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# Topological computation of local cohomology multiplicities

R. García López[1]

*Departament d'Àlgebra i Geometria, Universitat de Barcelona, Gran Via, 585*
*E-08007 Barcelona, Spain*

C. Sabbah[2]

*URA 169 du CNRS, Centre de Mathématiques, École Polytechnique*
*F-91128 Palaiseau cedex, France*

*En memoria de Fernando Serrano*

Abstract

We express the Lyubeznik numbers of the local ring of a complex isolated singularity in terms of Betti numbers of the associated real link.

## 1. Introduction

Let $A$ be a Noetherian local ring which contains a field. In [9], G. Lyubeznik introduces a set of numerical invariants for $A$. The purpose of this note is to compute these invariants when $A$ is the local ring of an isolated singularity of a complex space in terms of topological invariants attached to it, more precisely in terms of some of the Betti numbers of the associated real link.

In order to introduce Lyubeznik's numbers, we recall first some definitions: let $R$ be a commutative Noetherian ring, $\mathfrak{p} \subseteq R$ a prime ideal, $M$ a $R$-module and $p \geq 0$ a positive integer. The $p$-th Bass number of a $R$-module $M$ with respect to $\mathfrak{p}$ is the number

$$\mu_p(\mathfrak{p}, M) := \dim_{k(\mathfrak{p})} \operatorname{Ext}^p_R(k(\mathfrak{p}), M_{\mathfrak{p}})$$

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where $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is the residue field at $\mathfrak{p}$ and $M_\mathfrak{p}$ denotes the localization of $M$ at $\mathfrak{p}$. Bass numbers were introduced in [1] and they describe the structure of the minimal injective resolution of $M$. Namely, if $0 \to M \to I^\bullet$ is a minimal injective resolution then the $p$-th term is of the form

$$I^p \cong \oplus_{\mathfrak{p}\in \operatorname{Spec} R}\big(\mathrm{E}_R(R/\mathfrak{p})^{\oplus \mu_p(\mathfrak{p},M)}\big)$$

where $\mathrm{E}_R(R/\mathfrak{p})$ denotes the injective envelope of $R/\mathfrak{p}$ in the category of $R$-modules. We remark that, in general, Bass numbers are not necessarily finite. If $R$ is a Gorenstein ring, the Bass numbers of $R$ describe the structure of the Cousin complex of $R$ (cf. [5, Chapter IV, §2]), since in this case it is a minimal injective resolution. See [13], [14] for further details and results on these topics.

We recall as well that if $I \subseteq R$ is an ideal, $M$ a $R$-module and $j \geq 0$ an integer, then the $j$-th local cohomology module of $M$ with supports on $I$ is the $R$-module

$$H^j_I(M) := \operatorname{indlim}_k \operatorname{Ext}^j_R(R/I^k, M).$$

Now we come to the definition of Lyubeznik's numbers: let $(A, \mathfrak{m}_A)$ be a Noetherian local ring which contains a field. Because of Cohen's theorem there is an epimorphism of rings $\pi : R \to \widehat{A}$ where $R = k[\![x_1, \dots, x_n]\!]$ is a ring of power series with coefficients in a field $k$ and $\widehat{A}$ is the completion of $A$ with respect to the $\mathfrak{m}_A$-adic topology. Denote $\mathfrak{m} \subset R$ the maximal ideal and set $I = \ker \pi$. Following: [9], for given integers $p \geq 0$ and $i \leq n$ put

$$\lambda_{p,i}(A) = \mu_p(\mathfrak{m}, H^{n-i}_I(R)) = \dim_k \operatorname{Ext}^p_R\big(k,\, (H^{n-i}_I(R))_\mathfrak{m}\big).$$

It is proved in [9] (see also [7] for the case $\mathrm{char}(k) = p > 0$) that these numbers are always finite and that they depend only on the ring $A$ and not on the chosen presentation of $\widehat{A}$ as a quotient of a power series ring. According to [9, (1.4) and (3.4a)], the number $\lambda_{p,i}(A)$ is equal to $\dim_k \operatorname{Hom}_R(k,\, H^p_\mathfrak{m}(H^{n-i}_I(R)))$. See [9, §4] for further properties of these invariants and [15] for an algorithm which allows to compute the numbers $\lambda_{p,i}(R/I)$ from a set of generators for the ideal $I$. In this paper we prove the following theorem:

