. We have $i_z^* \tilde{\mathcal{D}}_X = \text{gr}^F \mathcal{D}_X$. We extend i_z^* with the same notation as a functor $\text{Mod}_{\text{gr}}(\tilde{\mathcal{D}}_X) \rightarrow \text{Mod}(\text{gr}^F \mathcal{D}_X)$ and $\text{D}_{\text{gr}}^b(\tilde{\mathcal{D}}_X) \rightarrow \text{D}^b(\text{gr}^F \mathcal{D}_X)$. We also use the same notation when considering $\mathcal{D}_X(\log D)$ instead of \mathcal{D}_X .

We say that $\tilde{\mathcal{N}}$ is *strict* if $z : \tilde{\mathcal{N}} \rightarrow \tilde{\mathcal{N}}$ is injective (equivalently, since $\tilde{\mathcal{N}}$ is graded, $\tilde{\mathcal{N}}$ is $k[z]$ -flat). A graded $\tilde{\mathcal{D}}_X(\log D)$ -module $\tilde{\mathcal{N}}$ is strict if and only if there exists an $F\mathcal{D}_X(\log D)$ -filtration $F_\bullet \mathcal{N}$ of the associated \mathcal{D}_X -module \mathcal{N} defined by (B.1) such that $\tilde{\mathcal{N}} = \bigoplus_p F_p \mathcal{N} z^p =: R_F \mathcal{N}$ (the *Rees module* of \mathcal{N} with respect to $F_\bullet \mathcal{N}$). Indeed, strictness is equivalent to the property that, for each p ,

$$(B.2) \quad z : (\tilde{\mathcal{N}})_p \longrightarrow (\tilde{\mathcal{N}})_{p+1} \text{ is injective.}$$

Setting $F_p \mathcal{N} = (\tilde{\mathcal{N}})'_p z^{-p}$ one has $\tilde{\mathcal{N}} = \bigoplus_p F_p \mathcal{N} z^p$. Moreover, by strictness,

$$Li_z^* \tilde{\mathcal{N}} = i_z^* \tilde{\mathcal{N}} = \text{gr}^F \mathcal{N}.$$

A similar characterization of strictness holds for graded $\tilde{\mathcal{D}}_X(\log D)$ -modules.

Lemma B.1. *Assume $\tilde{\mathcal{N}}$ is a strict graded $\tilde{\mathcal{D}}_X(\log D)$ -module $R_F \mathcal{N}$. Then the quotient of the graded module $\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{D}}_X(\log D)} \tilde{\mathcal{D}}_X$ by its z -torsion is the Rees module of the filtration $F_\bullet(\mathcal{N} \otimes_{\mathcal{D}_X(\log D)} \mathcal{D}_X)$ defined by*

$$F_p(\mathcal{N} \otimes_{\mathcal{D}_X(\log D)} \mathcal{D}_X) = \text{im} \left[F_p(\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \longrightarrow (\mathcal{N} \otimes_{\mathcal{D}_X(\log D)} \mathcal{D}_X) \right],$$

with

$$F_p(\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{D}_X) = \sum_q F_q \mathcal{N} \otimes_{\mathcal{O}_X} F_{p-q} \mathcal{D}_X \subset \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{D}_X.$$

Moreover, we have

$$(B.3) \quad Li_z^* \left[(\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{D}}_X(\log D)} \tilde{\mathcal{D}}_X) / z\text{-torsion} \right] \simeq \text{gr}^F(\mathcal{N} \otimes_{\mathcal{D}_X(\log D)} \mathcal{D}_X)$$

and

$$(B.4) \quad Li_z^* (\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{D}}_X(\log D)}^L \tilde{\mathcal{D}}_X) \simeq \text{gr}^F \mathcal{N} \otimes_{\text{gr}^F \mathcal{D}_X(\log D)}^L \text{gr}^F \mathcal{D}_X.$$

Proof. Since $\tilde{\mathcal{D}}_X$ is $\tilde{\mathcal{O}}_X$ -locally free and since $\tilde{\mathcal{N}}$ is strict, $\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{D}}_X$ is also strict, hence is the Rees module associated to a filtration $F_\bullet(\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{D}_X)$. This is nothing but the filtration defined in the lemma.

There is a natural surjective composed morphism

$$\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{D}}_X} \tilde{\mathcal{D}}_X \longrightarrow \tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{D}}_X(\log D)} \tilde{\mathcal{D}}_X \longrightarrow (\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{D}}_X(\log D)} \tilde{\mathcal{D}}_X)/z\text{-torsion}.$$

The last term is isomorphic to the Rees module of a filtration $F_\bullet(\mathcal{N} \otimes_{\mathcal{D}_X(\log D)} \mathcal{D}_X)$ for which (B.3) holds and, by considering the component of degree p for each p , one checks that this is the filtration defined in the lemma.

