## CHAPTER 8

## SPECIALIZATION OF FILTERED $\mathscr{D}$-MODULES


#### Abstract

Summary. In this chapter, we take up the notion of specialization and the compatibility property with proper pushforward for filtered $\mathscr{D}_{X}$-modules. Compared with the approach of Sections 7.3-7.8, we insist in keeping the strictness property, that is, we only work with filtered $\mathscr{D}_{X}$-modules, not graded modules over the Rees ring $R_{F} \mathscr{D}_{X}$. We will compare the two approaches in Section 8.8.


### 8.1. Introduction

One can introduce the notion of filtered $\mathscr{D}$-module by keeping the data of the $\mathscr{D}$-module and its filtration. The advantage is to keep a hand on the filtration at each step. The main goal of this chapter is to give a proof of the criterion given in Theorem 7.8.5 from this point of view. One should be careful since the category is not abelian anymore. As a consequence, dealing with derived categories, as needed when considering pushforward, needs some care, as well as strictness for bi-filtered complexes.

On the way, we will introduce the notion of compatible filtrations, which will be important in Chapter 11. The comparison between the present approach and that of Chapter 7 will be done in Section 8.8. Of particular interest is the property that, for a strict graded $R_{F} \mathscr{D}_{X}$-module, strict $\mathbb{R}$-specializability along a smooth divisor $H$ implies a regularity property, which has not been emphasized up to now, but which is essential for the approach in this chapter. In particular, the approach of Section 7.8 does not give as a result the strictness of the pushforward, only its strict specializability. We will show in Section 8.8 how to recover strictness properties from this point of view. On the other hand, the advantage of the approach of Section 7.8 is to allow generalization to cases where the regularity property is not fulfilled (twistor $\mathscr{D}$-modules), since strictness is not used for proving Theorem 7.8.5, only strict specializability is used. Lastly, localization and maximalization also have a natural formalism in the framework of graded $R_{F} \mathscr{D}_{X}$-module. We will not take up the corresponding formalism in the setting of filtered $\mathscr{D}_{X}$-modules.

### 8.2. Strict and bistrict complexes

In this section we review the definition and basic properties of strictness for filtered and bi-filtered complexes. We will consider the case of several filtrations in Section 8.3. In particular, when dealing with at least three filtrations, an important role is played by the compatibility condition on filtrations. However, this condition does not arise when dealing with one or two filtrations and the strictness condition on complexes is also very easy to treat directly.

Convention 8.2.1. We work in an abelian category A in which all filtered direct limits exist and are exact. Given an object $A$ in this category, we only consider increasing filtrations $F_{\bullet} A$ that are indexed by $\mathbb{Z}$ and satisfy ${\underset{\longrightarrow}{\lim }}_{k} F_{k} A=A$. We write a filtered object in A as $(A, F)$, where $F=\left(F_{k} A\right)_{k \in \mathbb{Z}}$.

Note that if $(A, F)$ is a filtered object, then a subobject $B$ of $A$ carries the induced filtration $\left(F_{k} A \cap B\right)_{k \in \mathbb{Z}}$, while a quotient object $A / A^{\prime}$ carries the induced filtration $\left(\left(F_{k} A+A^{\prime}\right) / A^{\prime}\right)_{k \in \mathbb{Z}}$. It is easy to see that the two possible induced filtrations on a subquotient $B / A^{\prime}$ of $A$ agree.

Definition 8.2.2 (Strictness of filtered complexes). Consider a complex $\left(C^{\bullet}, F\right)$ of filtered objects in A. This is a strict complex if all morphisms $d: C^{i} \rightarrow C^{i+1}$ are strict, in the sense that the isomorphism $\operatorname{Coim}(d) \rightarrow \operatorname{Im}(d)$ is an isomorphism of filtered objects, that is, we have

$$
d\left(F_{k} C^{i}\right)=F_{k} C^{i+1} \cap d\left(C^{i}\right) \quad \text { for all } k, i \in \mathbb{Z}
$$

Exercise 8.2.3. Show that a complex $\left(C^{\bullet}, F\right)$ which is bounded from above is strict if and only if the associated Rees complex $R_{F} C^{\bullet}$ is strict in the sense of Definition A.2.7.

Exercise 8.2.4. Show that $\left(C^{\bullet}, F\right)$ is strict if and only if the canonical morphism $\mathscr{H}^{i}\left(F_{k} C^{\bullet}\right) \rightarrow \mathscr{H}^{i}\left(C^{\bullet}\right)$ is a monomorphism for all $k, i \in \mathbb{Z}$.

Exercise 8.2.5. By considering the long exact sequence in cohomology for the exact sequence

$$
0 \longrightarrow F_{k} C^{\bullet} \longrightarrow C^{\bullet} \longrightarrow C^{\bullet} / F_{k} C^{\bullet} \longrightarrow 0
$$

show that if $\left(C^{\bullet}, F\right)$ is strict, then for every $i$ and $k$ we have a short exact sequence

$$
0 \longrightarrow \mathscr{H}^{i}\left(F_{k} C^{\bullet}\right) \longrightarrow \mathscr{H}^{i}\left(C^{\bullet}\right) \longrightarrow \mathscr{H}^{i}\left(C^{\bullet} / F_{k} C^{\bullet}\right) \longrightarrow 0
$$

Furthermore, show also that the map $\mathscr{H}^{i}\left(F_{k} C^{\bullet}\right) \rightarrow \mathscr{H}^{i}\left(F_{\ell} C^{\bullet}\right)$ is a monomorphism for every $k<\ell$, by considering the long exact sequence in cohomology corresponding to

$$
0 \longrightarrow F_{k} C^{\bullet} \longrightarrow F_{\ell} C^{\bullet} \longrightarrow F_{\ell} C^{\bullet} / F_{k} C^{\bullet} \longrightarrow 0
$$

and obtain a short exact sequence

$$
0 \longrightarrow \mathscr{H}^{i}\left(F_{k} C^{\bullet}\right) \longrightarrow \mathscr{H}^{i}\left(F_{\ell} C^{\bullet}\right) \longrightarrow \mathscr{H}^{i}\left(F_{\ell} C^{\bullet} / F_{k} C^{\bullet}\right) \longrightarrow 0
$$

for every $i \in \mathbb{Z}$.

Exercise 8.2.6. Show that if $\left(C^{\bullet}, F\right)$ is a strict complex, then for every $k \in \mathbb{Z}$, the complexes $\left(F_{k} C^{\bullet}, F\right)$ and $\left(C^{\bullet} / F_{k} C^{\bullet}, F\right)$, with the induced filtrations, are strict. In particular, using the second complex and Exercise 8.2.5, deduce that for every $k<$ $\ell<m$ and every $i$, we have short exact sequences

$$
0 \longrightarrow \mathscr{H}^{i}\left(F_{\ell} C^{\bullet} / F_{k} C^{\bullet}\right) \longrightarrow \mathscr{H}^{i}\left(C^{\bullet} / F_{k} C^{\bullet}\right) \longrightarrow \mathscr{H}^{i}\left(C^{\bullet} / F_{\ell} C^{\bullet}\right) \longrightarrow 0
$$

and

$$
0 \longrightarrow \mathscr{H}^{i}\left(F_{\ell} C^{\bullet} / F_{k} C^{\bullet}\right) \longrightarrow \mathscr{H}^{i}\left(F_{m} C^{\bullet} / F_{k} C^{\bullet}\right) \longrightarrow \mathscr{H}^{i}\left(F_{m} C^{\bullet} / F_{\ell} C^{\bullet}\right) \longrightarrow 0
$$

Exercise 8.2.7. Show that the complex $\left(C^{\bullet}, F\right)$ is strict if and only if all canonical morphisms $\mathscr{H}^{i}\left(F_{k} C^{\bullet}\right) \rightarrow \mathscr{H}^{i}\left(F_{k+1} C^{\bullet}\right)$ are monomorphisms. [Hint: It is clear that this condition is necessary; prove sufficiency by showing that the condition implies that $\mathscr{H}^{i}\left(F_{k} C^{\bullet}\right) \rightarrow \mathscr{H}^{i}\left(F_{\ell} C^{\bullet}\right)$ is a monomorphism for every $k<\ell$; use the exhaustivity of the filtration and the exactness of filtering direct limits to prove that $\mathscr{H}^{i}\left(C^{\bullet}\right) \simeq$ $\underline{\lim }_{\ell} \mathscr{H}^{i}\left(F_{\ell} C^{\bullet}\right)$.]

We will be interested in complexes of bi-filtered objects in A. These are objects of A carrying two filtrations $\left(A, F^{\prime}, F^{\prime \prime}\right)$. We write

$$
\begin{equation*}
F_{k}^{\prime} F_{\ell}^{\prime \prime} A:=F_{k}^{\prime} A \cap F_{\ell}^{\prime \prime} A \tag{8.2.8}
\end{equation*}
$$

The morphisms in this case are required to be compatible with each of the two filtrations.

Definition 8.2.9. Let $\left(C^{\bullet}, F^{\prime}, F^{\prime \prime}\right)$ be a complex of bi-filtered objects. We say that the complex is strict (or bistrict, if we want to emphasize the fact that we consider two filtrations) if for every $i, p$, and $q$, the natural maps in the commutative square

are injective, and furthermore, the square is Cartesian, that is, $\mathscr{H}^{i}\left(F_{k}^{\prime} F_{\ell}^{\prime \prime} C^{\bullet}\right)=$ $\mathscr{H}^{i}\left(F_{k}^{\prime} C^{\bullet}\right) \cap \mathscr{H}^{i}\left(F_{\ell}^{\prime \prime} C^{\bullet}\right)$.

Remark 8.2.10. It follows from Remark 8.2.4 that $\left(C^{\bullet}, F^{\prime}, F^{\prime \prime}\right)$ is strict if and only if all canonical morphisms

$$
\mathscr{H}^{i}\left(F_{k}^{\prime} C^{\bullet}\right) \longrightarrow F_{k}^{\prime} \mathscr{H}^{i}\left(C^{\bullet}\right), \quad \mathscr{H}^{i}\left(F_{\ell}^{\prime \prime} C^{\bullet}\right) \longrightarrow F_{\ell}^{\prime \prime} \mathscr{H}^{i}\left(C_{\bullet}\right),
$$

and

$$
\mathscr{H}^{i}\left(F_{k}^{\prime} F_{\ell}^{\prime \prime} C^{\bullet}\right) \longrightarrow F_{k}^{\prime} F_{\ell}^{\prime \prime} \mathscr{H}^{i}\left(C^{\bullet}\right)
$$

are isomorphisms.
Lemma 8.2.11. If $\left(C^{\bullet}, F^{\prime}, F^{\prime \prime}\right)$ is a strict complex of bi-filtered objects, then the complexes $\left(C^{\bullet}, F^{\prime}\right)$ and $\left(F_{k}^{\prime} C^{\bullet}, F^{\prime \prime}\right)$ are strict for every $k \in \mathbb{Z}$. In particular, we have a short exact sequence

$$
0 \longrightarrow \mathscr{H}^{i}\left(F_{k}^{\prime} F_{\ell}^{\prime \prime} C^{\bullet}\right) \longrightarrow \mathscr{H}^{i}\left(F_{k}^{\prime} F_{m}^{\prime \prime} C^{\bullet}\right) \longrightarrow \mathscr{H}^{i}\left(F_{k}^{\prime}\left(F_{m}^{\prime \prime} C^{\bullet} / F_{\ell}^{\prime \prime} C^{\bullet}\right)\right) \longrightarrow 0
$$

for every $\ell<m$ and every $i$. Furthermore, for every $k$, every $\ell<m<n$, and every $i$, we have short exact sequences
$0 \rightarrow \mathscr{H}^{i}\left(F_{k}^{\prime}\left(F_{m}^{\prime \prime} C^{\bullet} / F_{\ell}^{\prime \prime} C^{\bullet}\right)\right) \longrightarrow \mathscr{H}^{i}\left(F_{k}^{\prime}\left(C^{\bullet} / F_{\ell}^{\prime \prime} C^{\bullet}\right)\right) \longrightarrow \mathscr{H}^{i}\left(F_{k}^{\prime}\left(C^{\bullet} / F_{m}^{\prime \prime} C^{\bullet}\right)\right) \rightarrow 0$
and

$$
\begin{aligned}
& 0 \rightarrow \mathscr{H}^{i}\left(F_{k}^{\prime}\left(F_{m}^{\prime \prime} C^{\bullet} / F_{\ell}^{\prime \prime} C^{\bullet}\right)\right) \longrightarrow \mathscr{H}^{i}\left(F_{k}^{\prime}\left(F_{n}^{\prime \prime} C^{\bullet} / F_{\ell}^{\prime \prime} C^{\bullet}\right)\right) \\
& \longrightarrow \mathscr{H}^{i}\left(F_{k}^{\prime}\left(F_{n}^{\prime \prime} C^{\bullet} / F_{m}^{\prime \prime} C^{\bullet}\right)\right) \rightarrow 0
\end{aligned}
$$

Proof. The first assertion is an immediate consequence of the definition, while the exact sequences follow from the strictness of $\left(F_{k}^{\prime} C^{\bullet}, F^{\prime \prime}\right)$, using Remarks 8.2.5 and 8.2.6.

Lemma 8.2.12. If $\left(C^{\bullet}, F^{\prime}, F^{\prime \prime}\right)$ is a strict complex of bi-filtered objects, then for every $k<q$, the complex $\left(F_{k}^{\prime \prime} C^{\bullet} / F_{\ell}^{\prime \prime} C^{\bullet}, F^{\prime}\right)$ is strict. In particular, each complex $\left(\operatorname{gr}_{k}^{F^{\prime \prime}}\left(C^{\bullet}\right), F^{\prime}\right)$ is strict.

Proof. It follows from Lemma 8.2.11 that for every $s$ and $i$, in the following commutative diagram

the rows are exact. Furthermore, since $\left(C^{\bullet}, F^{\prime}, F^{\prime \prime}\right)$ is a strict complex, it follows that $u$ and $v$ are injective and the left square is Cartesian (this follows by describing all the objects that appear in that square as subobjects of $\left.\mathscr{H}^{i}\left(C^{\bullet}\right)\right)$. This implies that $w$ is injective, hence $\left(F_{\ell}^{\prime \prime} C^{\bullet} / F_{k}^{\prime \prime} C^{\bullet}, F^{\prime}\right)$ is a strict complex.

### 8.3. Compatible filtrations and strictness

We keep Convention 8.2.1.
8.3.a. Compatible filtrations. Suppose that $A$ is an object of our category A, and $A_{1}, \ldots, A_{n} \subseteq A$ are finitely many subobjects. When $n=1$, we have a short exact sequence of the form

$$
A_{1} \longrightarrow A \longrightarrow *
$$

where $*$ is of course just an abbreviation for the quotient $A / A_{1}$. When $n=2$, we similarly have a commutative diagram of the form

in which all rows and all columns are short exact sequences. (For example, the entry in the upper-right corner is $A /\left(A_{1}+A_{2}\right)$, the entry in the lower-left corner $A_{1} \cap A_{2}$.) Once $n \geqslant 3$, such a diagram no longer exists in general; if it does exist, one says that $A_{1}, \ldots, A_{n}$ are compatible subobjects of $A$. More precisely, the condition is the following: there should exist an $n$-dimensional commutative diagram $C\left(A_{1}, \ldots, A_{n} ; A\right)$, consisting of $3^{n}$ objects placed at the points $\{-1,0,1\}^{n}$ and $2 n \cdot 3^{n-1}$ morphisms corresponding to the line segments connecting those points, such that $A$ sits at the point $(0, \ldots, 0)$, each $A_{i}$ sits at the point $(0, \ldots,-1, \ldots, 0)$ on the $i$-th coordinate axis, and all lines parallel to the coordinate axes form short exact sequences in the abelian category. It is easy to see that the objects at points in $\{-1,0\}^{n}$ are just intersections: if the $i$-th coordinate of such a point is -1 for $i \in I \subset\{1, \ldots, n\}$ and 0 for $i \notin I$, then the exactness of the diagram forces the corresponding object to be

$$
\bigcap_{i \in I} A_{i},
$$

with the convention that the intersection equals $A$ when $I$ is empty. In particular, the object $A_{1} \cap \cdots \cap A_{n}$ always sits at the point with coordinates $(-1, \ldots,-1)$.

On the other hand, given a subset $I \subset\{1, \ldots, n\}$, fixing the coordinate $\varepsilon_{i}^{o} \in$ $\{-1,0,1\}$ for every $i \in I$ produces a sub-diagram of size $n-\# I$, hence $n-\# I$ compatible sub-objects of the term placed at $\left(\varepsilon_{i \in I}^{o}, 0_{i \notin I}\right)$, that we denote by $A\left(\varepsilon_{i \in I}^{o}, 0_{i \notin I}\right)$. For example, fixing $\varepsilon_{i}^{o}=0$ shows that the sub-family $\left(A_{i}\right)_{i \notin I}$ is a compatible family.

Exercise 8.3.1. Show similarly that the object $A\left(1_{i \in I}, 0_{i \notin I}\right)$ is equal to $A / \sum_{i \in I} A_{i}$.
As another example, fix $\varepsilon_{n}^{o}=-1$. Then the induced family $\left(A_{i} \cap A_{n}\right)_{i \in\{0, \ldots, n-1\}}$ of sub-objects of $A_{n}$ is also compatible.

As still another example, let us fix $\varepsilon_{n}^{o}=1$. We have an exact sequence

$$
A_{n}=A(0, \ldots, 0,-1) \longrightarrow A=A(0, \ldots, 0) \longrightarrow A / A_{n}=A(0, \ldots, 0,1)
$$

Our new diagram has central term $A / A_{n}$ and the term placed at $\left(0, \ldots,(-1)_{i}, \ldots, 0,1\right)$ is $A_{i} / A_{i} \cap A_{n}$. This means that the induced family $\left(A_{i} / A_{i} \cap A_{n}\right)_{i \in\{0, \ldots, n-1\}}$ is also compatible.

In the definition of compatibility, the object $A$ does not play a relevant role and one can replace it by a sub-object provided that all $A_{i}$ are contained in it. Similarly one can replace it by a sup-object. This is shown in the exercise below.

