CHAPTER 9

LOCALIZATION AND MAXIMAL EXTENSION

Summary. We introduce the localization functor along a divisor $D \subset X$. Although it only consists in tensoring with $\mathscr{O}_X(*D)$ in the case of $\mathscr{D}_X$-modules, the definition for modules over $R_F\mathscr{D}_X$ is more subtle. It strongly uses the Kashiwara-Malgrange filtration. This construction can also be made for the dual localization functor, and this leads to the notion of middle extension along D. On the other hand, the maximal extension functor enables one to describe a $\widetilde{\mathscr{D}}_X$-module in terms of the localized object along D and of a $\widetilde{\mathscr{D}}_X$-module supported on D.

In this chapter, we keep the notation and setting as in Chapter 7. In particular, we keep Notation 7.0.1, and Remarks 7.0.2 and 7.0.3 continue to be applied. We continue to treat the case of right $\widetilde{\mathscr{D}}_X$-modules.

***Remark 9.0.1* (The case of left $\widetilde{\mathscr{D}}_X$-modules).** The case of left $\widetilde{\mathscr{D}}_X$-modules is very similar, and the only changes to be made are the following:

- to consider $V^{>-1}$ instead of $V_{<0}$,
- to modify the definition of $\psi_{t,\lambda}$ with a shift,
- to change the definition of can (with a sign).

9.1. Introduction

We consider the following question in this chapter: given a coherent $\widetilde{\mathscr{D}}_X$-module, to classify all coherent $\widetilde{\mathscr{D}}_X$-modules which coincide with it on the complement of a divisor D. This has to be understood in the algebraic sense, i.e., the $\widetilde{\mathscr{D}}_X$-modules coincide after tensoring with the sheaf $\mathscr{O}_X(*D)$ of meromorphic functions with poles along D.

For every $\mathscr{D}_X$-module $\mathcal{M}$ which is $\mathbb{R}$-specializable along D, the localized $\mathscr{D}_X$-module $\mathcal{M}(*D) := \mathscr{O}_X(*D) \otimes_{\mathscr{O}_X} \mathcal{M}$ is $\mathscr{D}_X$-coherent and specializable along D. There is a dual notion, giving rise to $\mathcal{M}(!D)$, and we get natural morphisms

$$\mathcal{M}(!D) \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}(*D).$$

The notion of localization is subtler when taking into account the coherent F-filtration. Indeed, for a coherent graded $R_F\mathscr{D}_X$-module $\mathscr{M}$, we cannot just consider $\mathscr{M}(*D)$, since this would correspond to tensoring each term of the underlying coherent filtration by $\mathscr{O}_X(*D)$, which produces a non-coherent $\mathscr{O}_X$-module. If $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along a *smooth* hypersurface H, one can construct a substitute to the "stupid" localized module $\mathscr{M}(*H)$, that we call the *localized* $\widetilde{\mathscr{D}}_X$*-module*, denoted by $\mathscr{M}[*H]$, and a dual version $\mathscr{M}[!H]$. Both are $\widetilde{\mathscr{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along H, and we have natural morphisms

$$\mathscr{M}[!H] \longrightarrow \mathscr{M} \longrightarrow \mathscr{M}[*H].$$

Due to the possible failure of Kashiwara's equivalence for $R_F\mathscr{D}_X$-modules, the trick of considering the graph inclusion ι_g when $D = (g)$ is not enough to ensure localizability for arbitrary D, so we are forced to considering the possibly smaller category of strictly $\mathbb{R}$-specializable $\widetilde{\mathscr{D}}_X$-modules along D which are *localizable* along D, in order to have well-defined functors $[!D]$ and $[*D]$, and a sequence

$$\mathscr{M}[!D] \longrightarrow \mathscr{M} \longrightarrow \mathscr{M}[*D].$$

The purpose of this chapter reduces to recovering any strictly $\mathbb{R}$-specializable $\widetilde{\mathscr{D}}_X$-module $\mathscr{M}$ from a pair of $\widetilde{\mathscr{D}}_X$-modules and of morphisms between them, one of them being supported on D and the other one being localizable along D. This leads to the construction of the *maximal extension* $\Xi\mathscr{M}$ of $\mathscr{M}$ along D. It can be done when $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along D, at least when $D = H$ is a smooth hypersurface (with multiplicity one). For a general divisor D, we encounter the same problem as for the localization, and the existence of the maximal extension is not guaranteed by the strict specializability condition only. We say that $\mathscr{M}$ is maximalizable along D when this maximal extension exists.

Assume that $D = (g)$. Given a strictly $\mathbb{R}$-specializable, localizable and maximalizable (along D) $\widetilde{\mathscr{D}}_X(*D)$-module $\mathscr{M}_*$, we will construct a functor $\mathrm{G}_{\mathscr{M}_*}$ from the category consisting of triples $(\mathscr{N}, \mathrm{c}, \mathrm{v})$, where $\mathscr{N}$ is strictly $\mathbb{R}$-specializable along D and supported on D, and c, v are morphisms

$$\begin{tikzcd} \psi_{g,1}\mathscr{M}_* \arrow[r, bend left, "\mathrm{c}"] & \mathscr{N} \arrow[l, bend left, "\mathrm{v}", "(-1)"'] \end{tikzcd}$$

to that of strictly $\mathbb{R}$-specializable and localizable $\widetilde{\mathscr{D}}_X$-modules, so that

(a) $\mathrm{G}_{\mathscr{M}_*}(\mathscr{N}, \mathrm{c}, \mathrm{v})(*D) = \mathscr{M}_*$,

(b) the diagram above is isomorphic to the specialization diagram

$$\begin{tikzcd} \psi_{g,1}\mathrm{G}_{\mathscr{M}_*}(\mathscr{N}, \mathrm{c}, \mathrm{v}) \arrow[r, bend left, "\mathrm{can}"] & \phi_{g,1}\mathrm{G}_{\mathscr{M}_*}(\mathscr{N}, \mathrm{c}, \mathrm{v}) \arrow[l, bend left, "\mathrm{var}", "(-1)"'] \end{tikzcd}.$$

This classifies all such $\widetilde{\mathscr{D}}_X$-modules $\mathscr{M}'$ such that $\mathscr{M}'(*D) = \mathscr{M}_*$. A first approximation of this construction was obtained in Exercise 7.3.33.

9.2. Localization of $\mathscr{D}_X$-modules

Let us forget the filtration F in this subsection. Let us start with a smooth divisor, that we denote by H, with ideal $\mathscr{I}_H$. For $\mathscr{D}_X$-modules which are $\mathbb{R}$-specializable along H, the V-filtration enables us to control the localization functor.

Proposition 9.2.1. *Let $\mathcal{M}$ be a right $\mathscr{D}_X$-module. Assume that $\mathcal{M}$ is coherent and $\mathbb{R}$-specializable along H. Then*

(1) *its localization $\mathcal{M}(*H)$ along H is also $\mathscr{D}_X$-coherent and $\mathbb{R}$-specializable along H,*

(2) *the natural morphism $\mathcal{M} \to \mathcal{M}(*H)$ induces an isomorphism*

$$V_{<0}\mathcal{M} \longrightarrow V_{<0}\mathcal{M}(*H),$$

and, if $X \simeq H \times \Delta_t$, its kernel (resp. cokernel) is isomorphic to that of

$$_{\mathrm{D}}\iota_{H*}t : {_{\mathrm{D}}\iota_{H*}}\mathrm{gr}_0^V\mathcal{M} \longrightarrow {_{\mathrm{D}}\iota_{H*}}\mathrm{gr}_{-1}^V\mathcal{M}.$$

(3) *We have $V_0\mathcal{M}(*H) = V_{-1}\mathcal{M} \cdot \mathscr{I}_H^{-1}$ and $\mathcal{M}(*H) = V_0\mathcal{M}(*H) \cdot \mathscr{D}_X$.*

Let s be a new variable. Consider the sheaf $\mathscr{D}_X[s]$ of differential operators with coefficients in $\mathscr{O}_X[s]$ and set $\mathcal{M}(*H)[s] = \mathcal{M}(*H) \otimes_{\mathscr{D}_X} \mathscr{D}_X[s]$. This is a right $\mathscr{D}_X[s]$-module. We will now twist this structure, keeping fixed however the underlying $\mathscr{O}_X(*H)[s]$-structure.

Lemma 9.2.2. *Assume that we have a local decomposition $X \simeq H \times \Delta_t$. Then following rule defines a right $\mathscr{D}_X[s]$-module structure on the $\mathscr{O}_X[s]$-module $\mathcal{M}(*H)[s]$: for every $\ell \in \mathbb{N}$ and any local section m of $\mathcal{M}(*H)$, in local coordinates $(x_2, \dots, x_n, t)$ where $H = \{t = 0\}$,*

$$\begin{aligned} ms^\ell \partial_{x_j} &= m\partial_{x_j} s^\ell, \\ ms^\ell \partial_t &= [m\partial_t - mt^{-1}s]s^\ell. \end{aligned}$$

Proof. Use Exercise A.3.2. □

In this local setting, it will be convenient to denote by $\mathcal{M}(*H)[s]t^s$ the $\mathscr{O}_X(*H)[s]$-module $\mathcal{M}(*H)[s]$ equipped with this twisted structure. That is, we formally write the new action as $t^s \circ \mathscr{D}_X[s] \circ t^{-s}$. Be careful however that "t^s" is nothing but a symbol which enables one to remember, by means of the Leibniz rule, the right $\mathscr{D}_X[s]$ structure.

Exercise 9.2.3 **(Specialization to $s = k$).** Let k be any integer.

(1) Show that $t^k \mathscr{D}_X t^{-k}$ defines a right $\mathscr{D}_X$-structure on the $\mathscr{O}_X(*H)$-module $\mathcal{M}(*H)$, denoted by $\mathcal{M}(*H)t^k$.

(2) Show that $(\mathcal{M}(*H)t^k) \simeq \mathcal{M}(*H)[s]t^s/(s-k)\mathcal{M}(*H)[s]t^s$.

***Exercise 9.2.4* (Bernstein's functional equation).** Let $g : X \to \mathbb{C}$ be a holomorphic function and let $\mathcal{M}$ be a right $\mathscr{D}_X$-module. We denote by $\mathcal{M}(*g)$ its localization $\mathcal{M} \otimes_{\mathscr{O}_X} \mathscr{O}_X(*g)$, and from now on we assume that $\mathcal{M} = \mathcal{M}(*g)$.

(1) Show that the $\mathscr{O}_X(*g)[s]$-module $\mathcal{M}[s] \simeq \mathcal{M}[s] \otimes g^s$ endowed with the right action of ∂_{x_i} defined by

$$(m \otimes g^s)\partial_{x_j} = (m\partial_{x_j} \otimes g^s) - (m\partial_{x_i} \log f \otimes g^s)s$$

is a right $\mathscr{D}_X[s]$-module. [*Hint*: Use Exercise A.3.2.]

(2) Let τ be a new variable and define the right action of τ by the formula

$$(m \otimes g^s)h(x,s)\tau := (mg \otimes g^s)h(x, s+1).$$

Show that $[\tau, s]$ acts as τ, and conclude that, via the identification $s = \tau\partial_\tau$, $\mathcal{M}[s] \otimes g^s$ is naturally endowed with a right action of $\mathscr{D}_X[\tau]\langle\tau\partial_\tau\rangle = V_0\mathscr{D}_X[\tau]\langle\partial_\tau\rangle$.

(3) Let $\iota_g : X \hookrightarrow X \times \mathbb{C}$ denote the graph embedding of g, with coordinate τ on the second factor. Identify as in Example A.8.9 ${}_{\mathrm{D}}\iota_{g*}\mathcal{M}$ with $\iota_{g*}\mathcal{M}[\partial_\tau]$ and let $U_0({}_{\mathrm{D}}\iota_{g*}\mathcal{M})$ denote the $V_0\mathscr{D}_X[\tau]\langle\partial_\tau\rangle$-submodule generated by $\iota_{g*}\mathcal{M}$. Show that $U_0({}_{\mathrm{D}}\iota_{g*}\mathcal{M}) = \iota_{g*}\mathcal{M}[\tau\partial_\tau]$.

(4) Show that $\mathcal{M}[s] \otimes g^s \simeq U_0({}_{\mathrm{D}}\iota_{g*}\mathcal{M})$ as $V_0\mathscr{D}_X[\tau]\langle\partial_\tau\rangle$-modules.

(5) Let m be a local section of $\mathcal{M}(*H)$ and let $b(s) \in \mathbb{C}[s]$. Show that the following conditions are equivalent:

(a) $(m \otimes g^s)b(s) \in (m \otimes g^s)g\mathscr{D}_X[s]$,

(b) $\iota_{g*}m \cdot b(\tau\partial_\tau) \in \iota_{g*}m \cdot V_{-1}\mathscr{D}_X[\tau]\langle\partial_\tau\rangle$.

(6) Assume now, as in Lemma 9.2.2, that $X = H \times \Delta_t$ and set $g(x,t) = t$. Conclude from Exercise 7.3.37(4) that the following conditions are equivalent:

(7) $mb(t\partial_t) \in mV_{-1}(\mathscr{D}_X)$ (V-filtration with respect to t),

(8) $mt^s b(s) \in mt^{s+1}\mathscr{D}_X[s]$.

Proof of Proposition 9.2.1.

(1) Let $\mathcal{M}$ be a coherent $\mathscr{D}_X$-module which is $\mathbb{R}$-specializable along H. Let us first show the coherence of $\mathcal{M}(*H)$. This is a local problem; moreover, by induction on the cardinal of a generators system of $\mathcal{M}$, we can assume that $\mathcal{M}$ is generated by one section $m \in \mathcal{M}$. After Exercise 9.2.4, there exists a nonzero polynomial $b(s) \in \mathbb{C}[s]$ such that $mt^s b(s) \in mt^{s+1}\mathscr{D}_X[s]$.

Let $k_0 \in \mathbb{N}$ be an integer, such that $b(k) \neq 0$ for every $k \geqslant k_0+1$. Then, by specializing to $s = k, \dots, k_0+1$ the previous relation, we find $mt^{-k} \in mt^{-k_0}\mathscr{D}_X$, for $k \geqslant k_0+1$. From the identity $(m\partial_t)t^{-k} = (mt^{-k})\partial_t - kmt^{-k-1}$, we get $\mathcal{M}(*H) = \mathscr{D}_X \cdot mt^{-k_0}$. The filtration $mt^{-k_0} \cdot F_\ell\mathscr{D}_X$ ($\ell \in \mathbb{N}$) is a coherent filtration (see Exercise A.10.3), hence the $\mathscr{D}_X$-module $\mathcal{M}(*H)$ is coherent.

Let m' be a local section of $\mathcal{M}(*H)$. It can be written as $m' = mt^{-k}$ for some local section m of $\mathcal{M}$. As $\mathcal{M}$ is $\mathbb{R}$-specializable along H, there exists a nonzero polynomial $b(s)$ such that $mb(\mathrm{E}) \in mV_{-1}(\mathscr{D}_X)$. From this, we deduce a Bernstein's identity for

$m' \in \mathcal{M}(*H)$:

$$mt^{-k}b(\mathrm{E}+k) \in mt^{-k}V_{-1}(\mathscr{D}_X).$$

Therefore, $\mathcal{M}(*H)$ is $\mathbb{R}$-specializable along H.