For (B.4), we construct a graded resolution of $\tilde{\mathcal{D}}_X$ as a left graded $\tilde{\mathcal{D}}_X(\log D)$ -module by flat graded $\tilde{\mathcal{D}}_X(\log D)$ -modules L^\bullet . Since $\tilde{\mathcal{D}}_X$ is $k[z]$ -flat, one checks inductively that each differential $d_i : L^i \rightarrow L^{i-1}$ is strict, i.e., its cokernel is $k[z]$ -flat. It follows that $i_z^* L^\bullet$ is a resolution of $i_z^* \tilde{\mathcal{D}}_X = \text{gr}^F \mathcal{D}_X$ by flat $\text{gr}^F \mathcal{D}_X(\log D)$ -modules. We interpret $Li_z^*(\bullet)$ as $(k[z]/zk[z]) \otimes_{k[z]}^L \bullet$. The associativity property as given e.g. in [Kas03, p. 240] leads to

$$Li_z^*(\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{D}}_X(\log D)}^L \tilde{\mathcal{D}}_X) \simeq (Li_z^* \tilde{\mathcal{N}}) \otimes_{\tilde{\mathcal{D}}_X(\log D)}^L \tilde{\mathcal{D}}_X \simeq \text{gr}^F \mathcal{N} \otimes_{\tilde{\mathcal{D}}_X(\log D)}^L \tilde{\mathcal{D}}_X \simeq \text{gr}^F \mathcal{N} \otimes_{\tilde{\mathcal{D}}_X(\log D)} L^\bullet.$$

Since z acts by zero on $\text{gr}^F \mathcal{N}$, we have $\text{gr}^F \mathcal{N} \otimes_{\tilde{\mathcal{D}}_X(\log D)} L^\bullet \simeq \text{gr}^F \mathcal{N} \otimes_{\tilde{\mathcal{D}}_X(\log D)} i_z^* L^\bullet$, and the latter term is nothing but $\text{gr}^F \mathcal{N} \otimes_{\text{gr}^F \mathcal{D}_X(\log D)} i_z^* L^\bullet$. The choice of L^\bullet implies that this is a realization of $\text{gr}^F \mathcal{N} \otimes_{\text{gr}^F \mathcal{D}_X(\log D)}^L \text{gr}^F \mathcal{D}_X$. \square

As a consequence, we obtain the following criterion for a right graded $\mathcal{D}_X(\log D)$ -module $\tilde{\mathcal{N}}$.

Lemma B.2. *Assume that*

- (1) $\tilde{\mathcal{N}}$ is strict,
- (2) $\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{D}}_X(\log D)}^L \tilde{\mathcal{D}}_X \simeq \tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{D}}_X(\log D)} \tilde{\mathcal{D}}_X$,
- (3) $\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{D}}_X(\log D)} \tilde{\mathcal{D}}_X$ is strict.

Then we have

$$\text{gr}^F(\mathcal{N} \otimes_{\mathcal{D}_X(\log D)} \mathcal{D}_X) \simeq \text{gr}^F \mathcal{N} \otimes_{\text{gr}^F \mathcal{D}_X(\log D)} \text{gr}^F \mathcal{D}_X \simeq \text{gr}^F \mathcal{N} \otimes_{\text{gr}^F \mathcal{D}_X(\log D)}^L \text{gr}^F \mathcal{D}_X.$$

Proof. It is enough to prove the isomorphism between the first and the third term, since this would imply that the third term is isomorphic to its \mathcal{H}^0 , that is, the second term.

We write

$$\begin{aligned} \text{gr}^F \mathcal{N} \otimes_{\text{gr}^F \mathcal{D}_X(\log D)}^L \text{gr}^F \mathcal{D}_X & \\ (B.5) \quad & \simeq Li_z^*(\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{D}}_X(\log D)}^L \tilde{\mathcal{D}}_X) \quad (\text{by (B.4)}) \\ & \simeq Li_z^*(\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{D}}_X(\log D)} \tilde{\mathcal{D}}_X) \quad (\text{by Assumption (2)}) \\ & \simeq Li_z^*\left[(\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{D}}_X(\log D)} \tilde{\mathcal{D}}_X)/z\text{-torsion}\right] \quad (\text{by Assumption (3)}) \\ & \simeq \text{gr}^F(\mathcal{N} \otimes_{\mathcal{D}_X(\log D)} \mathcal{D}_X) \quad (\text{by (B.3)}). \end{aligned}$$

\square

We now relax Condition B.2(3), but we assume that $\tilde{\mathcal{N}}$ is $\tilde{\mathcal{D}}_X(\log D)$ -coherent. Let us denote by $\tilde{\mathcal{T}}$ the z -torsion of $\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{D}}_X(\log D)} \tilde{\mathcal{D}}_X$. It is a coherent graded $\tilde{\mathcal{D}}_X$ -module, the sections of which are annihilated by some power of z . It has thus a finite filtration such that

each graded piece $\mathrm{gr}_\ell \tilde{\mathcal{T}}$ is a coherent graded $\tilde{\mathcal{D}}_X$ -module annihilated by z , hence a coherent graded $\mathrm{gr}^F \mathcal{D}_X$ -module. By the first lines of (B.5), we find

