CHAPTER 7

NEARBY AND VANISHING CYCLES OF $\widetilde{\mathscr{D}}$ -MODULES

Summary. We introduce the Kashiwara-Malgrange filtration for a \mathcal{D}_X -module, and the notion of strict \mathbb{R} -specializability. This leads to the construction of the nearby and vanishing cycle functors. One of the main results is a criterion for the compatibility of this functor with the proper pushforward functor of \mathcal{D} -modules.

Throughout this chapter we use the following notation.

Notation 7.0.1.

- X denotes a complex manifold.
- H denotes a smooth hypersurface in X.

• Locally on H, we choose a decomposition $X = H \times \Delta_t$, where Δ_t is a small disc in \mathbb{C} with coordinate t. We have the corresponding z-vector field \mathfrak{d}_t .

• *D* denotes an effective divisor on *X*. Locally on *D*, we choose a holomorphic function $g: X \to \mathbb{C}$ such that D = (g). We then set $X_0 = g^{-1}(0)$ (this is the support of *D* in the local setting).

• Recall that $\widetilde{\mathscr{D}}_X$ means \mathscr{D}_X or $R_F \mathscr{D}_X$ and, in the latter case, $\widetilde{\mathscr{D}}_X$ -modules mean graded $\widetilde{\mathscr{D}}_X$ -modules (see Appendix A). We then use (k) for the shift by k of the grading (see Section A.2.a). When the information on the grading is not essential, we just omit to indicate the corresponding shift. We use the convention that, whenever $\widetilde{\mathscr{D}}_X$ means \mathscr{D}_X , all conditions and statements relying on gradedness or strictness are understood to be empty or tautological.

Remark 7.0.2 (Left and right $\widetilde{\mathscr{D}}$ -modules). For various purposes, it is more convenient to work with right $\widetilde{\mathscr{D}}$ -modules. However, left $\widetilde{\mathscr{D}}$ -modules are more commonly used in applications. We will therefore mainly treat right $\widetilde{\mathscr{D}}$ -modules and give the corresponding formulas for left $\widetilde{\mathscr{D}}$ -modules in various remarks.

Remark 7.0.3 (Restriction to z = 1). Throughout this chapter we keep the Convention A.2.19. All the constructions can be done either for \mathscr{D}_X -modules or for graded $R_F \mathscr{D}_X$ -modules, in which case a strictness assumption (strict \mathbb{R} -specializability) is most often needed. By "good behaviour with respect to the restriction z = 1", we

mean that the restriction functor $\mathcal{M} \mapsto \mathcal{M} := \mathcal{M}/(z-1)\mathcal{M}$ is compatible with the constructions. We will see that many, *but not all*, of the constructions in this chapter have good behaviour with respect to setting z = 1. We will make this precise for each such construction.

7.1. Introduction

This chapter has one main purpose: Given a coherent $\widehat{\mathscr{D}}_X$ -module, to give a sufficient condition such that the restriction functor to a divisor D, producing a complex of $\widehat{\mathscr{D}}_X$ -modules supported on the divisor D which corresponds to the functor ${}_{\mathsf{D}}\iota_{H*\mathsf{D}}\iota_H^*$ when $\iota_H: H \hookrightarrow X$ is the inclusion of a smooth hypersurface, gives rise to a complex of $\widehat{\mathscr{D}}_X$ -modules with coherent cohomology.

The property of being *specializable* along D will answer this first requirement. However, in the case where $\widetilde{\mathscr{D}}_X = R_F \mathscr{D}_X$, strictness comes into play in a fundamental way in order to ensure a good behaviour. This leads to the notion of *strict specializability* along D. When forgetting the F-filtration, i.e., when considering \mathscr{D}_X -modules, the strictness condition is empty.

Given any holomorphic function g on X with associated divisor D and for every strictly \mathbb{R} -specializable $\widetilde{\mathscr{D}}_X$ -module \mathscr{M} along D, we introduce the nearby cycle $\widetilde{\mathscr{D}}_X$ -modules $\psi_{g,\lambda}\mathscr{M}$ ($\lambda \in \mathbb{C}^*$ with $|\lambda| = 1$) and the vanishing cycle functor $\phi_{g,1}\mathscr{M}$. They are the "generalized restriction functors", which the usual restriction functors can be deduced from.

The construction is possible when the Kashiwara-Malgrange V-filtration exists on a given $\widetilde{\mathscr{D}}_X$ -module. More precisely, the notion of V-filtration is well-defined in the case when D is a smooth divisor. We reduce to this case by considering, when more generally D = (g), the graph inclusion $\iota_g : X \hookrightarrow X \times \mathbb{C}$. The V-filtration can exist on the pushforward ${}_{\mathsf{D}}\iota_*\mathscr{M}$. We then say that \mathscr{M} is strictly specializable along D.

Kashiwara's equivalence is an equivalence (via the pushforward functor $\iota_Y : Y \hookrightarrow X$) between the category of coherent \mathscr{D}_Y -modules and that of coherent \mathscr{D}_X -modules supported on the submanifold Y. When Y has codimension one in X, this equivalence can be extended as an equivalence between strict coherent $\widetilde{\mathscr{D}}_Y$ -modules and coherent $\widetilde{\mathscr{D}}_X$ -modules which are strictly \mathbb{R} -specializable along Y.

Complex Hodge modules will satisfy a property of semi-simplicity with respect to their support that we introduce in this chapter under the name of *strict Sdecomposability* ("S" is for "support"). The support of a coherent $\widetilde{\mathscr{D}}_X$ -module \mathscr{M} is a closed analytic subspace in X. It may have various irreducible components. We introduce a condition which ensures first that \mathscr{M} decomposes as the direct sum of $\widetilde{\mathscr{D}}_X$ -modules, each of which supported by a single component. However, we wish that each such summand decomposes itself as the direct sum of $\widetilde{\mathscr{D}}_X$ -modules, each of which supported on an irreducible closed analytic subset of the support of the given summand, in order to satisfy a "geometric simplicity property", namely each such new summand has no coherent sub-module supported on a strictly smaller closed analytic subset. We then say that such a summand has *pure support*.

In Section 7.8, we give a criterion in order that the functors $\psi_{g,\lambda}$ and $\phi_{g,1}$ commute with proper pushforward. This will be an essential step in the theory of complex Hodge modules (see Chapter 13), where we need to prove that the property of strict S-decomposability (i.e., geometric semi-simplicity) is preserved by proper pushforward.