(2) Let $T(\mathcal{M}) = \Gamma_{[H]}\mathcal{M}$ be the $\mathscr{D}_X$-submodule in $\mathcal{M}$ of sections supported by H. We have the exact sequence:

$$0 \longrightarrow T(\mathcal{M}) \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}(*H) \longrightarrow C(\mathcal{M}) \longrightarrow 0.$$

The modules $\mathcal{M}$ and $\mathcal{M}(*H)$ are $\mathbb{R}$-specializable along H. It follows from Exercise 7.3.37 (here the strictness property is empty) that the $\mathscr{D}_X$-modules $T(\mathcal{M})$ and $C(\mathcal{M})$ are so. On the other hand, these modules are supported by H, so that $V_{<0}(T(\mathcal{M})) = 0$ and $V_{<0}(C(\mathcal{M})) = 0$ and we deduce from Exercise 7.3.37 the natural isomorphism:

$$V_{<0}(\mathcal{M}) \longrightarrow V_{<0}(\mathcal{M}(*H)).$$

We apply Example 7.3.38 to get the second assertion.

(3) Let us check that the filtration of $\mathcal{M}(*H)$ defined by

$$V_\alpha\mathcal{M}(*H) := \begin{cases} V_\alpha\mathcal{M} & \text{for } \alpha < 0, \\ V_{\alpha-k}\mathcal{M}t^{-k} & \text{for } \alpha \geqslant 0,\ k \in \mathbb{Z} \text{ and } \alpha - k \in [-1, 0) \end{cases}$$

is a coherent $V_\bullet\mathscr{D}_X$-filtration (it is a priori a coherent $V_\bullet\mathscr{D}_X(*H)$-filtration). Let us check for example that $V_1\mathcal{M}(*H) = V_0\mathcal{M}(*H) + V_0\mathcal{M}(*H)\partial_t$, that is, $V_{-1}\mathcal{M}(*H)t^{-2} = V_{-1}\mathcal{M}(*H)t^{-1} + V_{-1}\mathcal{M}(*H)t^{-1}\partial_t$ and equivalently, since t acts in an invertible way on $\mathcal{M}(*H)$, $V_{-1}\mathcal{M} = V_{-1}\mathcal{M}t + V_{-1}\mathcal{M}t^{-1}\partial_t t^2$, which in turn reads

$$V_{-1}\mathcal{M} = V_{-1}\mathcal{M}t + V_{-1}\mathcal{M}(t\partial_t + 2).$$

The inclusion $\supset$ is clear. The inclusion $\subset$ amounts to the surjectivity of $(t\partial_t + 2)$: $\mathrm{gr}^V_{-1}\mathcal{M} \to \mathrm{gr}^V_{-1}\mathcal{M}$, which follows from the property that -2 is not an eigenvalue of E on $\mathrm{gr}^V_{-1}\mathcal{M} = V_{-1}\mathcal{M}/V_{-2}\mathcal{M}$. One shows similarly, for every $k \geqslant 0$, the equality $V_k\mathcal{M} = \sum_{j=0}^k V_0\mathcal{M}\partial_t^j$, hence the last statement of the proposition. $\square$

Let now $g : X \to \mathbb{C}$ be a holomorphic function and set $X_0 = g^{-1}(0)$ and $D = (g)$. We have $\mathscr{O}_X(*D) = \mathscr{O}_X(*X_0)$. The following result is easily obtained.

Corollary 9.2.5 **(Properties of the localization along D).** *Let $\mathcal{M}$ be $\mathscr{D}_X$-coherent and $\mathbb{R}$-specializable along X_0. Set $H = X \times \{0\} \subset X \times \mathbb{C}$.*

(1) *We have*

$$_{\mathrm{D}}\iota_{g*}\big(\mathcal{M}(*D)\big) = ({}_{\mathrm{D}}\iota_{g*}\mathcal{M})(*H).$$

(2) *The $\mathscr{D}_X$-module $\mathcal{M}(*D)$ is strictly $\mathbb{R}$-specializable along D and*

$$\mathrm{var} : \phi_{g,1}\big(\mathcal{M}(*D)\big) \longrightarrow \psi_{g,1}\big(\mathcal{M}(*D)\big)$$

is an isomorphism.

(3) *There is a natural morphism* $\mathcal{M} \to \mathcal{M}(*D)$. *This morphism induces isomorphisms*

$$\psi_{g,\lambda}\mathcal{M} \xrightarrow{\sim} \psi_{g,\lambda}\big(\mathcal{M}(*D)\big)$$

for every λ, *and its kernel (resp. cokernel) is isomorphic to the kernel (resp. cokernel) of* $\mathrm{var}: \psi_{g,1}\mathcal{M} \to \phi_{g,1}\mathcal{M}$. □

9.3. Localization of $\widetilde{\mathscr{D}}_X$-modules

Let us now return to the case of graded $\widetilde{\mathscr{D}}_X = R_F\mathscr{D}_X$-modules.

9.3.a. "Stupid" localization. Let D be an effective divisor in X. The sheaf $\mathscr{O}_X(*D)$ of meromorphic functions on X with arbitrary poles along the support of D at most is a coherent sheaf of ring. So are the sheaves $\mathscr{D}_X(*D) := \mathscr{O}_X(*D) \otimes_{\mathscr{O}_X} \mathscr{D}_X = \mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{O}_X(*D)$, and $\widetilde{\mathscr{O}}_X(*D), \widetilde{\mathscr{D}}(*D)$ defined similarly. Given a coherent $\widetilde{\mathscr{D}}_X$-module $\mathscr{M}$, its "stupid" localization $\mathscr{M}(*D) := \mathscr{M} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{O}}_X(*D)$ is a coherent $\widetilde{\mathscr{D}}_X(*D)$-module.

Assume that D is smooth. We then denote it by H, and we keep the notation of Section 7.2. The $\mathscr{I}_H$-adic filtration of $\widetilde{\mathscr{O}}_X(*H)$ is now indexed by $\mathbb{Z}$, and the corresponding V-filtration (7.2.1) of $\widetilde{\mathscr{D}}_X(*H)$ is nothing but the corresponding $\mathscr{I}_H$-adic filtration. We can then define the notion of a coherent V-filtration for a coherent $\widetilde{\mathscr{D}}_X(*H)$-module, and the notion of strict $\mathbb{R}$-specializability of Definition 7.3.25 can be adapted in the following way: we replace both conditions 7.3.25(2) and (3) by the only condition 7.3.25(2) which should hold for every for every $\alpha \in \mathbb{R}$. By using a local graph embedding, one defines similarly, for every effective divisor D, the notion of strict $\mathbb{R}$-specializability along D.

The following lemma is then mostly obvious.

Lemma 9.3.1. *Let* $\mathscr{M}$ *be a coherent* $\widetilde{\mathscr{D}}_X$*-module which is strictly* $\mathbb{R}$*-specializable along* D. *Then the coherent* $\widetilde{\mathscr{D}}_X(*D)$*-module* $\mathscr{M}(*D)$ *is strictly* $\mathbb{R}$*-specializable along* D. □

Our aim in the next subsections is to define a localization functor with values in the category of strictly $\mathbb{R}$-specializable $\widetilde{\mathscr{D}}_X$-modules along D.

9.3.b. Localization along a smooth hypersurface for $R_F\mathscr{D}_X$-modules

If $\mathscr{M}$ is coherent and strictly $\mathbb{R}$-specializable, we cannot assert that $\mathscr{M}(*H)$ is coherent. However, the natural morphism $V_{<0}\mathscr{M} \to \mathscr{M}(*H)$ is injective since $V_{<0}\mathscr{M}$ has no $\mathscr{I}_H$-torsion. For $\alpha \in [-1, 0)$ and $k \geqslant 1$, let us set

$$V_{\alpha+k}\mathscr{M}(*H) = V_\alpha\mathscr{M}t^{-k} \subset \mathscr{M}(*H),$$

where t is any local reduced equation of H. Each $V_\gamma\mathscr{M}(*H)$ is a coherent $V_0\widetilde{\mathscr{D}}_X$-submodule of $\mathscr{M}(*H)$, which satisfies $V_\gamma\mathscr{M}(*H)t = V_{\gamma-1}\mathscr{M}(*H)$ and $V_\gamma\mathscr{M}(*H)\eth_t \subset$

$V_{\gamma+1}\mathscr{M}(*H)$ (multiply both terms by t). Lastly, each $\mathrm{gr}_\gamma^V\mathscr{M}(*H)$ is strict, being isomorphic to $\mathrm{gr}^V_{\gamma-[\gamma]-1}\mathscr{M}$ if $\gamma\geqslant 0$.

***Definition 9.3.2* (Localization of strictly $\mathbb{R}$-specializable $\widetilde{\mathscr{D}}_X$-modules)**
For a coherent $\widetilde{\mathscr{D}}_X$-module which is strictly $\mathbb{R}$-specializable along H, the localized module is (see 7.3.31(b))

$$\mathscr{M}[*H]=V_0(\mathscr{M}(*H))\cdot\widetilde{\mathscr{D}}_X\subset\mathscr{M}(*H).$$

Remark 9.3.3. The construction of $\mathscr{M}[*H]$ only depends on the $\widetilde{\mathscr{D}}_X(*H)$-module $\mathscr{M}(*H)$, provided it is strictly $\mathbb{R}$-specializable in the sense given in Section 9.3.a. In Proposition 9.3.4 below, we could start from such a module.

***Proposition 9.3.4* (Properties of the localization along H).** *Assume that $\mathscr{M}$ is $\widetilde{\mathscr{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along H. Then we have the following properties.*

(1) *$\mathscr{M}[*H]$ is $\widetilde{\mathscr{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along H.*

(2) *The natural morphism $\mathscr{M}\to\mathscr{M}(*H)$ factorizes through $\mathscr{M}[*H]$, so defines a morphism $\iota^\vee:\mathscr{M}\to\mathscr{M}[*H]$ and induces an isomorphism*

$$V_{<0}\mathscr{M}\longrightarrow V_{<0}\mathscr{M}[*H],$$

and in particular

$$\mathrm{gr}_\gamma^V\iota^\vee:\mathrm{gr}_\gamma^V\mathscr{M}\xrightarrow{\sim}\mathrm{gr}_{-1}^V\mathscr{M}[*H]\quad\text{for any }\gamma\in[-1,0).$$

Moreover, if $X\simeq H\times\Delta_t$, $\operatorname{Ker}\iota^\vee$ (resp. $\operatorname{Coker}\iota^\vee$) is isomorphic to the kernel (resp. cokernel) of ${}_{\mathrm{D}}\iota_{H*}t:{}_{\mathrm{D}}\iota_{H*}\mathrm{gr}_0^V\mathscr{M}\to{}_{\mathrm{D}}\iota_{H*}\mathrm{gr}_{-1}^V\mathscr{M}$.

(3) *For every γ, we have $V_\gamma\mathscr{M}[*H]=V_\gamma\mathscr{M}(*H)\cap\mathscr{M}[*H]$ and, for $\gamma\leqslant 0$, we have $V_\gamma\mathscr{M}[*H]=V_\gamma\mathscr{M}(*H)$.*

(4) *We have, with respect to a local product decomposition $X\simeq H\times\Delta_t$,*

$$V_\gamma\mathscr{M}[*H]=\begin{cases}V_\gamma\mathscr{M} & \text{if }\gamma<0,\\ V_0\mathscr{M}(*H)=V_{-1}\mathscr{M}\cdot t^{-1} & \text{if }\gamma=0,\\ V_{\gamma-[\gamma]-1}\mathscr{M}\partial_t^{[\gamma]+1}+\sum_{j=0}^{[\gamma]}V_0\mathscr{M}(*H)\partial_t^j & \text{in }\mathscr{M}(*H),\text{ if }\gamma>0.\end{cases}$$

(5) *$(\mathscr{M}[*H]/(z-1)\mathscr{M}[*H])=(\mathscr{M}/(z-1)\mathscr{M})(*H)$, and $\mathscr{M}[*H][z^{-1}]=\mathscr{M}(*H)[z^{-1}]$.*

(6) *If t is a local generator of $\mathscr{I}_H$, the multiplication by t induces an isomorphism $\mathrm{gr}_0^V\mathscr{M}[*H]\xrightarrow{\sim}\mathrm{gr}_{-1}^V\mathscr{M}[*H]$.*

(7) *$\mathscr{M}[*H]=V_0(\mathscr{M}(*H))\otimes_{V_0\widetilde{\mathscr{D}}_X}\widetilde{\mathscr{D}}_X$.*

(8) *Assume $\mathscr{M}\to\mathscr{N}$ is a morphism between strictly $\mathbb{R}$-specializable coherent $\widetilde{\mathscr{D}}_X$-modules which induces an isomorphism $\mathscr{M}(*H)\to\mathscr{N}(*H)$ (i.e., whose restriction to $V_{<0}$ is an isomorphism). Assume moreover that $\mathscr{N}$ satisfies* (6), *i.e., the multiplication by t induces an isomorphism $\mathrm{gr}_0^V\mathscr{N}\xrightarrow{\sim}\mathrm{gr}_{-1}^V\mathscr{N}$. Then $\mathscr{N}\simeq\mathscr{M}[*H]$. More precisely, the induced morphism $\mathscr{M}[*H]\to\mathscr{N}[*H]$ is an isomorphism, as well as $\mathscr{N}\to\mathscr{N}[*H]$.*

(9) *Let $\mathscr{M}, \mathscr{N}$ be as in* (8). *Then any morphism $\mathscr{N} \to \mathscr{M}[*H]$ factorizes through $\mathscr{N}[*H]$. In particular, if $\mathscr{N}$ is supported on H, such a morphism is zero.*

(10) *If $\mathscr{M}$ is strict, then so is $\mathscr{M}[*H]$.*

(11) *Let $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ be an exact sequence of coherent strictly $\mathbb{R}$-specializable $\widetilde{\mathscr{D}}_X$-modules. Then the sequence*

$$0 \longrightarrow \mathscr{M}'[*H] \longrightarrow \mathscr{M}[*H] \longrightarrow \mathscr{M}''[*H] \longrightarrow 0$$

is exact.

Proof. The $\widetilde{\mathscr{D}}_X$-coherence of $\mathscr{M}[*H]$ is clear, by definition. Let us set $U_\alpha\mathscr{M}[*H] = V_\alpha(\mathscr{M}(*H)) \cap \mathscr{M}[*H]$ as in (3). Our first goal is to show both that $\mathscr{M}[*H]$ is strictly $\mathbb{R}$-specializable and that $U_\bullet\mathscr{M}[*H]$ is its Kashiwara-Malgrange filtration.

Note that $U_\alpha\mathscr{M}[*H]$ is a coherent $V_0\widetilde{\mathscr{D}}_X$-submodule of $\mathscr{M}[*H]$ (locally, $\mathscr{M}[*H]$ has a coherent V-filtration, which induces on $V_\alpha(\mathscr{M}(*H))$ a filtration by coherent $V_0\widetilde{\mathscr{D}}_X$-submodules, which is thus locally stationary since $V_\alpha(\mathscr{M}(*H))$ is $V_0\widetilde{\mathscr{D}}_X$-coherent). It satisfies in an obvious way the following local properties:

- $U_\alpha\mathscr{M}[*H]t \subset U_{\alpha-1}\mathscr{M}[*H]$,
- $U_\alpha\mathscr{M}[*H]\eth_t \subset U_{\alpha+1}\mathscr{M}[*H]$,
- $\mathrm{gr}^V_\alpha\mathscr{M}[*H] \subset \mathrm{gr}^V_\alpha\mathscr{M}(*H)$ is strict.