## Exercise 8.3.2.

(1) Let $A_{1}, \ldots, A_{n} \subset A$ be a compatible family of sub-objects of $A$ and let $B \supset A$. Show that $A_{1}, \ldots, A_{n}, A$ is a compatible family in $B$ (in particular, $A_{1}, \ldots, A_{n}$ is a compatible family in $B$ ). [Hint: note first that, for $\varepsilon=\left(\varepsilon_{1}, \ldots, \varepsilon_{n}\right)$ with $\varepsilon_{i} \geqslant 0$ forall $i$, $A$ surjects to $A(\varepsilon)$ and set $A(\varepsilon)=A / I(\varepsilon)$, with $I(0)=0$; define then $B\left(\varepsilon, \varepsilon_{n+1}\right)$ by

$$
\begin{aligned}
B(\varepsilon,-1) & =A(\varepsilon) \quad \forall \varepsilon, \\
B(\varepsilon, 0) & = \begin{cases}A(\varepsilon) & \text { if } \exists i, \varepsilon_{i}=-1, \\
B / I(\varepsilon) & \text { if } \forall i, \varepsilon_{i} \geqslant 0\end{cases} \\
B(\varepsilon, 1) & = \begin{cases}0 & \text { if } \exists i, \varepsilon_{i}=-1 \\
B / A & \text { if } \forall i, \varepsilon_{i} \geqslant 0\end{cases}
\end{aligned}
$$

check the exactness of sequences like $B\left(-1, \varepsilon^{\prime}, 0\right) \rightarrow B\left(0, \varepsilon^{\prime}, 0\right) \rightarrow B\left(1, \varepsilon^{\prime}, 0\right)$.]
(2) Let $A_{1}, \ldots, A_{n} \subset A$ be a family of sub-objects of $A$ which is compatible in $B$, for some $B \supset A$. Then this family is compatible in $A$. [Hint: set $A(\varepsilon)=B(\varepsilon)$ if $\varepsilon_{i}=-1$ for some $i$, and if $\varepsilon_{i} \geqslant 0$ for all $i$, set $A(\varepsilon)=A / I(\varepsilon)$, where $B(\varepsilon)=B / I(\varepsilon)$ and show first that $I(\varepsilon) \subset \sum_{i} A_{i}$ by using Exercise 8.3.1.]
(3) Let $A_{0}, A_{1}, \ldots, A_{n} \subset A$ be a family of sub-objects of $A$. Assume that $A_{1}, \ldots, A_{n-1} \subset A_{n}$. Show that the family $A_{0}, A_{1}, \ldots, A_{n}$ is compatible if and only if the family $A_{0} \cap A_{n}, A_{1}, \ldots, A_{n}$ of sub-objects of $A_{n}$ is compatible. [Hint: if the diagram $C\left(A_{0}, \ldots, A_{n} ; A\right)$ exists, there should be an exact sequence
$0 \rightarrow C\left(A_{0} \cap A_{n}, \ldots, A_{n} ; A_{n}\right) \rightarrow C\left(A_{0}, \ldots, A_{n} ; A\right) \rightarrow C\left(\frac{A_{0}}{A_{0} \cap A_{n}}, 0, \ldots, 0 ; \frac{A}{A_{n}}\right) \rightarrow 0$, corresponding to exact sequences

$$
0 \longrightarrow A\left(\varepsilon_{0}, \varepsilon^{\prime},-1\right) \longrightarrow A\left(\varepsilon_{0}, \varepsilon^{\prime}, 0\right) \longrightarrow A\left(\varepsilon_{0}, \varepsilon^{\prime}, 1\right)
$$

show that $A\left(\varepsilon_{0}, \varepsilon^{\prime}, 1\right)=0$ if $\varepsilon_{i}^{\prime}=-1$ for some $i=1, \ldots, n-1$; set thus $A\left(\varepsilon_{0}, \varepsilon^{\prime}, 0\right):=$ $A\left(\varepsilon_{0}, \varepsilon^{\prime},-1\right)$ for such an $\varepsilon^{\prime}$; to determine $A\left(\varepsilon_{0}, \varepsilon^{\prime}, 0\right)$ for $\varepsilon_{i}^{\prime} \geqslant 0$ for all $i$, use Exercise 8.3.1 if $\varepsilon_{0} \geqslant 0$ and deduce the case $\varepsilon_{0}=-1$; end by checking that all possibly exact sequences are indeed exact.]

Lemma 8.3.3. Let $A_{1}, \ldots, A_{n} \subset A$ be a family of sub-objects of $A$. Assume the following properties:
(1) $A_{1} \subset A_{2}$.
(2) Both sub-families $A_{1}, A_{3}, \ldots, A_{n}$ and $A_{2}, A_{3}, \ldots, A_{n}$ are compatible.

Then the family $A_{1}, \ldots, A_{n}$ is compatible. Moreover, the family $\left(A_{i} \cap A_{2}\right) /\left(A_{i} \cap A_{1}\right)$ ( $i=3, \ldots, n$ ) of sub-objects of $A_{2} / A_{1}$ is also compatible.

Proof. We wish to define a diagram with vertices $A\left(\varepsilon_{1}, \varepsilon_{2}, \ldots, \varepsilon_{n}\right)\left(\varepsilon_{i} \in\{-1,0,1\}\right)$ satisfying the properties above. The second assumption means that we have the diagrams with vertices $A\left(\varepsilon_{1}, 0, \varepsilon_{3}, \ldots, \varepsilon_{n}\right)$ and $A\left(0, \varepsilon_{2}, \ldots, \varepsilon_{n}\right)$. On the other hand,
if the diagram we search for exists, the inclusion $A_{1} \cap A_{2}=A_{1} \subset A_{2}$ is satisfied for all terms of the diagram, namely

$$
\begin{equation*}
A\left(-1,-1, \varepsilon_{\geqslant 3}\right)=A\left(-1,0, \varepsilon_{\geqslant 3}\right) \subset A\left(0,-1, \varepsilon_{\geqslant 3}\right) . \tag{8.3.4}
\end{equation*}
$$

We are thus forced to set

$$
\begin{align*}
A\left(1,-1, \varepsilon_{\geqslant 3}\right) & :=A\left(0,-1, \varepsilon_{\geqslant 3}\right) / A\left(-1,-1, \varepsilon_{\geqslant 3}\right) \\
A\left(1,1, \varepsilon_{\geqslant 3}\right) & :=A\left(0,1, \varepsilon_{\geqslant 3}\right) . \tag{8.3.5}
\end{align*}
$$

In such a way, we obtain a commutative diagram where the columns are exact sequences (by assumption for the middle one, by our setting for the left and right ones), as well as the middle horizontal line


It is then easy to check that the upper horizontal line is exact. This shows that, in the diagram of size $n$, the lines where $\varepsilon_{1}$ varies in $\{-1,0,1\}$ and all other $\varepsilon_{i}$ fixed, as well as the lines where $\varepsilon_{2}$ varies and all other $\varepsilon_{i}$ are fixed, are exact. Let us now vary $\varepsilon_{3}$, say, by fixing all other $\varepsilon_{i}$ and let us omit $\varepsilon_{i}$ for $i \geqslant 4$ in the notation. From the diagram above, we see that the only possibly non-obvious exact sequence has terms $A\left(1,-1, \varepsilon_{3}\right)_{\varepsilon_{3}=-1,0,1}$. We now consider the commutative diagram where the columns are exact and only the upper horizontal line is possibly non-exact:


But the snake lemma shows its exactness. We conclude that the family $A_{1}, A_{2}, \ldots, A_{n}$ is compatible. We now remark that

$$
A_{2} / A_{1}=A_{2} /\left(A_{1} \cap A_{2}\right)=A(1,-1,0, \ldots, 0)
$$

The compatibility of the family $\left(A_{i} \cap A_{2} / A_{i} \cap A_{1}\right)_{i=3, \ldots, n}$ will be proved if we prove $\left(A_{3} \cap A_{2}\right) /\left(A_{3} \cap A_{1}\right)=A(1,-1,-1,0, \ldots, 0)$, and similarly for $i \geqslant 4$. Let us consider the previous diagram when fixing $\varepsilon_{i}=0$ for $i \geqslant 4$. The left vertical inclusion reads $A_{1} \cap A_{2} \cap A_{3} \hookrightarrow A_{2} \cap A_{3}$, hence the desired equality.

The previous lemma can be taken the other way round, which can be used for an inductive construction of compatible families.

Lemma 8.3.8. Let $A_{1}, \ldots, A_{n} \subset A$ be a family of sub-objects of $A$. Assume the following properties:
(1) $A_{1} \subset A_{2}$.
(2) Both families $A_{1}, A_{3}, \ldots, A_{n}$ in $A$ and $A_{2} / A_{1}, A_{3} /\left(A_{3} \cap A_{1}\right), \ldots, A_{n} /\left(A_{n} \cap A_{1}\right)$ in $A / A_{1}$ are compatible.
Then the family $A_{1}, A_{2}, \ldots, A_{n}$ is compatible in $A$.
Proof. We argue as in Lemma 8.3.3, from which we keep the notation. We have the diagrams of exact sequences with $A\left(\varepsilon_{1}, 0, \varepsilon_{\geqslant 3}\right)$ and $A\left(1, \varepsilon_{2}, \varepsilon \geqslant 3\right)$. We must also have (8.3.4). It remains to determine $A\left(0, \varepsilon_{2}, \varepsilon_{\geqslant 3}\right)$ and check that the sequences, when $\varepsilon_{i}$ varies in $\{-1,0,1\}$ and all other $\varepsilon_{j}$ fixed, are exact. We know $A\left(0,0, \varepsilon_{\geqslant 3}\right)$, and as for (8.3.5), we must set

$$
A\left(0,1, \varepsilon_{\geqslant 3}\right):=A\left(1,1, \varepsilon_{\geqslant 3}\right),
$$

the latter term being known. We then search for a diagram similar to (8.3.6):

where $A\left(0,-1, \varepsilon_{\geqslant 3}\right)$ has to be chosen so that the left column is exact (the other ones being so), and we must then show that the middle line is exact (the other ones being so). Clearly $b$ is onto, so we are forced to set $A\left(0,-1, \varepsilon_{\geqslant 3}\right)=\operatorname{Ker} b$. The exactness of the left column is then left as an exercise. We now vary $\varepsilon_{3}$, say, and consider the corresponding sequences. The only possibly non-exact ones have $\varepsilon_{1}=0$ and we are left to examining the diagram


By the exactness of the middle line in (8.3.9), all columns are exact, and by assumption the middle line is exact. On the other hand, the upper line is identified with the similar line with $\varepsilon_{1}=1$, so is exact. Therefore, the lower line is also exact.

Definition 8.3.11 (Compatible filtrations). Given finitely many increasing filtrations $F_{\bullet}^{1} A, \ldots, F_{\bullet}^{n} A$ of an object $A$ in the abelian category, we call them compatible if

$$
F_{k_{1}}^{1} A, \ldots, F_{k_{n}}^{n} A \subseteq A
$$

are compatible sub-objects for every choice of $k_{1}, \ldots, k_{n} \in \mathbb{Z}$.

## Remark 8.3.12.

(1) As a consequence of our previous remarks, any sub-family of filtrations of a compatible family remains compatible. Moreover, by Lemma 8.3.3, any finite family of sub-objects consisting of terms of the filtrations $F_{\bullet}^{1} A, \ldots, F_{\bullet}^{n} A$ is compatible, and the last assertion of this lemma implies that the induced filtrations $F_{\bullet}^{1}, \ldots, F_{\bullet}^{n-1}$ on each $\operatorname{gr}_{\ell}^{F^{n}} A$ are compatible.
(2) Let $B=F_{j_{1}}^{1} \cap \cdots \cap F_{j_{n}}^{n}$ for some $j_{1}, \ldots, j_{n}$. Then the family of filtrations $F_{\bullet}^{1} B, \ldots, F_{\bullet}^{n} B$ naturally induced on $B$ is compatible, as follows from the compatibility of the family of $2 n$ sub-objects $F_{k_{1}}^{1} A, \ldots, F_{k_{n}}^{n} A, F_{j_{1}}^{1} A, \ldots, F_{j_{n}}^{n} A$ and that of the induced family on $B$.
8.3.b. Reformulation in terms of flatness. Let $A$ be an object with $n$ filtrations $F_{\bullet}^{1} A, \ldots, F_{\bullet}^{n} A$. As usual, we can pass from filtered to graded objects by the Rees construction. Let $R=\mathbb{C}\left[z_{1}, \ldots, z_{n}\right]$ denote the polynomial ring in $n$ variables, with the $\mathbb{Z}^{n}$-grading that gives $z_{i}$ the weight $e_{i}=(0, \ldots, 1, \ldots, 0)$. For $k \in \mathbb{Z}^{n}$, we define

$$
M_{k}=M_{k_{1}, \ldots, k_{n}}=F_{k_{1}}^{1} A \cap \cdots \cap F_{k_{n}}^{n} A \subseteq A
$$

We then obtain a $\mathbb{Z}^{n}$-graded module $M$ over the ring $R$ by taking the direct sum

$$
R_{F^{1}, \ldots, F^{n}} A:=M=\bigoplus_{k \in \mathbb{Z}^{n}} M_{k},
$$

with the obvious $\mathbb{Z}^{n}$-grading: for $m \in M_{k}$, the product $z_{i} m$ is simply the image of $m$ under the inclusion $M_{k} \subseteq M_{k+e_{i}}$. From now on, we use the term "graded" to mean " $\mathbb{Z}^{n}$-graded".

Theorem 8.3.13. A graded $R$-module comes from an object with $n$ compatible filtrations if and only if it is flat over $R$.

Before giving the proof, we recall a few general facts about flatness. For any commutative ring $R$, flatness of an $R$-module $M$ is equivalent to the condition that

$$
\operatorname{Tor}_{1}^{R}(M, R / I)=0
$$

for every finitely generated ideal $I \subseteq R$; when $R$ is Noetherian, it is enough to check this for all prime ideals $P \subseteq R$. In our setting, the ring $R$ is graded, and by a similar argument as in the ungraded case, flatness is equivalent to

$$
\operatorname{Tor}_{1}^{R}(M, R / P)=0
$$

for every graded prime ideal $P \subseteq R$. Of course, there are only finitely many graded prime ideals in $R=\mathbb{C}\left[z_{1}, \ldots, z_{n}\right]$, namely those that are generated by the $2^{n}$ possible subsets of the set $\left\{z_{1}, \ldots, z_{n}\right\}$. Moreover, the quotient $R / P$ always has a canonical free resolution given by the Koszul complex.

Example 8.3.14. For $n=1$, a graded $R$-module $M$ is flat if and only if $z_{1}: M \rightarrow M$ is injective. For $n=2$, a graded $R$-module $M$ is flat if and only if $z_{1}: M \rightarrow M$ and $z_{2}: M \rightarrow M$ are both injective and the Koszul complex

$$
M \xrightarrow{\left(-z_{2}, z_{1}\right)} M \oplus M \xrightarrow{z_{1} \bullet+z_{2} \bullet} M
$$

is exact in the middle. (Here we are ignoring the grading in the notation.) The Koszul complex is just the simple complex associated to the double complex

with Deligne's sign conventions. The exactness of the Koszul complex in the middle can be read on each graded term as $M_{k_{1}-1, k_{2}} \cap M_{k_{1}, k_{2}-1}=M_{k_{1}-1, k_{2}-1}$. In this way, it is clear that two filtrations give rise to a flat $R$-module, illustrating thereby Theorem 8.3.13.

## Exercise 8.3.15 (Applications of the flatness criterion).

(1) Let $A$ be an object with $n$ compatible filtrations $F_{\bullet}^{1} A, \ldots, F_{\bullet}^{n} A$. Show that any family of filtrations $G_{\bullet}^{1} A, \ldots, G_{\bullet}^{m} A$ where each $G_{\bullet}^{i} A$ is obtained by convolution of some of the filtrations $F_{\bullet}^{j} A$, i.e.,

$$
G_{p}^{i} A=\sum_{q_{1}+\cdots+q_{k}=p} F_{q_{1}}^{j_{1}} A+\cdots+F_{q_{k}}^{j_{k}} A
$$

(also denoted by $G_{\bullet}^{i} A=F_{\bullet}^{j_{1}} A \star \cdots \star F_{\bullet}^{j_{1}} A$ ) is also a compatible family. [Hint: express the Rees module $R_{G}^{i} A$ as obtained by base change from $R_{F^{j_{1}}, \ldots, F^{j_{k}}} A$ and, more generally express $R_{G^{1}, \ldots, G^{m}} A$ as obtained by base change from $R_{F^{1}, \ldots, F^{n}} A$; conclude by using that flatness is preserved by base change.]
(2) Let $F_{\bullet}^{1} A, \ldots, F_{\bullet}^{n} A$ be filtrations on $A$. Let $B$ be a sub-object of $A$ and let $F_{\bullet}^{i} B$ and $F_{\bullet}^{i}(A / B)$ be the induced filtrations. Assume that
(a) the families $\left(F_{\bullet}^{i} B\right)_{i}$ and $\left(F_{\bullet}^{i}(A / B)\right)_{i}$ are compatible,
(b) for all $k_{1}, \ldots, k_{n}$, the following sequence is exact:

$$
0 \longrightarrow \bigcap_{i=1}^{n} F_{k_{i}}^{i} B \longrightarrow \bigcap_{i=1}^{n} F_{k_{i}}^{i} A \longrightarrow \bigcap_{i=1}^{n} F_{k_{i}}^{i}(A / B) \longrightarrow 0
$$

Then the family $\left(F_{\bullet}^{i} A\right)_{i}$ is compatible. [Hint: show that there is an exact sequence of the associated Rees modules, and use that flatness of the extreme terms implies flatness of the middle term.]

Exactness of the Koszul complex is closely related to the concept of regular sequences. Recall that $z_{1}, \ldots, z_{n}$ form a regular sequence on $M$ if multiplication by $z_{1}$ is injective on $M$, multiplication by $z_{2}$ is injective on $M / z_{1} M$, multiplication by $z_{3}$ is injective on $M /\left(z_{1}, z_{2}\right) M$, and so on.

Lemma 8.3.16. A graded $R$-module $M$ is flat over $R$ if and only if any permutation of $z_{1}, \ldots, z_{n}$ is a regular sequence on $M$.

Proof. This is one of the basic properties of the Koszul complex. The point is that multiplication by $z_{1}$ is injective on $M$ if and only if the Koszul complex

$$
M \xrightarrow{z_{1}} M
$$

is a resolution of $M / z_{1} M$. If this is the case, multiplication by $z_{2}$ is injective on $M / z_{1} M$ if and only if the Koszul complex

$$
M \xrightarrow{\left(-z_{2}, z_{1}\right)} M \oplus M \xrightarrow{z_{1} \cdot+z_{2} \bullet} M
$$

is a resolution of $M /\left(z_{1}, z_{2}\right) M$, etc.
Proof of Theorem 8.3.13. Let us first show that if $F_{\bullet}^{1} A, \ldots, F_{\bullet}^{n} A$ are compatible filtrations, then the associated Rees module $M$ is flat over $R$. Because of the inherent symmetry, it is enough to prove that $z_{n}, \ldots, z_{1}$ form a regular sequence on $M$. Because $M$ comes from a filtered object, multiplication by $z_{n}$ is injective and

$$
M / z_{n} M=\bigoplus_{k \in \mathbb{Z}^{n}} M_{k_{1}, \ldots, k_{n}} / M_{k_{1}, \ldots, k_{n-1}, k_{n}-1}
$$

This is now a $\mathbb{Z}^{n}$-graded module over the polynomial ring $\mathbb{C}\left[z_{1}, \ldots, z_{n-1}\right]$. We remarked, after Definition 8.3.11, that for every $\ell \in \mathbb{Z}$, the $n-1$ induced filtrations on

$$
A_{\ell}=\operatorname{gr}_{\ell}^{F^{n}} A=F_{\ell}^{n} A / F_{\ell-1}^{n} A
$$

are still compatible, and that

$$
F_{k_{1}}^{1} A_{\ell} \cap \cdots \cap F_{k_{n-1}}^{n-1} A_{\ell} \simeq M_{k_{1}, \cdots, k_{n-1}, \ell} / M_{k_{1}, \ldots, k_{n-1}, \ell-1}
$$

By induction, this implies that $z_{n-1}, \ldots, z_{1}$ form a regular sequence on $M / z_{n} M$, which is what we wanted to show.