**Theorem**

*Let $V$ denote a complex space with an isolated singularity at $x \in V$ and of pure dimension $d \geq 2$ at $x$. Let $A = \mathcal{O}_{V,x}$ and denote by $H^i_{\{x\}}(V, \mathbf{C})$ the singular cohomology groups of $V$ with complex coefficients and support on $\{x\}$. Then one has:*

*(a)* $\lambda_{0,i}(A) = \dim_\mathbf{C} H^i_{\{x\}}(V, \mathbf{C})$ *for* $1 \leq i \leq d-1$.

*(b)* $\lambda_{p,d}(A) = \dim_\mathbf{C} H^{p+d}_{\{x\}}(V, \mathbf{C})$ *for* $2 \leq p \leq d$.

*(c) All other* $\lambda_{p,i}(A)$ *vanish.*

*If* $d = 1$, *all* $\lambda_{p,i}(A)$ *vanish except* $\lambda_{1,1}(A)$ *which is equal* 1.

*Remark 1.* It follows from the theorem that the $d-1$ numbers $\lambda_{0,i}$ $(1 \leq i \leq d-1)$ determine all the others. Indeed, there is a compact real orientable $(2d-1)$-manifold $K_{(V,x)}$ (the link of the singularity germ $(V,x)$, see e.g. [3, Chapter I, §5]) such that $H^i_{\{x\}}(V,\mathbf{C}) \simeq \widetilde{H}^{i-1}(K_{(V,x)},\mathbf{C})$, where $\widetilde{H}$ denotes reduced cohomology. Then, because of Poincaré duality applied to $K_{(V,x)}$, one has the relations $\lambda_{0,i} = \lambda_{d-i+1,d}$ $(2 \leq i \leq d-1)$, $\lambda_{0,1} + 1 = \lambda_{d,d}$.

*Remark 2.* It is pointed out in [9, p. 54] that statement (a) above follows from a theorem of Ogus ([11, 2.3]) relating local cohomology and algebraic de Rham cohomology and the comparison theorem between algebraic de Rham cohomology and singular cohomology proved by Hartshorne in [6, IV.3.1]. The proof given here replaces these arguments with the use of the theory of $\mathcal{D}$-modules (in particular duality) and the regularity property of the local cohomology (see e.g. [10, Chapter II]). When $d = 2$ the theorem follows also from the results of Walther in [14].

## 2. Proof of the theorem

From now on we fix $k = \mathbf{C}$, $\lambda_{p,i}$ will stand for $\lambda_{p,i}(\mathcal{O}_{V,x})$. For some $n \geq 1$ we will have $\mathcal{O}_{V,x} = \mathbf{C}\{x_1,\ldots,x_n\}/I_c$, where $\mathbf{C}\{x_1,\ldots,x_n\}$ is the ring of germs at $0 \in \mathbf{C}^n$ of convergent power series and $I_c \subseteq \mathbf{C}\{x_1,\ldots,x_n\}$ is an ideal. Set $R_c = \mathbf{C}\{x_1,\ldots,x_n\}$, $\mathfrak{m}_c \subseteq R_c$ its maximal ideal. Setting $R = \mathbf{C}[\![x_1,\ldots,x_n]\!]$, $\mathfrak{m} = (x_1,\ldots,x_n) \subseteq R$ and $I = I_cR$; it follows from the flatness of $R$ over $R_c$ and from [9, Lemma 3.1] that we have:

$$\begin{aligned}\lambda_{p,i} &= \dim_{\mathbf{C}} \operatorname{Hom}_R(\mathbf{C}, H^p_{\mathfrak{m}}(H^{n-i}_I(R))) \\ &= \dim_{\mathbf{C}} \operatorname{Hom}_R(\mathbf{C}, H^p_{\mathfrak{m}_c}(H^{n-i}_{I_c}(R_c)) \otimes_{R_c} R) \\ &= \dim_{\mathbf{C}} \operatorname{Hom}_{R_c}(\mathbf{C}, H^p_{\mathfrak{m}_c}(H^{n-i}_{I_c}(R_c)))\,,\end{aligned}$$