Also obvious is that $U_\alpha\mathscr{M}[*H] = V_\alpha\mathscr{M}(*H)$ for $\alpha \leqslant 0$, and thus $U_\alpha\mathscr{M}[*H]t = U_{\alpha-1}\mathscr{M}[*H]$ for such an α. To prove our assertion, we will check that $U_\alpha\mathscr{M}[*H] = U_{<\alpha}\mathscr{M}[*H] + U_{\alpha-1}\mathscr{M}[*H]\eth_t$ for $\alpha > 0$, i.e., $\eth_t : \mathrm{gr}^U_{\alpha-1}\mathscr{M}[*H] \to \mathrm{gr}^U_\alpha\mathscr{M}[*H]$ is onto. We will prove the following assertion, which is enough for our purpose:

For every $\alpha \in [-1, 0)$ and $k \geqslant 1$, if $m := \sum_{j=0}^N m_j\eth_t^j \in V_{\alpha+k}\mathscr{M}(*H)$ with $m_j \in V_0\mathscr{M}(*H)$ $(j = 0, \dots, N)$, then one can re-write m as a similar sum with $N \leqslant k$ and $m_k \in V_\alpha\mathscr{M}(*H)$.

Let us first reduce to $N \leqslant k$. If $N > k$, we have $m_N\eth_t^N \in V_{N-1}\mathscr{M}(*H)$, which is equivalent to $m_N\eth_t^N t^N \in V_{-1}\mathscr{M}(*H)$ by definition. We note that, by strictness, $\eth_t^N t^N$ is injective on $\mathrm{gr}^V_\delta\mathscr{M}(*H)$ for $\delta > -1$. We conclude that $m_N \in V_{-1}\mathscr{M}(*H)$. We can set $m'_{N-1} = m_{N-1} + m_N\eth_t \in V_0\mathscr{M}(*H)$ and decrease N by one. We can thus assume that $N = k$.

If $m_k \in V_\gamma\mathscr{M}(*H)$ with $\gamma > \alpha$, we argue as above that $m_k t^k\eth_t^k \in V_\alpha\mathscr{M}(*H)$, hence $m_k \in V_{<\gamma}\mathscr{M}(*H)$ by the same argument as above, and we finally find $m_k \in V_\alpha\mathscr{M}(*H)$. Now, (1) and (3) are proved, and (2) is then clear (according to Example 7.3.38 for the last statement), as well as (4). Then (5) means that, for $\mathscr{D}_X$-modules, there is no difference between $\mathscr{M}[*H]$ and $\mathscr{M}(*H)$, which is true since $\mathscr{M}(*H)$ is $\mathbb{R}$-specializable, so $\mathscr{D}_X$-generated by $V_0\mathscr{M}(*H)$.

For (6), we note that, by (3), $\mathrm{gr}^V_0\mathscr{M}[*H] = \mathrm{gr}^V_0\mathscr{M}(*H)$ and $\mathrm{gr}^V_{-1}\mathscr{M}[*H] = \mathrm{gr}^V_{-1}\mathscr{M}(*H)$, and by definition $t : \mathrm{gr}^V_0\mathscr{M}(*H) \xrightarrow{\sim} \mathrm{gr}^V_{-1}\mathscr{M}(*H)$ is an isomorphism.

Let us now prove (7). Set $\mathscr{M}' = V_0(\mathscr{M}(*H)) \otimes_{V_0\widetilde{\mathscr{D}}_X} \widetilde{\mathscr{D}}_X$. By definition, we have a natural surjective morphism $\mathscr{M}' \to \mathscr{M}[*H]$ and the composition $V_0(\mathscr{M}(*H)) \to \mathscr{M}' \to \mathscr{M}[*H]$ is injective, where the first morphism is defined by $m \mapsto m \otimes 1$. We thus have $V_0(\mathscr{M}(*H)) \subset \mathscr{M}'$ and we set $V_k\mathscr{M}' = \sum_{j=0}^k V_0\mathscr{M}(*H)\eth_t^j$ for $k \geqslant 0$. Let us check that, for $k \geqslant 1$, $\eth^k : \mathrm{gr}_0^V\mathscr{M}' \to \mathrm{gr}_k^V\mathscr{M}'$ is injective. We have a commutative diagram (here gr_k^V means V_k/V_{k+1})

$$\begin{tikzcd}
\mathrm{gr}_0^V\mathscr{M}' \arrow[r, "\eth_t^k"] \arrow[d, "\wr"'] & \mathrm{gr}_k^V\mathscr{M}' \arrow[d] \\
\mathrm{gr}_0^V\mathscr{M}[*H] \arrow[r, "\eth_t^k", "\sim"'] & \mathrm{gr}_k^V\mathscr{M}[*H]
\end{tikzcd}$$

Therefore, the upper horizontal arrow is injective. Note that it is onto by definition. Therefore, all arrows are isomorphisms, and it follows, by taking the inductive limit on k, that $\mathscr{M}' \to \mathscr{M}[*H]$ is an isomorphism.

For (8) we notice that, since $V_0\mathscr{M}(*H) \xrightarrow{\sim} V_0\mathscr{N}(*H)$ and according to (7), we have $\mathscr{M}[*H] \xrightarrow{\sim} \mathscr{N}[*H]$. Since $\mathscr{N}$ is strictly $\mathbb{R}$-specializable and satisfies (6), we have $\mathscr{N} \subset \mathscr{N}(*H)$ and $V_0\mathscr{N} = V_0\mathscr{N}(*H)$. Still due to the strict $\mathbb{R}$-specializability, $\mathscr{N}$ is generated by $V_0\mathscr{N}$, hence we conclude by Definition 9.3.2.

For (9), we remark that a morphism $\mathscr{N} \to \mathscr{M}[*H]$ induces a morphism $\mathscr{N}(*H) \to \mathscr{M}[*H](*H) = \mathscr{M}(*H)$ and thus $V_0\mathscr{N}(*H) \to V_0\mathscr{M}(*H)$, hence the first assertion follows (7). The second assertion is then clear, since $\mathscr{N}[*H] \subset \mathscr{N}(*H)$.

(10) holds since, if $\mathscr{M}$ is strict, then $\mathscr{M}(*H)$ is also strict, and thus so is $\mathscr{M}[*H]$.

It remains to prove (11). By flatness of $\widetilde{\mathscr{O}}_X(*H)$ over $\widetilde{\mathscr{O}}_X$, the sequence

$$0 \longrightarrow \mathscr{M}'(*H) \longrightarrow \mathscr{M}(*H) \longrightarrow \mathscr{M}''(*H) \longrightarrow 0$$

is exact, and by Exercise 7.3.37(2), the sub-sequence

$$0 \longrightarrow V_{-1}\mathscr{M}' \longrightarrow V_{-1}\mathscr{M} \longrightarrow V_{-1}\mathscr{M}'' \longrightarrow 0$$

is also exact. It follows that the sequence

$$0 \longrightarrow V_0\mathscr{M}'(*H) \longrightarrow V_0\mathscr{M}(*H) \longrightarrow V_0\mathscr{M}''(*H) \longrightarrow 0$$

is exact. By (7) we conclude that the sequence

$$\mathscr{M}'[*H] \longrightarrow \mathscr{M}[*H] \longrightarrow \mathscr{M}''[*H] \longrightarrow 0$$

is exact. Since $\mathscr{M}[*H] \subset \mathscr{M}(*H)$, the injectivity of $\mathscr{M}'[*H] \to \mathscr{M}[*H]$ is clear. $\square$

Remark 9.3.5. In the local setting $X = H \times \Delta_t$, if $t : \mathrm{gr}_0^V\mathscr{M} \to \mathrm{gr}_{-1}^V\mathscr{M}$ is injective, then $\iota^\vee : \mathscr{M} \to \mathscr{M}[*H]$ is injective. Indeed, the assumption implies that the t-torsion of $\mathscr{M}$ is zero, hence $\mathscr{M} \to \mathscr{M}(*H)$ is injective (see Proposition 7.7.2(1)).

9.3.c. Localization along a principal divisor

Let $g : X \to \widetilde{\mathbb{C}}$ be a holomorphic function. Let $\mathscr{M}$ be a coherent $\widetilde{\mathscr{D}}_X$-module which is strictly $\mathbb{R}$-specializable along (g). We say that $\mathscr{M}$ is *localizable along* (g) if there exists a coherent $\widetilde{\mathscr{D}}_X$-module $\mathscr{N}$ such that $({}_{\mathrm{D}}\iota_{g*}\mathscr{M})[*H] = {}_{\mathrm{D}}\iota_*\mathscr{N}$. Recall indeed that Kashiwara's equivalence is not strong enough in the filtered case in order to ensure the existence of $\mathscr{N}$. Nevertheless, by full faithfulness, if $\mathscr{N}$ exists, it is unique, and we denote it by $\mathscr{M}[*g]$. At this point, some checks are in order.

- Assume that g is smooth. Then one can check that $\mathscr{M}[*g]$ as defined by 9.3.2 satisfies the defining property above, so there is no discrepancy between Definition 9.3.2 and the definition above.
- By uniqueness, the local existence of $\mathscr{M}[*g]$ implies its global existence.
- Let u be an invertible holomorphic function on X. We denote by $\varphi_u : X \times \mathbb{C} \to X \times \mathbb{C}$ the isomorphism defined by $(x,t) \mapsto (x, u(x)t)$, so that $\iota_{ug} = \varphi_u \circ \iota_g$. We continue to set $H = X \times \{0\}$, so that φ_u induces the identity on H.

Let $\mathscr{M}$ be a coherent $\widetilde{\mathscr{D}}_X$-module which is strictly $\mathbb{R}$-specializable along (g). If $\mathscr{M}$ is localizable along (g), then it is so along (ug) and we have $\mathscr{M}[*g] = \mathscr{M}[*ug]$. Indeed, one checks that

$$ {}_{\mathrm{D}}\varphi_{u*}\big(({}_{\mathrm{D}}\iota_{g*}\mathscr{M})[*H]\big) = ({}_{\mathrm{D}}\iota_{ug*}\mathscr{M})[*H], $$

and this implies $({}_{\mathrm{D}}\iota_{ug*}\mathscr{M})[*H] = {}_{\mathrm{D}}\iota_{ug*}(\mathscr{M}[*g])$, hence the assertion by uniqueness.

This enables us to define $\mathscr{M}[*D]$ when $\mathscr{M}$ is a coherent $\widetilde{\mathscr{D}}_X$-module which is strictly $\mathbb{R}$-specializable along the support of D and such that $\mathscr{M}[*g]$ exists locally for some (or any) local equation g defining the divisor D. We then say that $\mathscr{M}$ is localizable along D.

***Corollary 9.3.6* (Properties of the localization along g).** *Let $g : X \to \mathbb{C}$ be a holomorphic function and let $\mathscr{M}$ be $\widetilde{\mathscr{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along (g). Set $H = X \times \{0\} \subset X \times \mathbb{C}$. Assume moreover that $\mathscr{M}$ is localizable along (g).*

(1) *The $\widetilde{\mathscr{D}}_X$-module $\mathscr{M}[*g]$ is strictly $\mathbb{R}$-specializable along (g) and*

$$ \mathrm{var} : \phi_{g,1}\big(\mathscr{M}[*g]\big) \longrightarrow \psi_{g,1}\big(\mathscr{M}[*g]\big)(-1) $$

is an isomorphism.

(2) *There is a natural morphism $\iota^\vee : \mathscr{M} \to \mathscr{M}[*g]$. This morphism induces an isomorphism*

$$ \mathscr{M}(*g) \xrightarrow{\ \sim\ } \big(\mathscr{M}[*g]\big)(*g), $$

and isomorphisms

$$ \psi_{g,\lambda}\mathscr{M} \xrightarrow{\ \sim\ } \psi_{g,\lambda}\big(\mathscr{M}[*g]\big) \quad \textit{for every } \lambda. $$

Moreover, we have a commutative diagram

$$ \begin{tikzcd} \phi_{g,1}\mathscr{M} \arrow[r, "\phi_{g,1}\iota^\vee"] \arrow[d, "\mathrm{var}_{\mathscr{M}}"'] & \phi_{g,1}\big(\mathscr{M}[*g]\big) \arrow[d, "\wr"', "\mathrm{var}_{\mathscr{M}[*g]}"] \\ \psi_{g,1}\mathscr{M}(-1) \arrow[r, "\psi_{g,1}\iota^\vee", "\sim"'] & \psi_{g,1}\big(\mathscr{M}[*g]\big)(-1) \end{tikzcd} $$

and $\operatorname{Ker}\iota^\vee$ *(resp.* $\operatorname{Coker}\iota^\vee$*) is identified with* $\operatorname{Ker}\mathrm{var}_{\mathscr{M}}$ *(resp.* $\operatorname{Coker}\mathrm{var}_{\mathscr{M}}$*).*

(3) *Given a short exact sequence of coherent $\widetilde{\mathscr{D}}_X$-modules which are strictly $\mathbb{R}$-specializable and localizable along (g), the $[*g]$ sequence is exact.*

Proof. This follows from Proposition 9.3.4 by using full faithfulness of ${}_{\mathrm{D}}\iota_{g*}$ (Proposition 7.6.2) and Proposition 7.6.6. □

Remark 9.3.7. The proof gives in particular that ${}_{\mathrm{D}}\iota_{g*}\iota_g^\vee = \iota_t^\vee$.

Remark 9.3.8 **(Remark 9.3.3 continued).** One checks easily that ${}_{\mathrm{D}}\iota_{g*}(\mathscr{M}(*g)) = ({}_{\mathrm{D}}\iota_{g*}\mathscr{M})(*H)$, so that, in Corollary 9.3.6, we could start from a coherent $\widetilde{\mathscr{D}}_X(*g)$-module $\mathscr{M}_*$ which is strictly $\mathbb{R}$-specializable. One deduces that the construction $\mathscr{M}[*g]$ only depends on the stupidly localized module $\mathscr{M}_*$. Similarly, for an effective divisor D, $\mathscr{M}[*D]$ (when it exists) only depends on $\mathscr{M}(*D)$.

Remark 9.3.9 **(Restriction to $z=1$).** Assume that $\mathscr{M}$ is $\widetilde{\mathscr{D}}_X$-coherent and is strictly $\mathbb{R}$-specializable and locaizable along (g). Then

$$({}_{\mathrm{D}}\iota_{g*}\mathscr{M})(*H)/(z-1)({}_{\mathrm{D}}\iota_{g*}\mathscr{M})(*H) = ({}_{\mathrm{D}}\iota_{g*}\mathcal{M})(*H),$$

the same holds for V_0, and thus $({}_{\mathrm{D}}\iota_{g*}\mathscr{M})[*H]/(z-1)({}_{\mathrm{D}}\iota_{g*}\mathscr{M})[*H] = ({}_{\mathrm{D}}\iota_{g*}\mathcal{M})(*H)$. As a consequence,

$$\mathscr{M}[*g]/(z-1)\mathscr{M}[*g] = \mathcal{M}(*g).$$

9.4. Dual localization

In this section, we treat simultaneously the case of $\mathscr{D}_X$-modules and that of $R_F\mathscr{D}_X$-modules. The Kashiwara-Malgrange filtration enables one to give a comprehensive definition of the dual localization functor, which should be thought of as the adjoint of the localization functor by the $\widetilde{\mathscr{D}}_X$-module duality functor. We will give a more direct definition and we will not need the duality functor.