7.2. The filtration $V_{\bullet} \widetilde{\mathscr{D}}_X$ relative to a smooth hypersurface

Let $H \subset X$ be a smooth hypersurface ⁽¹⁾ of X with defining ideal $\mathscr{I}_H \subset \mathscr{O}_X$. We first define a canonical increasing filtration of $\widetilde{\mathscr{D}}_X$ indexed by \mathbb{Z} . Let us set $\widetilde{\mathscr{I}}_H^{\ell} = \widetilde{\mathscr{O}}_X$ for $\ell < 0$ and $\widetilde{\mathscr{I}}_H^{\ell} = \mathscr{I}_H^{\ell} \widetilde{\mathscr{O}}_X$ for $\ell \ge 0$. For every $k \in \mathbb{Z}$, the subsheaf $V_k \widetilde{\mathscr{D}}_X \subset \widetilde{\mathscr{D}}_X$ $(k \in \mathbb{Z})$ consists of operators P such that $\widetilde{\mathscr{I}}_H^j P \subset \widetilde{\mathscr{I}}_H^{j-k}$ for every $j \in \mathbb{Z}$. For every open set U of X we thus have

(7.2.1)
$$V_k \widetilde{\mathscr{D}}_X(U) = \{ P \in \widetilde{\mathscr{D}}_X(U) \mid \widetilde{\mathscr{I}}_H^j(U) \cdot P \subset \widetilde{\mathscr{I}}_H^{j-k}(U), \, \forall j \in \mathbb{Z} \}.$$

This defines an increasing filtration $V_{\bullet}\widetilde{\mathscr{D}}_X$ of $\widetilde{\mathscr{D}}_X$ indexed by \mathbb{Z} . Note that one can also define $V_k\widetilde{\mathscr{D}}_X(U)$ as the set of $Q \in \widetilde{\mathscr{D}}_X(U)$ such that $Q \cdot \widetilde{\mathscr{I}}_H^j(U) \subset \widetilde{\mathscr{I}}_H^{j-k}(U)$, $\forall j \in \mathbb{Z}$.

Exercise 7.2.2. Show the following properties.

(1) Let us fix a local decomposition $X \simeq H \times \Delta_t$ (where $\Delta_t \subset \mathbb{C}$ is a disc with coordinate t). With respect to this decomposition,

$$V_0 \widetilde{\mathscr{D}}_X = \widetilde{\mathscr{O}}_X \langle \eth_x, t \eth_t \rangle, \quad V_{-j} \widetilde{\mathscr{D}}_X = V_0 \widetilde{\mathscr{D}}_X \cdot t^j, \quad V_j \widetilde{\mathscr{D}}_X = \sum_{k=0}^j V_0 \widetilde{\mathscr{D}}_X \cdot \eth_t^k \quad (j \ge 0).$$

(2) For every $k, V_k \widetilde{\mathscr{D}}_X$ is a locally free $V_0 \widetilde{\mathscr{D}}_X$ -module.

- (3) $\widetilde{\mathscr{D}}_X = \bigcup_k V_k \widetilde{\mathscr{D}}_X$ (the filtration is exhaustive).
- (4) $V_k \widetilde{\mathscr{D}}_X \cdot V_\ell \widetilde{\mathscr{D}}_X \subset V_{k+\ell} \widetilde{\mathscr{D}}_X$ with equality for $k, \ell \leq 0$ or $k, \ell \geq 0$.
- (5) $V_0 \widetilde{\mathscr{D}}_X$ is a sheaf of subalgebras of $\widetilde{\mathscr{D}}_X$.
- (6) $V_k \mathscr{D}_{X|X \smallsetminus H} = \mathscr{D}_{X|X \smallsetminus H}$ for all $k \in \mathbb{Z}$.
- (7) $\operatorname{gr}_{k}^{V} \widetilde{\mathscr{D}}_{X}$ is supported on H for all $k \in \mathbb{Z}$,

(8) The induced filtration $V_k \widetilde{\mathscr{D}}_X \cap \widetilde{\mathscr{O}}_X = \widetilde{\mathscr{I}}_H^{-k} \widetilde{\mathscr{O}}_X$ is the $\widetilde{\mathscr{I}}_H$ -adic filtration of $\widetilde{\mathscr{O}}_X$ made increasing.

(9)
$$\left(\bigcap_{k} V_{k} \widetilde{\mathscr{D}}_{X}\right)_{\mid H} = \{0\}.$$

Exercise 7.2.3 (Euler vector field).

(1) Show that the class E of $t \eth_t$ in $\operatorname{gr}_0^V \widetilde{\mathscr{D}}_X$ in some local product decomposition as above does not depend on the choice of such a local product decomposition.

^{1.} Other settings can be considered, for example a smooth subvariety, or a finite family of smooth subvarieties, but they will not be needed for our purpose.

- (2) Show that if H has a global equation g, then $\operatorname{gr}_0^V \widetilde{\mathscr{D}}_X \simeq \widetilde{\mathscr{D}}_H[\mathrm{E}].$
- (3) Conclude that $\operatorname{gr}_0^V \widetilde{\mathscr{D}}_X$ is a sheaf of rings and that E belongs to its center.

Remark 7.2.4 (Structure of $\operatorname{gr}_0^V \widetilde{\mathscr{D}}_X$). While $\widetilde{\mathscr{D}}_H$ can be identified to the quotient $\operatorname{gr}_0^V \widetilde{\mathscr{D}}_X / \operatorname{Egr}_0^V \widetilde{\mathscr{D}}_X = \operatorname{gr}_0^V \widetilde{\mathscr{D}}_X / \operatorname{gr}_0^V \widetilde{\mathscr{D}}_X$ E, it is not identified with a subsheaf of $\operatorname{gr}_0^V \widetilde{\mathscr{D}}_X$, except when $N_H X$ is trivial. When H is globally defined by a holomorphic function g, or more generally for any holomorphic function $g : X \to \mathbb{C}$, we will often use the trick of the graph inclusion $\iota_g : X \hookrightarrow X \times \mathbb{C}$ and we will then consider the filtration $V_{\bullet} \widetilde{\mathscr{D}}_{X \times \mathbb{C}}$ with respect to $X \times \{0\}$, so that we will be able to identify $\operatorname{gr}_0^V \widetilde{\mathscr{D}}_{X \times \mathbb{C}}$ with the ring $\widetilde{\mathscr{D}}_X[\mathrm{E}]$.