Moreover, we can choose an open neighborhood of $x$ in $V$ (that we will denote by $V$ as well) and an embedding $j : V \hookrightarrow X$ in an open polydisk $X \subset \mathbf{C}^n$ sending $x \in V$ to $0 \in X$ such that if $\mathcal{O}_X$ denotes the ring of analytic functions on $X$ then

$$\lambda_{p,i} = \dim_{\mathbf{C}} \operatorname{Hom}_{\mathcal{O}_X}\left(\mathbf{C}, H^p_{[0]}(H^{n-i}_{[V]}(\mathcal{O}_X))\right).$$

(If $Y \subset X$ is a closed analytic subset $H^*_{[Y]}$ will stand for the algebraic local cohomology with support on $Y$, thus in more algebraic terms $H^*_{[Y]} = H^*_{I_Y}$, where $I_Y \subset \mathcal{O}_X$ is the ideal of analytic functions vanishing on $Y$).

Denote by $\mathcal{D}_X$ the ring of analytic differential operators on $X$ (cf. e.g. [10, Chapter I, §2]). Then, because $H^p_{[0]}(H^{n-i}_{[V]}(\mathcal{O}_X))$ is a coherent left $\mathcal{D}_X$-module supported only at $0 \in X$ one has (see e.g. [loc. cit., Chapter I, (2.3.1)]):

$$H^p_{[0]}(H^{n-i}_{[V]}(\mathcal{O}_X)) = \frac{\mathcal{D}_X}{\mathcal{D}_X \mathfrak{m}} \oplus \cdots \oplus \frac{\mathcal{D}_X}{\mathcal{D}_X \mathfrak{m}}\,,$$

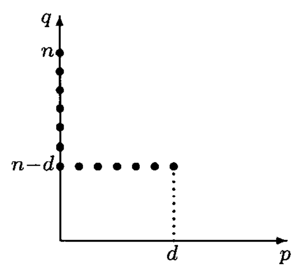
where $\mathcal{D}_X\mathfrak{m} \subset \mathcal{D}_X$ is the left ideal generated by the coordinate functions $x_1, \ldots, x_n$ (this follows also from the results in [9]). It is not difficult to see (e.g. using [9, Proposition 2.3]) that the invariant $\lambda_{p,i}$ is exactly the number of copies of $\mathcal{D}_X/\mathcal{D}_X\mathfrak{m}$ appearing in this decomposition. Therefore, if we denote by $e(\cdot)$ the multiplicity at $0 \in T^*X$ of a $\mathcal{D}_X$-module (see [10, Chapter I, (2.4)]), we have

$$\lambda_{p,i} = e\big(H^p_{[0]}(H^{n-i}_{[V]}(\mathcal{O}_X))\big)\,.$$

Now we come to the proof of the theorem. Consider first the spectral sequence (see e.g. [12, Theorem 11.38])

$$E_2^{p,q} = H^p_{[0]}(H^q_{[V]}(\mathcal{O}_X)) \quad \Rightarrow \quad H^{p+q}_{[0]}(\mathcal{O}_X)\,. \tag{1}$$

Notice that $H^{p+q}_{[0]}(\mathcal{O}_X) = 0$ for $p+q \neq n$ and also $H^q_{[V]}(\mathcal{O}_X) = 0$ if $q < \operatorname{codim}(V, X) = n - d$ or $q > n$. Moreover, since the singularity of $V$ at $x$ is isolated, the modules $H^q_{[V]}(\mathcal{O}_X)$ are supported at 0 for $q \neq n - d$, thus $H^p_{[0]}(H^q_{[V]}(\mathcal{O}_X)) = 0$ if $p > 0$ and $q \neq n - d$. Therefore $\lambda_{p,n-q} = 0$ in this range, and the only possibly non-zero terms in this spectral sequence are those on the thick dots in the picture below:



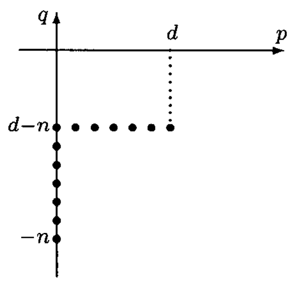
This implies that if $d > 1$ then $\lambda_{0,d} = \lambda_{1,d} = 0$ (actually this is a general fact) and proves part (c) of the theorem.