9.4.a. Dual localization along a smooth hypersurface

Definition 9.4.1 **(Dual localization along a smooth hypersurface)**

Let $H \subset X$ be a smooth hypersurface and let $\mathscr{M}$ be $\widetilde{\mathscr{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along H. The dual localization of $\mathscr{M}$ along H is defined as

$$\mathscr{M}[!H] := V_{<0}\mathscr{M} \otimes_{V_0\widetilde{\mathscr{D}}_X} \widetilde{\mathscr{D}}_X.$$

Proposition 9.4.2 **(Properties of the dual localization along H)**

Assume that $\mathscr{M}$ is $\widetilde{\mathscr{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along H. Then the following properties hold.

(1) *$\mathscr{M}[!H]$ is $\widetilde{\mathscr{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along H.*

(2) *The natural morphism $\iota : \mathscr{M}[!H] \to \mathscr{M}$ induces an isomorphism*

$$V_{<0}\mathscr{M}[!H] \xrightarrow{\sim} V_{<0}\mathscr{M},$$

and in particular

$$\mathrm{gr}^V_{-1}\iota : \mathrm{gr}^V_{-1}\mathscr{M}[!H] \xrightarrow{\sim} \mathrm{gr}^V_{-1}\mathscr{M}.$$

(3) *With respect to a local decomposition* $X \simeq H \times \Delta_t$,

$$\eth_t : \mathrm{gr}_{-1}^V \mathscr{M}[!H] \longrightarrow \mathrm{gr}_0^V \mathscr{M}[!H](-1)$$

is an isomorphism, and $\mathrm{Ker}\,\mathrm{gr}_{-1}^V \iota$ *(resp.* $\mathrm{Coker}\,\mathrm{gr}_{-1}^V \iota$*) is isomorphic to the kernel (resp. cokernel) of* $\eth_t : \mathrm{gr}_{-1}^V \mathscr{M} \to \mathrm{gr}_0^V \mathscr{M}$.

(4) *Assume* $\mathscr{N} \to \mathscr{M}$ *is a morphism between strictly* $\mathbb{R}$*-specializable coherent* $\widetilde{\mathscr{D}}_X$*-modules which induces an isomorphism* $\mathscr{N}(*H) \to \mathscr{M}(*H)$ *(i.e., whose restriction to* $V_{<0}$ *is an isomorphism). Assume moreover that* $\mathscr{N}$ *satisfies* (3)*, i.e., the action of* $\eth_t$ *induces an isomorphism* $\mathrm{gr}_{-1}^V \mathscr{N} \xrightarrow{\sim} \mathrm{gr}_0^V \mathscr{N}(-1)$. *Then* $\mathscr{N} \simeq \mathscr{M}[!H]$. *More precisely, the induced morphism* $\mathscr{N}[!H] \to \mathscr{M}[!H]$ *is an isomorphism, as well as* $\mathscr{N}[!H] \to \mathscr{N}$.

(5) *Let* $\mathscr{M}, \mathscr{N}$ *be as in* (4). *Then any morphism* $\mathscr{M}[!H] \to \mathscr{N}$ *factorizes through* $\mathscr{N}[!H]$. *In particular, if* $\mathscr{N}$ *is supported on* H*, such a morphism is zero.*

(6) *If* $\mathscr{M}$ *is strict, then so is* $\mathscr{M}[!H]$.

(7) *Let* $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ *be an exact sequence of coherent strictly* $\mathbb{R}$*-specializable* $\widetilde{\mathscr{D}}_X$*-modules. Then the sequence*

$$0 \longrightarrow \mathscr{M}'[!H] \longrightarrow \mathscr{M}[!H] \longrightarrow \mathscr{M}''[!H] \longrightarrow 0$$

is exact.

Proof. We first construct locally a $\widetilde{\mathscr{D}}_X$-module $\mathscr{M}_!$ which satisfies all properties described in Proposition 9.4.2, and we then identify it with the globally defined $\widetilde{\mathscr{D}}_X$-module $\mathscr{M}[!H]$. The question is therefore local on X and we can assume that $X \simeq H \times \Delta_t$. We will use the notation and results of Exercise 7.3.36.

Step one. We search for $\mathscr{M}_!$ with a morphism $\mathscr{M}_! \to \mathscr{M}$ inducing an isomorphism $V_{<0}\mathscr{M}_! \to V_{<0}\mathscr{M}$, hence $\psi_{t,\lambda}\mathscr{M}_! \xrightarrow{\sim} \psi_{t,\lambda}\mathscr{M}$ for every $\lambda \in S^1$, and such that $\phi_{t,1}\mathscr{M}_!$ is naturally identified with the graph of $\mathrm{can}_{\mathscr{M}} : \psi_{t,1}\mathscr{M} \to \phi_{t,1}\mathscr{M}$, hence to $\psi_{t,1}\mathscr{M}$, so that $\psi_{t,1}\mathscr{M}_! \to \psi_{t,1}\mathscr{M}$ is the identity, while $\phi_{t,1}\mathscr{M}_! \to \psi_{t,1}\mathscr{M}$ is induced by the second projection $\psi_{t,1}\mathscr{M} \oplus \phi_{t,1}\mathscr{M} \to \phi_{t,1}\mathscr{M}$, hence can be identified with $\mathrm{can}_{\mathscr{M}}$.

We use the identification of Exercise 7.3.36(5) of $\mathscr{M}/V_{-1}\mathscr{M}$ with $\bigoplus_{\alpha\in(-1,0]} \mathrm{gr}_\alpha^V \mathscr{M}[s]$. On the other hand, we introduce a similar $V_0\widetilde{\mathscr{D}}_X$-module structure on $\mathrm{gr}_{-1}^V \mathscr{M}[s]$ by setting

$$\begin{aligned} \mu_{-1}^{(j)} s^j \cdot t &= \begin{cases} 0 & \text{if } j = 0, \\ \big(\mu_{-1}^{(j)}(\mathrm{E} + (j-1)z)\big) s^{j-1} & \text{if } j \geqslant 1, \end{cases} \\ (\mu_{-1}^{(j)} s^j) t\eth_t &= \big(\mu_{-1}^{(j)}(\mathrm{E} + (j-1)z)\big) s^j. \end{aligned}$$

One checks similarly that this is indeed a $V_0\widetilde{\mathscr{D}}_X$-module structure (i.e., $[t\eth_t, t]$ acts as zt), but the action of $\eth_t$, defined as the multiplication by s, does not extend this structure as a $\widetilde{\mathscr{D}}_X$-module structure (see Exercise 7.3.36(6)). We then notice that the morphism

$$\begin{aligned}\rho : \mathrm{gr}^V_{-1}\mathscr{M}[s] &\longrightarrow \mathrm{gr}^V_0\mathscr{M}[s] \subset \mathscr{M}/V_{-1}\mathscr{M}\\ \mu^{(j)}_{-1}s^j &\longmapsto (\mu^{(j)}_{-1}\eth_t)s^j\end{aligned}$$

is $V_0\widetilde{\mathscr{D}}_X$-linear.

Exercise 9.4.3. Show however that the action of $\eth_t$ induces a $\widetilde{\mathscr{D}}_X$-module structure on $\operatorname{Ker}\rho$ and on $\operatorname{Coker}\rho$, and identify these $\widetilde{\mathscr{D}}_X$-modules with $\operatorname{Ker}\mathrm{can}_{\mathscr{M}}$ and $\operatorname{Coker}\mathrm{can}_{\mathscr{M}}$ respectively. [*Hint*: Argue as in Example 7.3.38.]

Given a local section m of $\mathscr{M}$, we denote by $[m]$ its class in $\mathscr{M}/V_{-1}\mathscr{M} = \bigoplus_{\alpha\in(-1,0]}\mathrm{gr}_\alpha\mathscr{M}[s]$, and by $[m]_0 = \sum_{j\geqslant 0}[m]^{(j)}_0 s^j$ the component of this class in $\mathrm{gr}^V_0\mathscr{M}[s]$. Let us consider the $V_0\widetilde{\mathscr{D}}_X$-submodule $\mathscr{M}_! \subset \mathscr{M}\oplus\mathrm{gr}^V_{-1}\mathscr{M}[s]$ consisting of pairs (m,μ_{-1}) of local sections such that $[m]_0 = \rho(\mu_{-1})$ (since the maps ρ and $m\mapsto[m]_0$ are $V_0\widetilde{\mathscr{D}}_X$-linear, $\mathscr{M}_!$ is indeed a $V_0\widetilde{\mathscr{D}}_X$-submodule). We will extend the $V_0\widetilde{\mathscr{D}}_X$-module structure on $\mathscr{M}_!$ to a $\widetilde{\mathscr{D}}_X$-module structure so that the natural morphism $\mathscr{M}_!\to\mathscr{M}$ induced by the first projection is $\widetilde{\mathscr{D}}_X$-linear.

We have a decomposition $\mathscr{M}/V_{<-1}\mathscr{M}\simeq\mathrm{gr}^V_{-1}\mathscr{M}\oplus\bigoplus_{\alpha\in(-1,0]}\mathrm{gr}^V_\alpha\mathscr{M}[s]$ and, for a local section m of $\mathscr{M}$, we can write

$$[m\eth_t]_0 = \mathrm{can}_{\mathscr{M}}[m]^{(0)}_{-1} + \sum_{j\geqslant 1}[m]^{(j-1)}_0 s^j = \mathrm{can}_{\mathscr{M}}[m]^{(0)}_{-1} + [m]_0 s,$$

where $[m]^{(0)}_{-1}$ obviously denotes the component of $m \bmod V_{<-1}\mathscr{M}$ in $\mathrm{gr}^V_{-1}\mathscr{M}$. For any local section (m,μ_{-1}) of $\mathscr{M}_!$ we define

$$(m,\mu_{-1})\eth_t := (m\eth_t, [m]^{(0)}_{-1} + \mu_{-1}s).$$

The right-hand term is easily checked to belong to $\mathscr{M}_!$. We now check that $(m,\mu_{-1})[\eth_t,t] = z(m,\mu_{-1})$. On the one hand, we have

$$(m,\mu_{-1})\eth_t t = \big(m\eth_t t, ([m]^{(0)}_{-1}+\mu_{-1}s)t\big) = \Big(m\eth_t t, \textstyle\sum_{j\geqslant 0}(\mathrm{N}+jz)\mu^{(j)}_{-1}s^j\Big) = (m\eth_t t, \mu_{-1}\eth_t t),$$

and, on the other hand,

$$\begin{aligned}(m,\mu_{-1})t\eth_t &= \Big(mt, \textstyle\sum_{j\geqslant 1}(\mathrm{N}+(j-1)z)\mu^{(j)}_{-1}s^{j-1}\Big)\eth_t\\ &= \Big(mt\eth_t, [mt]^{(0)}_{-1} + \textstyle\sum_{j\geqslant 1}(\mathrm{N}+(j-1)z)\mu^{(j)}_{-1}s^j\Big).\end{aligned}$$

Moreover, we have $[mt]^{(0)}_{-1} = \mathrm{var}_{\mathscr{M}}[m]^{(0)}_0 = \mathrm{var}_{\mathscr{M}}(\mathrm{can}_{\mathscr{M}}\,\mu^{(0)}_{-1}) = \mathrm{N}\mu^{(0)}_{-1}$. As a consequence,

$$(m,\mu_{-1})[\eth_t,t] = (zm, z\mu_{-1} + \mathrm{var}_{\mathscr{M}}[m]^{(0)}_0 - \mathrm{N}\mu^{(0)}_{-1}) = z(m,\mu_{-1}).$$

Since $\mathscr{M}$ is $\widetilde{\mathscr{D}}_X$-coherent and $\mathrm{gr}^V_{-1}\mathscr{M}$ is $\widetilde{\mathscr{D}}_H$-coherent, one concludes easily that $\mathscr{M}_!$ is $\widetilde{\mathscr{D}}_X$-coherent.

We set

$$V_\alpha\big(\mathscr{M}\oplus\mathrm{gr}_{-1}^V\mathscr{M}[s]\big):=V_\alpha\mathscr{M}\oplus\bigoplus_{j=0}^{[\alpha]}\mathrm{gr}_{-1}^V\mathscr{M}s^j.$$

The induced filtration $V_\alpha\mathscr{M}_!:=\mathscr{M}_!\cap V_\alpha\big(\mathscr{M}\oplus\mathrm{gr}_{-1}^V\mathscr{M}[s]\big)$ satisfies $V_\alpha\mathscr{M}_!\xrightarrow{\sim}V_\alpha\mathscr{M}$ for $\alpha<0$ and

$$\mathrm{gr}_\alpha^V\mathscr{M}_!=\begin{cases}\mathrm{gr}_\alpha^V\mathscr{M} & \text{if } \alpha\notin\mathbb{N},\\ \{([m]_0^{(j)},\mu_{-1}^{(j)})\in\mathrm{gr}_0^V\mathscr{M}\oplus\mathrm{gr}_{-1}^V\mathscr{M}\mid [m]_0^{(j)}=\mathrm{can}_{\mathscr{M}}\,\mu_{-1}^{(j)}\}\cdot s^j & \text{if } \alpha=j\\ \qquad\simeq\mathrm{gr}_{-1}^V\mathscr{M}s^j.\end{cases}$$

It is clear that this is a coherent V-filtration and that $\mathscr{M}_!$ satisfies 9.4.2(1)–(3).

Identification between $\mathscr{M}[!H]$ and $\mathscr{M}_!$. Since $V_{<0}\mathscr{M}_!\xrightarrow{\sim}V_{<0}\mathscr{M}$, the natural morphism $\mathscr{M}_![!H]\to\mathscr{M}[!H]$ is an isomorphism, and we will prove that the natural morphism

$$\mathscr{M}_![!H]=V_{<0}\mathscr{M}_!\otimes_{V_0\widetilde{\mathscr{D}}_X}\widetilde{\mathscr{D}}_X\longrightarrow\mathscr{M}_! \tag{9.4.4}$$

is an isomorphism. For any coherent $\widetilde{\mathscr{D}}_X$-module $\mathscr{N}$ which is strictly $\mathbb{R}$-specializable along H, the natural morphism $V_0\mathscr{N}\otimes_{V_0\widetilde{\mathscr{D}}_X}\widetilde{\mathscr{D}}_X\to\mathscr{N}$ is onto, and if $\mathrm{can}_{\mathscr{N}}$ is onto, then $V_{<0}\mathscr{N}\otimes_{V_0\widetilde{\mathscr{D}}_X}\widetilde{\mathscr{D}}_X\to\mathscr{N}$ is also onto. Since $\mathrm{can}_{\mathscr{M}_!}$ is an isomorphism, (9.4.4) is onto.

The composition $V_{<0}\mathscr{M}_!\simeq V_{<0}\mathscr{M}_![!H]\to\mathscr{M}_![!H]\to\mathscr{M}_!$, so (9.4.4) is injective when restricted to the $V_{<0}$ part. We V-filter $\mathscr{M}_![!H]$ by setting $U_{<k}\mathscr{M}_![!H]=\sum_{j\leqslant k}V_{<0}\mathscr{M}_!\eth_t^j$, so that $U_{<0}\mathscr{M}_![!H]=V_{<0}\mathscr{M}_!$. For $k\geqslant1$ we have a commutative diagram

$$\begin{tikzcd}(U_{<0}/U_{<-1})\mathscr{M}_![!H] \arrow[r, "\sim"] \arrow[d, "\eth_t^k"'] & (V_{<0}/V_{<-1})\mathscr{M}_! \arrow[d, "\eth_t^k", "\wr"'] \\ (U_{<k}/U_{<k-1})\mathscr{M}_![!H] \arrow[r] & (V_{<k}/V_{<k-1})\mathscr{M}_!\end{tikzcd}$$

The left down arrow is onto by definition, and since the right down arrow is an isomorphism by the properties of $\mathscr{M}_!$, the left down arrow is also an isomorphism, as well as the lower horizontal arrow, showing by induction on k that $\mathscr{M}_![!H]\to\mathscr{M}_!$ is an isomorphism, so $\mathscr{M}_![!H]=\mathscr{M}[!H]$ satisfies 9.4.2(1)–(3).