Exercise 7.2.5. Show the equivalence between the category of $\widetilde{\mathscr{O}}_X$ -modules with integrable logarithmic connection $\widetilde{\nabla} : \mathscr{M} \to \widetilde{\Omega}^1_X(\log H) \otimes \mathscr{M}$ and the category of left $V_0 \widetilde{\mathscr{O}}_X$ -modules. Show that the residue Res $\widetilde{\nabla}$ corresponds to the induced action of E on $\mathscr{M} / \widetilde{\mathscr{I}}_H \mathscr{M}$.

Let $\nu : N_H X \to H$ denote the normal bundle of H in X and set $\widetilde{\mathscr{D}}_{[N_H X]} := \nu_* \widetilde{\mathscr{D}}_{N_H X}$ (where ν_* is taken in the algebraic sense) with its filtration $V_{\bullet} \widetilde{\mathscr{D}}_{[N_H X]}$. Then there is a canonical isomorphism (as graded objects) $\operatorname{gr}^V \widetilde{\mathscr{D}}_X \simeq \operatorname{gr}^V \widetilde{\mathscr{D}}_{[N_H X]}$, and the latter sheaf is isomorphic (forgetting the grading) to $\widetilde{\mathscr{D}}_{[N_H X]}$.

Exercise 7.2.6 (The Rees sheaf of rings $R_V \widetilde{\mathscr{D}}_X$). Introduce the Rees sheaf of rings $R_V \widetilde{\mathscr{D}}_X := \bigoplus_k V_k \widetilde{\mathscr{D}}_X \cdot v^k \subset \widetilde{\mathscr{D}}_X[v, v^{-1}]$ associated to the filtered sheaf $(\widetilde{\mathscr{D}}_X, V_{\bullet} \widetilde{\mathscr{D}}_X)$ (see Definition A.2.3), and similarly $R_V \widetilde{\mathscr{O}}_X = \bigoplus_k V_k \widetilde{\mathscr{O}}_X \cdot v^k \subset \widetilde{\mathscr{O}}_X[v, v^{-1}]$, which is the Rees ring associated to the $\widetilde{\mathscr{I}}_H$ -adic filtration of $\widetilde{\mathscr{O}}_X$.

- (1) Show that $R_V \widetilde{\mathcal{O}}_X = \widetilde{\mathcal{O}}_X[v, tv^{-1}]$, where t = 0 is a local equation of H.
- (2) Show that $R_V \widetilde{\mathscr{D}}_X = \widetilde{\mathscr{O}}_X[v, tv^{-1}] \langle v \eth_t, \eth_{x_2}, \dots, \eth_{x_n} \rangle.$
- (3) Conclude that $R_V \widetilde{\mathscr{D}}_X$ is locally free over $R_V \widetilde{\mathscr{O}}_X$.

Remark 7.2.7 (V-filtration indexed by $A + \mathbb{Z}$). The following construction of extending the set of indices will prove useful. Let $A \subset [0,1)$ be a finite subset containing 0. Let us fix the numbering of $A + \mathbb{Z} = \{\ldots, \alpha_{-1}, \alpha_o, \alpha_1, \ldots\}$ which respect the order and such that $\alpha_o = 0$. We thus have $1 = \alpha_{\#A}$. We denote by ${}^{A}V_{\bullet}\widetilde{\mathscr{D}}_X$ the filtration indexed by $A + \mathbb{Z}$ defined by ${}^{A}V_{\alpha}\widetilde{\mathscr{D}}_X := V_{[\alpha]}\widetilde{\mathscr{D}}_X$. We consider it as a filtration indexed by \mathbb{Z} by using the previous order-preserving bijection. Since $[\alpha] + [\beta] \leq [\alpha + \beta]$, we have ${}^{A}V_{\alpha}\widetilde{\mathscr{D}}_X \cdot {}^{A}V_{\beta}\widetilde{\mathscr{D}}_X \subset {}^{A}V_{\alpha+\beta}\widetilde{\mathscr{D}}_X$, and on the other hand, ${}^{A}V_{\alpha_o}\widetilde{\mathscr{D}}_X = V_0\widetilde{\mathscr{D}}_X$. The Rees ring is $R_{AV}\widetilde{\mathscr{D}}_X := \bigoplus_{k \in \mathbb{Z}} {}^{A}V_{\alpha_k}\widetilde{\mathscr{D}}_X v^k$. Note also that

$$\operatorname{gr}^{A_{V}}\widetilde{\mathscr{D}}_{X} = \bigoplus_{k \in \mathbb{Z}} \operatorname{gr}^{A_{V}}_{\alpha_{k}} \widetilde{\mathscr{D}}_{X} = \bigoplus_{k \in \#A \cdot \mathbb{Z}} \operatorname{gr}^{V}_{(k/\#A)} \widetilde{\mathscr{D}}_{X}.$$

It will sometimes be convenient to write, for short, $R_{AV} \widetilde{\mathscr{D}}_X := \bigoplus_{\alpha \in A + \mathbb{Z}} {}^{A}V_{\alpha} \widetilde{\mathscr{D}}_X v^{\alpha}$.

Exercise 7.2.8. Define similarly ${}^{A}V_{\alpha}\widetilde{\mathcal{O}}_{X}$ and show that $R_{AV}\widetilde{\mathscr{D}}_{X}$ is locally free over $R_{AV}\widetilde{\mathscr{O}}_{X}$.

Remark 7.2.9 (Restriction to z = 1). The V-filtration restricts well when setting z = 1, that is, $V_k \mathscr{D}_X = V_k \widetilde{\mathscr{D}}_X / (z-1) V_k \widetilde{\mathscr{D}}_X = V_k \widetilde{\mathscr{D}}_X / (z-1) \widetilde{\mathscr{D}}_X \cap V_k \widetilde{\mathscr{D}}_X$.

7.3. Specialization of coherent $\widetilde{\mathscr{D}}_X$ -modules

In this section, H denotes a smooth hypersurface of a complex manifold X and we denote by t a local generator of \mathscr{I}_{H} . We use the definitions and notation of Section 7.2.

Caveat 7.3.1. In Subsections 7.3.a–7.3.c, when $\widetilde{\mathscr{D}}_X = R_F \mathscr{D}_X$, we will forget about the grading of the $\widetilde{\mathscr{D}}_X$ -modules and morphisms involved, in order to keep the notation similar to the case of \mathscr{D}_X -modules. From Section 7.4, we will remember the shift of grading for various morphisms, in the case of $R_F \mathscr{D}_X$ -modules (this shift has no influence in the case of \mathscr{D}_X -modules).