$$\begin{aligned} \mathcal{H}^0(\mathrm{gr}^F \mathcal{N} \otimes_{\mathrm{gr}^F \mathcal{D}_X(\log D)}^L \mathrm{gr}^F \mathcal{D}_X) &= \mathrm{gr}^F \mathcal{N} \otimes_{\mathrm{gr}^F \mathcal{D}_X(\log D)} \mathrm{gr}^F \mathcal{D}_X = i_z^*(\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{D}}_X(\log D)} \tilde{\mathcal{D}}_X) \\ \mathcal{H}^{-1}(\mathrm{gr}^F \mathcal{N} \otimes_{\mathrm{gr}^F \mathcal{D}_X(\log D)}^L \mathrm{gr}^F \mathcal{D}_X) &= \ker z, \end{aligned}$$

and $\mathcal{H}^i(\mathrm{gr}^F \mathcal{N} \otimes_{\mathrm{gr}^F \mathcal{D}_X(\log D)}^L \mathrm{gr}^F \mathcal{D}_X) = 0$ for $i \neq 0, -1$. We also have an exact sequence

$$\tilde{\mathcal{T}}/z\tilde{\mathcal{T}} \longrightarrow \mathrm{gr}^F \mathcal{N} \otimes_{\mathrm{gr}^F \mathcal{D}_X(\log D)} \mathrm{gr}^F \mathcal{D}_X \longrightarrow \mathrm{gr}^F(\mathcal{N} \otimes_{\mathcal{D}_X(\log D)} \mathcal{D}_X) \longrightarrow 0,$$

but one does not expect in general an isomorphism between $\mathrm{gr}^F \mathcal{N} \otimes_{\mathrm{gr}^F \mathcal{D}_X(\log D)} \mathrm{gr}^F \mathcal{D}_X$ and $\mathrm{gr}^F(\mathcal{N} \otimes_{\mathcal{D}_X(\log D)} \mathcal{D}_X)$.

Lemma B.3. *Let $\tilde{\mathcal{N}}$ be a coherent graded $\tilde{\mathcal{D}}_X(\log D)$ -module. Assume that Properties B.2(1) and B.2(2) hold. Then we have the equality in $K_0(\mathcal{O}_{T^*X})$:*

$$[\mathrm{gr}^F \mathcal{N} \otimes_{\mathrm{gr}^F \mathcal{D}_X(\log D)}^L \mathrm{gr}^F \mathcal{D}_X] = [\mathrm{gr}^F(\mathcal{N} \otimes_{\mathcal{D}_X(\log D)} \mathcal{D}_X)].$$

Proof. If we do not assume B.2(3), we can nevertheless write in $K_0(\mathcal{O}_{T^*X})$, with the notation as above:

$$[\mathrm{gr}^F \mathcal{N} \otimes_{\mathrm{gr}^F \mathcal{D}_X(\log D)}^L \mathrm{gr}^F \mathcal{D}_X] = [\mathrm{gr}^F(\mathcal{N} \otimes_{\mathcal{D}_X(\log D)} \mathcal{D}_X)] + \sum_{\ell} [Li_z^* \mathrm{gr}_\ell \tilde{\mathcal{T}}].$$

On the other hand, we have

$$[Li_z^* \mathrm{gr}_\ell \tilde{\mathcal{T}}] = [\mathrm{gr}_\ell \tilde{\mathcal{T}}] - [\mathrm{gr}_\ell \tilde{\mathcal{T}}] = 0$$

in $K_0(\mathcal{O}_{T^*X})$. □

B.2. Flatness properties.

Assumption B.4. We denote by \hat{X} a formal neighbourhood of a closed point $x \in D$ and we assume that there exists a regular system of parameters (x_1, \dots, x_n) in $\mathcal{O}_{\hat{X}, x}$ such that $\hat{D} = \{x_1 \cdots x_\ell = 0\}$.

For the sake of simplicity, we still denote \hat{X} by X and \hat{D} by D .

It is straightforward to check that the results of Section B.1 apply in this setting. The following proposition and its proof are inspired by [Wei17].

Proposition B.5. *Let $\tilde{\mathcal{N}}$ be a right graded $\tilde{\mathcal{D}}_X(\log D)$ -module. Assume that any subsequence of the sequence (x_1, \dots, x_ℓ) is a regular sequence for $\tilde{\mathcal{N}}$. Then*

$$\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{D}}_X(\log D)}^L \tilde{\mathcal{D}}_X \simeq \tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{D}}_X(\log D)} \tilde{\mathcal{D}}_X.$$

If moreover every quotient $\tilde{\mathcal{N}}/\sum_{i \in I} \tilde{\mathcal{N}}x_i$ ($I \subset \{1, \dots, \ell\}$) is strict, then $\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{D}}_X(\log D)} \tilde{\mathcal{D}}_X$ is strict.