We now prove (4). Since $V_{<0}\mathscr{N}\xrightarrow{\sim}V_{<0}\mathscr{M}$, Definition 9.4.1 implies $\mathscr{N}[!H]\xrightarrow{\sim}\mathscr{M}[!H]$. It remains to check that $\mathscr{N}[!H]\to\mathscr{N}$ is an isomorphism. Since the question is local, it is enough to check that the morphism $\mathscr{N}_!\to\mathscr{N}$ is an isomorphism, which is straightforward from the construction of $\mathscr{N}_!$, with the assumption that $\mathrm{can}_{\mathscr{N}}$ is an isomorphism.

For (5), we remark that the morphism $\mathscr{M}[!H]\to\mathscr{N}$ restricts to a morphism $V_{<0}\mathscr{M}[!H]=V_{<0}\mathscr{M}\to V_{<0}\mathscr{N}$, so the first assertion follows from Definition 9.4.1. The second one is then obvious since $V_{<0}\mathscr{N}=0$ if $\mathscr{N}$ is supported on H.

Let us now check (6), that is, the strictness of $\mathscr{M}[!H]$. One check it locally for $\mathscr{M}_!$, for which it is clear since $\mathscr{M}_!\subset\mathscr{M}\oplus\mathrm{gr}_{-1}^V\mathscr{M}[s]$.

It remains to prove (7). The argument is the same as for 9.3.4(11) except for the injectivity of $\mathscr{M}'[!H] \to \mathscr{M}[!H]$. In order to prove this property, we notice that $V_{<0}\mathscr{M}'[!H] \to V_{<0}\mathscr{M}[!H]$ is injective, according to (2). It is then enough to check the injectivity of $\mathrm{gr}_\alpha^V \mathscr{M}'[!H] \to \mathrm{gr}_\alpha^V \mathscr{M}[!H]$ for every $\alpha \geqslant 0$. Due to the strict $\mathbb{R}$-specializability of $\mathscr{M}'[!H], \mathscr{M}[!H]$, injectivity holds for every $\alpha \notin \mathbb{Z}$ since $\mathrm{gr}_\alpha^V \mathscr{M}' \to \mathrm{gr}_\alpha^V \mathscr{M}$ is injective. Similarly, if α is a nonnegative integer, the injectivity at α holds if and only if it holds at $\alpha = 0$. Now, (3) reduces this check to the case $\alpha = -1$, where the injectivity holds since $\mathrm{gr}_{-1}^V \mathscr{M}' \to \mathrm{gr}_{-1}^V \mathscr{M}$ is injective. □

Remark 9.4.5 **(The case of $\mathscr{D}_X$-modules).** In case of $\mathbb{R}$-specializable $\mathscr{D}_X$-modules, we simply denote $\mathcal{M}[!H]$ by $\mathcal{M}(!H)$ for the symmetry with the notation in Section 9.2.

Remark 9.4.6 **(Remark 9.3.3 continued).** Clearly, $\mathscr{M}[!H]$ only depends on $\mathscr{M}(*H)$, so that, in Proposition 9.4.2, we could start from a coherent $\widetilde{\mathscr{D}}_X(*H)$-module $\mathscr{M}$ which is strictly $\mathbb{R}$-specializable.

Remark 9.4.7 **(Uniqueness of the morphism ι).** Let $\iota' : \mathscr{M}[!H] \to \mathscr{M}$ be a morphism whose stupid localization $\iota'_{(*H)} : \mathscr{M}[!H](*H) \to \mathscr{M}(*H)$ coincides with the stupid localization $\iota_{(*H)}$ of ι. Then $\iota' = \iota$. Indeed, the assumption implies that ι' coincides with ι when restricted to $V_{<0}\mathscr{M}[!H] = V_{<0}\mathscr{M}$. Both induce then the same morphism $\mathscr{M}[!H] = V_{<0}\mathscr{M} \otimes_{V_0\widetilde{\mathscr{D}}_X} \widetilde{\mathscr{D}}_X \to \mathscr{M}$.

Remark 9.4.8. In the local setting $X = H \times \Delta_t$, if $\eth_t : \mathrm{gr}_{-1}^V \mathscr{M} \to \mathrm{gr}_0^V \mathscr{M}$ is onto, then $\iota : \mathscr{M}[!H] \to \mathscr{M}$ is onto. Indeed, the assumption implies that $\mathscr{M} = V_{<0}\mathscr{M} \cdot \widetilde{\mathscr{D}}_X$ (in general, we only have $\mathscr{M} = V_0\mathscr{M} \cdot \widetilde{\mathscr{D}}_X$).

9.4.b. Dual localization along an arbitrary effective divisor

We keep the same notation as in Section 9.3.c. Let $\mathscr{M}$ be $\widetilde{\mathscr{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along D. We say that $\mathscr{M}$ is *dual-localizable along D* if for any local equation g defining D, there exists a coherent $\widetilde{\mathscr{D}}_X$-module $\mathscr{M}[!g]$ such that ${}_{\mathrm{D}}\iota_{g*}(\mathscr{M}[!g]) = ({}_{\mathrm{D}}\iota_{g*}\mathscr{M})[!H]$. The various checks done in Section 9.3.c can be done similarly here in order to fully justify this definition.

Corollary 9.4.9 **(Properties of the dual localization along g).** *Let $g : X \to \mathbb{C}$ be a holomorphic function and let $\mathscr{M}$ be $\widetilde{\mathscr{D}}_X$-coherent, strictly $\mathbb{R}$-specializable and dual-localizable along (g). Set $H = X \times \{0\} \subset X \times \mathbb{C}$.*

(1) *The $\widetilde{\mathscr{D}}_X$-module $\mathscr{M}[!g]$ is strictly $\mathbb{R}$-specializable along g and*

$$\mathrm{can} : \psi_{g,1}\big(\mathscr{M}[!g]\big) \longrightarrow \phi_{g,1}\big(\mathscr{M}[!g]\big)$$

is an isomorphism.

(2) *There is a natural morphism $\iota : \mathscr{M}[!g] \to \mathscr{M}$. This morphism induces an isomorphism*

$$\big(\mathscr{M}[!g]\big)(*g) \xrightarrow{\sim} \mathscr{M}(*g),$$

and therefore isomorphisms

$$\psi_{g,\lambda}\big(\mathscr{M}[!g]\big) \xrightarrow{\sim} \psi_{g,\lambda}\mathscr{M} \quad \textit{for every } \lambda.$$

Moreover, we have a commutative diagram

$$\begin{tikzcd}
\psi_{g,1}\big(\mathscr{M}[!g]\big) \arrow[r, "\psi_{g,1}\iota", "\sim"'] \arrow[d, "\mathrm{can}_{\mathscr{M}[!g]}"', "\wr"] & \psi_{g,1}\mathscr{M} \arrow[d, "\mathrm{can}_{\mathscr{M}}"] \\
\phi_{g,1}\big(\mathscr{M}[!g]\big) \arrow[r, "\phi_{g,1}\iota"] & \phi_{g,1}\mathscr{M}
\end{tikzcd}$$

and $\operatorname{Ker}\iota$ *(resp.* $\operatorname{Coker}\iota$*) is identified with* $\operatorname{Ker}\mathrm{can}_{\mathscr{M}}$ *(resp.* $\operatorname{Coker}\mathrm{can}_{\mathscr{M}}$*).*

(3) *Given a short exact sequence of coherent $\widetilde{\mathscr{D}}_X$-modules which are strictly $\mathbb{R}$-specializable and dual-localizable along g, the $[!g]$ sequence is exact.*

Proof. Similar to that of Corollary 9.3.6. □

Remark 9.4.10. The proof gives in particular that ${}_{\mathrm{D}}\iota_{g*}\iota_g = \iota_t$.

Remark 9.4.11 **(Remark 9.3.3 continued).** In Corollary 9.4.9, we could start from a coherent $\widetilde{\mathscr{D}}_X(*g)$-module $\mathscr{M}$ which is strictly $\mathbb{R}$-specializable and, globally, we could start from a coherent $\widetilde{\mathscr{D}}_X(*D)$-module $\mathscr{M}$ which is strictly $\mathbb{R}$-specializable.

Remark 9.4.12 **(Restriction to $z=1$).** One proves as in Remark 9.3.9 that dual localization behaves well with respect to the restriction $z=1$.

9.5. D-localizable $\widetilde{\mathscr{D}}_X$-modules and middle extension

9.5.a. D-localizable $\widetilde{\mathscr{D}}_X$-modules. Let D be an arbitrary effective divisor.

Definition 9.5.1 **(D-localizable $\widetilde{\mathscr{D}}_X$-modules).** Assume that $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along D. We say that it is *D-localizable* if it is localizable and dual-localizable along D. The localized (resp. dual localized) module $\mathscr{M}[\star D]$ ($\star = *$, resp. $\star = !$) is then well-defined and is strictly $\mathbb{R}$-specializable along D.

Recall that, if $D = H$ is smooth, any $\mathscr{M}$ which is $\widetilde{\mathscr{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along D is D-localizable. On the other hand, for $\mathscr{D}_X$-modules, $\mathbb{R}$-specializability implies D-localizability, whatever D is.

Exercise 9.5.2. We work within the full subcategory of $\widetilde{\mathscr{D}}_X$-modules which are strictly $\mathbb{R}$-specializable and localizable along D.

(1) Show that $\mathscr{M}[*D]$ and $\mathscr{M}[!D]$ are localizable along D and

(a) the morphisms $\mathscr{M}[!D][*D] \to \mathscr{M}[*D]$ and $\mathscr{M}[!D][!D] \to \mathscr{M}[!D]$ induced by $\mathscr{M}[!D] \to \mathscr{M}$ are isomorphisms,

(b) the morphisms $\mathscr{M}[!D] \to \mathscr{M}[*D][!D]$ and $\mathscr{M}[*D] \to \mathscr{M}[*D][*D]$ induced by $\mathscr{M} \to \mathscr{M}[*D]$ are isomorphisms.

(2) Let g be a local equation of D. Show that there are isomorphisms of diagrams (see Definition 7.7.3)

$$\begin{tikzcd} \psi_{g,1}\mathscr{M}[*g] \arrow[rr, bend left, "\mathrm{can}"] && \phi_{g,1}\mathscr{M}[*g] \arrow[ll, bend left, "\mathrm{var}", "\sim"', "(-1)" near end] \end{tikzcd} \simeq \begin{tikzcd} \psi_{g,1}\mathscr{M} \arrow[rr, bend left, "\mathrm{N}", "(-1)" near end] && \psi_{g,1}\mathscr{M} \arrow[ll, bend left, "\mathrm{Id}"] \end{tikzcd}$$

and

$$\begin{tikzcd} \psi_{g,1}\mathscr{M}[!g] \arrow[rr, bend left, "\mathrm{can}", "\sim"'] && \phi_{g,1}\mathscr{M}[!g] \arrow[ll, bend left, "\mathrm{var}", "(-1)" near end] \end{tikzcd} \simeq \begin{tikzcd} \psi_{g,1}\mathscr{M} \arrow[rr, bend left, "\mathrm{Id}"] && \psi_{g,1}\mathscr{M} \arrow[ll, bend left, "\mathrm{N}", "(-1)" near end] \end{tikzcd}.$$

9.5.b. Middle extension along an arbitrary effective divisor

Definition 9.5.3 **(Middle extension).** Assume that $\mathscr{M}$ is $\widetilde{\mathscr{D}}_X$-coherent, strictly $\mathbb{R}$-specializable and localizable along an effective divisor D. The image of the composed morphism $\mathscr{M}[!D] \to \mathscr{M} \to \mathscr{M}[*D]$ is called the *middle extension of $\mathscr{M}$ along D* and denoted by $\mathscr{M}[!*D]$.

Note however that we do not assert that $\mathscr{M}[!*D]$ is strictly $\mathbb{R}$-specializable along D. Nevertheless, if $D = (g)$, ${}_{\mathrm{D}}\iota_{g*}\mathscr{M}[!*D]$ is the image of ${}_{\mathrm{D}}\iota_{g*}\mathscr{M}[!D] \to {}_{\mathrm{D}}\iota_{g*}\mathscr{M}[*D]$, that is, the image of $({}_{\mathrm{D}}\iota_{g*}\mathscr{M})[!H] \to ({}_{\mathrm{D}}\iota_{g*}\mathscr{M})[*H]$, and it is $\mathbb{R}$-specializable along H with strict V-graded objects, according to Exercise 7.3.37(3). We will still use the notation $\psi_{g,\lambda}\mathscr{M}[!*D]$ and $\phi_{g,1}\mathscr{M}[!*D]$ for $\mathrm{gr}^V_\alpha\,{}_{\mathrm{D}}\iota_{g*}\mathscr{M}[!*D](1)$ for $\alpha \in [-1,0)$ and $\mathrm{gr}^V_0\,{}_{\mathrm{D}}\iota_{g*}\mathscr{M}[!*D]$ respectively.

Remark 9.5.4 **(Minimal extension and middle extension).** Assume that $D = (g)$ and that $\mathscr{M}$ is strictly $\mathbb{R}$-specializable and localizable along D (if $D = H$ is smooth, the latter condition holds if the former holds). Assume moreover that can is onto and var is injective, that is, $\mathscr{M}$ is a minimal extension along g. Then, according to Remarks 9.3.5 and 9.4.8, $\mathscr{M}[!D] \to \mathscr{M}$ is onto and $\mathscr{M} \to \mathscr{M}[*D]$ is injective, so $\mathscr{M} = \mathscr{M}[!*D]$, and in particular $\mathscr{M}[!*D]$ is strictly $\mathbb{R}$-specializable along D.

Exercise 9.5.5. We keep the assumptions as in Definition 9.5.3 and we also assume also that $D = (g)$. Recall that $\iota^\vee$ (resp. ι) have been defined in 9.3.4(2) (resp. 9.4.2(2)).

(1) Show that the kernel and cokernel of the natural morphism

$$\iota^\vee \circ \iota : \mathscr{M}[!g] \longrightarrow \mathscr{M}[*g]$$

are equal respectively to the kernel and cokernel of

$$\phi_{g,1}(\iota^\vee \circ \iota) : \phi_{g,1}\mathscr{M}[!g] \longrightarrow \phi_{g,1}\mathscr{M}[*g],$$

and also to the kernel and cokernel of

$$\mathrm{N} : \psi_{g,1}\mathscr{M} \longrightarrow \psi_{g,1}\mathscr{M}(-1).$$

[*Hint*: Show that $\iota^\vee\circ\iota$ induces an isomorphism on $V_{<0}$ and argue as in Example 7.3.38 for ${}_{\mathrm{D}}\iota_{g*}(\mathscr{M}[*g])$.]

(2) Identify $\psi_{g,\lambda}\mathscr{M}[!{*}g]$ with $\psi_{g,\lambda}\mathscr{M}$ and $\phi_{g,1}\mathscr{M}[!{*}g]$ with image(N).