For the converse, suppose that $M$ is now an arbitrary graded $R$-module that is flat over $R$. We need to construct from $M$ an object with $n$ compatible filtrations. We consider the graded components $M_{k}$ as a directed system, indexed by $k \in \mathbb{Z}^{n}$, with morphisms given by multiplication by $z_{1}, \ldots, z_{n}$; since $M$ is flat, all these morphisms are injective. Since we are working in an abelian category in which all filtered direct limits exist and are exact, we can define

$$
A=\lim _{k \in \mathbb{Z}^{n}} M_{k}
$$

If we hold the $i$-th index fixed, the resulting direct limit determines a subobject $F_{k_{i}}^{i} A$, and in fact an increasing filtration $F_{\bullet}^{i} A$. We can use the flatness of $M$ to prove that these $n$ filtrations are compatible, and that

$$
\begin{equation*}
M_{k_{1}, \ldots, k_{n}}=F_{k_{1}}^{1} A \cap \cdots \cap F_{k_{n}}^{n} A \tag{8.3.17}
\end{equation*}
$$

as subobjects of $A$.

Fix $k, \ell \in \mathbb{Z}^{n}$. Observe that because $R$ is graded, the graded submodules $z_{1}^{\ell_{1}} R, \ldots, z_{n}^{\ell_{n}} R$ are trivially compatible; in fact, the required $n$-dimensional commutative diagram exists in the category of graded $R$-modules. If we tensor this diagram by $M$, it remains exact everywhere, due to the fact that $M$ is flat. Take the graded piece of degree $k+\ell$ everywhere; for $n=2$, for example, the result looks like this:


Apply the direct limit over $\ell \in \mathbb{Z}^{n}$; this operation preserves exactness. For $n=2$, for example, the resulting $n$-dimensional commutative diagram looks like this:


The existence of such a diagram proves that $F_{k_{1}}^{1} A, \ldots, F_{k_{n}}^{n} A$ are compatible subobjects of $A$, and also that (8.3.17) holds.

## Remark 8.3.18 (Interpretation of flatness in terms of multi-grading)

Lemma 8.3.16 has the following practical consequence: for compatible filtrations $F_{\bullet}^{1} A, \ldots, F_{\bullet}^{n} A$, the $n$-graded object obtained by inducing iteratively the filtrations on the $j$-graded object $\operatorname{gr}_{k_{i_{j}}}^{F^{i}{ }_{j}} \cdots \operatorname{gr}_{k_{i_{1}}}^{F_{1}} A(j=1, \ldots, n)$ does not depend on the order $\left\{i_{1}, \ldots, i_{n}\right\}=\{1, \ldots, n\}$, and is equal to

$$
\frac{F_{k_{1}}^{1} A \cap \cdots \cap F_{k_{n}}^{n} A}{\sum_{j} F_{k_{1}}^{1} A \cap \cdots \cap F_{k_{j}-1}^{j} A \cap \cdots \cap F_{k_{n}}^{n} A} .
$$

Remark 8.3.19 (Multi-filtered morphisms). Let $\left(A,\left(F_{\bullet}^{i} A\right)_{i=1, \ldots, n}\right)$ and $\left(B,\left(F_{\bullet}^{i} B\right)_{i=1, \ldots, n}\right)$ be two multi-filtered objects in A and let $\varphi: A \rightarrow B$ be a morphism compatible with the filtrations. It induces various morphisms $\operatorname{gr}_{k_{i_{j}}}^{F_{j}} \cdots \operatorname{gr}_{k_{i_{1}}}^{F_{1}} \varphi$. Assume that the filtrations in $A$ and in $B$ are compatible. Then the source and the target of these morphisms are independent of the order of multi-grading, as remarked above. We claim that the morphisms $\operatorname{gr}_{k_{i_{j}}}^{F^{i j}} \cdots \operatorname{gr}_{k_{i_{1}}}^{F^{i}} \varphi$ are also independent of the order of multi-grading. Indeed, $\varphi$ induces a graded morphism $R_{F} \varphi: M \rightarrow N$ between the associated Rees objects, and due to the compatibility assumption, we are led to
checking that the restriction of $R_{F} \varphi$ to $M /\left(z_{k_{i_{1}}}, \ldots, z_{k_{i_{j}}}\right) M$ is independent of the order, which is clear.

Exercise 8.3.20. We keep the notation as in Lemma 8.3.16.
(1) Show that the sequence $z_{1}, \ldots, z_{n}$ is a regular sequence on $M$ if and only if for every $k=1, \ldots, n$, the Koszul complex $K\left(z_{1}, \ldots, z_{k} ; M\right)$ is a graded resolution of $M /\left(z_{1}, \ldots, z_{k}\right) M$.
(2) Deduce that the following properties are equivalent:
(a) any permutation of $z_{1}, \ldots, z_{n}$ is a regular sequence on $M$,
(b) any subsequence of $z_{1}, \ldots, z_{n}$ is a regular sequence on $M$,
(c) for every subset $J \subset\{1, \ldots, n\}$ the Koszul complex $K\left(\left(z_{j}\right)_{j \in J} ; M\right)$ is a graded resolution of $M /\left(z_{j}\right)_{j \in J} M$.
8.3.c. Flatness criterion. Under certain conditions on the graded $R$-module $M$, one can deduce flatness from the vanishing of the single $R$-module

$$
\operatorname{Tor}_{1}^{R}\left(M, R /\left(z_{1}, \ldots, z_{n}\right) R\right)
$$

In the case of local rings, this kind of result is usually called the "local criterion for flatness". The simplest example is when $M$ is finitely generated as an $R$-module, which is to say that all the filtrations are bounded from below.

Proposition 8.3.21. If $M$ is a finitely generated graded $R$-module, then the vanishing of $\operatorname{Tor}_{1}^{R}\left(M, R /\left(z_{1}, \ldots, z_{n}\right) R\right)$ implies that $M$ is flat.

Proof. This is a general result in commutative algebra. To show what is going on, let us give a direct proof in the case $n=2$. By assumption, the Koszul complex

$$
M \xrightarrow{\left(-z_{2}, z_{1}\right)} M \oplus M \xrightarrow{z_{1} \bullet+z_{2} \bullet} M
$$

is exact in the middle. It follows quite easily that multiplication by $z_{1}$ is injective. Indeed, if there is an element $m \in M_{i, j}$ with $z_{1} m=0$, then the pair $(m, 0)$ is in the kernel of the differential $\left(z_{1}, z_{2}\right)$, and therefore $m=-z_{2} m^{\prime}$ and $0=z_{1} m^{\prime}$ for some $m^{\prime} \in M_{i, j-1}$. Continuing in this way, we eventually arrive at the conclusion that $m=0$, because $M_{i, j}=0$ for $j \ll 0$. For the same reason, multiplication by $z_{2}$ is injective; but now we have checked the condition in the definition of flatness for all graded prime ideals in $R$.
8.3.d. Strictness. Let $A$ and $B$ be two objects in our abelian category A, each with $n$ compatible filtrations $F_{\bullet}^{1} A, \ldots, F_{\bullet}^{n} A$ respectively $F_{\bullet}^{1} B, \ldots, F_{\bullet}^{n} B$. Denote by $M$ and $N$ the graded $R$-modules that are obtained by the Rees construction; both are flat by Theorem 8.3.13. Now consider a filtered morphism $\varphi: A \rightarrow B$. It induces an $R$-linear morphism $R_{F} \varphi: M \rightarrow N$ between the two Rees modules.

Definition 8.3.22. We say that $\varphi: A \rightarrow B$ is strict if $\operatorname{Coker} R_{F} \varphi$ is again a flat $R$-module.

Flatness of Coker $R_{F} \varphi$ implies also that $\operatorname{Ker} R_{F} \varphi$ and $\operatorname{Im} R_{F} \varphi$ are flat: the reason is that we have two short exact sequences
and

$$
\begin{gathered}
0 \longrightarrow \operatorname{Ker} R_{F} \varphi \longrightarrow M \longrightarrow \operatorname{Im} R_{F} \varphi \longrightarrow 0 \\
0 \longrightarrow \operatorname{Im} R_{F} \varphi \longrightarrow N \longrightarrow \operatorname{Coker} R_{F} \varphi \longrightarrow 0
\end{gathered}
$$

and because $M$ and $N$ are both flat, flatness of Coker $R_{F} \varphi$ implies that of $\operatorname{Im} R_{F} \varphi$, which implies that of $\operatorname{Ker} R_{F} \varphi$. Note that $\operatorname{Ker} \varphi$ and $\operatorname{Coker} \varphi$ are equipped with filtrations $F_{\bullet}^{1} \operatorname{Ker} \varphi, \ldots, F_{\bullet}^{n} \operatorname{Ker} \varphi$ respectively $F_{\bullet}^{1} \operatorname{Coker} \varphi, \ldots, F_{\bullet}^{n} \operatorname{Coker} \varphi$ naturally induced from those on $A$ and $B$. If $\varphi$ is strict, we have

$$
\operatorname{Ker} R_{F} \varphi=R_{F} \operatorname{Ker} \varphi, \quad \operatorname{Im} R_{F} \varphi=R_{F} \operatorname{Im} \varphi, \quad \operatorname{Coker} R_{F} \varphi=R_{F} \operatorname{Coker} \varphi
$$

Indeed, we know by Theorem 8.3.13 that the graded modules $\operatorname{Ker} R_{F} \varphi, \operatorname{Im} R_{F} \varphi$ and Coker $R_{F} \varphi$ are attached to compatible filtrations, and (for Coker $\varphi$ for example) the term in degree $k \in \mathbb{Z}^{n}$ is $\left(F_{k_{1}}^{1} B \cap \cdots \cap F_{k_{n}}^{n} B\right)+\operatorname{Im} \varphi / \operatorname{Im} \varphi$, so that the compatible filtrations on Coker $\varphi$ given by the theorem are nothing but the filtrations induced by $F_{\bullet}^{i} B$.

For example, in the case of two filtrations $F^{\prime}, F^{\prime \prime}$ as considered in Definition 8.2.9, the last equality in bi-degree $k, \ell$ gives

$$
F_{k}^{\prime} F_{\ell}^{\prime \prime} B / \varphi\left(F_{k}^{\prime} F_{\ell}^{\prime \prime} A\right)=\left(F_{k}^{\prime} B+\operatorname{Im} \varphi\right) \cap\left(F_{\ell}^{\prime \prime} B+\operatorname{Im} \varphi\right) / \operatorname{Im} \varphi
$$

which corresponds to the condition of Definition 8.2.9.
Caveat 8.3.23. The strictness of $\varphi$ implies that the induced filtrations (on $\operatorname{Ker} \varphi, \operatorname{Im} \varphi$ and Coker $\varphi$ ) are compatible. However, the latter condition is not enough for strictness. For example, two filtrations are always compatible, while a morphism between bi-filtered objects need not be strict.

Example 8.3.24 (Strict inclusions). The composition of strict morphisms need not be strict in general. However, the composition of strict monomorphisms $i_{1}, i_{2}$ between objects with compatible filtrations remains a strict monomorphism since $\operatorname{Coker} R_{F}\left(i_{2} \circ i_{1}\right)=\operatorname{Coker}\left(R_{F} i_{2} \circ R_{F} i_{1}\right)$ is an extension of Coker $R_{F} i_{2}$ by Coker $R_{F} i_{1}$, and flatness is preserved by extensions.

Given $n$ compatible filtrations $F_{\bullet}^{1} A, \ldots, F_{\bullet}^{n} A$, they induce compatible filtrations on $M_{k}:=F_{k_{1}}^{1} A \cap \cdots \cap F_{k_{n}}^{n} A$ for every $k=\left(k_{1}, \ldots, k_{n}\right) \in \mathbb{Z}^{n}$ (see Remark 8.3.12). Moreover, for $k \leqslant \ell \in \mathbb{Z}^{n}$ (i.e., $k_{i} \leqslant \ell_{i}$ for all $i=1, \ldots, n$ ), the inclusion $M_{k} \hookrightarrow M_{\ell}$ is strict. Indeed, by the preliminary remark, it is enough to show that the inclusion $M_{k-1_{i}} \hookrightarrow M_{k}$ is strict for all $i$. This has been explained in the first part of the proof of Theorem 8.3.13.
Lemma 8.3.25 (A criterion for strictness of inclusions). Let $\left(A, F_{\bullet}^{0} A, F_{\bullet}^{1} A, \ldots, F_{\bullet}^{n} A\right)$ and $\left(B, F_{\bullet}^{0} B, F_{\bullet}^{1} B, \ldots, F_{\bullet}^{n} B\right)$ be multi-filtered objects of A and let $\varphi$ be a multi-filtered monomorphism between them. Assume the following properties:
(a) $F_{p}^{0} B=0$ for $p \ll 0$,
(b) $\varphi$ is $F^{i}$-strict for $i=0,1, \ldots, n$ (i.e., $F_{p}^{i} A=F_{p}^{i} B \cap A$ ),
(c) for each $p$, the induced filtrations $F^{1}, \ldots, F^{n}$ on $\operatorname{gr}_{p}^{F^{0}} A, \operatorname{gr}_{p}^{F^{0}} B$ are compatible and $\operatorname{gr}_{p}^{F^{0}} \varphi: \operatorname{gr}_{p}^{F^{0}} A \rightarrow \operatorname{gr}_{p}^{F^{0}} B$ is an $n$-strict monomorphism.

Then the filtrations $F^{1}, \ldots, F^{n}$ on $A, B$ are compatible and $\varphi$ is an $n$-strict monomorphism.

Proof. We denote by $F^{\prime}$ the $n$-multi-filtration forgetting $F^{0}$ and by $C$ the cokernel of $\varphi: A \rightarrow B$, equipped with the induced filtrations. By the second assumption, the sequence

$$
0 \longrightarrow R_{F^{\prime}} A \longrightarrow R_{F^{\prime}} B \longrightarrow R_{F^{\prime}} C
$$

is exact, and we wish to complete it into a short exact sequence. By the same assumption, Coker $\operatorname{gr}_{p}^{F^{0}} \varphi=\operatorname{gr}_{p}^{F^{0}} C$ for every $p$. Then the third assumption says that $R_{F^{\prime}} \operatorname{gr}_{p}^{F^{0}} B \rightarrow R_{F^{\prime}} \operatorname{gr}_{p}^{F^{0}} C$ is an epimorphism for every $p, \operatorname{and}_{\operatorname{gr}}^{p}{ }_{p}^{F^{0}} R_{F^{\prime}} C$ is $\mathbb{C}\left[z_{1}, \ldots, z_{n}\right]$ flat. Let us also note that $R_{F^{\prime}} \mathrm{gr}_{p}^{F^{0}}=\operatorname{gr}_{p}^{F^{0}} R_{F^{\prime}}$.

By induction on $p$ and because of the first assumption, we have a diagram where all terms except possibly those of the middle line are $\mathbb{C}\left[z_{1}, \ldots, z_{n}\right]$-flat and all sequences are exact:


It follows that all terms of the middle line are $\mathbb{C}\left[z_{1}, \ldots, z_{n}\right]$-flat, since flatness is preserved by extensions, and the middle line can be completed as an exact sequence. Taking the limit for $p \rightarrow \infty$ implies that the sequence

$$
0 \longrightarrow R_{F^{\prime}} A \longrightarrow R_{F^{\prime}} B \longrightarrow R_{F^{\prime}} C \longrightarrow 0
$$

is exact and all its terms are $\mathbb{C}\left[z_{1}, \ldots, z_{n}\right]$-flat.
If we have a complex of objects with $n$ compatible filtrations and differentials that preserve the filtrations, we consider the associated complex of flat graded $R$ modules; if all of its cohomology modules are again flat over $R$, we say that the original filtered complex is strict. At least if the complex is bounded from above, a similar argument with short exact sequences proves that strictness of a complex is equivalent to strictness of the individual differentials.

### 8.4. Specializability of filtered $\mathscr{D}_{X}$-modules

In the remaining part of this chapter, we will work with right $\mathscr{D}_{X}$-modules, since we are mainly interested in the pushforward theorem. Accordingly, we will consider increasing $V$-filtrations.

We assume that $X=H \times \Delta_{t}$, where $\Delta_{t}$ is a disc with coordinate $t$ and we set $X_{0}=H \times\{0\} \subset X$. We use the notion of a coherent $V$-filtration for a coherent $\mathscr{D}_{X}$-module $\mathcal{M}$ as defined in Section 7.3, as well as the notion of $\mathbb{R}$-specializability. Since we are dealing with $\mathscr{D}_{X}$-modules, the strictness property is not involved.

We now turn to $\mathbb{R}$-specializability for filtered $\mathscr{D}_{X}$-modules. Suppose that $(\mathcal{M}, F)$ is a coherent filtered $\mathscr{D}_{X}$-module (the coherence condition means that the $\operatorname{gr}^{F}\left(\mathscr{D}_{X}\right)$ module $\operatorname{gr}{ }^{F} \mathcal{M}:=\oplus_{m} F_{m} \mathcal{M} / F_{m-1} \mathcal{M}$ is coherent). One says that $(\mathcal{M}, F)$ is $\mathbb{R}$-specializable along $H$ if the following hold:
(a) $\mathcal{M}$ is $\mathbb{R}$-specializable along $H$ with $V$-filtration denoted by $V$ • $\mathcal{M}$.
(b) $\left(F_{p} V_{\alpha} \mathcal{M}\right) \cdot t=F_{p} V_{\alpha-1} \mathcal{M}$ for all $p \in \mathbb{Z}$ and $\alpha<0$.
(c) $\left(F_{p} \operatorname{gr}_{\alpha}^{V} \mathcal{M}\right) \cdot \partial_{t}=F_{p+1} \operatorname{gr}_{\alpha+1}^{V} \mathcal{M}$ for all $p \in \mathbb{Z}$ and $\alpha>-1$.
(We do not use the terminology "strictly $\mathbb{R}$-specializable" since strictness is included in the fact that we only consider filtered $\mathscr{D}_{X}$-modules.)

In the above, we have set $F_{p} V_{\alpha} \mathcal{M}:=F_{p} \mathcal{M} \cap V_{\alpha} \mathcal{M}$ as in (8.2.8), and $F_{p} \operatorname{gr}_{\alpha}^{V} \mathcal{M}:=$ $F_{p} V_{\alpha} \mathcal{M} / F_{p} V_{<\alpha} \mathcal{M}$. Note that, arguing as in Proposition 7.3.28, Condition (b) says that multiplication by $t$ induces an isomorphism $F_{p} V_{\alpha} \mathcal{M} \rightarrow F_{p} V_{\alpha-1} \mathcal{M}$ and Condition (c) implies that multiplication by $\partial_{t}$ gives an isomorphism $F_{p} \operatorname{gr}_{\alpha}^{V} \mathcal{M} \rightarrow F_{p+1} \operatorname{gr}_{\alpha+1}^{V} \mathcal{M}$. Of course, the inclusions " $\subseteq$ " in (b) and (c) always hold for every $\alpha \in \mathbb{R}$. We also note that each $\left(\operatorname{gr}_{\alpha}^{V} \mathcal{M}, F\right)$ is a filtered $\mathscr{D}_{X_{0}}$-module.