Proof. Recall that $\mathrm{Sp} \tilde{\mathcal{D}}_X(\log D)$ is the complex having $\tilde{\mathcal{D}}_X(\log D) \otimes_{\tilde{\mathcal{O}}_X} \wedge^k \tilde{\Theta}_X(\log D)$ as its term in degree $-k$, and differential the left $\tilde{\mathcal{D}}_X(\log D)$ -linear morphism

$$\tilde{\mathcal{D}}_X(\log D) \otimes_{\tilde{\mathcal{O}}_X} \wedge^k \tilde{\Theta}_X(\log D) \xrightarrow{\delta} \tilde{\mathcal{D}}_X(\log D) \otimes_{\tilde{\mathcal{O}}_X} \wedge^{k-1} \tilde{\Theta}_X(\log D)$$

given, for $\boldsymbol{\theta} = \theta_1 \wedge \cdots \wedge \theta_k$

$$\delta(P \otimes \boldsymbol{\theta}) = \sum_{i=1}^k (-1)^{i-1} (P \cdot \theta_i) \otimes \widehat{\boldsymbol{\theta}}_i + \sum_{i < j} (-1)^{i+j} P \otimes ([\theta_i, \theta_j] \wedge \widehat{\boldsymbol{\theta}}_{i,j}),$$

with $\widehat{\boldsymbol{\theta}}_i = \theta_1 \wedge \cdots \wedge \theta_{i-1} \wedge \theta_{i+1} \wedge \cdots \wedge \theta_k$, and a similar meaning for $\widehat{\boldsymbol{\theta}}_{i,j}$. Since $\text{Spec}(\widetilde{\mathcal{D}}_X(\log D))$ is a resolution of $\widetilde{\mathcal{O}}_X$ by locally free left $\widetilde{\mathcal{D}}_X(\log D)$ -modules which are thus $\widetilde{\mathcal{O}}_X$ -locally free, we have

$$\widetilde{\mathcal{N}} \simeq \widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \text{Sp } \widetilde{\mathcal{D}}_X(\log D),$$

with their right $\widetilde{\mathcal{D}}_X(\log D)$ -module structure, by using the tensor right structure on the right-hand side. The complex $\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \text{Sp } \widetilde{\mathcal{D}}_X(\log D)$ has

$$\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{D}}_X(\log D) \otimes_{\widetilde{\mathcal{O}}_X} \wedge^k \widetilde{\Theta}_X(\log D))$$

as its term in degree $-k$, and differential $\text{id} \otimes \delta$, which is right $\widetilde{\mathcal{D}}_X(\log D)$ -linear for the tensor right structure.

We recall that there are two natural structures of right $\widetilde{\mathcal{D}}_X(\log D)$ -module on the tensor product

$$\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X(\log D) \otimes_{\widetilde{\mathcal{O}}_X} \wedge^k \widetilde{\Theta}_X(\log D).$$

The *tensor structure* is obtained by using the right structure on $\widetilde{\mathcal{N}}$ and the left structure on $\widetilde{\mathcal{D}}_X(\log D)$. On the other hand, the *trivial structure*, for which we rather denote the tensor product as

$$(\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \wedge^k \widetilde{\Theta}_X(\log D)) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X(\log D),$$

is obtained by using the right structure on $\widetilde{\mathcal{D}}_X(\log D)$ and by completely forgetting the right action of derivations on $\widetilde{\mathcal{N}}$ while only remember its $\widetilde{\mathcal{O}}_X$ -structure. However, there exists a unique involution of right $\widetilde{\mathcal{D}}_X$ -modules

$$\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{D}}_X(\log D) \otimes_{\widetilde{\mathcal{O}}_X} \wedge^k \widetilde{\Theta}_X(\log D)) \xrightarrow{\sim} (\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \wedge^k \widetilde{\Theta}_X(\log D)) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X(\log D)$$

extending the natural involution of $\widetilde{\mathcal{O}}_X$ -modules

$$\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{O}}_X \otimes_{\widetilde{\mathcal{O}}_X} \wedge^k \widetilde{\Theta}_X(\log D)) \xrightarrow{\sim} (\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \wedge^k \widetilde{\Theta}_X(\log D)) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{O}}_X.$$

Let us make explicit the differential δ . For $P \in \widetilde{\mathcal{D}}_X(\log D)$, the element $[n \otimes (1 \otimes \boldsymbol{\theta})] \cdot P$ (tensor structure) is complicated to express, but we must have, by right $\widetilde{\mathcal{D}}_X(\log D)$ -linearity of $\text{id} \otimes \delta$,

$$\begin{aligned} (\text{id} \otimes \delta)[(n \otimes (1 \otimes \boldsymbol{\theta})) \cdot P] &= [(\text{id} \otimes \delta)(n \otimes (1 \otimes \boldsymbol{\theta}))] \cdot P \\ &= \left[n \otimes \left[\sum_{i=1}^k (-1)^{i-1} \theta_i \otimes \widehat{\boldsymbol{\theta}}_i + \sum_{i < j} (-1)^{i+j} 1 \otimes ([\theta_i, \theta_j] \wedge \widehat{\boldsymbol{\theta}}_{i,j}) \right] \right] \cdot P. \end{aligned}$$