(3) Show that if $\mathrm{N}:\psi_{g,1}\mathscr{M}\to\psi_{g,1}\mathscr{M}(-1)$ is strict, then $\iota^\vee\circ\iota:\mathscr{M}[!g]\to\mathscr{M}[*g]$ is strictly $\mathbb{R}$-specializable.

Proposition 9.5.6 **(A criterion for the strict $\mathbb{R}$-specializability of $\mathscr{M}[!{*}g]$)**

Assume that $\mathscr{M}$ is $\widetilde{\mathscr{D}}_X$-coherent, strictly $\mathbb{R}$-specializable and localizable along (g). If $\mathrm{N}=\mathrm{var}\circ\mathrm{can}:\psi_{g,1}\mathscr{M}\to\psi_{g,1}\mathscr{M}(-1)$ is a strict morphism, then $\mathscr{M}[!{*}g]$ is strictly $\mathbb{R}$-specializable along g.

Proof. This follows from Exercise 9.5.5. □

Exercise 9.5.7. With the assumptions of Proposition 9.5.6, show similarly that the morphism $\mathscr{M}\to\mathscr{M}[*g]$ (resp. $\mathscr{M}[!g]\to\mathscr{M}$) is strictly $\mathbb{R}$-specializable along g if and only if the morphism $\mathrm{var}:\phi_{g,1}\mathscr{M}\to\psi_{g,1}\mathscr{M}(-1)$ (resp. $\mathrm{can}:\psi_{g,1}\mathscr{M}\to\phi_{g,1}\mathscr{M}$) is strict.

9.6. Beilinson's maximal extension and applications

In this section, we continue working with right $\widetilde{\mathscr{D}}_X$-modules.

Remark 9.6.1 **(The case of left $\widetilde{\mathscr{D}}_X$-modules).** The same changes as in Remark 9.0.1 have to be made for left $\widetilde{\mathscr{D}}_X$-modules.

9.6.a. Properties of Beilinson's maximal extension. Let $g:X\to\mathbb{C}$ be a holomorphic function. Let $\mathscr{M}$ be a coherent $\widetilde{\mathscr{D}}_X$-module which is strictly $\mathbb{R}$-specializable along $D:=(g)$. When D is not smooth, we also assume that $\mathscr{M}$ is D-localizable, and *maximalizable* (see Definition 9.6.13 below). We aim at constructing a *coherent* $\widetilde{\mathscr{D}}_X$-module $\Xi_g\mathscr{M}$, called *Beilinson's maximal extension of $\mathscr{M}$ along D*, which is also strictly $\mathbb{R}$-specializable along D. It comes with two exact sequences

$$0\longrightarrow\mathscr{M}[!g]\xrightarrow{\;a\;}\Xi_g\mathscr{M}\xrightarrow{\;b\;}\psi_{g,1}\mathscr{M}(-1)\longrightarrow0,\tag{9.6.2 !}$$

$$0\longrightarrow\psi_{g,1}\mathscr{M}\xrightarrow{\;b^\vee\;}\Xi_g\mathscr{M}\xrightarrow{\;a^\vee\;}\mathscr{M}[*g]\longrightarrow0,\tag{9.6.2 $*$}$$

such that $b\circ b^\vee=-\mathrm{N}$ and $a^\vee\circ a=\iota^\vee\circ\iota$, where $\iota,\iota^\vee$ are the natural morphisms (see Corollaries 9.3.6(2) and 9.4.9(2))

$$\mathscr{M}[!g]\xrightarrow{\;\iota\;}\mathscr{M}\quad\text{and}\quad\mathscr{M}\xrightarrow{\;\iota^\vee\;}\mathscr{M}[*g].$$

The construction and the exact sequences only depend on the stupidly localized module $\mathscr{M}(*D)$ (recall also that $\mathscr{M}[!g]$ and $\mathscr{M}[*g]$ only depend on $\mathscr{M}(*D)$). It can be done for any coherent $\widetilde{\mathscr{D}}_X(*D)$-module $\mathscr{M}_*$ which is strictly $\mathbb{R}$-specializable along D and gives rise nevertheless to a coherent $\widetilde{\mathscr{D}}_X$-module which is strictly $\mathbb{R}$-specializable along D. The usefulness of Beilinson's maximal extension comes from Corollary 9.6.5 below, which enables one to treat some questions on $\widetilde{\mathscr{D}}_X$-modules which are strictly

$\mathbb{R}$-specializable along D by reducing to the case of $\widetilde{\mathscr{D}}_X(*D)$-modules strictly $\mathbb{R}$-specializable along D on the one hand, and to the case of $\widetilde{\mathscr{D}}_X$-modules supported on D and strictly $\mathbb{R}$-specializable along D on the other hand, the latter case being subject to an induction argument.

Theorem 9.6.3 **(Gluing construction).** *Let $\mathscr{M}_*$ be a coherent $\widetilde{\mathscr{D}}_X(*D)$-module which is strictly $\mathbb{R}$-specializable, D-localizable and maximalizable along $D = (g)$. Let $(\mathscr{N}, \mathrm{c}, \mathrm{v})$ be a triple consisting of a coherent $\widetilde{\mathscr{D}}_X$-module supported on D and strictly $\mathbb{R}$-specializable along D, and a pair morphisms $\mathrm{c} : \psi_{g,1}\mathscr{M}_* \to \mathscr{N}$ and $\mathrm{v} : \mathscr{N} \to \psi_{g,1}\mathscr{M}_*(-1)$ such that $\mathrm{v} \circ \mathrm{c} = \mathrm{N}$. Then the complex*

$$(9.6.3\,*)\qquad \psi_{g,1}\mathscr{M}_* \xrightarrow{\;b^\vee \oplus \mathrm{c}\;} \Xi_g\mathscr{M}_* \oplus \mathscr{N} \xrightarrow{\;b + \mathrm{v}\;} \psi_{g,1}\mathscr{M}_*(-1)$$

has nonzero cohomology in degree one at most, its $\mathscr{H}^1$ is a coherent $\widetilde{\mathscr{D}}_X$-module $\mathrm{G}(\mathscr{M}_*, \mathscr{N}, \mathrm{c}, \mathrm{v})$ which is strictly $\mathbb{R}$-specializable along D and we have an isomorphism of diagrams

$$\left[\begin{tikzcd} \psi_{g,1}\mathrm{G}(\mathscr{M}_*, \mathscr{N}, \mathrm{c}, \mathrm{v}) \arrow[r, bend left, "\mathrm{can}"] & \phi_{g,1}\mathrm{G}(\mathscr{M}_*, \mathscr{N}, \mathrm{c}, \mathrm{v}) \arrow[l, bend left, "\mathrm{var}", "(-1)"'] \end{tikzcd}\right] \simeq \left[\begin{tikzcd} \psi_{g,1}\mathscr{M}_* \arrow[r, bend left, "\mathrm{c}"] & \mathscr{N} \arrow[l, bend left, "\mathrm{v}", "(-1)"'] \end{tikzcd}\right].$$

Remarks 9.6.4.

(1) We obviously have $\mathrm{G}(\mathscr{M}_*, \mathscr{N}, \mathrm{c}, \mathrm{v})(*D) = (\Xi_g\mathscr{M}_*)(*D) = \mathscr{M}_*$.

(2) If $D = H$ is smooth and g is a projection, the conditions "D-localizable" and "maximalizable" along D follow from the condition "strictly $\mathbb{R}$-specializable along D".

Set $D = (g)$ and consider the category $\mathrm{Glue}(X, D)$ whose objects consist of data $(\mathscr{M}_*, \mathscr{N}, \mathrm{c}, \mathrm{v})$ satisfying the properties as in the theorem, and whose morphisms are pairs of morphisms $\mathscr{M}_* \to \mathscr{M}'_*$ and $\mathscr{N} \to \mathscr{N}'$ which are naturally compatible with c, v and c', v'.

We have a natural functor

$$\mathscr{M} \longmapsto \mathrm{G}(\mathscr{M}(*D), \phi_{g,1}\mathscr{M}, \mathrm{can}, \mathrm{var}).$$

from the category of $\widetilde{\mathscr{D}}_X$-coherent modules which are strictly $\mathbb{R}$-specializable, localizable and maximalizable along D, to the category $\mathrm{Glue}(X, D)$.

Corollary 9.6.5. *This functor is an equivalence of categories.*

We start with the case of a projection $t : X \simeq H \times \Delta_t \to \Delta_t$ in Sections 9.6.b–9.6.c.

9.6.b. A construction of $\psi_{t,1}$ starting from localization. We will give another construction of $\psi_{t,1}\mathscr{M}_*$ for a strictly $\mathbb{R}$-specializable $\widetilde{\mathscr{D}}_X(*H)$-module $\mathscr{M}_*$ (see Section 9.3.a for this notion).

Let k be a non-negative integer, set $\varepsilon = 0, 1$, and let $\mathcal{J}^{(\varepsilon,k)}$ denote the upper Jordan block of size $k+\varepsilon$, that is, the filtered vector space $\mathbb{C}e_\varepsilon \oplus \cdots \oplus \mathbb{C}e_k$, where $e_i \in F^{i-\varepsilon}$ ($i \geqslant \varepsilon$), so $\mathcal{J}^{(\varepsilon,k)}$ is in fact graded, with the nilpotent endomorphism

$$\begin{aligned} \mathcal{J}^{(\varepsilon,k)} &\xrightarrow{\mathrm{J}^{(\varepsilon,k)}} \mathcal{J}^{(\varepsilon,k)}(-1) \\ e_i &\longmapsto e_{i-1} \qquad \text{(convention: } e_{\varepsilon-1} = 0). \end{aligned}$$

Similarly, we denote by $\mathcal{J}_{(\varepsilon,k)}$ the lower Jordan block $\mathbb{C}e_\varepsilon \oplus \cdots \oplus \mathbb{C}e_k$ increasingly filtered (in fact graded) so that $e_i \in F_{i-\varepsilon}$, with the nilpotent endomorphism

$$\begin{aligned} \mathcal{J}_{(\varepsilon,k)} &\xrightarrow{\mathrm{J}_{(\varepsilon,k)}} \mathcal{J}_{(\varepsilon,k)}(-1) \\ e_i &\longmapsto e_{i+1} \qquad \text{(convention: } e_{k+1} = 0). \end{aligned}$$

We have natural morphisms (graded of degree zero and compatible with the nilpotent endomorphisms):

$$\begin{array}{ccccc} \mathcal{J}^{(1,k)}(-1) & \twoheadleftarrow & \mathcal{J}^{(0,k)} & \hookrightarrow & \mathcal{J}^{(0,k+1)} \\ e_i & \longleftarrow\!\shortmid & e_{i=1,\dots,k} & \longmapsto & e_i \\ 0 & \longleftarrow\!\shortmid & e_0 & \longmapsto & e_0 \end{array} \qquad \begin{array}{ccccc} \mathcal{J}_{(1,k)}(1) & \hookrightarrow & \mathcal{J}_{(0,k)} & \twoheadleftarrow & \mathcal{J}_{(0,k+1)} \\ e_{i=1,\dots,k} & \longmapsto & e_i & \longleftarrow\!\shortmid & e_{i=0,\dots,k} \\ & & 0 & \longleftarrow\!\shortmid & e_{k+1} \end{array}$$

***Exercise 9.6.6* (Linear algebra 1).** Let (M, N) be a graded $\mathbb{C}$-vector space with a nilpotent endomorphism $\mathrm{N} : M \to M(-1)$. For $\varepsilon = 0, 1$, set $M^{(\varepsilon,k)} = M \otimes_{\mathbb{C}} \mathcal{J}^{(\varepsilon,k)}$ with nilpotent endomorphism

$$\mathrm{N}^{(\varepsilon,k)} := \mathrm{N} \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathrm{J}^{(\varepsilon,k)} : M^{(\varepsilon,k)} \longrightarrow M^{(\varepsilon,k)}(-1)$$

and similarly for $\mathrm{N}_{(\varepsilon,k)}$. Show the following properties.

(1) The morphism

$$\begin{aligned} M &\longrightarrow M^{(\varepsilon,k)} \\ m &\longmapsto \textstyle\sum_{i=\varepsilon}^{k} (-\mathrm{N})^{i-\varepsilon} m \otimes e_i \end{aligned}$$

induces an isomorphism $\operatorname{Ker} \mathrm{N}^{k+1-\varepsilon} \xrightarrow{\sim} \operatorname{Ker} \mathrm{N}^{(\varepsilon,k)}$ with respect to which the natural morphism $\operatorname{Ker} \mathrm{N}^{(\varepsilon,k)} \to \operatorname{Ker} \mathrm{N}^{(\varepsilon,k+1)}$ correspond to the natural morphism $\operatorname{Ker} \mathrm{N}^{k+1-\varepsilon} \hookrightarrow \operatorname{Ker} \mathrm{N}^{k+2-\varepsilon}$ and the natural morphism $\operatorname{Ker} \mathrm{N}^{(0,k)} \to \operatorname{Ker} \mathrm{N}^{(1,k)}(-1)$ correspond to the natural morphism $\operatorname{Ker} \mathrm{N}^{k+1} \xrightarrow{-\mathrm{N}} \operatorname{Ker} \mathrm{N}^{k}(-1)$. In particular, if N has finite order on M, show that have natural commutative diagrams

$$\begin{array}{ccccc} \varinjlim_k \operatorname{Ker} \mathrm{N}^{(0,k)} & \xleftarrow{\ \sim\ } & \varinjlim_k \operatorname{Ker} \mathrm{N}^{k+1} & \xrightarrow{\ \sim\ } & M \\ \downarrow & & \downarrow{\scriptstyle -\mathrm{N}} & & \downarrow{\scriptstyle -\mathrm{N}} \\ \varinjlim_k \operatorname{Ker} \mathrm{N}^{(1,k)}(-1) & \xleftarrow{\ \sim\ } & \varinjlim_k \operatorname{Ker} \mathrm{N}^{k}(-1) & \xrightarrow{\ \sim\ } & M(-1) \end{array}$$

and the limits are achieved for $k > \operatorname{ord}(\mathrm{N})$.