As above, in the presence of a nonzero $g \in \mathscr{O}(X)$, we consider the graph embedding $\iota_{g}: X \rightarrow X \times \mathbb{A}_{\mathbb{C}}^{1}$. Given a filtered $\mathscr{D}_{X}$-module $(\mathcal{M}, F)$ on $X$, we say that $(\mathcal{M}, F)$ is $\mathbb{R}$-specializable along $(g)$ if $\iota_{g_{*}}(\mathcal{M}, F)$ is so along $H \subset X \times \mathbb{A}_{\mathbb{C}}^{1}$. One can show that if $(g=0)$ is smooth, then this condition holds if and only if $(\mathcal{M}, F)$ is $\mathbb{R}$-specializable along (g) (see Exercise 7.3.31(4)).

### 8.5. Strictness criterion for complexes of filtered $\mathscr{D}$-modules

8.5.a. Setup. Assume that $X=H \times \Delta_{t}$ and set $X_{0}=X \times\{0\}$. We consider a bounded complex

$$
\cdots \longrightarrow \mathcal{M}^{i-1} \xrightarrow{d} \mathcal{M}^{i} \xrightarrow{d} \mathcal{M}^{i+1} \longrightarrow \cdots
$$

of $\mathscr{D}_{X}$-modules. We set $X=H \times \Delta_{t}$. We make the following assumptions:
(a) Each $\mathcal{N}^{i}$ has an increasing filtration $F_{\cdot} \mathcal{N}^{i}$ by $\mathscr{O}_{X}$-submodules, exhaustive, locally bounded below, and compatible with the order filtration on $\mathscr{D}_{X}$.
(b) Each $\mathcal{M}^{i}$ has an increasing filtration $V_{\bullet} \mathcal{N}^{i}$ by $\mathscr{O}_{X}$-submodules, discretely indexed by $\mathbb{R}$, on which $t$ and $\partial_{t}$ act in the usual way.
(c) The differentials $d: \mathcal{M}^{i} \rightarrow \mathcal{M}^{i+1}$ respect both filtrations $F_{\bullet} \mathcal{N}^{i}$ and $V_{\bullet} \mathcal{M}^{i}$.
(d) The $\mathscr{O}_{X}$-modules $\mathscr{H}^{i}\left(F_{p} V_{\alpha} \mathcal{N}{ }^{\bullet}\right)$ are coherent for every $i, p \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$.
(e) The morphism $t: F_{p} V_{\alpha} \mathcal{M}^{i} \rightarrow F_{p} V_{\alpha-1} \mathcal{M}^{i}$ is an isomorphism for $i, p \in \mathbb{Z}$ and $\alpha<0$.
(f) The morphism $\partial_{t}: F_{p} \operatorname{gr}_{\alpha}^{V} \mathcal{M}^{i} \rightarrow F_{p+1} \operatorname{gr}_{\alpha+1}^{V} \mathcal{M}^{i}$ is an isomorphism for $i, p \in \mathbb{Z}$ and $\alpha>-1$.
(g) For every $\alpha \in \mathbb{R}$, the operator $t \partial_{t}-\alpha$ acts nilpotently on $\mathscr{H}^{i}\left(\operatorname{gr}_{\alpha}^{V} \mathcal{N}^{\bullet}\right)$.
(h) For every $\alpha \in[-1,0]$, the complex $\operatorname{gr}_{\alpha}^{V} \mathcal{M}^{\bullet}$, with the induced differential and the filtration induced by $F_{\bullet} \mathcal{M}^{\bullet}$, is strict.
(i) For every $i \in \mathbb{R}$, the Rees module $\bigoplus_{p \in \mathbb{Z}} \mathscr{H}^{i}\left(F_{p} \mathcal{N}^{\bullet}\right)$ is coherent over $R_{F} \mathscr{D}_{X}$.

Let us denote by $\left(M^{\bullet}, d\right)$ the resulting complex of graded modules over the ring $R=\mathbb{C}[z, v]$; here the $z$-variable goes with the filtration $F_{\bullet} \mathcal{M}^{i}$, and the $v$-variable with the filtration $V_{\bullet} \mathcal{N}^{i}$. Since the latter is indexed by $\mathbb{R}$, this needs a little bit of care. Because we are dealing with a bounded complex, we can choose an increasing sequence of real numbers $\alpha_{k} \in \mathbb{R}$, indexed by $k \in \mathbb{Z}$, such that all the jumps in the filtrations $V_{\bullet} \mathcal{N}^{b}$ happen at some $\alpha_{k}$; we then define

$$
M_{j, k}^{i}=F_{j} V_{\alpha_{k}} \mathcal{M}^{i}
$$

for $i, j, k \in \mathbb{Z}$. This makes each

$$
M^{i}=\bigoplus_{j, k \in \mathbb{Z}} M_{j, k}^{i}
$$

into a $\mathbb{Z}^{2}$-graded module over the ring $R$; since the differentials in the original complex are compatible with both filtrations, they induce morphisms of graded $R$-modules $d: M^{i} \rightarrow M^{i+1}$.

Theorem 8.5.1. The complex $\left(\mathcal{M}^{\bullet}, d\right)$, endowed with the two filtrations $F_{0} \mathcal{M}^{\bullet}$ and $V_{\bullet} \mathcal{M}^{\bullet}$, is strict on an open neighborhood of $X_{0}$.

In contrast with the analogous proposition 7.8.7, the proof we give here does not use completions.
8.5.b. Proof of Theorem 8.5.1. Note first that each $M^{i}$ is a flat $R$-module. Using the above definition of the complex $\left(M^{\bullet}, d\right)$, we clearly have

$$
\left(M^{\bullet} / y M^{\bullet}\right)_{j, k}=\frac{F_{j} V_{\alpha_{k}} \mathcal{M}^{\bullet}}{F_{j} V_{\alpha_{k-1}} \mathcal{M} \mathcal{M}^{\bullet}}=F_{j} \operatorname{gr}_{\alpha_{k}}^{V} \mathcal{M}^{\bullet} .
$$

The condition in (h) has the following interpretation.
Lemma 8.5.2. All cohomology modules of the complex $\left(M^{\bullet} / y M^{\bullet}, d\right)$ are flat over the ring $R / y R=\mathbb{C}[x]$.

Proof. Together with (e) and (f), the condition in (h) says that the complex $\operatorname{gr}_{\alpha}^{V} \mathcal{N}^{\bullet}$ is strict for every $\alpha \in \mathbb{R}$. In terms of graded modules, this means that multiplication by $x$ is injective on the cohomology of the complex $M^{\bullet} / y M^{\bullet}$, which is equivalent to flatness over the ring $\mathbb{C}[z]$.

The next step in the proof involves a local argument, and so we fix a point $x \in X_{0}$ and localize everything at $x$. Although we keep the same notation as above, in the remainder of this section, each $\mathcal{N}^{i}$ is a $\mathscr{D}_{X, x}$-module, the condition in (d) reads $\mathscr{H}^{i}\left(F_{p} V_{\alpha} \mathcal{M}^{\bullet}\right)$ is a finitely generated $\mathscr{O}_{X, x}$-module, etc. With this convention in place, consider the short exact sequence of complexes

$$
0 \longrightarrow M^{\bullet} \longrightarrow M^{\bullet} \longrightarrow M^{\bullet} / v M^{\bullet} \longrightarrow 0
$$

in which the morphism from $M^{\bullet}$ to $M^{\bullet}$ is multiplication by $v$. (To keep the notation simple, we are leaving out the change in the grading.) The resulting long exact sequence in cohomology looks like this:
$\cdots \longrightarrow H^{i}\left(M^{\bullet}\right) \longrightarrow H^{i}\left(M^{\bullet}\right) \longrightarrow H^{i}\left(M^{\bullet} / v M^{\bullet}\right) \longrightarrow H^{i+1}\left(M^{\bullet}\right) \longrightarrow H^{i+1} M^{\bullet} \longrightarrow \cdots$
The following result constitutes the heart of the proof.
Proposition 8.5.3. The connecting homomorphisms $\delta: H^{i}\left(M^{\bullet} / v M^{\bullet}\right) \rightarrow H^{i+1}\left(M^{\bullet}\right)$ in the long exact sequence are trivial.

Once we have proved the proposition, we will know that the multiplication morphisms $v: H^{i}\left(M^{\bullet}\right) \rightarrow H^{i}\left(M^{\bullet}\right)$ are injective and that

$$
\frac{H^{i}\left(M^{\bullet}\right)}{v H^{i}\left(M^{\bullet}\right)} \simeq H^{i}\left(M^{\bullet} / v M^{\bullet}\right)
$$

Together with Lemma 8.5.2, this will tell us that $v, z$ is a regular sequence on $H^{i}\left(M^{\bullet}\right)$, which is two thirds of what we need to prove that $H^{i}\left(M^{\bullet}\right)$ is a flat $R$-module.

In preparation for the proof, let us consider the graded pieces in a fixed bidegree $(j, k)$ in the long exact sequence; to simplify the notation, set $p=j$ and $\alpha=\alpha_{k}$. We then have the following commutative diagram with exact rows and columns:


Here $\beta<\alpha$, and the notation $V_{(\beta, \alpha)} \mathcal{M}^{\bullet}$ is an abbreviation for $V_{<\alpha} \mathcal{M}^{\bullet} / V_{\beta} \mathcal{M}^{\bullet}$. We observe that the morphism $\varepsilon$ is trivial because the source and the target have different "weights" with respect to the action of the operator $t \partial_{t}$.

Lemma 8.5.4. With notation as above, the morphism

$$
\varepsilon: \mathscr{H}^{i}\left(F_{p} \operatorname{gr}_{\alpha}^{V} \mathcal{M}^{\bullet}\right) \longrightarrow \mathscr{H}^{i+1}\left(F_{p} V_{(\beta, \alpha)} \mathcal{M}^{\bullet}\right)
$$

is trivial.

Proof. We have a commutative diagram

in which the two vertical morphisms are injective because of (h). Now the operator $t \partial_{t}$ acts on the $\mathscr{O}_{X}$-module in the lower left corner with $\alpha$ as its only eigenvalue, and on the $\mathscr{O}_{X}$-module in the lower right corner with eigenvalues contained in the interval $(\beta, \alpha)$; this is a consequence of $(\mathrm{g})$. Since the bottom arrow is compatible with the action of $t \partial_{t}$, it must be zero; but then $\varepsilon$ is also zero.

We conclude from the lemma that the image of

$$
\delta: \mathscr{H}^{i}\left(F_{p} \operatorname{gr}_{\alpha}^{V} \mathcal{M}^{\bullet}\right) \rightarrow \mathscr{H}^{i+1}\left(F_{p} V_{<\alpha} \mathcal{M}^{\bullet}\right)
$$

is contained in the intersection

$$
\bigcap_{\beta<\alpha} \operatorname{Im}\left(\mathscr{H}^{i+1}\left(F_{p} V_{\beta} \mathcal{M}^{\bullet}\right) \rightarrow \mathscr{H}^{i+1}\left(F_{p} V_{<\alpha} \mathcal{M} \bullet\right)\right)
$$

We can now use (e) and Krull's intersection theorem to prove that this intersection is trivial (in the local ring $\mathscr{O}_{X, x}$ ).

Lemma 8.5.5. We have

$$
\bigcap_{\beta<\alpha} \operatorname{Im}\left(\mathscr{H}^{i}\left(F_{p} V_{\beta} \mathcal{M}^{\bullet}\right) \rightarrow \mathscr{H}^{i}\left(F_{p} V_{\alpha} \mathcal{M}^{\bullet}\right)\right)=\{0\}
$$

Proof. Consider the following commutative diagram:


Suppose that we have an element $m \in F_{p} V_{\alpha} \mathcal{N}^{i}$ with $d m=0$ that belongs to the image of $\mathscr{H}^{i}\left(F_{p} V_{\beta} \mathcal{M}^{\bullet}\right)$. Then

$$
m=d m_{0}+m_{1}
$$

for some $m_{0} \in F_{p} V_{\alpha} \mathcal{M}^{i-1}$ and some $m_{1} \in F_{p} V_{\beta} \mathcal{M}^{i}$. Now if $\beta<-1$, then by (e), we have $m_{1}=m_{2} t$ for a unique $m_{2} \in F_{p} V_{\beta+1} \mathcal{N}^{i}$. Since multiplication by $t$ is injective on $F_{p} V_{\beta+1} \mathcal{N}^{i+1}$, the fact that $d m_{1}=0$ implies that $d m_{2}=0$. As long as $\beta+1 \leqslant \alpha$, we also have

$$
m_{2} t \in\left(F_{p} V_{\beta+1} \mathcal{N}^{i}\right) \cdot t \subseteq\left(F_{p} V_{\alpha} \mathcal{M}^{i}\right) \cdot t
$$

and therefore $m \in d\left(F_{p} V_{\alpha} \mathcal{M}^{i-1}\right)+\left(F_{p} V_{\alpha} \mathcal{M}^{i}\right) \cdot t$. By this type of argument, one shows more generally that

$$
\bigcap_{\beta<\alpha} \operatorname{Im}\left(\mathscr{H}^{i}\left(F_{p} V_{\beta} \mathcal{M}^{\bullet}\right) \rightarrow \mathscr{H}^{i}\left(F_{p} V_{\alpha} \mathcal{M}^{\bullet}\right)\right) \subseteq \bigcap_{m \in \mathbb{N}} \mathscr{H}^{i}\left(F_{p} V_{\alpha} \mathcal{M} \cdot \bullet\right) \cdot t^{m}
$$

Since $\mathscr{H}^{i}\left(F_{p} V_{\alpha} \mathcal{M}^{\bullet}\right)$ is finitely generated as an $\mathscr{O}_{X, x}$-module by (d), Krull's intersection theorem implies that the right-hand side is equal to zero.

The conclusion is that $\delta=0$, and hence that $v, z$ form a regular sequence on $H^{i}\left(M^{\bullet}\right)$. Together with the following result, this proves that $H^{i}\left(M^{\bullet}\right)$ is flat as an $R$-module.

Lemma 8.5.6. The morphism $z: H^{i}\left(M^{\bullet}\right) \rightarrow H^{i}\left(M^{\bullet}\right)$ is injective.
Proof. Since $v, z$ form a regular sequence on $H^{i}\left(M^{\bullet}\right)$, the corresponding Koszul complex is exact. By the same argument as in the proof of Proposition 8.3.21, every element in the kernel of $z: H^{i}\left(M^{\bullet}\right) \rightarrow H^{i}\left(M^{\bullet}\right)$ can be written as $v$ times another element in the kernel; consequently,

$$
\operatorname{Ker}\left(z: H^{i}\left(M^{\bullet}\right) \rightarrow H^{i}\left(M^{\bullet}\right)\right) \subseteq \bigcap_{m \geqslant 1} v^{m} H^{i}\left(M^{\bullet}\right)
$$

Looking at a fixed bidegree $(j, k)$ and setting $p=j$ and $\alpha=\alpha_{k}$ as above, the righthand side equals

$$
\bigcap_{\beta<\alpha} \operatorname{Im}\left(\mathscr{H}^{i}\left(F_{p} V_{\beta} \mathcal{M}^{\bullet}\right) \rightarrow \mathscr{H}^{i}\left(F_{p} V_{\alpha} \mathcal{M}^{\bullet}\right)\right)
$$

which is equal to zero by Lemma 8.5.5.
In summary, we have shown that for every point $x \in X_{0}$, the localization of the complex $\left(\mathcal{M}^{\bullet}, d\right)$ is strict (as a complex of $\mathscr{D}_{X, x}$-modules with two filtrations). Now it remains to prove that the complex $\left(\mathcal{M}^{\bullet}, d\right)$ is strict on an open neighborhood of $X_{0}$, using the coherence condition in (i). This will end the proof of Theorem 8.5.1.

Lemma 8.5.7. If $\left(\mathcal{M}^{\bullet}, F, V\right)$ is a complex of bi-filtered $\mathscr{D}_{X}$-modules whose restriction to $X_{0}$ is strict and which satisfies the following two conditions:
(1) for every $j$, the $R_{F} \mathscr{D}_{X}$-module $\oplus_{p \in \mathbb{Z}} \mathscr{H}^{j}\left(F_{p} \mathcal{M}^{\bullet}\right)$ is coherent;
(2) we have $\mathscr{H}^{j}\left(F_{p} \mathcal{M} \cdot\right)=0$ for $|j| \gg 0$ and all $p$.

Then $\left(\mathcal{M}^{\bullet}, F, V\right)$ is strict in a neighborhood of $X_{0}$.
Proof. Note that over $X \backslash X_{0}$ we have $V_{\alpha} \mathscr{D}_{X}=\mathscr{D}_{X}$ for every $\alpha$. Since $\bigcup_{\alpha} V_{\alpha} \mathcal{M}=\mathcal{M}$, it is easy to deduce that over this open subset, $V_{\alpha} \mathcal{M}=\mathcal{M}$ for every $\alpha$. Therefore $\left(\mathcal{M}^{\bullet}, F, V\right)$ is strict over an open subset $U \subseteq X \backslash X_{0}$ if and only if $\left(\mathcal{M}^{\bullet}, F\right)$ is strict over $U$.

By assumption, $\left(\mathcal{M}^{\bullet}, F, V\right)$ is strict at the points $x \in X_{0}$, hence in order to complete the proof of the lemma, it is enough to show that if $\left(\mathcal{M}^{\bullet}, F\right)$ is strict at a point $x \in X$, then it is strict in an open neighborhood of $x$. Since the $F$-filtration on $\mathcal{M}^{\bullet}$ is exhaustive, it follows from Exercise 8.2.7 that $\left(\mathcal{M}^{\bullet}, F\right)$ is strict at $x \in X$ if and only if the natural map $\mathscr{H}^{j}\left(F_{p} \mathcal{M}{ }^{\bullet}\right)_{x} \rightarrow \mathscr{H}^{j}\left(F_{p+1} \mathcal{N}^{\bullet}\right)_{x}$ is injective for all $p$ and $j$. For every $j$, consider the coherent $\mathscr{B}_{X}$-module $\mathscr{M}_{j}:=\oplus_{p \in \mathbb{Z}} \mathscr{H}^{j}\left(F_{p} \mathcal{M} \mathcal{M}^{\bullet}\right)$. We see that $\left(\mathcal{M}^{\bullet}, F\right)$ is strict at $x \in X$ if and only if the map $u_{j}: \mathscr{M}_{j} \rightarrow \mathscr{M}_{j}$ given by multiplication with $z$ is injective for all $j$. Furthermore, by (2) we only need to consider finitely
many $j$. Since $\mathscr{M}_{j}$ is a coherent $\mathscr{B}_{X}$-module, it follows that $\operatorname{Ker}\left(u_{j}\right)$ is a coherent $R_{F} \mathscr{D}_{X}$-module. In the neighborhood of a given point $x \in X$, we have a finite set of generators $s_{1}, \ldots, s_{r}$ of $\operatorname{Ker}\left(u_{j}\right)$ over $R_{F} \mathscr{D}_{X}$. If all the $s_{i}$ vanish at $x$, then they also vanish in an open neighborhood of $x$ and $u_{j}$ is injective in this neighborhood. Since we can argue in this way simultaneously for finitely many $j$, this concludes the proof of the lemma.