7.3.a. Coherent V-filtrations

Exercise 7.3.2 (Coherence of $R_V \widetilde{\mathscr{D}}_X$). We consider the Rees sheaf of rings $R_V \widetilde{\mathscr{D}}_X := \bigoplus_k V_k \widetilde{\mathscr{D}}_X \cdot v^k$ as in Exercise 7.2.6. The aim of this exercise is to show the coherence of the sheaf of rings $R_V \widetilde{\mathscr{D}}_X$. Since the problem is local, we can assume that there are coordinates (t, x_2, \ldots, x_n) such that $H = \{t = 0\}$.

(1) Let K be a compact polycylinder in X. Show that $R_V \widetilde{\mathscr{O}}_X(K) = R_V(\widetilde{\mathscr{O}}_X(K))$ is Noetherian, being the Rees ring of the $\widetilde{\mathscr{I}}_H$ -adic filtration on the ring $\widetilde{\mathscr{O}}_X(K)$ (which is Noetherian, by a theorem of Frisch). Similarly, as $\widetilde{\mathscr{O}}_{X,x}$ is flat on $\widetilde{\mathscr{O}}_X(K)$ for every $x \in K$, show that the ring $(R_V \widetilde{\mathscr{O}}_X)_x = R_V \widetilde{\mathscr{O}}_X(K) \otimes_{\widetilde{\mathscr{O}}_X(K)} \widetilde{\mathscr{O}}_{X,x}$ is flat on $R_V \widetilde{\mathscr{O}}_X(K)$.

(2) Show that $R_V \widetilde{\mathscr{O}}_X$ is coherent on X by following the strategy developed in [GM93]. [*Hint*: Let $\widetilde{\Omega}$ be any open set in X and let $\varphi : (R_V \widetilde{\mathscr{O}}_X)_{|\widetilde{\Omega}}^q \to (R_V \widetilde{\mathscr{O}}_X)_{|\widetilde{\Omega}}^p$ be any morphism. Let K be a polycylinder contained in $\widetilde{\Omega}$. Show that $\operatorname{Ker} \varphi(K)$ is finitely generated over $R_V \widetilde{\mathscr{O}}_X(K)$ and, if K° is the interior of K, show that $\operatorname{Ker} \varphi_{|K^\circ} = \operatorname{Ker} \varphi(K) \otimes_{R_V \widetilde{\mathscr{O}}_X(K)} (R_V \widetilde{\mathscr{O}}_X)_{|K^\circ}$. Conclude that $\operatorname{Ker} \varphi_{|K^\circ}$ is finitely generated, whence the coherence of $R_V \widetilde{\mathscr{O}}_X$.]

(3) Consider the sheaf $\mathscr{O}_X[\tau,\xi_2,\ldots,\xi_n]$ equipped with the V-filtration for which τ has degree 1, the variables ξ_2,\ldots,ξ_n have degree 0, and inducing the V-filtration (i.e., t-adic in the reverse order) on $\widetilde{\mathscr{O}}_X$. Firstly, forgetting τ , Show that $R_V(\widetilde{\mathscr{O}}_X[\xi_2,\ldots,\xi_n]) = (R_V\widetilde{\mathscr{O}}_X)[\xi_2,\ldots,\xi_n]$. Secondly, using $V_k(\widetilde{\mathscr{O}}_X[\tau,\xi_2,\ldots,\xi_n]) =$ $\sum_{j\geq 0} V_{k-j}(\widetilde{\mathscr{O}}_X[\xi_2,\ldots,\xi_n])\tau^j$ for every $k \in \mathbb{Z}$, show that we have a surjective morphism

$$R_V \widetilde{\mathscr{O}}_X[\xi_2, \dots, \xi_n] \otimes_{\widetilde{\mathbb{C}}} \widetilde{\mathbb{C}}[\tau'] \longrightarrow R_V (\widetilde{\mathscr{O}}_X[\tau, \xi_2, \dots, \xi_n])$$
$$V_\ell \widetilde{\mathscr{O}}_X[\xi_2, \dots, \xi_n] q^\ell \tau'^j \longmapsto V_\ell \widetilde{\mathscr{O}}_X[\xi_2, \dots, \xi_n] \tau^j q^{\ell+j}.$$

If $K \subset X$ is any polycylinder show that $R_V(\widetilde{\mathscr{O}}_X[\tau,\xi_2,\ldots,\xi_n])(K)$ is Noetherian, by using that $(R_V\widetilde{\mathscr{O}}_X(K))[\tau',\xi_2,\ldots,\xi_n]$ is Noetherian.

(4) As $R_V \widetilde{\mathscr{D}}_X$ can be filtered (by the degree of the operators) in such a way that, locally on X, $\operatorname{gr} R_V \widetilde{\mathscr{D}}_X$ is isomorphic to $R_V(\widetilde{\mathscr{O}}_X[\tau, \xi_2, \ldots, \xi_n])$, conclude that, if K is any sufficiently small polycylinder, then $R_V \widetilde{\mathscr{D}}_X(K)$ is Noetherian.

(5) Use now arguments similar to that of [GM93] to concludes that $R_V \mathscr{D}_X$ is coherent.

Definition 7.3.3 (Coherent V-filtrations). Let \mathscr{M} be a coherent right $\widehat{\mathscr{D}}_X$ -module. A V-filtration indexed by \mathbb{Z} is an increasing filtration $U_{\bullet}\mathscr{M}$ which satisfies $U_{\ell}\mathscr{M} \cdot V_k \widehat{\mathscr{D}}_X \subset U_{\ell+k}\mathscr{M}$ for every $k, \ell \in \mathbb{Z}$. In particular, each $U_{\ell}\mathscr{M}$ is a right $V_0 \widehat{\mathscr{D}}_X$ -module. We say that it is a *coherent* V-filtration if each $U_{\ell}\mathscr{M}$ is $V_0 \widehat{\mathscr{D}}_X$ -coherent, locally on X, there exists $\ell_o \geq 0$ such that, for all $k \geq 0$,

$$U_{-k-\ell_0}\mathscr{M} = U_{-\ell_o}\mathscr{M} \cdot t^k$$
 and $U_{k+\ell_0}\mathscr{M} = \sum_{j=0}^k U_{\ell_o}\mathscr{M} \eth_t^j$.