We now write

$$n \otimes (\theta_i \otimes \widehat{\boldsymbol{\theta}}_i) = n \theta_i \otimes (1 \otimes \widehat{\boldsymbol{\theta}}_i) - [n \otimes (1 \otimes \widehat{\boldsymbol{\theta}}_i)] \cdot \theta_i,$$

so the previous formula reads, after the involution transforming the tensor structure to the trivial one, by denoting δ_{triv} the corresponding differential:

$$\begin{aligned}
\text{(B.6)} \quad \delta_{\text{triv}}[(n \otimes \boldsymbol{\theta}) \otimes P] &= \sum_{i=1}^k (-1)^{i-1} (n\theta_i \otimes \widehat{\boldsymbol{\theta}}_i) \otimes P \\
&\quad - \sum_{i=1}^k (-1)^{i-1} (n \otimes \widehat{\boldsymbol{\theta}}_i) \otimes (\theta_i P) + \sum_{i < j} (-1)^{i+j} (n \otimes ([\theta_i, \theta_j] \wedge \widehat{\boldsymbol{\theta}}_{i,j})) \otimes P \\
&= [\delta_{\widetilde{\mathcal{N}}}(n \otimes \boldsymbol{\theta})] \otimes P - \sum_{i=1}^k (-1)^{i-1} (n \otimes \widehat{\boldsymbol{\theta}}_i) \otimes (\theta_i P),
\end{aligned}$$

where $\delta_{\widetilde{\mathcal{N}}}$ is the differential of the Spencer complex $\text{Sp}_{\log} \widetilde{\mathcal{N}}$ of $\widetilde{\mathcal{N}}$ as a right $\widetilde{\mathcal{D}}_X(\log D)$ -module.

We obtain, due to the local $\widetilde{\mathcal{O}}_X$ -freeness of $\widetilde{\mathcal{D}}_X(\log D)$ and $\widetilde{\mathcal{D}}_X$,

$$\begin{aligned}
\text{(B.7)} \quad \widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{D}}_X(\log D)}^L \widetilde{\mathcal{D}}_X &\simeq (\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \text{Sp } \widetilde{\mathcal{D}}_X(\log D)) \otimes_{\widetilde{\mathcal{D}}_X(\log D)}^L \widetilde{\mathcal{D}}_X \\
&\simeq ((\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \wedge^{-\bullet} \widetilde{\Theta}_X(\log D)) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X(\log D), \delta_{\text{triv}}) \otimes_{\widetilde{\mathcal{D}}_X(\log D)}^L \widetilde{\mathcal{D}}_X \\
&\simeq ((\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \wedge^{-\bullet} \widetilde{\Theta}_X(\log D)) \otimes_{\widetilde{\mathcal{O}}_X}^L \widetilde{\mathcal{D}}_X(\log D), \delta_{\text{triv}}) \otimes_{\widetilde{\mathcal{D}}_X(\log D)}^L \widetilde{\mathcal{D}}_X \\
&\simeq ((\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \wedge^{-\bullet} \widetilde{\Theta}_X(\log D)) \otimes_{\widetilde{\mathcal{O}}_X}^L \widetilde{\mathcal{D}}_X, \delta_{\text{triv}}) \\
&\simeq ((\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \wedge^{-\bullet} \widetilde{\Theta}_X(\log D)) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X, \delta_{\text{triv}}).
\end{aligned}$$

In the last line, δ_{triv} is given by (B.6), where P is now a local section of $\widetilde{\mathcal{D}}_X$. We have thus realized $\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{D}}_X(\log D)}^L \widetilde{\mathcal{D}}_X$ as a complex $(\mathcal{F}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X, \delta_{\text{triv}})$, where each term \mathcal{F}^k is an $\widetilde{\mathcal{O}}_X$ -module.

With respect to the filtration $\mathcal{F}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} F_k \widetilde{\mathcal{D}}_X$, δ_{triv} has degree one, and the differential $\text{gr}_1^F \delta_{\text{triv}}$ of the graded complex $\mathcal{F}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} \text{gr}^F \widetilde{\mathcal{D}}_X$ is expressed as

$$\text{gr}_1^F \delta_{\text{triv}}[(n \otimes \boldsymbol{\theta}) \otimes Q] = \sum_{i=1}^k (-1)^i (n \otimes \widehat{\boldsymbol{\theta}}_i) \otimes (\theta_i \cdot Q)$$

for a local section Q of $\text{gr}^F \widetilde{\mathcal{D}}_X$. The filtration $F_p(\mathcal{F}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X, \delta_{\text{triv}})$ whose term in degree $-k$ is $\mathcal{F}^{-k} \otimes_{\widetilde{\mathcal{O}}_X} F_{p-k} \widetilde{\mathcal{D}}_X$ satisfies $F_p(\mathcal{F}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X, \delta_{\text{triv}}) = 0$ for $p < 0$ and we have

$$\text{(B.8)} \quad \text{gr}^F(\mathcal{F}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X, \delta_{\text{triv}}) = (\mathcal{F}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} \text{gr}^F \widetilde{\mathcal{D}}_X, \text{gr}_1^F \delta_{\text{triv}}),$$

compatible with the grading, if the grading on the right-hand side takes into account the cohomology degree.