### 8.6. Induced bi-filtered $\mathscr{D}_{X}$-modules

We assume that we are in the setting of Section 8.5. In particular, we consider the $F$ and $V$-filtrations on $\mathscr{D}_{X}$ corresponding to $X_{0} \subset X$.

Let $\mathscr{L}$ be an $\mathscr{O}_{X}$-module and suppose that $p \in \mathbb{Z}$ and $\alpha \in[-1,0]$. We assume that the following holds: $\mathscr{L}$ has no $t$-torsion, unless $\alpha=0$, and in this case

$$
\left\{u \in \mathscr{L} \mid u t^{j}=0 \text { for some } j \geqslant 1\right\}=\{u \in \mathscr{L} \mid u t=0\} .
$$

Given this data, we define $\mathscr{L} \otimes\left(\mathscr{D}_{X}, F[p], V[\alpha]\right)$ to be the right $\mathscr{D}_{X}$-module $\mathscr{L} \otimes_{\mathscr{O}_{X}} \mathscr{D}_{X}$, with the corresponding filtrations given by

$$
\begin{aligned}
& F_{q}\left(\mathscr{L} \otimes \mathscr{D}_{X}\right)=\operatorname{Im}\left(\mathscr{L} \otimes F_{q-p} \mathscr{D}_{X} \longleftrightarrow \mathscr{L} \otimes \mathscr{D}_{X}\right) \\
& V_{\beta}\left(\mathscr{L} \otimes \mathscr{D}_{X}\right)=\operatorname{Im}\left(\mathscr{L} \otimes V_{\beta-\alpha} \mathscr{D}_{X} \longleftrightarrow \mathscr{L} \otimes \mathscr{D}_{X}\right) .
\end{aligned}
$$

In order to study the properties of such objects, it is convenient to treat separately the case when $\mathscr{L}$ has no $t$-torsion and when $\mathscr{L} t=0$, the general case following using the existence of an exact sequence

$$
0 \longrightarrow \mathscr{L}^{\prime} \longrightarrow \mathscr{L} \longrightarrow \mathscr{L}^{\prime \prime} \longrightarrow 0
$$

with $\mathscr{L}^{\prime} t=0$ and $\mathscr{L}^{\prime \prime}$ without $t$-torsion.
It is useful to note that since $\mathscr{L}^{\prime} t=0$, we have

$$
\begin{align*}
\mathscr{L}^{\prime} \otimes \mathscr{D}_{X} & =\mathscr{L}^{\prime} \otimes_{\mathscr{O}_{X_{0}}} \mathscr{D}_{X_{0}}\left[\partial_{t}\right], \\
F_{q}\left(\mathscr{L}^{\prime} \otimes \mathscr{D}_{X}\right) & =\oplus_{j \geqslant 0}\left(\mathscr{L}^{\prime} \otimes_{\mathscr{O}_{X_{0}}}\left(F_{q-p-j} \mathscr{D}_{X_{0}}\right) \partial_{t}^{j}\right)  \tag{8.6.1}\\
V_{\beta}\left(\mathscr{L}^{\prime} \otimes \mathscr{D}_{X}\right) & =\oplus_{j=0}^{\llcorner\beta\rfloor}\left(\mathscr{L}^{\prime} \otimes_{\mathscr{O}_{X_{0}}} \mathscr{D}_{X_{0}} \partial_{t}^{j}\right) .
\end{align*}
$$

Lemma 8.6.2. With the above notation, for every $q$ and $\beta$, we have
(i) $F_{q} V_{\beta}\left(\mathscr{L} \otimes \mathscr{D}_{X}\right)=\operatorname{Im}\left(\mathscr{L} \otimes F_{q-p} V_{\beta-\alpha} \mathscr{D}_{X} \rightarrow \mathscr{L} \otimes \mathscr{D}_{X}\right)$.
(ii) There is an exact sequence

$$
0 \longrightarrow F_{q} V_{\beta}\left(\mathscr{L}^{\prime} \otimes \mathscr{D}_{X}\right) \longrightarrow F_{q} V_{\beta}\left(\mathscr{L} \otimes \mathscr{D}_{X}\right) \longrightarrow F_{q} V_{\beta}\left(\mathscr{L}^{\prime \prime} \otimes \mathscr{D}_{X}\right) \longrightarrow 0
$$

Furthermore, we have $\mathscr{L} \otimes\left(\mathscr{D}_{X}, F[p], V[\alpha]\right) \in \mathrm{FV}\left(\mathscr{D}_{X}\right)$.
Proof. The assertion in (i) follows easily when $\mathscr{L}$ has no $t$-torsion, using the fact that the following maps are injective:

$$
\begin{gathered}
\mathscr{L} \otimes V_{\beta-\alpha} \mathscr{D}_{X} \longrightarrow \mathscr{L} \otimes \mathscr{D}_{X}, \quad \mathscr{L} \otimes F_{q-p} V_{\beta-\alpha} \mathscr{D}_{X} \longrightarrow \mathscr{L} \otimes \mathscr{D}_{X}, \\
\mathscr{L} \otimes\left(V_{\beta-\alpha} \mathscr{D}_{X} / F_{q-p} V_{\beta-\alpha} \mathscr{D}_{X}\right) \longrightarrow \mathscr{L} \otimes \mathscr{D}_{X} / F_{q-p} \mathscr{D}_{X} .
\end{gathered}
$$

and

When $\mathscr{L} t=0$, we deduce (i) from the explicit description in (8.6.1).
We now note that we have exact sequences

$$
0 \longrightarrow F_{q}\left(\mathscr{L}^{\prime} \otimes \mathscr{D}_{X}\right) \longrightarrow F_{q}\left(\mathscr{L} \otimes \mathscr{D}_{X}\right) \longrightarrow F_{q}\left(\mathscr{L}^{\prime \prime} \otimes \mathscr{D}_{X}\right) \longrightarrow 0
$$

and $\quad 0 \longrightarrow V_{\beta}\left(\mathscr{L}^{\prime} \otimes \mathscr{D}_{X}\right) \longrightarrow V_{\beta}\left(\mathscr{L} \otimes \mathscr{D}_{X}\right) \longrightarrow V_{\beta}\left(\mathscr{L}^{\prime \prime} \otimes \mathscr{D}_{X}\right) \longrightarrow 0$
(exactness follows from definition and the fact that the maps

$$
\mathscr{L}^{\prime \prime} \otimes F_{q-p} \mathscr{D}_{X} \longrightarrow \mathscr{L}^{\prime \prime} \otimes \mathscr{D}_{X} \quad \text { and } \quad \mathscr{L}^{\prime \prime} \otimes V_{\beta-\alpha} \mathscr{D}_{X} \longrightarrow \mathscr{L}^{\prime \prime} \otimes \mathscr{D}_{X}
$$

are injective. Let

$$
M=\operatorname{Im}\left(\mathscr{L} \otimes F_{q-p} V_{\beta-\alpha} \mathscr{D}_{X} \longrightarrow \mathscr{L} \otimes \mathscr{D}_{X}\right)
$$

and we similarly define $M^{\prime}$ and $M^{\prime \prime}$. We deduce that we have a commutative diagram with exact rows and injective vertical maps

(for the exactness of the top row we use the fact that the map

$$
\mathscr{L}^{\prime \prime} \otimes F_{q-p} V_{\beta-\alpha} \mathscr{D}_{X} \longrightarrow \mathscr{L}^{\prime \prime} \otimes \mathscr{D}_{X}
$$

is injective; the exactness of the bottom row follows from the above two exact sequences). Since we know that $j^{\prime}$ and $j^{\prime \prime}$ are surjective, it follows that $j$ is also surjective. This completes the proof of both (i) and (ii). The last assertion in the lemma is easy to check for $\mathscr{L}^{\prime}$ and $\mathscr{L}^{\prime \prime}$, and we deduce it also for $\mathscr{L}$ using (ii).

We consider the category $\mathrm{FV}\left(\mathscr{D}_{X}\right)$ consisting of triples $(\mathcal{M}, F, V)$, where $\mathcal{M}$ is a right $\mathscr{D}_{X}$-module, $F$ is a (usual) filtration and $V$ is a $V$-filtration indexed by $A+\mathbb{Z}$ on $\mathcal{M}$, for some finite set $A \in(-1,0]$, both of them exhaustive and compatible with the $F$ and $V$-filtrations on $\mathscr{D}_{X}$, such that $F_{p} \mathcal{M}=0$ for $p \ll 0$, and the following two conditions are satisfied (no coherence assumption is made here):
(i) Multiplication by $t$ induces an isomorphism $F_{p} V_{\alpha} \mathcal{M} \simeq F_{p} V_{\alpha-1} \mathcal{M}$ whenever $\alpha<0$.
(ii) Multiplication by $\partial_{t}$ induces an isomorphism $F_{p} \operatorname{gr}_{\alpha}^{V} \mathcal{M} \simeq F_{p+1} \operatorname{gr}_{\alpha+1}^{V} \mathcal{M}$ whenever $\alpha>-1$.

We note that Condition (i) implies, in particular, that $V_{\alpha} \mathcal{M} \cdot t=V_{\alpha-1} \mathcal{M}$ for all $\alpha<0$. However, we do not assume that $\left(V_{\alpha} \mathcal{M}\right)_{\alpha \in \mathbb{R}}$ is a coherent $V$-filtration with respect to $H$ (more precisely, we do not require any coherence condition or the fact that $t \partial_{t}-\alpha$ is nilpotent on $\left.\operatorname{gr}_{\alpha}^{V} \mathcal{M}\right)$. The morphisms in $\operatorname{FV}\left(\mathscr{D}_{X}\right)$ are morphisms of right $\mathscr{D}_{X}$-modules that are compatible with both filtrations. We usually refer to the objects of $\mathrm{FV}\left(\mathscr{D}_{X}\right)$ simply as bi-filtered $\mathscr{D}_{X}$-modules.

Remark 8.6.3. It is not true that a morphism $\varphi$ in $\mathrm{FV}\left(\mathscr{D}_{X}\right)$ has kernels and cokernels (it is not necessarily true that the induced filtrations on the $\mathscr{D}_{X}$-modules kernels or cokernels satisfy conditions (i) and (ii) above). However, this is the case if $\varphi$ is strict, and in this case we have an isomorphism $\operatorname{Coim}(\varphi) \simeq \operatorname{Im}(\varphi)$.

An induced bi-filtered $\mathscr{D}_{X}$-module is an object of $\mathrm{FV}\left(\mathscr{D}_{X}\right)$ that is isomorphic to a direct sum

$$
\oplus_{i}\left(\mathscr{L}_{i} \otimes_{\mathscr{O}_{X}}\left(\mathscr{D}_{X}, F\left[p_{i}\right], V\left[\alpha_{i}\right]\right)\right)
$$

for some $\mathscr{L}_{i}, p_{i}$, and $\alpha_{i}$ as above. The full subcategory of $\operatorname{FV}\left(\mathscr{D}_{X}\right)$ consisting of induced objects is denoted $\mathrm{FV}_{i}\left(\mathscr{D}_{X}\right)$.

Remark 8.6.4. Given $(\mathcal{M}, F, V) \in \mathrm{FV}\left(\mathscr{D}_{X}\right)$, note that for every $\alpha \in[-1,0]$ and every $p \in \mathbb{Z}$, we can define $F_{p} V_{\alpha} \mathcal{M} \otimes\left(\mathscr{D}_{X}, F[p], V[\alpha]\right)$. Indeed, we know that $F_{p} V_{\alpha} \mathcal{M}$ has no $t$-torsion when $\alpha<0$. Furthermore, if $u \in F_{p} V_{0} \mathcal{M}$ is such that $t^{j} u=0$ for some $j \geqslant 2$, then $t u \in F_{p} V_{-1} \mathcal{M}$, hence $t u=0$. We have a strict surjective morphism

$$
\underset{\substack{p \in \mathbb{Z} \\ \alpha \in[0,1]}}{\bigoplus_{p}} F_{p} V_{\alpha} \mathcal{M} \otimes\left(\mathscr{D}_{X}, F[p], V[\alpha]\right) \longrightarrow(\mathcal{M}, F, V)
$$

(in this case strictness simply means that the filtrations on the target are induced by the ones on the source). Indeed, the surjectivity is a consequence of Conditions (i) and (ii) in the definition of the category $\mathrm{FV}\left(\mathscr{D}_{X}\right)$. By applying the same argument to the kernel, with the induced filtrations (note that this lies in $\mathrm{FV}\left(\mathscr{D}_{X}\right)$ ), we obtain a (possibly infinite) resolution of ( $\mathcal{M}, F, V)$ by induced objects.

We consider the category of complexes $\mathrm{C}^{*}\left(\mathrm{FV}\left(\mathscr{D}_{X}\right)\right)$, where $*$ stands for,,$+- b$, or for the empty set. We assume that all complexes $C^{\bullet}$ in this category satisfy the following assumptions:
(i) For $p \ll 0$, we have $F_{p} C^{\bullet}=0$.
(ii) There exists a finite set $A$ suitable for each term of $C^{\bullet}$.

We have a corresponding homotopic category $\mathrm{K}^{*}\left(\mathrm{FV}\left(\mathscr{D}_{X}\right)\right)$. A morphism $C_{1}^{\boldsymbol{\bullet}} \rightarrow C_{2}^{\boldsymbol{\bullet}}$ in $\mathrm{K}\left(\mathrm{FV}\left(\mathscr{D}_{X}\right)\right)$ is a filtered quasi-isomorphism if $\mathscr{H}^{i}\left(F_{p} V_{\alpha} C_{1}^{\bullet}\right) \rightarrow \mathscr{H}^{i}\left(F_{p} V_{\alpha} C_{2}^{\bullet}\right)$ is an isomorphism for all $p \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$. Note that since we work with exhaustive filtrations, every filtered quasi-isomorphism is, in particular, a quasi-isomorphism.

We obtain the filtered derived categories $\mathrm{D}^{*}\left(\mathrm{FV}\left(\mathscr{D}_{X}\right)\right)$ by localizing $\mathrm{K}^{*}\left(\mathrm{FV}\left(\mathscr{D}_{X}\right)\right)$ at the class of filtered quasi-isomorphisms. As in the case of the derived category of an abelian category, one shows that each $\mathrm{D}^{*}\left(\mathrm{FV}\left(\mathscr{D}_{X}\right)\right)$ is a triangulated category. It follows from the universal property of the localization that we get exact functors

$$
\mathscr{H}^{i} F_{p} V_{\alpha}(-): \mathrm{D}^{*}\left(\mathrm{FV}\left(\mathscr{D}_{X}\right)\right) \longrightarrow \mathrm{D}^{*}\left(\mathscr{O}_{X}\right)
$$

where $\mathrm{D}^{*}\left(\mathscr{O}_{X}\right)$ is the derived category of $\mathscr{O}_{X}$-modules, with the suitable boundedness condition.

Remark 8.6.5. Note that for every $\alpha \in \mathbb{R}$, taking $\left(C^{\bullet}, F, V\right)$ to $\left(\operatorname{gr}_{\alpha}^{V}\left(C^{\bullet}\right), F\right)$ defines an exact functor $\mathrm{D}^{*}\left(\mathrm{FV}\left(\mathscr{D}_{X}\right)\right) \rightarrow \mathrm{D}^{*}\left(\mathrm{~F}\left(\mathscr{D}_{X_{0}}\right)\right)$, where $\mathrm{D}^{*}\left(\mathrm{~F}\left(\mathscr{D}_{X_{0}}\right)\right)$ is the filtered derived category of filtered $\mathscr{D}_{X}$-modules (with suitable boundedness conditions).

Let $\mathrm{K}^{*}\left(\mathrm{FV}_{i}\left(\mathscr{D}_{X}\right)\right)$ be the homotopic category of complexes of induced objects in $\mathrm{FV}\left(\mathscr{D}_{X}\right)$, with suitable boundedness conditions. By localizing this with respect to filtered quasi-isomorphisms, we get $\mathrm{D}^{*}\left(\mathrm{FV}_{i}\left(\mathscr{D}_{X}\right)\right)$.

Lemma 8.6.6. The exact functor

$$
\mathrm{D}^{-}\left(\mathrm{FV}_{i}\left(\mathscr{D}_{X}\right)\right) \longrightarrow \mathrm{D}^{-}\left(\mathrm{FV}\left(\mathscr{D}_{X}\right)\right)
$$

induced by inclusion is an equivalence of categories.
Proof. For every $(\mathcal{M}, F, V) \in \mathrm{FV}\left(\mathscr{D}_{X}\right)$, we construct the resolution $\mathscr{I}^{\bullet}(\mathcal{M}, F, V)$ by induced bi-filtered $\mathscr{D}_{X}$-modules as in Remark 8.6.4. It is clear that this is functorial and we extend the construction to a functor $\mathrm{C}^{-}\left(\mathrm{FV}\left(\mathscr{D}_{X}\right)\right) \rightarrow \mathrm{C}^{-}\left(\mathrm{FV}_{i}\left(\mathscr{D}_{X}\right)\right)$, by mapping a complex $\left(\mathcal{M}^{\bullet}, F, V\right)$ to the total complex of the double complex $\mathscr{I}^{\bullet}\left(\mathcal{M}^{\bullet}, F, V\right)$. It is standard to check that this induces a functor between the corresponding filtered derived categories and that this gives an inverse for the functor induced by the inclusion.

Remark 8.6.7. If $(\mathcal{M}, F, V)$ is a bi-filtered $\mathscr{D}_{X}$-module, we can choose a finite subset $A \subset[-1,0]$ such that $\operatorname{gr}_{\alpha}^{V}(\mathcal{M})=0$ for all $\alpha \in[-1,0] \backslash A$. As in Remark 8.6.4, we obtain a strict surjective morphism

$$
\underset{\substack{p \in \mathbb{Z} \\ \alpha \in A}}{\bigoplus_{p}} F_{p} V_{\alpha} \mathcal{M} \otimes\left(\mathscr{D}_{X}, F[p], V[\alpha]\right) \longrightarrow(\mathcal{M}, F, V)
$$

and by iterating this construction, we obtain a resolution $\left(\mathscr{I}^{\bullet}, F, V\right)$ of $(\mathcal{M}, F, V)$ by induced objects such that each $\left(\mathscr{I}^{i}, F, V\right)$ is the direct sum of bi-filtered modules of the form $\mathscr{L} \otimes \mathscr{O}_{X}\left(\mathscr{D}_{X}, F\left[p_{i}\right], V\left[\alpha_{i}\right]\right)$, with the $\alpha_{i}$ varying over a finite set. In particular, since for every $q$ and $\beta$ we have

$$
F_{q} V_{\beta}\left(\mathscr{L} \otimes \mathscr{O}_{X}\left(\mathscr{D}_{X}, F\left[p_{i}\right], V\left[\alpha_{i}\right]\right)\right)=0 \text { unless } p_{i} \leqslant q
$$

we conclude that if $F_{p} V_{\alpha} \mathcal{M}$ is a coherent $\mathscr{O}_{X}$-module for every $p$ and $\alpha$, then $F_{p} V_{\alpha} \mathscr{I}^{j}$ is a coherent $\mathscr{O}_{X}$-module for every $p, \alpha$, and $j$.