Remark 7.3.4 (The case of left $\widetilde{\mathscr{D}}_X$ -modules). For left $\widetilde{\mathscr{D}}$ -modules, it is more usual to consider a decreasing filtration $U^{\bullet}\mathscr{M}$ which satisfies $V_k\widetilde{\mathscr{D}}_X \cdot U^{\ell}\mathscr{M} \subset U^{\ell-k}\mathscr{M}$ for every $k, \ell \in \mathbb{Z}$. We say that such a filtration is a *coherent V*-filtration if each $U^{\ell}\mathscr{M}$ is $V_0\widetilde{\mathscr{D}}_X$ -coherent, locally on X, there exists $\ell_o \geq 0$ such that, for all $k \geq 0$,

$$U^{k+\ell_0}\mathscr{M} = t^k U^{\ell_o}\mathscr{M} \text{ and } U^{-(k+\ell_0)}\mathscr{M} = \sum_{j=0}^k \eth_t^j U^{-\ell_o}\mathscr{M}.$$

Exercise 7.3.5 (Characterization of coherent *V*-filtrations). Let \mathscr{M} be a coherent right $\widetilde{\mathscr{D}}_X$ -module. Show that the following properties are equivalent for a *V*-filtration $U_{\bullet}\mathscr{M}$.

- (1) $U_{\bullet}\mathscr{M}$ is a coherent filtration.
- (2) The Rees module $R_U \mathscr{M} := \bigoplus_{\ell} U_{\ell} \mathscr{M} v^{\ell}$ is $R_V \widetilde{\mathscr{D}}_X$ -coherent.

(3) For every $x \in X$, replacing X with a small neighbourhood of x, there exist integers $\lambda_{j=1,\ldots,q}, \mu_{i=1,\ldots,p}, k_{i=1,\ldots,p}$ and a presentation (recall that [•] means a shift of the grading)

$$\bigoplus_{j=1}^{q} \widetilde{\mathscr{D}}_{X}[\lambda_{j}] \longrightarrow \bigoplus_{i=1}^{p} \widetilde{\mathscr{D}}_{X}[\mu_{i}] \longrightarrow \mathscr{M} \longrightarrow 0$$

such that $U_{\ell}\mathcal{M} = \operatorname{image}(\bigoplus_{i=1}^{p} V_{k_i+\ell} \widetilde{\mathscr{D}}_X[\mu_i]).$

Note that, as for $\widetilde{\mathscr{I}}_H$ -adic filtrations on coherent $\widetilde{\mathscr{O}}_X$ -modules, it is not enough to check the coherence of $\operatorname{gr}_U \mathscr{M}$ as a $\operatorname{gr}^V \widetilde{\mathscr{D}}_X$ -module in order to deduce that $U_{\bullet} \mathscr{M}$ is a coherent V-filtration.

Exercise 7.3.6 (From coherent $R_V \widetilde{\mathscr{D}}_X$ -modules to $\widetilde{\mathscr{D}}_X$ -modules with a coherent V-filtration)

(1) Show that a graded $R_V \widetilde{\mathscr{D}}_X$ -module \mathcal{M} can be written as $R_U \mathscr{M}$ for some V-filtration on some $\widetilde{\mathscr{D}}_X$ -module \mathscr{M} if and only if it has no v-torsion.

(2) Show that, if \mathcal{M} is a graded coherent $R_V \widetilde{\mathscr{D}}_X$ -module, then its v-torsion is a graded coherent $R_V \widetilde{\mathscr{D}}_X$ -module.

(3) Conclude that, for any graded coherent $R_V \widetilde{\mathscr{D}}_X$ -module \mathcal{M} , there exists a unique coherent $\widetilde{\mathscr{D}}_X$ -module and a unique coherent V-filtration $U^{\bullet}\mathscr{M}$ such that \mathcal{M}/v -torsion = $R_U \mathscr{M}$.

Exercise 7.3.7 (Some basic properties of coherent V-filtrations)

(1) Show that the filtration naturally induced by a coherent V-filtration on a coherent $\widetilde{\mathscr{D}}_X$ -module on a coherent sub or quotient $\widetilde{\mathscr{D}}_X$ -modules is a coherent V-filtration.

(2) Deduce that, locally on X, there exist integers $\lambda_{j=1,\ldots,q}$, $\ell_{j=1,\ldots,q}$, $\mu_{i=1,\ldots,p}$, $k_{i=1,\ldots,p}$ and a presentation $\bigoplus_{j=1}^{q} \widetilde{\mathscr{D}}_{X}[\lambda_{j}] \to \bigoplus_{i=1}^{p} \widetilde{\mathscr{D}}_{X}[\mu_{i}] \to \mathscr{M} \to 0$ inducing for every ℓ a presentation

$$\bigoplus_{j=1}^{q} V_{\ell_{j}+\ell} \widetilde{\mathscr{D}}_{X}[\lambda_{j}] \longrightarrow \bigoplus_{i=1}^{p} V_{k_{i}+\ell} \widetilde{\mathscr{D}}_{X}[\mu_{i}] \longrightarrow U_{\ell} \mathscr{M} \longrightarrow 0.$$

(3) Show that two coherent V-filtrations $U_{\bullet}\mathcal{M}$ and $U' \bullet \mathcal{M}$ are locally comparable, that is, locally on X there exists $\ell_o \ge 0$ such that, for every $\ell \in \mathbb{Z}$,

$$U_{\ell-\ell_o}\mathscr{M} \subset U'_{\ell}\mathscr{M} \subset U_{\ell+\ell_o}\mathscr{M}.$$

(4) If $U_{\bullet}\mathcal{M}$ is a coherent V-filtration, then for every $\ell_o \in \mathbb{Z}$, the filtration $U_{\bullet+\ell_o}\mathcal{M}$ is also coherent.

(5) If $U_{\bullet}\mathcal{M}$ and $U'_{\bullet}\mathcal{M}$ are two coherent V-filtrations, then the filtration $U''_{\ell}\mathcal{M} := U_{\ell}\mathcal{M} + U'_{\ell}\mathcal{M}$ is also coherent.

(6) Assume that H is defined by an equation t = 0. Prove that, locally on X, there exists k_0 such that, for every $k \leq k_0$, $t: U_k \to U_{k-1}$ is bijective. [*Hint*: Use (2) above.]