(2) Show that the morphism

$$\begin{aligned} M^{(\varepsilon,k)} &\longrightarrow M(\varepsilon-k)\\ \sum_{i=\varepsilon}^{k} m_i\otimes e_i &\longmapsto \sum_{i=\varepsilon}^{k}(-\mathrm{N})^{k-i}m_i \end{aligned}$$

induces an isomorphism

$$\operatorname{Coker}\mathrm{N}^{(\varepsilon,k)} := M^{(\varepsilon,k)}(-1)/\operatorname{Im}\mathrm{N}^{(\varepsilon,k)} \xrightarrow{\sim} M(\varepsilon-(k+1))/\operatorname{Im}\mathrm{N}^{k+1-\varepsilon},$$

and thus, if $k>\operatorname{ord}(\mathrm{N})$,

$$\operatorname{Coker}\mathrm{N}^{(\varepsilon,k)} \simeq M(\varepsilon-(k+1)).$$

(3) Show similar properties for the lower Jordan block. Note that the previous diagram becomes

$$\begin{tikzcd} M \arrow[r, "\sim"] \arrow[d, "-\mathrm{N}"'] & \varprojlim_k \operatorname{Coker}\mathrm{N}^k \arrow[d, "-\mathrm{N}"'] & \varprojlim_k \operatorname{Coker}\mathrm{N}_{(1,k)} \arrow[l, "\sim"'] \arrow[d] \\ M(-1) \arrow[r, "\sim"] & \varprojlim_k \operatorname{Coker}\mathrm{N}^{k+1}(-1) & \varprojlim_k \operatorname{Coker}\mathrm{N}_{(0,k)}(-1) \arrow[l, "\sim"'] \end{tikzcd}$$

***Exercise 9.6.7* (Linear algebra 2).** We keep the notation as in Exercise 9.6.6.

(1) Show that the two composed natural maps

$$M^{(0,k)} \longrightarrow M^{(1,k)}(-1) \xrightarrow{\mathrm{N}^{(1,k)}} M^{(1,k)}(-2)$$

and

$$M^{(0,k)} \xrightarrow{N^{(0,k)}} M^{(0,k)}(-1) \longrightarrow M^{(1,k)}(-2)$$

coincide. Let $\Xi^k M$ denote their kernel. In particular, $N^{(0,k)}$ induces a map

$$\mathrm{N}^{(0,k)}_{|\Xi^k M} : \Xi^k M \longrightarrow \operatorname{Ker}\big[M^{(0,k)}(-1)\to M^{(1,k)}(-2)\big] \simeq (M\otimes e_0)(-1) \simeq M(-1).$$

(2) Show that the map

$$\begin{aligned} M\oplus\operatorname{Ker}\mathrm{N}^k(-1) &\longrightarrow M^{(0,k)}\\ (n,m) &\longmapsto n\otimes e_0 + \textstyle\sum_{i=1}^{k}(-\mathrm{N})^{i-1}m\otimes e_i \end{aligned}$$

induces an isomorphism onto $\Xi^k M$.

(3) Show that, under this isomorphism, $\mathrm{N}^{(0,k)}_{|\Xi^k M} : \Xi^k M \to M(-1)$ is identified with $(n,m)\mapsto \mathrm{N}n+m$.

(4) Conclude that, if $\operatorname{ord}(\mathrm{N})$ is finite and $k>\operatorname{ord}(\mathrm{N})$, then the exact sequence

$$0\longrightarrow \operatorname{Ker}\big[M^{(0,k)}\to M^{(1,k)}(-1)\big] \longrightarrow \Xi^k M \longrightarrow \operatorname{Ker}\mathrm{N}^{(1,k)} \longrightarrow 0$$

is isomorphic to the naturally split sequence $0\to M\to M\oplus M(-1)\to M(-1)\to 0$ with respect to which the exact sequence

$$0\longrightarrow \operatorname{Ker}\mathrm{N}^{(0,k)} \longrightarrow \Xi^k M \longrightarrow \operatorname{Ker}\big[M^{(0,k)}(-1)\to M^{(1,k)}(-2)\big] \longrightarrow 0$$

corresponds to

$$0 \longrightarrow \mathrm{Ker}(\mathrm{N}+\mathrm{Id}) \longrightarrow M \oplus M(-1) \xrightarrow{\mathrm{N}+\mathrm{Id}} M(-1) \longrightarrow 0.$$

(5) Show similar properties for the lower Jordan block.

Let $\mathscr{M}_*$ be a strictly $\mathbb{R}$-specializable $\widetilde{\mathscr{D}}_X(*H)$-module. We set $\mathscr{M}_{*(\varepsilon,k)} = \mathscr{M}_* \otimes_{\mathbb{C}} \mathfrak{I}_{(\varepsilon,k)}$ with the action of $t\eth_t$ given by

$$(m \otimes e_i)t\eth_t := (mt\eth_t) \otimes e_i + m \otimes \mathrm{J}_{(\varepsilon,k)}e_i,$$

and we define $\mathscr{M}_*^{(\varepsilon,k)}$ similarly. The following lemma is easy.

Lemma 9.6.8. *If $\mathscr{M}_*$ is strictly $\mathbb{R}$-specializable along H, then so are $\mathscr{M}_*^{(\varepsilon,k)}$ and $\mathscr{M}_{*(\varepsilon,k)}$, we have $V_\bullet\mathscr{M}_*^{(\varepsilon,k)} = (V_\bullet\mathscr{M}_*)^{(\varepsilon,k)}$ and the lower similar equalities, and for every λ, $\psi_{t,\lambda}(\mathscr{M}_*^{(\varepsilon,k)}) \simeq (\psi_{t,\lambda}\mathscr{M}_*)^{(\varepsilon,k)}$, and other similar equalities with $\phi_{t,1}$, together with the lower similar equalities.* □

Proposition 9.6.9. *Assume that $\mathscr{M}_*$ is strictly $\mathbb{R}$-specializable along H.*

(1) *The morphisms*

$$(\iota^\vee \circ \iota)^{(\varepsilon,k)} : \mathscr{M}_*^{(\varepsilon,k)}[!H] \longrightarrow \mathscr{M}_*^{(\varepsilon,k)}[*H]$$

and

$$(\iota^\vee \circ \iota)_{(\varepsilon,k)} : \mathscr{M}_{*(\varepsilon,k)}[!H] \longrightarrow \mathscr{M}_{*(\varepsilon,k)}[*H]$$

are strictly $\mathbb{R}$-specializable for k large enough, locally on H.

(2) *We have functorial isomorphisms*

$$\varinjlim_k \mathrm{Ker}(\iota^\vee \circ \iota)^{(\varepsilon,k)} \simeq \psi_{t,1}\mathscr{M}_* \simeq \varprojlim_k \mathrm{Coker}(\iota^\vee \circ \iota)_{(\varepsilon,k)},$$

and the limits are achieved for k large enough, locally on H.

(3) *The composed natural morphisms*

$$\mathscr{M}_*^{(0,k)}[!H] \longrightarrow \mathscr{M}_*^{(0,k)}[*H] \longrightarrow \mathscr{M}_*^{(1,k)}[*H](-1)$$

and

$$\mathscr{M}_{*(1,k)}[!H](1) \longrightarrow \mathscr{M}_{*(0,k)}[!H] \longrightarrow \mathscr{M}_{*(0,k)}[*H]$$

are strictly $\mathbb{R}$-specializable for k large enough, locally on H.

Proof.

(1) Since the morphisms considered induce isomorphisms on $V_{<0}$, it is enough to check that their $\phi_{t,1}$ are strict for k large enough (Example 7.3.38). By Exercise 9.5.5(3), this amounts to the strictness of $\mathrm{N}^{(\varepsilon,k)} : \psi_{t,1}\mathscr{M}_*^{(\varepsilon,k)} \to \psi_{t,1}\mathscr{M}_*^{(\varepsilon,k)}(-1)$ and, by Lemma 9.6.8, to the strictness of $\mathrm{N}^{(\varepsilon,k)} : (\psi_{t,1}\mathscr{M}_*)^{(\varepsilon,k)} \to (\psi_{t,1}\mathscr{M}_*)^{(\varepsilon,k)}(-1)$, and similarly for $\mathrm{N}_{(\varepsilon,k)}$. For k large enough locally on H, the cokernel of $\mathrm{N}^{(\varepsilon,k)}$ is identified with $\psi_{t,1}\mathscr{M}_*(\varepsilon-(k+1))$, and similarly for $\mathrm{N}_{(\varepsilon,k)}$, according to Exercise 9.6.6, hence the strictness.

(2) By Exercise 9.5.5(1) and Lemma 9.6.8, we have

$$\operatorname{Ker}(\iota^\vee\circ\iota)^{(\varepsilon,k)}\simeq\operatorname{Ker}\bigl[\mathrm{N}^{(\varepsilon,k)}:(\psi_{t,1}\mathscr{M}_*)^{(\varepsilon,k)}\to(\psi_{t,1}\mathscr{M}_*)^{(\varepsilon,k)}(-1)\bigr],$$

which is identified with $\psi_{t,1}\mathscr{M}_*$ according to Exercise 9.6.6. We argue similarly for the lower case.

(3) Arguing as above, we are reduced to checking the strictness of $\phi_{t,1}$ of the composed morphisms. The upper one reads

$$(\psi_{t,1}\mathscr{M}_*)^{(0,k)}\xrightarrow{\mathrm{N}^{(0,k)}}(\psi_{t,1}\mathscr{M}_*)^{(0,k)}(-1)\longrightarrow(\psi_{t,1}\mathscr{M}_*)^{(1,k)}(-2)$$

and, according to Exercise 9.6.7(1), coincides with the composed morphism

$$(\psi_{t,1}\mathscr{M}_*)^{(0,k)}\longrightarrow(\psi_{t,1}\mathscr{M}_*)^{(1,k)}(-1)\xrightarrow{\mathrm{N}^{(1,k)}}(\psi_{t,1}\mathscr{M}_*)^{(1,k)}(-2)$$

whose cokernel, which is the cokernel of $\mathrm{N}^{(1,k)}$ since the first morphism is onto, is identified with $\psi_{t,1}\mathscr{M}_*(-k-1)$ for k large, hence the strictness. The argument for the lower one is similar. □

9.6.c. The maximal extension along $H\times\{0\}$

Definition 9.6.10 **(Maximal extension along H).** Let $\mathscr{M}_*$ be a coherent $\widetilde{\mathscr{D}}_X(*H)$-module which is strictly $\mathbb{R}$-specializable along H. We set

$$\Xi_t\mathscr{M}_*:=\varinjlim_k\operatorname{Ker}\bigl(\mathscr{M}_*^{(0,k)}[!H]\to\mathscr{M}_*^{(1,k)}[*H](-1)\bigr).$$

Proposition 9.6.11 **(The basic exact sequences).** *The limit in the definition of $\Xi_t\mathscr{M}_*$ is achieved for k large enough, locally on H, and $\Xi_t\mathscr{M}_*$ is a coherent $\widetilde{\mathscr{D}}_X$-module which is strictly $\mathbb{R}$-specializable along H. We have two functorial exact sequences*

$$0\longrightarrow\mathscr{M}_*[!H]\xrightarrow{a}\Xi_t\mathscr{M}_*\xrightarrow{b}\psi_{t,1}\mathscr{M}_*(-1)\longrightarrow 0,\tag{9.6.11 !}$$

$$0\longrightarrow\psi_{t,1}\mathscr{M}_*\xrightarrow{b^\vee}\Xi_t\mathscr{M}_*\xrightarrow{a^\vee}\mathscr{M}_*[*H]\longrightarrow 0,\tag{9.6.11 *}$$

with $b\circ b^\vee=-\mathrm{N}$ and $a^\vee\circ a=\iota^\vee\circ\iota$ (see Corollaries 9.3.6(2) *and* 9.4.9(2)*). Moreover, we also have*

$$\Xi_t\mathscr{M}_*:=\varprojlim_k\operatorname{Coker}\bigl(\mathscr{M}_{*(1,k)}[!H](1)\to\mathscr{M}_{*(0,k)}[*H]\bigr).$$

Proof. Arguing as in Proposition 7.3.40, one checks that the kernel of the morphism $\mathscr{M}_*^{(0,k)}[!H]\to\mathscr{M}_*^{(1,k)}[*H](-1)$ is strictly $\mathbb{R}$-specializable along H. We decompose this morphism either as

$$\mathscr{M}_*^{(0,k)}[!H]\longrightarrow\mathscr{M}_*^{(1,k)}[!H](-1)\longrightarrow\mathscr{M}_*^{(1,k)}[*H](-1)$$

or as

$$\mathscr{M}_*^{(0,k)}[!H]\longrightarrow\mathscr{M}_*^{(0,k)}[*H]\longrightarrow\mathscr{M}_*^{(1,k)}[*H](-1).$$

In the first case, its kernel is the middle term of a short exact sequence having the kernel of the right-hand morphism as right-hand term, that is, $\psi_{t,1}\mathscr{M}_*(-1)$ for k large enough locally, according to Proposition 9.6.9, and the kernel of the left-hand morphism as left-hand term, that is, $\mathscr{M}_*[!H]$, according to Proposition 9.4.2(7). The kernel is thus independent of k if k is large enough locally, and we have thus obtained (9.6.11 !).

In the second case, its kernel is the middle term of a short exact sequence having the kernel of the right-hand morphism as right-hand term, that is, $\mathscr{M}_*[*H]$, according to Proposition 9.3.4(11), and the kernel of the left-hand morphism as left-hand term, that is, $\psi_{t,1}\mathscr{M}_*$ for k large enough locally, according to Proposition 9.6.9. We have thus obtained (9.6.11 $*$).

The composed morphism $a^\vee \circ a$ is the composition

$$\mathscr{M}_*[!H] \simeq \mathscr{M}_*[!H]\otimes e_0 \hookrightarrow \mathscr{M}_*^{(0,k)}[!H] \xrightarrow{\iota^{\vee(0,k)}\circ\iota^{(0,k)}} \mathscr{M}_*^{(0,k)}[*H] \\ \longrightarrow \mathscr{M}_*[*H]\otimes e_0 \simeq \mathscr{M}_*[*H]$$

which is equal to $\iota^\vee\circ\iota$. On the other hand, the morphism $b\circ b^\vee : \psi_{t,1}\mathscr{M}_* \to \psi_{t,1}\mathscr{M}_*(-1)$ is identified with the natural morphism

$$\operatorname{Ker}(\iota^{\vee(0,k)}\circ\iota^{(0,k)}) \longrightarrow \operatorname{Ker}(\iota^{\vee(1,k)}\circ\iota^{(1,k)})$$

for k large enough locally. It is identified with the natural morphism

$$\operatorname{Ker}\big[\mathrm{N}^{(0,k)} : (\psi_{t,1}\mathscr{M}_*)^{(0,k)} \to (\psi_{t,1}\mathscr{M}_*)^{(0,k)}(-1)\big] \\ \longrightarrow \operatorname{Ker}\big[\mathrm{N}^{(1,k)} : (\psi_{t,1}\mathscr{M}_*)^{(1,k)} \to (\psi_{t,1}\mathscr{M}_*)^{(1,k)}(-1)\big],$$

which is identified, as in Exercise 9.6.6, to the morphism (k large enough locally)

$$-\mathrm{N} : \operatorname{Ker}\mathrm{N}^{k+1} \simeq \psi_{t,1}\mathscr{M}_* \longrightarrow \operatorname{Ker}\mathrm{N}^k(-1) \simeq \psi_{t,1}\mathscr{M}_*(-1). \qquad \square$$

Proposition 9.6.12 **(Nearby and vanishing cycles of the maximal extension)**

(1) *The morphisms $a : \mathscr{M}_*[!H] \to \Xi_t\mathscr{M}_*$ and $a^\vee : \Xi_t\mathscr{M}_* \to \mathscr{M}_*[*H]$ induce isomorphisms when restricted to $V_{<0}$, and thus isomorphisms of the $\psi_{t,\lambda}$ objects.*

(2) *The exact sequence $\phi_{t,1}$(9.6.11 !) is isomorphic to the naturally split exact sequence $0 \to \psi_{t,1}\mathscr{M}_* \xrightarrow{i_1} \psi_{t,1}\mathscr{M}_* \oplus \psi_{t,1}\mathscr{M}_*(-1) \xrightarrow{p_2} \psi_{t,1}\mathscr{M}_*(-1) \to 0$. With respect to this isomorphism, the exact sequence $\phi_{t,1}$(9.6.11 $*$) reads*

$$0 \longrightarrow \psi_{t,1}\mathscr{M}_* \xrightarrow{(\mathrm{Id},-\mathrm{N})} \psi_{t,1}\mathscr{M}_* \oplus \psi_{t,1}\mathscr{M}_*(-1) \xrightarrow{\mathrm{N}+\mathrm{Id}} \psi_{t,1}\mathscr{M}_*(-1) \longrightarrow 0.$$

Proof.