In order to prove parts (a) and (b) we will use duality for $\mathcal{D}$-modules. We refer to [10] for details. If $M$ is a holonomic left $\mathcal{D}_X$-module we denote $M^* = \mathrm{Hom}_{\mathcal{O}_X}(\omega_X, \mathrm{Ext}^n_{\mathcal{D}_X}(M, \mathcal{D}_X))$ its dual. We denote by $D^b_h(\mathcal{D}_X)$ the derived category of complexes of $\mathcal{D}_X$-modules with bounded and holonomic cohomology. We recall that the duality functor $(\cdot)^*$ defined above extends to a duality functor $(\cdot)^*$ defined on $D^b_h(\mathcal{D}_X)$. For $Y \subseteq X$ a closed analytic subspace we denote $\mathrm{R}\Gamma_{[Y]}(\mathcal{O}_X) \in D^b_h(\mathcal{D}_X)$ the derived functor of $\Gamma_{[Y]}(\cdot)$ applied to $\mathcal{O}_X$ (cf. [10, Chapter I, §6]). Finally, if $M \in D^b_h(\mathcal{D}_X)$ we denote by $\mathcal{H}^s(M)$ its $s$-th cohomology module (so that we have $H^s_{[Y]}(\mathcal{O}_X) = \mathcal{H}^s(\mathrm{R}\Gamma_{[Y]}(\mathcal{O}_X))$).

From the definition of the duality functor it is easy to see that if $M \in D^b_h(\mathcal{D}_X)$ then $\mathcal{H}^s(M)^* \simeq \mathcal{H}^{-s}(M^*)$. It follows that in our situation we have a Grothendieck spectral sequence ([12, Theorem 11.38])

$$E_2^{p,q} = H^p_{[0]}\big((H^{-q}_{[V]}(\mathcal{O}_X))^*\big) \quad \Rightarrow \quad \mathcal{H}^{p+q}\big(\mathrm{R}\Gamma_{[0]}((\mathrm{R}\Gamma_{[V]}(\mathcal{O}_X))^*)\big)\,. \tag{2}$$

Notice that, applying the same considerations as for spectral sequence (1), the only possibly non-zero terms of this sequence are those on the thick dots pictured below:




This implies that this sequence degenerates at the $E_2$-term, thus

$$H^p_{[0]}\big((H^{-q}_{[V]}(\mathcal{O}_X))^*\big) = \mathcal{H}^{p+q}\big(\mathrm{R}\Gamma_{[0]}((\mathrm{R}\Gamma_{[V]}(\mathcal{O}_X))^*)\big)\,. \tag{3}$$

Let DR denote the de Rham functor $\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \cdot)[\dim X]$. Notice that the $\mathcal{D}_X$-module $\mathcal{H}^{p+q}(\mathrm{R}\Gamma_{[0]}((\mathrm{R}\Gamma_{[V]}(\mathcal{O}_X))^*))$ is supported only at zero. Using this fact and (3) above we have:

$$\begin{aligned} e\big(H^p_{[0]}((H^{-q}_{[V]}(\mathcal{O}_X))^*)\big) &= e\big(\mathcal{H}^{p+q}(\mathrm{R}\Gamma_{[0]}((\mathrm{R}\Gamma_{[V]}(\mathcal{O}_X))^*))\big) \\ &= \dim_{\mathbf{C}} \mathcal{H}^{p+q}\mathrm{DR}\big((\mathrm{R}\Gamma_{[0]}((\mathrm{R}\Gamma_{[V]}(\mathcal{O}_X))^*))\big)_0, \end{aligned} \tag{4}$$