Exercise 7.3.8. Let \mathscr{U} be a coherent left $V_0 \widetilde{\mathscr{D}}_X$ -module and let \mathscr{T} be its *t*-torsion subsheaf, i.e., the subsheaf of local sections locally killed by some power of *t*. Show that, locally on *X*, there exists ℓ such that $\mathscr{T} \cap \mathscr{U}t^{\ell} = 0$. [*Hint*: Consider the *t*-adic filtration on $V_0 \widetilde{\mathscr{D}}_X$, i.e., the filtration $V_{-j} \widetilde{\mathscr{D}}_X$ with $j \ge 0$. Show that the filtration $\mathscr{U}t^j$ is coherent with respect to it, and locally there is a surjective morphism $(V_0 \widetilde{\mathscr{D}}_X)^n \to \mathscr{U}$ which is strict with respect to the *V*-filtration. Deduce that its kernel \mathscr{K} is coherent and comes equipped with the induced *V*-filtration, which is coherent. Conclude that, locally on *X*, there exists $j_0 \ge 0$ such that $V_{j_0-j}\mathscr{K} = V^{j_0}\mathscr{K} \cdot t^j$ for every $j \ge 0$. Show that, for every $j \ge 0$ there is locally an exact sequence (up to shifting the grading on each $V_{\bullet} \widetilde{\mathscr{D}}_X$ summand)

$$(V_{-j}\widetilde{\mathscr{D}}_X)^m \longrightarrow (V_{-(j+j_0)}\widetilde{\mathscr{D}}_X)^n \longrightarrow \mathscr{U}t^{(j+j_0)} \longrightarrow 0.$$

As $t: V_k \widetilde{\mathscr{D}}_X \to V_{k-1} \widetilde{\mathscr{D}}_X$ is bijective for $k \leq 0$, conclude that $t: \mathscr{U} t^{j_0} \to \mathscr{U} t^{j_0+1}$ is so, hence $\mathscr{T} \cap \mathscr{U} t^{j_0} = 0$.]

Exercise 7.3.9 (Coherent V-filtration indexed by $A + \mathbb{Z}$). Extend the previous properties to coherent V-filtrations indexed by $A + \mathbb{Z}$, where $A \subset [0, 1)$ is some finite set (see Remark 7.2.7).

7.3.b. Specializable coherent $\widetilde{\mathscr{D}}_X$ -modules. Let $H \subset X$ be a smooth hypersurface. Let \mathscr{M} be a left (resp. right) coherent $\widetilde{\mathscr{D}}_X$ -module and let m be a germ of section of \mathscr{M} . In the following, we abuse notation by denoting $E \in V_0 \widetilde{\mathscr{D}}_X$ any local lifting of the Euler operator $E \in \operatorname{gr}_0^V \widetilde{\mathscr{D}}_X$, being understood that the corresponding formula does not depend on the choice of such a lifting.

Definition 7.3.10.

(1) A weak Bernstein equation for m is a relation

(7.3.10*)
$$m \cdot (z^{\ell}b(\mathbf{E}) - P) = 0,$$

where

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• ℓ is some nonnegative integer,

• b(s) is a nonzero polynomial in a variable s with coefficients in \mathbb{C} , which takes the form $\prod_{\alpha \in A} (s - \alpha z)^{\nu_{\alpha}}$ for some finite subset $A \in \mathbb{C}$ (depending on m),

• P is a germ in $V_{-1}\widetilde{\mathscr{D}}_X$, i.e., P = tQ = Q't with Q, Q' germs in $V_0\widetilde{\mathscr{D}}_X$.

(2) We say that \mathscr{M} is *specializable along* H if any germ of section of \mathscr{M} is the solution of some weak Bernstein equation (7.3.10*).

Exercise 7.3.11. Show that a coherent $\widetilde{\mathscr{D}}_X$ -module \mathscr{M} is specializable along H if and only if one of the following properties holds:

(1) locally on X, some coherent V-filtration $U_{\bullet}\mathscr{M}$ has a weak Bernstein polynomial, i.e., there exists a nonzero b(s) and a nonnegative integer ℓ such that

(7.3.11*)
$$\forall k \in \mathbb{Z}, \quad \operatorname{gr}_{k}^{U} \mathscr{M} \cdot z^{\ell} b(\mathbf{E} - kz) = 0;$$

(2) locally on X, any coherent V-filtration $U_{\bullet}\mathscr{M}$ has a weak Bernstein polynomial. [*Hint*: in one direction, take the V-filtration generated by a finite number of local generators of \mathscr{M} ; in the other direction, use that two coherent filtrations are locally comparable.]

Exercise 7.3.12. Assume that \mathscr{M} is $\widetilde{\mathscr{D}}_X$ -coherent and specializable along H.

(1) Fix $\ell_o \in \mathbb{Z}$ and set $U'_{\ell}\mathcal{M} = U_{\ell+\ell_o}\mathcal{M}$. Show that $b_{U'}(s)$ can be chosen as $b_U(s-\ell_o z)$.

(2) Set $b_U = b_1 b_2$ where b_1 and b_2 have no common root. Show that the filtration $U'_k \mathscr{M} := U_{k-1} \mathscr{M} + b_2 (E - kz) U_k \mathscr{M}$ is a coherent filtration and compute a polynomial $b_{U'}$ in terms of b_1, b_2 .

(3) Conclude that there exists locally a coherent filtration $U_{\bullet}\mathscr{M}$ for which $b_U(s) = \prod_{\alpha \in A} (s - \alpha z)^{\nu_{\alpha}}$ and $\operatorname{Re}(A) \subset (-1, 0]$.

Assume that \mathscr{M} is $\widetilde{\mathscr{D}}_X$ -coherent and specializable along H. According to Bézout, for every local section m of \mathscr{M} , there exists a minimal polynomial

$$b_m(s) = \prod_{\alpha \in R(m)} (s - \alpha z)^{\nu_\alpha}$$

giving rise to a weak Bernstein equation (7.3.10*). We say that \mathscr{M} is \mathbb{R} -specializable along H if for every local section m, we have $R(m) \subset \mathbb{R}$. We then set:

(7.3.13)
$$\operatorname{ord}_H(m) = \max R(m).$$

Exercise 7.3.14. Assume that \mathscr{M} is an \mathbb{R} -specializable coherent $\widetilde{\mathscr{D}}_X$ -module. Show that, for $m \in \mathscr{M}_{x_o}$ and $P \in V_k \widetilde{\mathscr{D}}_{X,x_o}$, we have

$$\operatorname{ord}_{H,x_o}(m \cdot P) \leqslant \operatorname{ord}_{H,x_o}(m) + k$$

[*Hint*: use that $[\mathbb{E}, V_{-1}\widetilde{\mathscr{D}}_X] \subset V_0\widetilde{\mathscr{D}}_X$ and that the coherent V-filtrations $\widetilde{\mathscr{D}}_X(mP) \cap V_{\bullet}\widetilde{\mathscr{D}}_X \cdot m$ and $V_{\bullet}\widetilde{\mathscr{D}}_X \cdot mP$ of $\widetilde{\mathscr{D}}_X \cdot (mP)$ are locally comparable.]