Lemma B.6. *If the first condition of Proposition B.5 is fulfilled, the graded complex $(\mathcal{F}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} \text{gr}^F \widetilde{\mathcal{D}}_X, \text{gr}_1^F \delta_{\text{triv}})$ has zero cohomology in any degree $i \neq 0$. If the second condition is also fulfilled, then the cohomology in degree zero is strict.*

Proof. Let us consider the basis

$$x_1 \widetilde{\partial}_{x_1}, x_2 \widetilde{\partial}_{x_2}, \dots, x_\ell \widetilde{\partial}_{x_\ell}, \widetilde{\partial}_{x_{\ell+1}}, \dots, \widetilde{\partial}_{x_n}$$

of $\tilde{\Theta}_X(\log D)$, so that, by replacing $x_i \tilde{\partial}_{x_i}$ with $\tilde{\partial}_{x_i}$ we obtain a basis of $\tilde{\Theta}_X$. Let $\tilde{\xi}_1, \tilde{\xi}_2, \dots, \tilde{\xi}_n$ resp. $x_1 \tilde{\xi}_1, x_2 \tilde{\xi}_2, \dots, \tilde{\xi}_n$ be the corresponding basis of $\mathrm{gr}_1^F \tilde{\mathcal{D}}_X$ resp. $\mathrm{gr}_1^F \tilde{\mathcal{D}}_X(\log D)$. Then $\mathrm{gr}^F(\mathcal{F}^\bullet \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{D}}_X, \delta_{\mathrm{triv}})$ is identified with a Koszul complex. More precisely, it is isomorphic to the simple complex associated to the n -cube with vertices

$$\tilde{\mathcal{N}} \otimes_{\tilde{\mathcal{O}}_X} \mathrm{gr}^F \tilde{\mathcal{D}}_X = \tilde{\mathcal{N}} \otimes_k k[\tilde{\xi}_1, \dots, \tilde{\xi}_n]$$

and arrows in the i -th direction all equal to multiplication by $\tilde{\xi}_i$ if $i > \ell$ and by $x_i \otimes \tilde{\xi}_i$ if $i \leq \ell$. In such a way we obtain that $(\mathcal{F}^\bullet \otimes_{\tilde{\mathcal{O}}_X} \mathrm{gr}^F \tilde{\mathcal{D}}_X, \mathrm{gr}_1^F \delta_{\mathrm{triv}})$ is quasi-isomorphic to the simple complex attached to the ℓ -cube having

$$\tilde{\mathcal{N}} \otimes_k k[\tilde{\xi}_1, \dots, \tilde{\xi}_\ell] \xrightarrow{x_i \otimes \tilde{\xi}_i} \tilde{\mathcal{N}} \otimes_k k[\tilde{\xi}_1, \dots, \tilde{\xi}_\ell]$$

as its arrow in the i -th directions.

Let us prove by induction on ℓ that, under the first assumption of Proposition B.5, this complex has cohomology in degree zero only, and this cohomology is isomorphic to a direct sum of terms, each of which is isomorphic to $\tilde{\mathcal{N}} / \sum_{i \in I} \tilde{\mathcal{N}} x_i$ for some $I \subset \{1, \dots, \ell\}$. This will give the first part of the lemma.

Indeed, since x_1 is injective on $\tilde{\mathcal{N}}$, each arrow $x_1 \otimes \tilde{\xi}_1$ is injective with cokernel isomorphic to $\tilde{\mathcal{N}}_1 \otimes_k k[\tilde{\xi}_2, \dots, \tilde{\xi}_\ell]$, where $\tilde{\mathcal{N}}_1 = \tilde{\mathcal{N}} \oplus \bigoplus_{k \geq 1} (\tilde{\mathcal{N}} / \tilde{\mathcal{N}} x_1)[\tilde{\xi}_1^k]$ and $[\tilde{\xi}_1^k]$ is the image of $\tilde{\xi}_1^k$ in the cokernel. It follows that the complex we consider is quasi-isomorphic to the simple complex attached to the $(\ell - 1)$ -cube with vertices $\tilde{\mathcal{N}}_1 \otimes_k k[\tilde{\xi}_2, \dots, \tilde{\xi}_\ell]$ and arrows $x_i \otimes \tilde{\xi}_i$ ($i = 2, \dots, \ell$). Now, any subsequence of (x_2, \dots, x_ℓ) is a regular sequence for $\tilde{\mathcal{N}}_1$ by our assumption, and we obtain the desired assertion by induction on ℓ .

For the second part of the lemma, we note that, by the second assumption of Proposition B.5, each term $\tilde{\mathcal{N}} / \sum_{i \in I} \tilde{\mathcal{N}} x_i$ is strict, so the cohomology in degree zero is strict. \square

End of the proof of Proposition B.5. By (B.8), the lemma applies to the graded complex $\mathrm{gr}^F(\mathcal{F}^\bullet \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{D}}_X, \delta_{\mathrm{triv}})$ and therefore each $\mathrm{gr}_p^F(\mathcal{F}^\bullet \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{D}}_X, \delta_{\mathrm{triv}})$ has cohomology in degree zero at most and, if the supplementary strictness assumption on the quotients of $\tilde{\mathcal{N}}$ is fulfilled, the cohomology in degree zero is strict. It follows that each $F_p(\mathcal{F}^\bullet \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{D}}_X, \delta_{\mathrm{triv}})$ satisfies the same property since $F_{-1}(\mathcal{F}^\bullet \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{D}}_X, \delta_{\mathrm{triv}}) = 0$. Passing to the inductive limit, we conclude that so does the complex $(\mathcal{F}^\bullet \otimes_{\tilde{\mathcal{O}}_X} \tilde{\mathcal{D}}_X, \delta_{\mathrm{triv}})$. \square

B.3. A criterion for having $\mathcal{N} \otimes_{\mathcal{D}_X(\log D)} \mathcal{D}_X \simeq \mathcal{N}(*D)$.