where the subscript 0 denotes the fiber at the origin. We want to compute now the de Rham complex of $\mathrm{R}\Gamma_{[0]}((\mathrm{R}\Gamma_{[V]}(\mathcal{O}_X))^*)$. If $Y$ is a locally compact topological space we denote by $\mathbf{C}_Y$ the constant sheaf $\mathbf{C}$ on the space $Y$ (notice that for the polydisk $X$ we have $\mathrm{DR}(\mathcal{O}_X) = \mathbf{C}_X[n]$). We denote by $D^b_c(\mathbf{C}_Y)$ the derived category of the category of complexes of sheaves of complex vector spaces on $Y$ with bounded and constructible cohomology. We denote by $D_Y : D^b_c(\mathbf{C}_Y) \to D^b_c(\mathbf{C}_Y)$ the Verdier duality functor (see e.g. [8, Chapter II and III] for details). If $f : Z \to Y$ is a proper continuous map between locally compact topological spaces we denote by $f_* : D^b_c(\mathbf{C}_Z) \to D^b_c(\mathbf{C}_Y)$ (resp. by $f^{-1} : D^b_c(\mathbf{C}_Y) \to D^b_c(\mathbf{C}_Z)$) the direct (resp. inverse) image functor, and we set $f^! = D_Z \cdot f^{-1} \cdot D_Y$. Finally, if $f : Z \hookrightarrow Y$ is the inclusion of a closed subspace and $\mathcal{F} \in D^b_c(\mathbf{C}_Y)$ we set

$$\mathrm{R}\Gamma_Z(\mathcal{F}) = f_* f^!(\mathcal{F}).$$

We recall that in our situation, $j : V \hookrightarrow X$ denotes the inclusion map. Set $k : \{x\} \hookrightarrow V$ the inclusion, $i = j \cdot k$. From the regularity of $\mathcal{O}_X$ as a $\mathcal{D}_X$-module and the local duality theorem (commutation $D_X \mathrm{DR} = \mathrm{DR} \cdot (\ )^*$, see e.g. [10, Chapter II]), we get

$$\begin{aligned}
&\mathrm{DR}\big(\mathrm{R}\Gamma_{[0]}(\mathrm{R}\Gamma_{[V]}(\mathcal{O}_X))^*\big) \\
&\quad = \mathrm{R}\Gamma_{\{0\}}\mathrm{DR}((\mathrm{R}\Gamma_{[V]}(\mathcal{O}_X))^*) && \text{by regularity of } (\mathrm{R}\Gamma_{[V]}(\mathcal{O}_X))^* \\
&\quad = \mathrm{R}\Gamma_{\{0\}}\, D_X\, \mathrm{DR}(\mathrm{R}\Gamma_{[V]}(\mathcal{O}_X)) && \text{by local duality} \\
&\quad = \mathrm{R}\Gamma_{\{0\}}\, D_X\, \mathrm{R}\Gamma_V(\mathbf{C}_X) && \text{by regularity of } \mathcal{O}_X \\
&\quad = \mathrm{R}\Gamma_{\{0\}} j_*(\mathbf{C}_V[n])
\end{aligned}$$

where the last equality follows from the definition of $\mathrm{R}\Gamma_V$, using that $D_X \cdot j_* = j_* \cdot D_V$ ($j$ is proper) and also that $D_V \cdot D_V = \mathrm{Id}$ (see e.g. [8, Chapter III] for more details). By the definition of $\mathrm{R}\Gamma_{\{0\}}$ and [8, Proposition 3.1.9, (ii)] this complex is equal to $i_* k^! \mathbf{C}_V[n]$ in $D^b_c(\mathbf{C}_X)$. It follows that we have:

$$\begin{aligned}
\mathrm{DR}\big(\mathrm{R}\Gamma_{[0]}(\mathrm{R}\Gamma_{[V]}(\mathcal{O}_X))^*\big)_0 &= \mathrm{R}\Gamma(\{x\}, k^!\, \mathbf{C}_V[n]) \\
&= \mathrm{R}\Gamma(V, k_*\, k^!\, \mathbf{C}_V[n]) \\
&= \mathrm{R}\Gamma(V, \mathrm{R}\Gamma_{\{x\}}(\mathbf{C}_V)[n]) \\
&= \mathrm{R}\Gamma_{\{x\}}(V, \mathbf{C}_V[n])\,.
\end{aligned}$$