The filtration by the order along H, also called the Kashiwara-Malgrange filtration of \mathscr{M} along H, is the increasing filtration $V_{\bullet}\mathscr{M}_{x_o}$ indexed by \mathbb{R} defined by

(7.3.15)
$$V_{\alpha}\mathscr{M}_{x_{o}} = \{ m \in \mathscr{M}_{x_{o}} \mid \operatorname{ord}_{H,x_{o}}(m) \leqslant \alpha \},$$

$$(7.3.16) V_{<\alpha}\mathscr{M}_{x_o} = \{m \in \mathscr{M}_{x_o} \mid \operatorname{ord}_{H,x_o}(m) < \alpha\}.$$

We do not claim that it is a coherent V-filtration. The order filtration satisfies, $\forall k \in \mathbb{Z}, \forall \alpha, \beta \in \mathbb{R}$

 $V_{\alpha}\mathscr{M}_{x_o} \cdot V_k \widetilde{\mathscr{D}}_{X, x_o} \subset V_{\alpha+k}\mathscr{M}_{x_o}.$

It is a filtration of \mathscr{M} by subsheaves $V_{\alpha}\mathscr{M}$ of $V_0\widetilde{\mathscr{D}}_X$ -modules. We set

(7.3.17)
$$\operatorname{gr}_{\alpha}^{V} \mathscr{M} := V_{\alpha} \mathscr{M} / V_{<\alpha} \mathscr{M}.$$

These are $\operatorname{gr}_0^V \widetilde{\mathscr{D}}_X$ -modules. In particular, they are endowed with an action of the Euler field E. We already notice, as a preparation to strict \mathbb{R} -specializability, that the satisfy part of the strictness condition.

Lemma 7.3.18. The $\operatorname{gr}_0^V \widetilde{\mathscr{D}}_X$ -module $\operatorname{gr}_\alpha^V \mathscr{M}$ has no z-torsion.

Proof. It is a matter of proving that, for a section m of $V_{\alpha}\mathcal{M}$, if mz^{j} is a section of $V_{<\alpha}\mathcal{M}$ for some $j \ge 0$, then so does m. But one checks in a straightforward way that, if P in Exercise 7.3.14 is equal to z^{j} , then the inequality there is an equality (with k = 0).

Remark 7.3.19 (The case of left $\widehat{\mathscr{D}}_X$ -modules). The order of a local section m is defined as $\operatorname{ord}_H(m) = \min R(m)$. In Exercise 7.3.14 we have $\operatorname{ord}_{H,x_o}(Pm) \ge \operatorname{ord}_{H,x_o}(m) - k$. The filtration by the order along H is the decreasing filtration $V^{\bullet}\mathscr{M}_{x_o}$ indexed by \mathbb{R} defined by

$$V^{\beta}\mathcal{M}_{x_{o}} = \{ m \in \mathcal{M}_{x_{o}} \mid \operatorname{ord}_{H,x_{o}}(m) \geq \beta \},\$$
$$V^{>\beta}\mathcal{M}_{x_{o}} = \{ m \in \mathcal{M}_{x_{o}} \mid \operatorname{ord}_{H,x_{o}}(m) > \beta \}.$$

The order filtration satisfies, $\forall k \in \mathbb{Z}, \forall \alpha, \beta \in \mathbb{R}, V_k \widetilde{\mathscr{D}}_{X,x_o} \cdot V^{\beta} \mathscr{M}_{x_o} \subset V^{\beta-k} \mathscr{M}_{x_o}$. We set $\operatorname{gr}_V^{\beta} \mathscr{M} := V^{\beta} \mathscr{M}/V^{>\beta} \mathscr{M}$.

Exercise 7.3.20.

(1) Assume that \mathscr{M} is \mathbb{R} -specializable along H. Show that any sub- $\widetilde{\mathscr{D}}_X$ -module \mathscr{M}' and any quotient $\widetilde{\mathscr{D}}_X$ -module \mathscr{M}'' is also \mathbb{R} -specializable along H.

(2) Let $\varphi : \mathscr{M}_1 \to \mathscr{M}_2$ be a morphism between \mathbb{R} -specializable modules along H. Show that φ is compatible with the order filtrations along H. Conclude that, on the full sbucategory consisting of \mathbb{R} -specializable $\widetilde{\mathscr{D}}_X$ -modules of the category of $\widetilde{\mathscr{D}}_X$ -modules (and morphisms consist of all morphisms of $\widetilde{\mathscr{D}}_X$ -modules), $\operatorname{gr}^V_{\alpha}$ is a functor to the category of $\operatorname{gr}^V_0 \widetilde{\mathscr{D}}_X$ -modules.

Exercise 7.3.21 (Restriction to z = 1). Let \mathscr{M} be a coherent $R_F \mathscr{D}_X$ -module. Assume that \mathscr{M} is \mathbb{R} -specializable along H.

(1) Show that for every α ,

$$(z-1)\mathcal{M} \cap V_{\alpha}\mathcal{M} = (z-1)V_{\alpha}\mathcal{M}$$

[*Hint*: let m = (z-1)n be a local section of $(z-1)\mathcal{M} \cap V_{\alpha}\mathcal{M}$; then n is a local section of $V_{\gamma}\mathcal{M}$ for some γ ; if $\gamma > \alpha$, show that the class of n in $\operatorname{gr}_{\gamma}^{V}\mathcal{M}$ is a annihilated by z-1; conclude with Exercise A.2.5(1).]

(2) Conclude that $\mathcal{M} := \mathcal{M}/(z-1)\mathcal{M}$ is \mathbb{R} -specializable along H and that, for every α ,

$$V_{\alpha}\mathcal{M} = V_{\alpha}\mathcal{M}/(z-1)V_{\alpha}\mathcal{M} = V_{\alpha}\mathcal{M}/((z-1)\mathcal{M} \cap V_{\alpha}\mathcal{M}),$$

$$\mathrm{gr}_{\alpha}^{V}\mathcal{M} = \mathrm{gr}_{\alpha}^{V}\mathcal{M}/(z-1)\mathrm{gr}_{\alpha}^{V}\mathcal{M}.$$

(3) Show that
$$(V_{\alpha}\mathscr{M}) \otimes_{\mathbb{C}[z]} \mathbb{C}[z, z^{-1}] = V_{\alpha} \mathfrak{M}[z, z^{-1}].$$

Exercise 7.3.22 (Side changing). Define the side changing functor for $V_0 \widetilde{\mathscr{D}}_X$ -modules by replacing $\widetilde{\mathscr{D}}_X$ with $V_0 \widetilde{\mathscr{D}}_X$ in Definition A.3.10. Show that $\mathscr{M}^{\text{left}}$ is \mathbb{R} -specializable along H if and only if $\mathscr{M}^{\text{right}}$ is so and, for every $\beta \in \mathbb{R}$, $V^{\beta}(\mathscr{M}^{\text{left}}) = [V_{-\beta-1}(\mathscr{M}^{\text{right}})]^{\text{left}}$. [*Hint*: Use the local computation of Exercise A.3.17.]