Assumption B.4 and the corresponding simplifying notation remain in order in this section.

For $I \subset \{1, \dots, \ell\}$ we set $I^c = \{1, \dots, \ell\} \setminus I$, $D_I = \bigcap_{i \in I} D_i$ and, for each $j \in I^c$, $D_{I,j} = D_I \cap D_j$ and $D_I^{(I^c)} = \bigcup_{j \in \{1, \dots, \ell\} \setminus I} D_{I,j}$, so that $D_I^{(I^c)}$ is a divisor with normal crossings in D_I . If $I = \emptyset$, we have $D_I = X$ and $D_I^{(I^c)} = D$.

We identify $i_{D_I}^* \mathcal{O}_X$ with \mathcal{O}_{D_I} and $i_{D_I}^* \mathcal{D}_X(\log D)$ with $\mathcal{D}_{D_I}(\log D_I^{(I^c)})[(\mathrm{Eu}_i)_{i \in I}]$, where Eu_i is the class of $x_i \partial_{x_i}$. It is a central element in this ring.

The following result is inspired by [Moc15, Lem. 3.1.2].

Proposition B.7. *Let \mathcal{N} be a right $\mathcal{D}_X(\log D)$ -module. Together with the assumption above, assume the following property: for any (possibly empty) subset I of $\{1, \dots, \ell\}$,*

- (1) $i_{D_I}^* \mathcal{N} \subset (i_{D_I}^* \mathcal{N})(*D_I^{(I^c)})$,
- (2) for all $j \in I$ and all $k \geq 1$,

$$(\text{Eu}_j + k) : i_{D_I}^* \mathcal{N} \longrightarrow i_{D_I}^* \mathcal{N} \quad \text{is bijective.}$$

Then the natural morphism $\mathcal{N} \otimes_{\mathcal{D}_X(\log D)} \mathcal{D}_X \rightarrow \mathcal{N}(*D)$ is an isomorphism.

In order to argue by induction on $\#D$, we first consider how the properties in Proposition B.7 are preserved when

- $D = D_1 \cup D'$ and D is replaced with D' (in the local setting, we assume that $D_1 = \{x_1 = 0\}$ and $D' = \{x_2 \cdots x_\ell = 0\}$),
- correspondingly, \mathcal{N} is replaced with $\mathcal{N} \otimes_{\mathcal{D}_X(\log D)} \mathcal{D}_X(\log D')$.

In order to simplify the formulas, we use in this section the simplified notation below:

$$\mathcal{D}_{X,\log} := \mathcal{D}_X(\log D), \quad \mathcal{D}'_{X,\log} := \mathcal{D}_X(\log D'), \quad \mathcal{N}' = \mathcal{N} \otimes_{\mathcal{D}_{X,\log}} \mathcal{D}'_{X,\log}.$$

Since $\mathcal{D}_{X,\log}(*D_1) = \mathcal{D}'_{X,\log}(*D_1)$, we have $\mathcal{N}'(*D_1) \simeq \mathcal{N}(*D_1)$, and we have a natural localization morphism $\mathcal{N}' \rightarrow \mathcal{N}'(*D_1) = \mathcal{N}(*D_1)$.

Lemma B.8. *Let \mathcal{N} be a right $\mathcal{D}_{X,\log}$ -module such that $\mathcal{N} \subset \mathcal{N}(*D_1)$ and B.7(2) for $I = \{1\}$ holds. Then the localization morphism*

$$\mathcal{N}' \longrightarrow \mathcal{N}'(*D_1) = \mathcal{N}(*D_1)$$

is an isomorphism.

Proof. The statement is local. For $k \geq 0$, we set

$$\mathcal{D}'_{X,\log,\leq k} = \sum_{j=0}^k \partial_{x_1}^j \mathcal{D}_{X,\log} = \sum_{j=0}^k \mathcal{D}_{X,\log} \partial_{x_1}^j.$$

We will prove by induction on k that $\mathcal{N} \otimes_{\mathcal{D}_{X,\log}} \mathcal{D}'_{X,\log,\leq k}$ injects into $\mathcal{N}(*D_1)$ with image equal to $\mathcal{N}x_1^{-k}$, the case $k = 0$ being given by B.7(1) for $I = \emptyset$. The composition

$$\mathcal{N} \otimes_{\mathcal{D}_{X,\log}} \mathcal{D}'_{X,\log,\leq k-1} \longrightarrow \mathcal{N} \otimes_{\mathcal{D}_{X,\log}} \mathcal{D}'_{X,\log,\leq k} \longrightarrow \mathcal{N}(*D),$$