Thus, from this computation and (4) above we have

$$e\big(H^p_{[0]}((H^{-q}_{[V]}(\mathcal{O}_X))^*)\big) = \dim_{\mathbf{C}} \mathcal{H}^{p+q}\big(\mathrm{R}\Gamma_{\{x\}}(V, \mathbf{C}_V[n])\big) = \dim_{\mathbf{C}} H^{p+q+n}_{\{x\}}(V, \mathbf{C})\,.$$

If $M$ is an holonomic $\mathcal{D}_X$-module, then one has $e(M) = e(M^*)$. Therefore, if for $i < d$ we put $q = n - i$, we get

$$\begin{aligned} \lambda_{0,i} &= e\big(H^0_{[0]}(H^{n-i}_{[V]}(\mathcal{O}_X))\big) &&= e\big(H^{n-i}_{[V]}(\mathcal{O}_X)\big) \\ &= e\big(H^{n-i}_{[V]}(\mathcal{O}_X)^*\big) &&= e\big(H^0_{[0]}((H^{n-i}_{[V]}(\mathcal{O}_X))^*)\big) \\ &= \dim_{\mathbf{C}} H^i_{\{x\}}(V, \mathbf{C}) \,, \end{aligned}$$

which proves part (a) of the theorem.
In order to prove part (b) we have to show that for $p \geq 2$,

$$e\big(H^p_{[0]}((H^{n-d}_{[V]}(\mathcal{O}_X))^*)\big) = e\big(H^p_{[0]}(H^{n-d}_{[V]}(\mathcal{O}_X))\big) \,.$$

Set $\mathcal{K} = H^0_{[0]}((H^{n-d}_{[V]}(\mathcal{O}_X))^*)$, $\mathcal{L} = (H^{n-d}_{[V]}(\mathcal{O}_X))^*/\mathcal{K}$. On the one hand, since $\mathcal{K}$ is supported only at 0, the long exact sequence obtained by applying $H^*_{[0]}$ to

$$0 \to \mathcal{K} \to \big(H^{n-d}_{[V]}(\mathcal{O}_X)\big)^* \to \mathcal{L} \to 0 \tag{5}$$

gives

$$e\big(H^p_{[0]}((H^{n-d}_{[V]}(\mathcal{O}_X))^*)\big) \quad = \quad e\big(H^p_{[0]}(\mathcal{L})\big) \quad \text{for } p \geq 1 \,. \tag{6}$$

On the other hand, the singularity $(V, x)$ being isolated, the holonomic $\mathcal{D}_X$-module $\mathcal{L}$ is the one introduced in [2, Proposition (8.5) and its proof] if $d > 1$. In particular $\mathcal{L}$ is self-dual (i.e., $\mathcal{L}^* \cong \mathcal{L}$) and $\mathrm{DR}\mathcal{L}$ is the intersection complex $\mathrm{IC}_V$ (cf. [2, Theorem 8.6]).
Thus, dualizing (5) and applying $H^*_{[0]}$ again, we get

$$e\big(H^p_{[0]}(H^{n-d}_{[V]}(\mathcal{O}_X))\big) \quad = \quad e\big(H^p_{[0]}(\mathcal{L})\big) \quad \text{for } p \geq 2 \,.$$

Together with (6), this equality gives the desired result for $d \geq 2$. If $d = 1$ the spectral sequence (1) degenerates already at the $E_2$-term. From this fact follows that all Lyubeznik numbers are zero except $\lambda_{0,0}$ and $\lambda_{1,1}$ which verify $\lambda_{0,0} + \lambda_{1,1} = 1$. But $\lambda_{0,0} = \dim_{\mathbf{C}} H^0_{\{x\}}(V, \mathbf{C}) = 0$ and the claimed result follows. □

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