Lemma 8.6.8. Consider two induced bi-filtered $\mathscr{D}_{X}$-modules

$$
\left(\mathcal{M}_{i}, F, V\right)=\mathscr{L}_{i} \otimes\left(\mathscr{D}_{X}, F[p], V[\alpha]\right) \quad i=1,2,
$$

and consider the exact sequences

$$
0 \longrightarrow \mathscr{L}_{i}^{\prime} \longrightarrow \mathscr{L}_{i} \longrightarrow \mathscr{L}_{i}^{\prime \prime} \longrightarrow 0
$$

where $\mathscr{L}_{i}^{\prime} t=0$ and $\mathscr{L}_{i}^{\prime \prime}$ has no t-torsion. If $u: \mathscr{L}_{1} \rightarrow \mathscr{L}_{2}$ is an injective morphism such that the induced morphism $u^{\prime \prime}: \mathscr{L}_{1}^{\prime \prime} \rightarrow \mathscr{L}_{2}^{\prime \prime}$ has the property that $\operatorname{Coker}\left(u^{\prime \prime}\right)$ has no t-torsion, then the induced morphism $\bar{u}:\left(\mathcal{M}_{1}, F, V\right) \rightarrow\left(\mathcal{M}_{2}, F, V\right)$ is strict and $\operatorname{Coker}(\bar{u}) \simeq \operatorname{Coker}(u) \otimes\left(\mathscr{D}_{X}, F[p], V[\alpha]\right)$.

Proof. We need to show that if we consider on $\operatorname{Coker}(\bar{u}) \simeq \operatorname{Coker}(u) \otimes \mathscr{D}_{X}$ the induced filtrations, then for every $q$ and $\beta$, the sequence

$$
0 \longrightarrow F_{q} V_{\beta}\left(\mathscr{L}_{1} \otimes \mathscr{D}_{X}\right) \longrightarrow F_{q} V_{\beta}\left(\mathscr{L}_{2} \otimes \mathscr{D}_{X}\right) \longrightarrow F_{q} V_{\beta}\left(\operatorname{Coker}(u) \otimes \mathscr{D}_{X}\right) \longrightarrow 0
$$

is exact. This is easy to check when both $\mathscr{L}_{i}$ have no $t$-torsion and it follows from the explicit description in (8.6.1) when $\mathscr{L}_{i} t=0$ for $i=1,2$.

We now consider the general case. Let $u^{\prime}: \mathscr{L}_{1}^{\prime} \rightarrow \mathscr{L}_{2}^{\prime}$ be the morphism induced by $u$. Note first that the Snake lemma gives an exact sequence

$$
0 \longrightarrow \operatorname{Coker}\left(u^{\prime}\right) \longrightarrow \operatorname{Coker}(u) \longrightarrow \operatorname{Coker}\left(u^{\prime \prime}\right) \longrightarrow 0
$$

(since $\operatorname{Ker}\left(u^{\prime \prime}\right)$ has no $t$-torsion, it has to be zero). This exact sequence is the canonical one associated to Coker $(u)$ such that the first term is annihilated by $t$ and the third one has no $t$-torsion.

Consider the commutative diagram


The first and the third columns are exact by what we have already discussed. Moreover, the rows are all exact by Lemma 8.6.2. Therefore the middle column is also exact, which is what we had to prove.

In order to define functors between filtered derived categories, it will be convenient to use the Godement resolution (see Definition A.8.7), that we now extend to our bi-filtered setting.

For $(\mathcal{M}, F, V) \in \mathrm{FV}\left(\mathscr{D}_{X}\right)$, we define $\mathcal{C}^{0}(\mathcal{M}, F, V)$ to be the bi-filtered $\mathscr{D}_{X}$-module $\mathscr{N}=\bigcup_{p, \alpha} \mathcal{C}^{0}\left(F_{p} V_{\alpha} \mathcal{M}\right) \subseteq \mathcal{C}^{0}(\mathcal{M})$, with the filtrations given by $F_{p} \mathscr{N}=\bigcup_{\alpha} \mathcal{C}^{0}\left(F_{p} V_{\alpha} \mathcal{M}\right)$ and $V_{\alpha} \mathscr{N}=\bigcup_{p} \mathcal{C}^{0}\left(F_{p} V_{\alpha} \mathcal{M}\right)$ for $p \in \mathbb{Z}, \alpha \in \mathbb{R}$. One checks that

$$
\mathfrak{C}^{0}\left(F_{p} V_{\alpha} \mathcal{M}\right) \cap \mathfrak{C}^{0}\left(F_{q} V_{\beta} \mathcal{M}\right)=\mathcal{C}^{0}\left(F_{\min (p, q)} V_{\min (\alpha, \beta)} \mathcal{M}\right)
$$

It follows that $F_{p} V_{\alpha} \mathscr{N}=\mathcal{C}^{0}\left(F_{p} V_{\alpha} \mathcal{N}\right)$, hence each $F_{p} V_{\alpha} \mathscr{N}$ is flabby. We have a natural strict monomorphism $(\mathcal{M}, F, V) \hookrightarrow \mathcal{C}^{0}(\mathcal{M}, F, V)$, whose cokernel is also a bifiltered $\mathscr{D}_{X}$-module, and we can proceed inductively as in Definition A.8.7 to define the complex $\operatorname{God}^{\bullet}(\mathcal{M}, F, V)$ in $\mathrm{C}^{+}\left(\mathrm{FV}\left(\mathscr{D}_{X}\right)\right)$ that is filtered quasi-isomorphic to $(\mathcal{M}, F, V)$.

Lemma 8.6.9. Given an induced bi-filtered $\mathscr{D}_{X}$-module

$$
(\mathcal{M}, F, V) \simeq \mathscr{L} \otimes_{\mathscr{O}_{X}}\left(\mathscr{D}_{X}, F[p], V[\alpha]\right),
$$

we have

$$
\mathcal{C}^{0}(\mathcal{M}, F, V) \simeq \mathcal{C}^{0}(\mathscr{L}) \otimes_{\mathscr{O}_{X}}\left(\mathscr{D}_{X}, F[p], V[\alpha]\right) .
$$

Proof. If we consider the exact sequence

$$
0 \longrightarrow \mathscr{L}^{\prime} \longrightarrow \mathscr{L} \longrightarrow \mathscr{L}^{\prime \prime} \longrightarrow 0
$$

where $\mathscr{L}^{\prime} t=0$ and $\mathscr{L}^{\prime \prime}$ has no $t$-torsion, then we have an induced exact sequence

$$
0 \longrightarrow \mathcal{C}^{0}\left(\mathscr{L}^{\prime}\right) \longrightarrow \mathcal{C}^{0}(\mathscr{L}) \longrightarrow \mathcal{C}^{0}\left(\mathscr{L}^{\prime \prime}\right) \longrightarrow 0
$$

and $\mathcal{C}^{0}\left(\mathscr{L}^{\prime}\right) t=0$, while $\mathcal{C}^{0}\left(\mathscr{L}^{\prime \prime}\right)$ has no $t$-torsion. In particular, we see that every $t$-torsion element in $\mathcal{C}^{0}(\mathscr{L})$ is annihilated by $t$. We also deduce from this that it is enough to prove the lemma when either $\mathscr{L}$ has no $t$-torsion or when $\mathscr{L} t=0$.

Suppose first that $\mathscr{L}$ has no $t$-torsion. In this case we have

$$
F_{q} V_{\beta}\left(\mathfrak{C}^{0}(\mathscr{L}) \otimes \mathscr{D}_{X}\right)=\mathfrak{C}^{0}(\mathscr{L}) \otimes F_{q-p} V_{\beta-\alpha} \mathscr{D}_{X} \simeq \mathfrak{C}^{0}\left(\mathscr{L} \otimes F_{q-p} V_{\beta-\alpha} \mathscr{D}_{X}\right)
$$

since $F_{q-p} V_{\beta-\alpha} \mathscr{D}_{X}$ is a locally free $\mathscr{O}_{X}$-module, of finite type (see Exercise A.8.9(2)). This implies the isomorphism in the lemma. The case when $\mathscr{L} t=0$ follows similarly, using the explicit description in (8.6.1).

Corollary 8.6.10. If $(\mathcal{M}, F, V) \in \mathrm{FV}\left(\mathscr{D}_{X}\right)$ is induced, then its filtered resolution

$$
0 \longrightarrow(\mathcal{M}, F, V) \longrightarrow \mathcal{C}^{0}(\mathcal{M}, F, V) \longrightarrow \mathcal{C}^{1}(\mathcal{M}, F, V) \longrightarrow \ldots
$$

consists of induced objects and the morphisms are strict and they correspond to morphisms of $\mathscr{O}_{X}$-modules.

Proof. This follows by combining Lemmas 8.6 .8 and 8.6.9. The only thing to note is that if we have a short exact sequence of $\mathscr{O}_{X}$-modules

$$
0 \longrightarrow \mathscr{L}^{\prime} \longrightarrow \mathscr{L} \longrightarrow \mathscr{L}^{\prime \prime} \longrightarrow 0
$$

with $\mathscr{L}^{\prime} t=0$ and $\mathscr{L}^{\prime \prime}$ without $t$-torsion, then $\operatorname{Coker}\left(\mathscr{L}^{\prime \prime} \rightarrow \mathcal{C}^{0}\left(\mathscr{L}^{\prime \prime}\right)\right)$ has no $t$ torsion.

### 8.7. The direct image of bi-filtered $\mathscr{D}_{X}$-modules

Let $f: X \rightarrow X^{\prime}$ be a morphism between complex manifolds. We assume that $X^{\prime}=H^{\prime} \times \Delta_{t}$ and $X=H \times \Delta_{t}$ such that $f=f_{\mid H} \times \operatorname{Id}_{t}$. We set $X_{0}=H \times\{0\}$ and $X_{0}^{\prime}=H^{\prime} \times 0$. Our first goal is to define a functor $f_{*}: \mathrm{D}^{-}\left(\mathrm{FV}\left(\mathscr{D}_{X}\right)\right) \rightarrow \mathrm{D}^{-}\left(\mathrm{FV}\left(\mathscr{D}_{X^{\prime}}\right)\right)$.

In addition to the sheaf $\mathscr{D}_{X}$, we also have on $X$ the sheaf $f^{-1}\left(\mathscr{D}_{X^{\prime}}\right)$. This carries the $F$-filtration and the $V$-filtration induced from $\mathscr{D}_{X^{\prime}}$ (the $V$-filtration being the one with respect to $\left.X_{0}^{\prime}\right)$. In particular, we may consider the categories $\mathrm{FV}\left(f^{-1}\left(\mathscr{D}_{X^{\prime}}\right)\right)$ and $\mathrm{FV}_{i}\left(f^{-1}\left(\mathscr{D}_{X^{\prime}}\right)\right)$. For example, an object in $\mathrm{FV}_{i}\left(f^{-1}\left(\mathscr{D}_{X^{\prime}}\right)\right)$ is one that is isomorphic to a direct sum of objects of the form $\mathscr{L} \otimes_{f^{-1}\left(\mathscr{O}_{X^{\prime}}\right)}\left(f^{-1}\left(\mathscr{D}_{X^{\prime}}\right), F[p], V[\alpha]\right)$, where $\mathscr{L}$ is an $f^{-1}\left(\mathscr{O}_{X^{\prime}}\right)$-module that has no $t$-torsion, unless $\alpha=0$, in which case the all local sections of $\mathscr{L}$ that are annihilated by some power of $t$ are actually annihilated by $t$.

The same construction from before shows that the inclusion functor determines an equivalence of categories

$$
\mathrm{D}^{-}\left(\mathrm{FV}_{i}\left(f^{-1}\left(\mathscr{D}_{X^{\prime}}\right)\right)\right) \longrightarrow \mathrm{D}^{-}\left(\mathrm{FV}\left(f^{-1}\left(\mathscr{D}_{X^{\prime}}\right)\right)\right)
$$

As in the case of the direct image of non-filtered $\mathscr{D}_{X}$-modules, the key player in the definition of the direct image for bi-filtered $\mathscr{D}_{X}$-modules is

$$
\mathscr{D}_{X \rightarrow X^{\prime}}:=\mathscr{O}_{X} \otimes_{f^{-1}\left(\mathscr{O}_{X^{\prime}}\right)} f^{-1}\left(\mathscr{D}_{X^{\prime}}\right) .
$$

This has a structure of left $\mathscr{D}_{X}$-module and right $f^{-1}\left(\mathscr{D}_{X^{\prime}}\right)$-bimodule and carries an $F$-filtration and a $V$-filtration induced from $f^{-1}\left(\mathscr{D}_{X^{\prime}}\right)$. These are compatible not only with the $F$ and $V$-filtrations on $f^{-1}\left(\mathscr{D}_{X^{\prime}}\right)$, via right multiplication, but also with the $F$ and $V$-filtrations on $\mathscr{D}_{X}$, via left multiplication.

Example 8.7.1. The two main examples are when $f$ is smooth and when $f$ is a closed immersion. The typical case for $f$ being smooth is when $f: X=X^{\prime} \times W \rightarrow X^{\prime}$ is the projection onto the first factor. In this case we have a surjection $\mathscr{D}_{X^{\prime} \times W} \rightarrow$ $\mathscr{D}_{X^{\prime} \times W \rightarrow X^{\prime}}$ such that in local coordinates $w_{1}, \ldots, w_{r}$ on $W$, we get an isomorphism

$$
\mathscr{D}_{X^{\prime} \times W \rightarrow X^{\prime}} \simeq \mathscr{D}_{X^{\prime} \times W} / \mathscr{D}_{X^{\prime} \times W} \cdot\left(\partial_{w_{1}}, \ldots, \partial_{w_{r}}\right) .
$$

On the other hand, the typical case when $f$ is a closed immersion is when $f: X \hookrightarrow$ $X^{\prime}=X \times Z$ is given by $f(x)=\left(x, z_{0}\right)$. If we have coordinates $z_{1}, \ldots, z_{r}$ on $Z$, then

$$
\mathscr{D}_{X \rightarrow X \times Z} \simeq \mathscr{D}_{X} \otimes \mathbb{C}\left[\partial_{z_{1}}, \ldots, \partial_{z_{r}}\right] .
$$

We first define the functor $\mathrm{DR}_{X / X^{\prime}}: \mathrm{FV}_{i}\left(\mathscr{D}_{X}\right) \rightarrow \mathrm{FV}_{i}\left(f^{-1}\left(\mathscr{D}_{X^{\prime}}\right)\right)$ by

$$
\mathrm{DR}_{X / X^{\prime}}(\mathcal{M}, F, V)=(\mathcal{M}, F, V) \otimes_{\mathscr{D}_{X}}\left(\mathscr{D}_{X \rightarrow X^{\prime}}, F, V\right),
$$

with the tensor product of the filtrations from the two factors. Note that this is well-defined, since we have

$$
\mathscr{L} \otimes_{\mathscr{O}_{X}}\left(\mathscr{D}_{X}, F[p], V[\alpha]\right) \otimes_{\mathscr{D}_{X}}\left(\mathscr{D}_{X \rightarrow X^{\prime}}, F, V\right) \simeq \mathscr{L} \otimes_{f^{-1}\left(\mathscr{O}_{X^{\prime}}\right)}\left(f^{-1}\left(\mathscr{D}_{X^{\prime}}\right), F[p], V[\alpha]\right) .
$$

Lemma 8.7.2. The functor $\mathrm{DR}_{X / X^{\prime}}$ maps a filtered quasi-isomorphism in $\mathrm{K}\left(\mathrm{FV}_{i}\left(\mathscr{D}_{X}\right)\right)$ to a filtered quasi-isomorphism.

Proof. We need to prove that if $\left(C^{\bullet}, F, V\right)$ is a complex of bi-filtered $\mathscr{D}_{X}$-modules such that all complexes $F_{p} V_{\alpha} C^{\bullet}$ are exact, then $F_{p} V_{\alpha} \mathrm{DR}_{X / X^{\prime}}\left(C^{\bullet}\right)$ is exact for all $p \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$. By factoring $f$ as $X \xrightarrow{j} X \times X^{\prime} \xrightarrow{p} X^{\prime}$, where $p$ is the second projection and $j$ is the graph of $f$, we reduce the proof for $f$ to proving the assertion separately for $j$ and $p$ (note that $\mathscr{D}_{X \rightarrow X^{\prime}} \simeq \mathscr{D}_{X \rightarrow X \times X^{\prime}} \otimes_{j^{-1}\left(\mathscr{D}_{X \times X^{\prime}}\right)} j^{-1}\left(\mathscr{D}_{X \times X^{\prime} \rightarrow X^{\prime}}\right)$ ).

The assertion for $j$ is trivial since we may assume that we have coordinates $y_{1}, \ldots, y_{r}$ on $X^{\prime}$, so that $\mathrm{DR}_{X / X \times X^{\prime}}$ can be identified with $\mathbb{C}\left[\partial_{y_{1}}, \ldots, \partial_{y_{r}}\right] \otimes_{\mathbb{C}}(-)$. Let us prove now the assertion for the projection $p: X \times X^{\prime} \rightarrow X^{\prime}$. For every $(\mathcal{M}, F, V) \in \mathrm{FV}\left(\mathscr{D}_{X \times X^{\prime}}\right)$, consider the complex $\mathrm{DR}_{X}(\mathcal{M}, F, V)$ consisting of $\mathcal{M} \otimes_{\mathscr{D}_{X \times X^{\prime}}} \wedge^{-\bullet} \Theta_{X}$ (given local coordinates $x_{1}, \ldots, x_{n}$ on $X$, this complex can be
identified to the Koszul-type complex corresponding to $\left.\partial_{x_{1}}, \ldots, \partial_{x_{r}}\right)$. The filtrations are defined by

$$
F_{p}\left(\mathcal{M} \otimes \wedge^{-i} \Theta_{X}\right)=F_{p+i} \mathcal{M} \otimes \wedge^{-i} \Theta_{X}, \quad V_{\alpha}\left(\mathcal{M} \otimes \wedge^{-i} \Theta_{X}\right)=V_{\alpha} \mathcal{M} \otimes \wedge^{-i} \Theta_{X}
$$

Note that the morphism $\mathscr{D}_{X \times X^{\prime}} \otimes_{\mathscr{O}_{X}} \wedge^{-\bullet} \Theta_{X} \rightarrow \mathscr{D}_{X \rightarrow X^{\prime}}$ induces a morphism

$$
\mathrm{DR}_{X}(\mathcal{M}, F, V) \longrightarrow(\mathcal{M}, F, V) \otimes_{\mathscr{D}_{X}}\left(\mathscr{D}_{X \rightarrow X^{\prime}}, F, V\right)
$$

for every bi-filtered $\mathscr{D}_{X}$-module $(\mathcal{M}, F, V)$. This is a filtered quasi-isomorphism if $(\mathcal{M}, F, V) \simeq \mathscr{L} \otimes\left(\mathscr{D}_{X \times X^{\prime}}, F[p], V[\alpha]\right)$, hence for all induced bi-filtered $\mathscr{D}_{X \times X^{\prime-}}$ modules. Indeed, it is enough to check the assertion when either $\mathscr{L}$ has no $t$-torsion, or when $\mathscr{L} t=0$; in each case, the verification is straightforward.