being injective by induction, so is the first morphism, whose cokernel will be denoted by $\text{gr}_k(\mathcal{N} \otimes_{\mathcal{D}_{X,\log}} \mathcal{D}'_{X,\log})$. Since \mathcal{N} is acted on by $x_1 \partial_{x_1}$, hence by $\partial_{x_1}^k x_1^k = \prod_{j=1}^k (x_1 \partial_{x_1} + j)$, the second morphism factorizes through $\mathcal{N}x_1^{-k} \subset \mathcal{N}(*D_1)$: an element $m \otimes \partial_{x_1}^k$ has image $m \partial_{x_1}^k = (m \partial_{x_1}^k x_1^k) x_1^{-k}$, and $m \partial_{x_1}^k x_1^k$ is the image of $(m \partial_{x_1}^k x_1^k) \otimes 1$, hence belongs to \mathcal{N} . We consider the commutative diagram of exact sequences, where the first two terms of the lower line are regarded in $\mathcal{N}(*D_1)$,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{N} \otimes_{\mathcal{D}_{X,\log}} \mathcal{D}'_{X,\log,\leq k-1} & \longrightarrow & \mathcal{N} \otimes_{\mathcal{D}_{X,\log}} \mathcal{D}'_{X,\log,\leq k} & \longrightarrow & \text{gr}_k(\mathcal{N} \otimes_{\mathcal{D}_{X,\log}} \mathcal{D}'_{X,\log}) \longrightarrow 0 \\ & & \downarrow \wr & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{N}x_1^{-k+1} & \longrightarrow & \mathcal{N}x_1^{-k} & \longrightarrow & \mathcal{N}x_1^{-k} / \mathcal{N}x_1^{-k+1} \longrightarrow 0 \end{array}$$

and we aim at proving that the right vertical arrow is bijective. For that purpose, we consider the commutative diagram

$$(B.9) \quad \begin{array}{ccc} \mathcal{N}/\mathcal{N}x_1 & \xrightarrow{\partial_{x_1}^k} & \mathrm{gr}_k(\mathcal{N} \otimes_{\mathcal{D}_{X,\log}} \mathcal{D}'_{X,\log}) \\ \parallel & & \downarrow \\ \mathcal{N}/\mathcal{N}x_1 & \xrightarrow{\partial_{x_1}^k} & \mathcal{N}x_1^{-k}/\mathcal{N}x_1^{-k+1}. \end{array}$$

By definition, the upper horizontal morphism is onto. We claim that it is also injective. We will check that the lower horizontal morphism is an isomorphism. This will imply the desired injectivity, hence the bijectivity of the upper horizontal arrow, together with that of the right vertical one.

Composing the lower horizontal morphism with right multiplication by x_1^k :

$$\mathcal{N}x_1^{-k}/\mathcal{N}x_1^{-k+1} \xrightarrow{x_1^k} \mathcal{N}/\mathcal{N}x_1,$$

which is bijective, we find the morphism

$$\partial_{x_1}^k x_1^k : \mathcal{N}/\mathcal{N}x_1 \longrightarrow \mathcal{N}/\mathcal{N}x_1,$$

that we can write as $\prod_{j=1}^k (\mathrm{Eu}_1 + j)$. Property B.7(2) for $I = \{1\}$ gives its bijectivity, as wanted. \square

Lemma B.9. *Let \mathcal{N} be a $\mathcal{D}_{X,\log}$ -module satisfying B.7(1) and (2). Then so does $\mathcal{N}' = \mathcal{N}(*D_1)$ as a $\mathcal{D}'_{X,\log}$ -module.*

Proof. Let I be a subset of $\{2, \dots, \ell\}$ and let I^c be its complement in $\{2, \dots, n\}$, while $I^c = \{1\} \cup I^c$ is its complement in $\{1, \dots, n\}$. We wish to prove that the localization morphism

$$i_I^*(\mathcal{N}(*D_1)) \longrightarrow i_I^*(\mathcal{N}(*D_1))(*D_I^{(I^c)})$$

is injective. Since $x_1 : \mathcal{N}(*D_1) \rightarrow \mathcal{N}(*D_1)$ is bijective, so is $x_1 : i_I^*(\mathcal{N}(*D_1)) \rightarrow i_I^*(\mathcal{N}(*D_1))$, and thus $i_I^*(\mathcal{N}(*D_1)) = (i_I^*\mathcal{N})(*D_1)$. Then, since the localization morphism $i_I^*\mathcal{N} \rightarrow (i_I^*\mathcal{N})(*D_I^{(I^c)})$ is injective by B.7(1) for \mathcal{N} , it remains so after applying the functor $(*D_1)$, which gives the desired injectivity, hence B.7(1) for $\mathcal{N}' = \mathcal{N}(*D_1)$. Similarly, B.7(2) is obtained by applying $(*D_1)$, since Eu_j commutes with x_1 for $j \neq 1$. \square

End of the proof of Proposition B.7. We argue by induction on $\#D$. We write

$$\mathcal{N} \otimes_{\mathcal{D}_{X,\log}} \mathcal{D}_X = \mathcal{N} \otimes_{\mathcal{D}_{X,\log}} \mathcal{D}'_{X,\log} \otimes_{\mathcal{D}'_{X,\log}} \mathcal{D}_X = \mathcal{N}' \otimes_{\mathcal{D}'_{X,\log}} \mathcal{D}_X.$$

Due to Lemma B.9 and the induction hypothesis, we can apply Proposition B.7 to \mathcal{N}' with respect to the divisor D' , and we find

$$\mathcal{N} \otimes_{\mathcal{D}_{X,\log}} \mathcal{D}_X = \mathcal{N}'(*D') = \mathcal{N}(*D). \quad \square$$

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