(1) We notice that, since all modules in (9.6.11 !) and (9.6.11 $*$) are strictly $\mathbb{R}$-specializable, the morphisms a and $a^\vee$ are strictly $\mathbb{R}$-specializable, in the sense of Definition 7.3.39. The result follows from Proposition 7.3.40, since $\psi_{t,1}\mathscr{M}_*$ is supported on H.

(2) This follows from Exercise 9.6.7. $\square$

Proof of Theorem 9.6.3 *for the function t.* The complex $C^\bullet$ considered in the theorem has nonzero cohomology in degree one only, since $b^\vee$ is injective and b is onto. We show that $\psi_{t,\lambda}C^\bullet$ and $\phi_{t,1}C^\bullet$ are strict. We have $\psi_{t,\lambda}C^\bullet = \{0 \to \psi_{t,\lambda}\Xi_t\mathscr{M} \to 0\}$, so the strictness follows from Proposition 9.6.11. On the other hand, according to Proposition 9.6.12, $\phi_{t,1}C^\bullet$ is identified with the complex

$$\begin{aligned}\psi_{t,1}\mathscr{M} \longrightarrow \psi_{t,1}\mathscr{M} &\oplus \psi_{t,1}\mathscr{M}(-1) \oplus \mathscr{N} \longrightarrow \psi_{t,1}\mathscr{M}(-1)\\ e \longmapsto (e, -\mathrm{N}e, &\mathrm{c}e)\\ &(e, m, \varepsilon) \longmapsto m + \mathrm{v}\varepsilon.\end{aligned}$$

Its cohomology in degree one is then identified with $\mathscr{N}$. Since $\mathscr{N}$ is assumed to be strict, $\mathscr{H}^1\phi_{t,1}C^\bullet$ is strict, and we clearly have $\mathscr{H}^j\phi_{t,1}C^\bullet = 0$ for $j \neq 1$. We deduce from Corollary 7.3.41 that $\mathscr{H}^1C^\bullet$ is strictly $\mathbb{R}$-specializable along H and $\psi_{t,\lambda}\mathscr{H}^1C^\bullet = \mathscr{H}^1\psi_{t,\lambda}C^\bullet$, and $\phi_{t,1}\mathscr{H}^1C^\bullet = \mathscr{H}^1\phi_{t,1}C^\bullet$. □

Proof of Corollary 9.6.5 *for the function* t. The construction G of Theorem 9.6.3 gives a right inverse of the functor considered in Corollary 9.6.5, implying that the latter is essentially surjective. That it is fully faithful now follows from Corollary 7.3.34. □

9.6.d. The maximal extension along an arbitrary effective divisor

Definition 9.6.13. Let D be an arbitrary effective divisor in X and let $\mathscr{M}_*$ be $\widetilde{\mathscr{D}}_X(*D)$-coherent and strictly $\mathbb{R}$-specializable along D.

(1) If $D = (g)$, where $g : X \to \mathbb{C}$ is a holomorphic function, set $H = X \times \{0\} \subset X \times \mathbb{C}$. We say that $\mathscr{M}_*$ is *maximalizable* along (g) if $\mathscr{M}_*^{(\varepsilon,k)}$ is (g)-localizable for every k and $\varepsilon \in \{0, 1\}$ (see Definition 9.5.1).

(2) In general, we say that $\mathscr{M}_*$ is *maximalizable* along D if for each point $x_o \in D$ and some (or any) local equation g of D near x_o, $\mathscr{M}_*$ is *maximalizable* along (g).

Proposition 9.6.14. *Assume that* $\mathscr{M}_*$ *is maximalizable along* $D = (g)$. *Set*

$$\Xi_g\mathscr{M}_* := \varinjlim_k \mathrm{Ker}\big(\mathscr{M}_*^{(0,k)}[!D] \to \mathscr{M}_*^{(1,k)}[*D](-1)\big).$$

Then the analogues of Propositions 9.6.11 *and* 9.6.12 *hold for* $\Xi_g\mathscr{M}_*$.

Sketch of proof. One first checks that the analogue of Proposition 9.6.9 holds, by checking that it holds after applying ${}_{\mathrm{D}}\iota_{g*}$. This follows from the fact that the morphisms ι and $\iota^\vee$ behave well under ${}_{\mathrm{D}}\iota_{g*}$ (see Remarks 9.4.10 and 9.3.7). The remaining part of the proof is done with similar arguments. □

Remark 9.6.15. If we denote by $a_g, a_g^\vee, b_g, b_g^\vee$ and $a_t, a_t^\vee, b_t, b_t^\vee$ the morphisms $a, a^\vee, b, b^\vee$ given by (9.6.2 !), (9.6.2 ∗) and Proposition 9.6.11 respectively, we have $a_t = {}_{\mathrm{D}}\iota_{g*}a_g$, etc.

Proof of Theorem 9.6.3 *and Corollary* 9.6.5. Let us apply the exact functor ${}_{\mathrm{D}}\iota_{g*}$ to $(9.6.3\,*)_g$. Since $\mathscr{M}_*$ is maximalizable along D, this produces $(9.6.3\,*)_t$, to which we apply the theorem. Since $\mathscr{H}^j{}_{\mathrm{D}}\iota_{g*}(9.6.3\,*)_g \simeq {}_{\mathrm{D}}\iota_{g*}\mathscr{H}^j(9.6.3\,*)_t$, we deduce the theorem for $(9.6.3\,*)_g$, and thus the functor of Corollary 9.6.5 is essentially surjective. It is fully faithful because it is so when $g = t$ and ${}_{\mathrm{D}}\iota_{g*}$ is fully faithful by Proposition 7.6.2. □

Proposition 9.6.16 **(Recovering $\phi_{g,1}$ from localization and maximalization)**
Let $\mathscr{M}$ be as above and set $\mathscr{M}_* = \mathscr{M}(*D)$. Then the complex

$$\Phi_g^\bullet \mathscr{M} := \Big\{ \mathscr{M}_*[!g] \xrightarrow{\ a\oplus\iota\ } \Xi_g\mathscr{M}_* \oplus \mathscr{M} \xrightarrow{\ a^\vee - \iota^\vee\ } \mathscr{M}_*[*g] \Big\} \tag{9.6.16 $*$}$$

satisfies $\mathscr{H}^k\Phi_g^\bullet\mathscr{M} = 0$ for $k \neq 1$ and $\mathscr{H}^1\Phi_g^\bullet\mathscr{M} \simeq \phi_{g,1}\mathscr{M}$.

Proof. We first consider the case of $X = H \times \widetilde{\mathbb{C}}$ and $g = t$. Injectivity of $a \oplus \iota$ follows from that of a, and surjectivity of $a^\vee - \iota^\vee$ follows form that of $a^\vee$. Since, for every $\lambda \in S^1$, $\psi_{t,\lambda}a$ and $\psi_{t,\lambda}a^\vee$ are isomorphisms inverse one to the other, and the same property holds for $\psi_{t,\lambda}\iota$ and $\psi_{t,\lambda}\iota^\vee$, it follows that $\psi_{t,\lambda}\Phi_t^\bullet\mathscr{M} \simeq 0$. On the other hand, the complex $\phi_{t,1}\Phi_t^\bullet\mathscr{M}$ is isomorphic to the complex

$$\begin{array}{ccccccc} 0 \longrightarrow \psi_{t,1}\mathscr{M} & \longrightarrow & \psi_{t,1}\mathscr{M} \oplus \psi_{t,1}\mathscr{M}(-1) \oplus \phi_{t,1}\mathscr{M} & \longrightarrow & \psi_{t,1}\mathscr{M}(-1) \longrightarrow 0 \\ e & \longmapsto & (e, 0, \operatorname{can} e) & & \\ & & (e, n, \varepsilon) & \longmapsto & \mathrm{N}e + n - \operatorname{var}\varepsilon \end{array}$$

so $\mathscr{H}^1\phi_{t,1}\Phi_t^\bullet\mathscr{M} \simeq (\psi_{t,1}\mathscr{M} \oplus \phi_{t,1}\mathscr{M})/\operatorname{Im}(\mathrm{Id}\oplus\operatorname{can})$, and therefore the projection $\psi_{t,1}\mathscr{M} \oplus \phi_{t,1}\mathscr{M} \to \phi_{t,1}\mathscr{M}$ induces an isomorphism $\mathscr{H}^1\phi_{t,1}\Phi_t^\bullet\mathscr{M} \xrightarrow{\sim} \phi_{t,1}\mathscr{M}$. As a consequence of Corollary 7.3.41, the cohomology ot complex $\Phi_t^\bullet\mathscr{M}$ is strictly $\mathbb{R}$-specializable along H and in particular $\phi_{t,1}\mathscr{H}^1\Phi_t^\bullet\mathscr{M} \simeq \mathscr{H}^1\phi_{t,1}\Phi_t^\bullet\mathscr{M}$. The first part of the proof also shows that $\mathscr{H}^1\Phi_t^\bullet\mathscr{M} \simeq \phi_{t,1}\mathscr{H}^1\Phi_t^\bullet\mathscr{M}$, so $\mathscr{H}^1\Phi_t^\bullet\mathscr{M} \simeq \phi_{t,1}\mathscr{M}$.

The general case is obtained by using the exactness of ${}_{\mathrm{D}}\iota_{g*}$. □

9.7. Good behaviour of localizability and maximalizability by pushforward

Let us keep the notation and assumptions of Corollary 7.8.6.

Corollary 9.7.1.

(1) *Assume moreover that $\mathscr{M}$ is localizable along (g). Then $\mathscr{H}^i{}_{\mathrm{D}}f_*\mathscr{M}$ are so along (g') for all i, we have $(\mathscr{H}^i{}_{\mathrm{D}}f_*\mathscr{M})[\star g'] \simeq \mathscr{H}^i{}_{\mathrm{D}}f_*(\mathscr{M}[\star g])$ ($\star = *, !$) and the morphisms $\iota, \iota^\vee$ behave well under $\mathscr{H}^i{}_{\mathrm{D}}f_*$.*

(2) *Assume moreover that $\mathscr{M}$ is maximalizable along (g). Then $\mathscr{H}^i{}_{\mathrm{D}}f_*\mathscr{M}$ are so along (g') for all i, we have $\Xi_{g'}(\mathscr{H}^i{}_{\mathrm{D}}f_*\mathscr{M}) \simeq \mathscr{H}^i{}_{\mathrm{D}}f_*(\Xi_g\mathscr{M})$, and the exact sequences (9.6.2 !) and (9.6.2 $*$) behave well under $\mathscr{H}^i{}_{\mathrm{D}}f_*$.*

Proof.

(1) Assume first that f takes the form $f_H \times \mathrm{Id} : H \times \Delta_t \to H' \times \Delta_t$. Then from Theorem 7.8.5 one deduces that $\mathscr{H}^i{}_{\mathrm{D}}f_*(\mathscr{M}[\star H])$ satisfies the characteristic properties 9.3.4(8) or 9.4.2(4) for $(\mathscr{H}^i{}_{\mathrm{D}}f_*\mathscr{M})[\star H']$, so the statement holds in this case.

For the general case, we note that we have a cartesian diagram

$$\begin{tikzcd} X \arrow[r, hook, "\iota_g"] \arrow[d, "f"'] & X \times \Delta_t \arrow[d, "f \times \mathrm{Id}"] \\ X' \arrow[r, hook, "\iota_{g'}"] & X' \times \Delta_t \end{tikzcd}$$

and we set $H = X \times \{0\}$, $H' = X' \times \{0\}$. Then

$$\begin{aligned}(\mathscr{H}^i{}_{\mathrm{D}}(f \times \mathrm{Id})_*\mathscr{M})[\star H'] &\simeq \mathscr{H}^i{}_{\mathrm{D}}(f \times \mathrm{Id})_*\big(({}_{\mathrm{D}}\iota_{g*}\mathscr{M})[\star H]\big) \\ &\simeq \mathscr{H}^i{}_{\mathrm{D}}(f \times \mathrm{Id})_*\big({}_{\mathrm{D}}\iota_{g*}(\mathscr{M}[\star g])\big) \simeq {}_{\mathrm{D}}\iota_{g'*}(\mathscr{H}^i{}_{\mathrm{D}}f_*(\mathscr{M}[\star g])),\end{aligned}$$

and the assertion holds according to the first case.

(2) Let us indicate the proof in the case where $f = f_H \times \mathrm{Id}$, as above. We first notice that $\mathscr{H}^i{}_{\mathrm{D}}f_*(\mathscr{M}^{(\varepsilon,k)}) \simeq (\mathscr{H}^i{}_{\mathrm{D}}f_*\mathscr{M})^{(\varepsilon,k)}$, and since f is proper, we can locally on X' choose k big enough so that the limits involved are already obtained for k. Let us denote by φ_k the morphism $\mathscr{M}^{(0,k)}[!H] \to \mathscr{M}^{(1,k)}[*H]$. We have a natural morphism $\mathscr{H}^i{}_{\mathrm{D}}f_* \operatorname{Ker}\varphi_k \to \operatorname{Ker}\mathscr{H}^i{}_{\mathrm{D}}f_*\varphi_k$ and, according to (1), it induces a morphism between sequences

$$\begin{aligned}\mathscr{H}^i{}_{\mathrm{D}}f_*\big((9.6.11\,!)(\mathscr{M})\big) &\longrightarrow (9.6.11\,!)(\mathscr{H}^i{}_{\mathrm{D}}f_*\mathscr{M}),\\ \mathscr{H}^i{}_{\mathrm{D}}f_*\big((9.6.11\,*)(\mathscr{M})\big) &\longrightarrow (9.6.11\,*)(\mathscr{H}^i{}_{\mathrm{D}}f_*\mathscr{M}).\end{aligned}$$

The right-hand sequences are short exact, while the left-hand ones are a priori only exact in the middle. Moreover, the extreme morphisms between these sequences are isomorphisms, by the previous results. Let us show that the left-hand sequences are indeed short exact and that the morphisms (in the middle) are isomorphisms. We will treat (9.6.11 !) for example. The composed (diagonal) morphism

$$\begin{tikzcd} \mathscr{H}^i{}_{\mathrm{D}}f_*(\mathscr{M}[!H]) \arrow[r, "\mathscr{H}^i{}_{\mathrm{D}}f_*a"] \arrow[d, "\wr"'] \arrow[dr] & \mathscr{H}^i{}_{\mathrm{D}}f_*\Xi_g(\mathscr{M}) \arrow[d] \\ (\mathscr{H}^i{}_{\mathrm{D}}f_*\mathscr{M})[!H'] \arrow[r, hook, "a"] & \Xi_{g'}\mathscr{H}^i{}_{\mathrm{D}}f_*\mathscr{M} \end{tikzcd}$$

is injective by assumption, hence so is $\mathscr{H}^i{}_{\mathrm{D}}f_*a$, and by applying this with $i+1$, we find that $\mathscr{H}^i{}_{\mathrm{D}}f_*\Xi_g(\mathscr{M}) \to \mathscr{H}^i{}_{\mathrm{D}}f_*(\psi_{t,1}\mathscr{M})$ is onto, so that the sequence $\mathscr{H}^i{}_{\mathrm{D}}f_*\big((9.6.11\,!)(\mathscr{M})\big)$ is short exact. Now, it is clear that it is isomorphic to $(9.6.11\,!)(\mathscr{H}^i{}_{\mathrm{D}}f_*\mathscr{M})$. $\square$

9.8. Comments

Here come the references to the existing work which has been the source of inspiration for this chapter.