On the other hand, it is clear that if all $F_{p} V_{\alpha} C^{\bullet}$ are exact, then also all complexes $F_{p} V_{\alpha}\left(C^{\bullet} \otimes \wedge^{-i} \Theta_{W}\right)$ are exact, hence each $F_{p} V_{\alpha} \mathrm{DR}_{X / X^{\prime}}\left(C^{\bullet}\right)$ is exact by the above discussion. This completes the proof of the lemma.

As a consequence of the lemma, the functor $\mathrm{DR}_{X / X^{\prime}}$ we have defined induces an exact functor $\mathrm{DR}_{X / X^{\prime}}: \mathrm{D}^{*}\left(\mathrm{FV}_{i}\left(\mathscr{D}_{X}\right)\right) \rightarrow \mathrm{D}^{*}\left(\mathrm{FV}_{i}\left(f^{-1}\left(\mathscr{D}_{X^{\prime}}\right)\right)\right)$, where $*$ stands for + , ,$- b$, or the empty set.

We now introduce the topological direct image. We first define it at the level of bifiltered $D$-modules. Suppose that $f: X \rightarrow X^{\prime}$ is as above. If $(\mathcal{M}, F, V)$ is a bi-filtered $f^{-1}\left(\mathscr{D}_{X^{\prime}}\right)$-module, we define $f_{*}(\mathcal{M}, F, V) \in \mathrm{FV}\left(\mathscr{D}_{X^{\prime}}\right)$ to be given by $(\mathscr{N}, F, V)$, where $\mathscr{N}=\bigcup_{p, \alpha} f_{*}\left(F_{p} V_{\alpha} \mathcal{M}\right)$, with $F_{p} \mathscr{N}=\bigcup_{\alpha} f_{*}\left(F_{p} V_{\alpha} \mathcal{M}\right)$ and $V_{\alpha} \mathscr{N}=\bigcup_{p} f_{*}\left(F_{p} V_{\alpha} \mathcal{M}\right)$. We obtain in this way a functor $f_{*}: \mathrm{FV}\left(f^{-1}\left(\mathscr{D}_{X}\right)\right) \rightarrow \mathrm{FV}\left(\mathscr{D}_{X^{\prime}}\right)$. Note that if $\mathscr{L}$ is an $f^{-1}\left(\mathscr{O}_{X^{\prime}}\right)$-module, then $f_{*}\left(\mathscr{L} \otimes\left(f^{-1}\left(\mathscr{D}_{X^{\prime}}\right), F[p], V[\alpha]\right)\right) \simeq f_{*}(\mathscr{L}) \otimes\left(\mathscr{D}_{X^{\prime}}, F[p], V[\alpha]\right)$ by the projection formula (we use the fact that $\mathscr{D}_{X^{\prime}}$ is a locally free $\mathscr{O}_{X^{\prime}}$-module). Therefore we also have a functor $f_{*}: \mathrm{FV}_{i}\left(f^{-1}\left(\mathscr{D}_{X}\right)\right) \rightarrow \mathrm{FV}_{i}\left(\mathscr{D}_{X^{\prime}}\right)$.

We next define a version of the topological direct image functor at the level of filtered derived categories

$$
f_{*}: \mathrm{D}^{*}\left(\mathrm{FV}_{i}\left(f^{-1}\left(\mathscr{D}_{X^{\prime}}\right)\right)\right) \longrightarrow \mathrm{D}^{*}\left(\mathrm{FV}_{i}\left(\mathscr{D}_{X^{\prime}}\right)\right)
$$

as follows. By a variant of Corollary 8.6.10, we associate functorially to every $(\mathcal{M}, F, V) \in \operatorname{FV}_{i}\left(f^{-1}\left(\mathscr{D}_{X^{\prime}}\right)\right)$ a strict complex $\mathcal{C}^{\bullet}(\mathcal{M}, F, V)$

$$
0 \longrightarrow \mathfrak{C}^{0}(\mathcal{M}, F, V) \longrightarrow \mathcal{C}^{1}(\mathcal{M}, F, V) \longrightarrow \cdots
$$

that gives a filtered resolution of $(\mathcal{M}, F, V)$ by induced bi-filtered modules. It is convenient to replace this by a bounded complex, hence if $\operatorname{dim}(X)=n$, we consider the complex

$$
\widetilde{\mathfrak{C}}^{\bullet}(\mathcal{M}, F, V):\left\{0 \rightarrow \widetilde{\mathfrak{C}}^{0}(\mathcal{M}, F, V) \rightarrow \widetilde{\mathfrak{C}}^{1}(\mathcal{M}, F, V) \rightarrow \cdots \rightarrow \widetilde{\mathfrak{C}}^{n}(\mathcal{M}, F, V) \rightarrow 0\right\}
$$

where

$$
\widetilde{\mathfrak{C}}^{j}(\mathcal{M}, F, V)= \begin{cases}\mathcal{C}^{j}(\mathcal{M}, F, V), & 0 \leq j \leqslant n-1 ; \\ \operatorname{Coker}\left(\mathrm{C}^{n-2}(\mathcal{M}, F, V) \rightarrow \mathcal{C}^{n-1}(\mathcal{M}, F, V)\right), & j=n ; \\ 0, & j \geqslant n+1\end{cases}
$$

It follows from the construction that $\widetilde{\mathcal{C}}^{\bullet}(\mathcal{M}, F, V)$ is a strict complex, giving a filtered resolution of $(\mathcal{M}, F, V)$ by induced bi-filtered $f^{-1}\left(\mathscr{D}_{X^{\prime}}\right)$-modules. Moreover, since we truncated at the dimension of $X$, we have $R^{m} f_{*}\left(F_{p} V_{\alpha} \widetilde{\mathcal{C}}^{j}(\mathcal{M}, F, V)\right)=0$ for every $m \geqslant 1$ and every $j, p$, and $\alpha$. Given a complex $\left(\mathcal{M}^{\bullet}, F, V\right)$ in $\mathrm{FV}_{i}\left(f^{-1}\left(\mathscr{D}_{X^{\prime}}\right)\right)$, we consider the total complex of the double complex $\widetilde{\mathcal{C}}^{\bullet}\left(\mathcal{M}^{\bullet}, F, V\right)$. It is now standard to see that this induces exact functors $f_{*}: \mathrm{D}^{*}\left(\mathrm{FV}_{i}\left(f^{-1}\left(\mathscr{D}_{X^{\prime}}\right)\right)\right) \rightarrow \mathrm{D}^{*}\left(\mathrm{FV}_{i}\left(\mathscr{D}_{X^{\prime}}\right)\right)$, where $*$ stands for,,$+- b$, or for the empty set.

By composing $f_{*}$ and $\mathrm{DR}_{X / X^{\prime}}$, we obtain an exact functor

$$
f_{*}: \mathrm{D}^{*}\left(\mathrm{FV}_{i}\left(\mathscr{D}_{X}\right)\right) \longrightarrow \mathrm{D}^{*}\left(\mathrm{FV}_{i}\left(\mathscr{D}_{X^{\prime}}\right)\right)
$$

which in light of Lemma 8.6.6 also gives a functor $\mathrm{D}^{-}\left(\mathrm{FV}\left(\mathscr{D}_{X}\right)\right) \rightarrow \mathrm{D}^{-}\left(\mathrm{FV}\left(\mathscr{D}_{X^{\prime}}\right)\right)$.
It is easy to see that taking the direct image commutes with taking the graded pieces of the $V$-filtration. More precisely, if $f_{0}: X_{0} \rightarrow X_{0}^{\prime}$ is the restriction of $f$, then given any $\left(\mathcal{M}^{\bullet}, F, V\right) \in \mathrm{D}^{-}\left(\mathrm{FV}\left(\mathscr{D}_{X}\right)\right)$ and any $\alpha \in \mathbb{R}$, we have an isomorphism

$$
\begin{equation*}
\left(\operatorname{gr}_{\alpha}^{V} f_{*}\left(\mathcal{M}^{\bullet}, F, V\right), F\right) \simeq f_{0 *}\left(\operatorname{gr}_{\alpha}^{V}\left(\mathcal{M}^{\bullet}, F, V\right), F\right) \tag{8.7.3}
\end{equation*}
$$

in $\mathrm{D}^{-}\left(\mathrm{FV}\left(\mathscr{D}_{X_{0}^{\prime}}\right)\right)$ for every $\alpha \in \mathbb{R}$.
We consider two conditions on an object $C^{\bullet}$ of $\mathrm{C}\left(\mathrm{FV}\left(\mathscr{D}_{X}\right)\right)$ :
(a) The action of $t \partial_{t}-\alpha$ on $\mathscr{H}^{i}\left(\operatorname{gr}_{\alpha}^{V} C^{\bullet}\right)$ is nilpotent for all $i \in \mathbb{Z}, \alpha \in \mathbb{R}$,
(b) Each $\mathscr{H}^{i}\left(F_{p} V_{\alpha} C^{\bullet}\right)$ is a coherent $\mathscr{O}_{X}$-module.

Let $\mathrm{C}_{m}^{*}\left(\mathrm{FV}\left(\mathscr{D}_{X}\right)\right)$ and $\mathrm{C}_{c}^{*}\left(\mathrm{FV}\left(\mathscr{D}_{X}\right)\right)$ be the full subcategories of $\mathrm{C}^{*}\left(\mathrm{FV}\left(\mathscr{D}_{X}\right)\right)$ consisting of those objects that satisfy condition (a), respectively (b), and we similarly define $\mathrm{D}_{m}^{*}\left(\mathrm{FV}\left(\mathscr{D}_{X}\right)\right)$ and $\mathrm{D}_{c}^{*}\left(\mathrm{FV}\left(\mathscr{D}_{X}\right)\right)$ as full subcategories of $\mathrm{D}^{*}\left(\mathrm{FV}\left(\mathscr{D}_{X}\right)\right)$

Lemma 8.7.4. With the above notation, suppose also that $f$ is proper and $(\mathcal{M}, F, V) \in$ $\mathrm{FV}\left(\mathscr{D}_{X}\right)$.
(i) If $(\mathcal{M}, F, V) \in \mathrm{C}_{c}\left(\mathrm{FV}\left(\mathscr{D}_{X}\right)\right)$, then $f_{*}(\mathcal{M}, F, V) \in \mathrm{D}_{c}^{-}\left(\mathrm{FV}_{c}\left(\mathscr{D}_{X^{\prime}}\right)\right)$.
(ii) If $(\mathcal{M}, F, V) \in \mathrm{C}_{m}\left(\mathrm{FV}\left(\mathscr{D}_{X}\right)\right)$, then $f_{*}(\mathcal{M}, F, V) \in \mathrm{D}_{m}^{-}\left(\mathrm{FV}_{m}\left(\mathscr{D}_{X^{\prime}}\right)\right)$.

Proof. Let $\left(C^{\bullet}, F, V\right) \rightarrow(\mathcal{M}, F, V)$ be a filtered resolution by induced bi-filtered $\mathscr{D}_{X^{-}}$ modules constructed as in Remark 8.6.7. If $F_{p} V_{\alpha} \mathcal{N}$ is a coherent $\mathscr{O}_{X}$-module for every $p, \alpha$, then $F_{p} V_{\alpha} C^{k}$ is a coherent $\mathscr{O}_{X}$-module for every $p, \alpha$, and $k$. One can then deduce that all $\mathscr{H}^{k}\left(F_{p} V_{\alpha} \mathrm{DR}_{X / X^{\prime}}\left(C^{\bullet}\right)\right)$ are coherent $f^{-1}\left(\mathscr{O}_{X^{\prime}}\right)$-modules, and then that all $\mathscr{H}^{k}\left(F_{p} V_{\alpha} f_{*}\left(\mathrm{DR}_{X / X^{\prime}}\left(C^{\bullet}\right)\right)\right)$ are coherent $\mathscr{O}_{X^{\prime}}$-modules.

If the action of $\left(t \partial_{t}-\alpha\right)^{m}$ on $\operatorname{gr}_{\alpha}^{V}(\mathcal{M})$ is zero, then also its action on

$$
f_{0 *}\left(\operatorname{gr}_{\alpha}^{V}(\mathcal{M}), F\right) \simeq \operatorname{gr}_{\alpha}^{V} f_{*}(\mathcal{M}, F, V)
$$

is zero, hence the same holds for the action on $\mathscr{H}^{k}\left(\operatorname{gr}_{\alpha}^{V} f_{*}(\mathcal{M}, F, V)\right)$.
We now come to the main result of this chapter.

Theorem 8.7.5. Let $f: X \rightarrow X^{\prime}$ be a proper morphism as above and let $f_{0}: X_{0} \rightarrow X_{0}^{\prime}$ be the restriction of $f$. Suppose that $(\mathcal{M}, F)$ is a coherent filtered $\mathscr{D}_{X}$-module on $X$ which is $\mathbb{R}$-specializable, with $V$-filtration $V . \mathcal{M}$. If $f_{0 *}\left(\operatorname{gr}_{\alpha}^{V}(\mathcal{M}), F\right)$ is strict for every $\alpha \in \mathbb{R}$, then $f_{*}(\mathcal{M}, F, V)$ is strict in a neighborhood of $X_{0}^{\prime}$.

Proof. We will apply Theorem 8.5.1 to the bounded complex $\pi_{*}(\mathcal{M}, F, V)$. We first check that the conditions (a)-(i) of Section 8.5.a are fulfilled.

Since $(\mathcal{M}, F, V) \in \mathrm{FV}_{m}\left(\mathscr{D}_{X}\right)$, an application of Lemma 8.7.4 gives that $f_{*}(\mathcal{M}, F, V) \in$ $\mathrm{D}_{m}^{-}\left(\mathrm{FV}\left(\mathscr{D}_{X^{\prime}}\right)\right)$. Moreover, by hypothesis we have that

$$
\left(\operatorname{gr}_{\alpha}^{V} f_{*}\left(\mathcal{M}^{\bullet}, F, V\right), F\right) \simeq f_{0 *}\left(\operatorname{gr}_{\alpha}^{V}\left(\mathcal{M}^{\bullet}, F, V\right), F\right)
$$

is strict (the isomorphism is given by (8.7.3)). On the other hand, since ( $\mathcal{M}, F)$ is coherent, $(\mathcal{M}, F, V) \in \mathrm{FV}_{c}\left(\mathscr{D}_{X}\right)$. Therefore another application of Lemma 8.7.4 implies that $f_{*}(\mathcal{M}, F, V) \in \mathrm{D}_{c}^{-}\left(\mathrm{FV}\left(\mathscr{D}_{X^{\prime}}\right)\right)$. As a consequence, the conditions (a)-(h) are thus fulfilled by $f_{*}(\mathcal{M}, F, V)$. Lastly, the coherence condition (i) follows from the coherence theorem A.10.26. Therefore Theorem 8.5.1 implies that $f_{*}(\mathcal{M}, F, V)$ is strict in a neighborhood of $X_{0}^{\prime}$.

### 8.8. Strictness of strictly $\mathbb{R}$-specializable $R_{F} \mathscr{D}_{X}$-modules

In this section we compare the notion of specializability for filtered $\mathscr{D}_{X}$-modules, as developed in this chapter, and that for a strict $R_{F} \mathscr{D}_{X}$-module, as considered in Chapter 7.

Let $\mathscr{M}$ be a (left) coherent graded $R_{F} \mathscr{D}_{X}$-module which is strictly $\mathbb{R}$-specializable along $H$ and let $V^{\bullet} \mathscr{M}$ denotes its Kashiwara-Malgrange filtration. Then $\mathscr{M}$ is strict if and only if $V_{\alpha} \mathscr{M}$ is strict for some $\alpha$, since all $\mathrm{gr}_{\gamma}^{V} \mathscr{M}$ are assumed to be strict. The former property is equivalent to the existence of a coherent $\mathscr{D}_{X}$-module $\mathcal{M}$ equipped with a coherent $F$-filtration $F . \mathcal{M}$ such that $\mathscr{M}=R_{F} \mathcal{M}$, while the latter is equivalent to the existence of a coherent $V_{0} \mathscr{D}_{X}$-module $V_{\alpha} \mathcal{M}$ equipped with a coherent $F$-filtration $F_{\bullet} V_{\alpha} \mathcal{M}$ such that $V_{\alpha} \mathscr{M}=R_{F} V_{\alpha} \mathcal{M}$.

Lemma 8.8.1. Let $\mathscr{M}$ be as above. If $\mathscr{M}$ is strict, then the Kashiwara-Malgrange filtration of $\mathscr{M}$ satisfies

$$
\begin{equation*}
V_{\alpha} \mathscr{M}=\mathscr{M} \cap\left(V_{\alpha} \mathscr{M}\left[z^{-1}\right]\right) \tag{8.8.1*}
\end{equation*}
$$

where the intersection takes place in $\mathscr{M}\left[z^{-1}\right]$.

Proof. For $\gamma>\alpha$, we have a commutative diagram


The upper horizontal line is clearly exact, and the lower one is so because $\mathbb{C}\left[z, z^{-1}\right]$ is flat over $\mathbb{C}[z]$. The first two vertical maps are injective since $\mathscr{M}$ is strict. The third vertical map is injective since $\mathscr{M}$ is strictly $\mathbb{R}$-specializable. It follows that $V_{\alpha} \mathscr{M}=$ $V_{\gamma} \mathscr{M} \cap V_{\alpha} \mathscr{M}\left[z^{-1}\right]$ in $\mathscr{M}\left[z^{-1}\right]$. Taking the limit for $\gamma \rightarrow \infty$ gives the assertion.

Assume that $\mathscr{M}$ is strict and let $\left(\mathcal{M}, F_{\mathbf{\bullet}} \mathcal{M}\right)$ be the coherent filtered $\mathscr{D}_{X}$-module such that $\mathscr{M}=R_{F} \mathcal{M}$. Then $\mathcal{M}$ is $\mathbb{R}$-specializable and we have (see Exercise 7.3.20):

$$
V_{\alpha} \mathcal{M}=V_{\alpha} \mathscr{M} /(z-1) V_{\alpha} \mathscr{M} \quad \text { and } \quad V_{\alpha} \mathscr{M}\left[z^{-1}\right]=V_{\alpha} \mathcal{M}\left[z, z^{-1}\right] .
$$

Consider on $\mathcal{M}$ the bi-filtration $F_{p} V_{\alpha} \mathcal{M}:=F_{p} \mathcal{M} \cap V_{\alpha} \mathcal{M}$. Then (8.8.1*) means that the filtration $U \cdot \mathscr{M}$ defined by $U_{\alpha} \mathscr{M}:=\bigoplus_{p}\left(F_{p} V_{\alpha} \mathcal{M}\right) z^{p}$ satisfies the properties of the Kashiwara-Malgrange filtration of a strictly $\mathbb{R}$-specializable $R_{F} \mathscr{D}_{X}$-module. In particular we get, according to 7.3.28(a) and (d):

- $\forall p$ and $\forall \alpha<0, t: F_{p} V_{\alpha} \mathcal{M} \rightarrow F_{p} V_{\alpha-1} \mathcal{N}$ is an isomorphism,
- $\forall p$ and $\forall \alpha>-1, \partial_{t}: F_{p} \operatorname{gr}_{\alpha}^{V} \mathcal{M} \rightarrow F_{p+1} \operatorname{gr}_{\alpha+1}^{V} \mathcal{M}$ is an isomorphism.

The first condition can be called a regularity condition. Indeed, for a nonzero holonomic $\mathscr{D}_{X}$-module $\mathcal{M}$ with irregular singularities, we can have $V_{\alpha} \mathcal{M}=\mathcal{M}$ for every $\alpha$ (e.g. when $\operatorname{dim} X=1, \mathcal{M}=\mathscr{D}_{X} / \mathscr{D}_{X}\left(t^{2} \partial_{t}+1\right)$ ), and the condition $t F_{p} \mathcal{M}=F_{p} \mathcal{M}$ cannot be satisfied by a nonzero coherent $\mathscr{O}_{X}$-module $F_{p} \mathcal{M}$.

Proposition 8.8.2. Let $\mathcal{M}$ be a coherent $\mathscr{D}_{X}$-module which is $\mathbb{R}$-specializable along $H$, equipped with a coherent $F$-filtration. The following properties are equivalent:
(1) $R_{F} \mathcal{M}$ is strictly $\mathbb{R}$-specializable along $H$,
(2) the coherent filtration $F \cdot \mathcal{M}$ satisfies
(a) $\forall p$ and $\forall \alpha<0, t: F_{p} V_{\alpha} \mathcal{M} \rightarrow F_{p} V_{\alpha-1} \mathcal{M}$ is an isomorphism,
(b) $\forall p$ and $\forall \alpha>-1, \partial_{t}: F_{p} \operatorname{gr}_{\alpha}^{V} \mathcal{M} \rightarrow F_{p+1} \operatorname{gr}_{\alpha+1}^{V} \mathcal{M}$ is an isomorphism,
(c) for every $\alpha \in[-1,0]$ (equivalently after (2a) and (2b), for every $\alpha$ ), the induced filtration $F \cdot \operatorname{gr}_{\alpha}^{V} \mathcal{M}$ is a coherent $F_{\bullet} \operatorname{gr}_{0}^{V} \mathscr{D}_{X}$-filtration.
Moreover, when these conditions are fulfilled, the filtration $F_{\mathbf{\bullet}} \mathcal{M}$ induces in some neighbourhood of $H$ on each $V_{\alpha} \mathcal{M}$ a coherent $F_{\bullet} \mathscr{D}_{X / \mathbb{C}}$-filtration with respect to any local reduced equation $t: X \rightarrow \mathbb{C}$ of $H$, i.e., each $V_{\alpha} R_{F} \mathcal{M}=R_{F} V_{\alpha} \mathcal{M}$ is $R_{F} \mathscr{D}_{X / \mathbb{C}^{-}}$ coherent in some neighbourhood of $H$.

Proof. Let us assume (1). We have already seen that (2a) and (2b) are fulfilled since $R_{F} \mathcal{M}$ is strict, and clearly $F_{\bullet} \operatorname{gr}_{\alpha}^{V} \mathcal{M}$ is coherent for every $\alpha$, since $R_{F} \operatorname{gr}_{\alpha}^{V} \mathcal{M}=\operatorname{gr}_{\alpha}^{V} R_{F} \mathcal{M}$.

Conversely, let us assume (2). We first note that $F \cdot{ }_{\bullet}{ }_{\alpha}^{V} \mathcal{M}$ is coherent for every $\alpha$ if it is so for $\alpha \in[-1,0]$, according to (2a) and (2b). Let us set
$U_{\alpha} R_{F} \mathcal{M}=\bigoplus_{p}\left(F_{p} V_{\alpha} \mathcal{M}\right) z^{p}$. For a local section $m z^{p}$ of $\left(F_{p} V_{\alpha} \mathcal{M}\right) z^{p}$, we have $\left(t \partial_{t}-\alpha z\right)^{\nu_{m}} m z^{p} \in\left(F_{p+\nu_{m}} V_{<\alpha} \mathcal{M}\right) z^{p+\nu_{m}}$, showing the $\mathbb{R}$-specializability of $R_{F} \mathcal{M}$ and the fact that $U_{\alpha} R_{F} \mathcal{M} \subset V_{\alpha} R_{F} \mathcal{M}$. It is enough to show that $U \mathcal{N} \mathcal{N}$ is a coherent filtration indexed by $A+\mathbb{Z}$, since we obviously have $\operatorname{gr}_{\alpha}^{U} R_{F} \mathcal{M}=R_{F} \operatorname{gr}_{\alpha}^{V} \mathcal{M}$, hence the strictness. According to (2a) and (2b), it is enough to show the $V_{0} R_{F} \mathscr{D}_{X}$-coherence of $U_{\alpha} R_{F} \mathcal{M}$ for $\alpha \in[-1,0)$. For a local reduced equation $t: X \rightarrow \mathbb{C}$ of $H$, we will more precisely show the $R_{F} \mathscr{D}_{X / \mathbb{C}}$-coherence of $U_{\alpha} R_{F} \mathcal{M}$ in some neighbourhood of $H$, showing both the reverse implication $(2) \Rightarrow(1)$ and the last part of the proposition.

Since $V_{\bullet} \mathcal{M}$ is a coherent $V$-filtration of $\mathcal{M}$ (because $\mathcal{M}$ is assumed to be $\mathbb{R}$-specializable), each $V_{\alpha} \mathcal{M}$ is $V_{0} \mathscr{D}_{X}$-coherent, and therefore each $F_{p} V_{\alpha} \mathcal{N}$ is $\mathscr{O}_{X}$-coherent (consider an exhaustive increasing filtration $\left(V_{\alpha} \mathcal{M}\right)_{\ell}$ by $\mathscr{O}_{X}$-coherent submodules and use that, for $p$ fixed, $F_{p} \mathcal{M} \cap\left(V_{\alpha} \mathcal{M}\right)_{\ell}$ is locally stationary). It is thus enough to show that, locally on $X$, there exists $p_{o}$ such that $F_{p} \mathscr{D}_{X / \mathbb{C}} \cdot\left(F_{p_{o}} V_{\alpha} \mathcal{M}\right)=F_{p+p_{o}} V_{\alpha} \mathcal{M}$ for all $p \geqslant 0$. Since $\mathrm{E}-\alpha$ is nilpotent on $\operatorname{gr}_{\alpha}^{V} \mathcal{M}$, the filtration $F_{\bullet} \operatorname{gr}_{\alpha}^{V} \mathcal{M}$, being $F_{\bullet} \operatorname{gr}_{0}^{V} \mathscr{D}_{X}$-coherent for every $\alpha$ by assumption (2c), is also $F_{\bullet} \mathscr{D}_{H}$-coherent. The same argument applies to the induced filtration $\left(F_{\bullet} V_{\alpha} \mathcal{M}\right) /\left(F_{\bullet} V_{\alpha-1} \mathcal{N}\right)$ and therefore there exists locally $p_{o}$ such that

$$
F_{p} \mathscr{D}_{H} \cdot\left(F_{p_{o}} V_{\alpha} \mathcal{M}\right) /\left(F_{p_{o}} V_{\alpha-1} \mathcal{M}\right)=\left(F_{p+p_{o}} V_{\alpha} \mathcal{M}\right) /\left(F_{p+p_{o}} V_{\alpha-1} \mathcal{M}\right)
$$

Let us set $U_{\alpha, p}=F_{p} \mathscr{D}_{X / \mathbb{C}} \cdot\left(F_{p_{o}} V_{\alpha} \mathcal{M}\right)$. By (2a) and since $\alpha$ has been chosen in $[-1,0)$, the left-hand term above can be written as $U_{\alpha, p} / t U_{\alpha, p}$, while the right-hand term is

$$
\left(F_{p+p_{o}} V_{\alpha} \mathcal{M}\right) / t\left(F_{p+p_{o}} V_{\alpha} \mathcal{M}\right),
$$

so Nakayama's lemma implies finally $F_{p} \mathscr{D}_{X / \mathbb{C}} \cdot\left(F_{p_{o}} V_{\alpha} \mathcal{M}\right)=F_{p+p_{o}} V_{\alpha} \mathcal{M}$ in some neighbourhood of $H$, as wanted.

Corollary 8.8.3. Let $\mathscr{M}$ be a coherent graded $R_{F} \mathscr{D}_{X}$-module which is strictly $\mathbb{R}$-specializable along $H$. Then $\mathscr{M}$ strict in some neighbourhood of $H$ if and only if, for some $\alpha<0$ and all $p$, the pth graded component $\left(V_{\alpha} \mathscr{M}\right)_{p}$ is $\mathscr{O}_{X}$-coherent. In such a case, the properties of Proposition 8.8.2 hold true and in particular, $\mathscr{M}=R_{F} \mathcal{M}$ and $\left(V_{\alpha} \mathscr{M}\right)_{p}=F_{p} \mathcal{M} \cap V_{\alpha} \mathcal{M}$ for every $\alpha, p$, where $\mathcal{M}:=\mathscr{M} /(z-1) \mathscr{M}$ is a coherent $\mathscr{D}_{X}$-module which is $\mathbb{R}$-specializable along $H$ and $F_{\mathbf{\bullet}} \mathcal{M}$ is a coherent $F$-filtration of $\mathcal{M}$.

Proof. If $\mathscr{M}$ is strict, we can write $\mathscr{M}=R_{F} \mathcal{M}$ for some coherent $F$-filtration on $\mathcal{M}:=$ $\mathscr{M} /(z-1) \mathscr{M}$, and we have, according to Proposition 8.8.2, $\left(V_{\alpha} \mathscr{M}\right)_{p}=F_{p} \mathcal{M} \cap V_{\alpha} \mathcal{M}$, which is $\mathscr{O}_{X}$-coherent as we have seen in the proof of Proposition 8.8.2.

Conversely, since $\mathscr{M}$ is assumed to be strictly $\mathbb{R}$-specializable, each $\operatorname{gr}_{\gamma}^{V} \mathscr{M}$ is strict, and it is enough to prove that $V_{\alpha} \mathscr{M}$ is strict for some $\alpha<0$. For such an $\alpha$, $V_{\alpha} \mathscr{M} / t^{j} V_{\alpha} \mathscr{M}$ is also strict for every $j \geqslant 1$. By left exactness of $\lim _{\leftrightarrows_{j}}$, we deduce that $\varliminf_{j}{ }_{j}\left(V_{\alpha} \mathscr{M} / t^{j} V_{\alpha} \mathscr{M}\right)$ is also strict. It is thus enough to show that the natural morphism $V_{\alpha} \mathscr{M} \rightarrow \lim _{j}\left(V_{\alpha} \mathscr{M} / t^{j} V_{\alpha} \mathscr{M}\right)$ is injective.

We choose $\alpha<0$ as given by the assumption of the proposition, and we have $t^{j}\left(V_{\alpha} \mathscr{M}\right)_{p}=\left(V_{\alpha-j} \mathscr{M}\right)_{p}$ for $j \geqslant 0$ and any $p$, due to 7.3.28(a). Then

$$
\left(V_{\alpha} \mathscr{M} / t^{j} V_{\alpha} \mathscr{M}\right)_{p}=\left(V_{\alpha} \mathscr{M}\right)_{p} / t^{j}\left(V_{\alpha} \mathscr{M}\right)_{p}
$$

for every $j \geqslant 0$ and $p$, and therefore

$$
\left(\lim _{\underset{j}{ }}\left(V_{\alpha} \mathscr{M} / t^{j} V_{\alpha} \mathscr{M}\right)\right)_{p}=\underset{j}{\lim _{j}}\left(\left(V_{\alpha} \mathscr{M}\right)_{p} / t^{j}\left(V_{\alpha} \mathscr{M}\right)_{p}\right) .
$$

Since $\left(V_{\alpha} \mathscr{M}\right)_{p}$ is $\mathscr{O}_{X}$-coherent, $\lim _{j}\left(V_{\alpha} \mathscr{M}\right)_{p} / t^{j}\left(V_{\alpha} \mathscr{M}\right)_{p}=\mathscr{O}_{\widehat{X \mid H}} \otimes_{\mathscr{O}_{X \mid H}}\left(V_{\alpha} \mathscr{M}\right)_{p}$ and the natural morphism $\left(V_{\alpha} \mathscr{M}\right)_{p \mid H} \rightarrow \lim _{j}\left(V_{\alpha} \mathscr{M}\right)_{p} / t^{j}\left(V_{\alpha} \mathscr{M}\right)_{p}$ is injective. It follows that

$$
\left(V_{\alpha} \mathscr{M}\right)_{p \mid H} \longrightarrow\left(\varliminf_{\check{j}}\left(V_{\alpha} \mathscr{M} / t^{j} V_{\alpha} \mathscr{M}\right)\right)_{p}
$$

is injective for every $p$, and thus so is $\left(V_{\alpha} \mathscr{M}\right)_{\mid H} \rightarrow\left(\lim _{j}\left(V_{\alpha} \mathscr{M} / t^{j} V_{\alpha} \mathscr{M}\right)\right)$, as wanted.

We can now add the strictness property in Theorem 7.8.5, obtaining thus a complete analogue of Theorem 8.7.5.

Corollary 8.8.4. With the notation and assumptions of Theorem 7.8.5,
(4) if $\mathscr{M}$ is strict in the neightbourhood of $H$, then $\mathscr{H}^{i}{ }_{\mathrm{D}} f_{*} \mathscr{M}$ is strict in the neightbourhood of $H^{\prime}$.

Proof. We replace $X^{\prime}$ by a suitable neighbourhood of $H^{\prime}$ and $X$ by the pullback of this neighbourhood, so that $\mathscr{M}$ is strict on $X$. By Corollary 8.8.3 it is enough to show the $\mathscr{O}_{X}$-coherence of $U_{\alpha} \mathscr{H}^{i}\left({ }_{\mathrm{D}} f_{*} \mathscr{M}\right)_{p}=\mathscr{H}^{i}\left({ }_{\mathrm{D}} f_{*} V_{\alpha} \mathscr{M}\right)_{p}$ for some $\alpha<0$ and each $p, i$, where the equality holds according to 7.8.5(1).

If $f: X=X^{\prime} \times Z \rightarrow X^{\prime}$ is a projection, we have, in a way similar to Theorem A.8.11(6), ${ }_{\mathrm{D}} f_{*} V_{\alpha} \mathscr{M}=\boldsymbol{R} f_{*}\left(V_{\alpha} \mathscr{M} \otimes_{\widetilde{O}_{X}} \wedge^{-\bullet} \widetilde{\Theta}_{X / X^{\prime}}\right)$, and $\mathscr{H}^{i}\left({ }_{\mathrm{D}} f_{*} V_{\alpha} \mathscr{M}\right)_{p}$ is the $i$ th cohomology of the relative Spencer complex $\left(m=\operatorname{dim} X / X^{\prime}\right)$

$$
\boldsymbol{R} f_{*}\left(0 \rightarrow\left(V_{\alpha} \mathscr{M}\right)_{p-m} \otimes \wedge^{m} \Theta_{X / X^{\prime}} \rightarrow \cdots \rightarrow\left(V_{\alpha} \mathscr{M}\right)_{p-1} \otimes \Theta_{X / X^{\prime}} \rightarrow\left(V_{\alpha} \mathscr{M}\right)_{p} \rightarrow 0\right)
$$

whose differentials are $\mathscr{O}_{X^{\prime}}$-linear. Since each term of the complex is $\mathscr{O}_{X}$-coherent by our assumption of strictness of $\mathscr{M}$ and since $f$ is proper, Grauert's coherence theorem together with a standard spectral sequence argument in the category of $\mathscr{O}_{X^{\prime}}$-complexes show that $\mathscr{H}^{i}\left({ }_{\mathrm{D}} f_{*} V_{\alpha} \mathscr{M}\right)_{p}$ is $\mathscr{O}_{X^{\prime}}$-coherent.

If $f: X \hookrightarrow X^{\prime}$ is a closed immersion, it is locally of the form $\left(t, x_{2}, \ldots, x_{n}\right) \mapsto$ $\left(t, x_{2}, \ldots, x_{n}, 0, \ldots, 0\right)$. Then

$$
{ }_{\mathrm{D}} f_{*} V_{\alpha} \mathscr{M}=\mathscr{H}_{\mathrm{D}}^{0} f_{*} V_{\alpha} \mathscr{M}=f_{*} V_{\alpha} \mathscr{M}\left[\partial_{x_{1}^{\prime}}, \ldots, \partial_{x_{m}^{\prime}}\right]
$$

and

$$
\left(\mathscr{H}_{\mathrm{D}}^{0} f_{*} V_{\alpha} \mathscr{M}\right)_{p}=\sum_{|a| \leqslant p} f_{*}\left(V_{\alpha} \mathscr{M}\right)_{p-|a|} \partial_{x^{\prime}}^{a},
$$

which is $\widetilde{\mathscr{O}}_{X^{\prime}}$-coherent since $\left(V_{\alpha} \mathscr{M}\right)_{q}=0$ for $q \ll 0$ locally (use that $\left(V_{\alpha} \mathscr{M}\right)_{q}=$ $F_{q}(\mathscr{M} /(z-1) \mathscr{M}) \cap V_{\alpha}(\mathscr{M} /(z-1) \mathscr{M})$ according to Corollary 8.8.3, and apply Exercise A.10.5(3)).

Corollary 8.8.5. With the notation and assumptions of Corollary 7.8.6, if $\mathscr{M}$ is strict in the neighbourhood of $g^{-1}(0)$, so is $\mathscr{H}^{i}{ }_{\mathrm{D}} f_{*} \mathscr{M}$ in the neighbourhood of $g^{\prime-1}(0)$.

### 8.9. Comments

The aim of this chapter, which covers part of the content of [Sai88, §1\&3] and whose first sketch has been written by Mircea Mustaţă, is to give a proof of Theorem 8.7.5 which closely follows the original proof of Saito [Sai88, Prop.3.3.17], from which is extracted the formalism of bi-filtered derived categories (see also Appendix A. 11 which is inspired form [Sai89a]). However, the original argument using formal completions, which has been reproduced in the proof of Proposition 7.8.7, has been replaced here (Section 8.5.b) by an argument, due to Christian Schnell, using his interpretation of compatibility of a finite family of filtrations in terms of flatness, which somewhat clarifies [Sai88, §1.1]. This interpretation is explained with details in Section 8.3, ending with Exercise 8.3.20 due to Matthieu Kochersperger. The conclusion of Proposition 8.8.2 is an adaptation of [Sai88, Cor.3.4.7], and is inspired from [ESY15, Prop. 2.2.4].