7.3.c. Strictly \mathbb{R} -specializable coherent $\widehat{\mathscr{D}}_X$ -modules. A drawback of the setting of Section 7.3.b is that we cannot ensure that the order filtration is a coherent *V*-filtration.

Lemma 7.3.23 (Kashiwara-Malgrange V-filtration). Let \mathscr{M} be an \mathbb{R} -specializable coherent $\widetilde{\mathscr{D}}_X$ -module. Assume that, in the neighbourhood of $x_o \in X$ there exists a coherent V-filtration U. \mathscr{M} with the following two properties:

(1) its minimal weak Bernstein polynomial $b_U(s) = \prod_{\alpha \in A(U)} (s - \alpha z)^{\nu_{\alpha}}$ satisfies $A(U) \subset (-1, 0],$

(2) for every k, $U_k \mathcal{M}/U_{k-1} \mathcal{M}$ has no z-torsion.

Then such a filtration is unique and equal to the order filtration when considered indexed by integers, which is therefore a coherent V-filtration as such. It is called the Kashiwara-Malgrange filtration of \mathcal{M} .

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Proof. Assume $U_{\bullet}\mathcal{M}$ satisfies (1) and (2). Let m be a local section of $U_k\mathcal{M}$ and let $U_{\bullet}(m \cdot \widetilde{\mathscr{D}}_X)$ be the V-filtration induced by $U_{\bullet}\mathscr{M}$ on $m \cdot \widetilde{\mathscr{D}}_X$. By Exercise 7.3.7(1), it is a coherent V-filtration. There exists thus $k_o \ge 1$ such that $U_{k-k_o}(m \cdot \widehat{\mathscr{D}}_X) \subset$ $m \cdot V_{-1} \mathscr{D}_X$. It follows that $R(m) \subset (A(U) + k) \cup \cdots \cup (A(U) + k - k_o + 1)$ and thus $\operatorname{ord}_H m = \max R(m) \leq k$, so $m \subset V_k \mathcal{M}$.

Conversely, assume m is a local section of $V_k \mathcal{M}$. It is also a local section of $U_{k+k_o}\mathcal{M}$ for some $k_o \ge 0$. Its class in $\operatorname{gr}_{k+k_o}^U\mathcal{M}$ is annihilated both by $z^\ell b_m(\mathbf{E})$ and by $z^{\ell'} b_U(\mathbf{E} - (k + k_o)z)$ (for some $\ell, \ell' \ge 0$), so if $k_o > 0$, both polynomials have no common z-root, and this class is annihilated by some nonnegative power of z, according to Bézout. By Assumption (2), it is zero, and m is a local section of $U_{k+k_o-1}\mathcal{M}$, from which we conclude by induction that m is a local section of $U_k\mathcal{M}$, as wanted.

Exercise 7.3.24 (Indexing with \mathbb{Z} or with \mathbb{R}). The order filtration is naturally indexed by \mathbb{R} , while the notion of V-filtration considers filtrations indexed by \mathbb{Z} . The purpose of this exercise is to show how both notions match when the properties of Lemma 7.3.23 are satisfied. Let $U_{\bullet}\mathcal{M}$ be a filtration for which the properties of Lemma 7.3.23 are satisfied. Then we have seen that $U_{\bullet}\mathcal{M}$ coincides with the "integral part" of the order filtration $V_{\bullet}\mathcal{M}$. Show the following properties.

(1) The weak Bernstein equations (7.3.10*) and (7.3.11*) hold without any power of z, i.e., for every k the operator E - kz has a minimal polynomial on $U_k \mathcal{M} / U_{k-1} \mathcal{M} =$ $V_k \mathcal{M}/V_{k-1} \mathcal{M}$ which does not depend on k.

(2) The eigen module of E - kz on this quotient module corresponding to the eigenvalue αz isomorphic to $\operatorname{gr}_{\alpha+k}^V \mathscr{M}$ and the corresponding nilpotent endomorphism is

(7.3.24*)
$$N := (E - (k + \alpha)z).$$

In particular, each $\operatorname{gr}_{\alpha+k}^V \mathscr{M}$ is strict and we have a canonical identification

$$V_k \mathscr{M} / V_{k-1} \mathscr{M} = \bigoplus_{-1 < \alpha \leqslant 0} \operatorname{gr}_{\alpha+k}^V \mathscr{M}.$$

(3) For every $\alpha \in (-1, 0]$, identify $V_{\alpha+k}\mathcal{M}$ with the pullback of $\bigoplus_{-1 < \alpha' \leq \alpha} \operatorname{gr}_{\alpha'+k}^V \mathcal{M}$ by the projection $V_k \mathscr{M} \to V_k \mathscr{M} / V_{k-1} \mathscr{M}$, and show that the shifted order filtration indexed by integers $V_{\alpha+\bullet}\mathcal{M}$ is a coherent V-filtration.

(4) Conclude that there exists a finite set $A \subset (-1, 0]$ such that the order filtration is indexed by $A + \mathbb{Z}$, and is coherent as such (see Exercise 7.3.9).

Definition 7.3.25 (Strictly \mathbb{R} -specializable $\widetilde{\mathscr{D}}_X$ -modules). Assume that \mathscr{M} is \mathbb{R} -specializable along H. We say that it is strictly \mathbb{R} -specializable along H if

(1) there exists a finite set $A \subset (-1, 0]$ such that the filtration by the order along H is a coherent V-filtration indexed by $A + \mathbb{Z}$,

and for some (or any) local decomposition $X \simeq H \times \Delta_t$,

- (2) for every $\alpha < 0, t : \operatorname{gr}_{\alpha}^{V} \mathcal{M} \to \operatorname{gr}_{\alpha-1}^{V} \mathcal{M}$ is onto, (3) for every $\alpha > -1, \eth_{t} : \operatorname{gr}_{\alpha}^{V} \mathcal{M} \to \operatorname{gr}_{\alpha+1}^{V} \mathcal{M}(-1)$ is onto.

Proposition 7.3.26. Assume that \mathscr{M} is strictly \mathbb{R} -specializable along H. Then, every $\operatorname{gr}_{\alpha}^{V} \mathscr{M}$ is a graded $\operatorname{gr}_{0}^{V} \widetilde{\mathscr{D}}_{X}$ -module, and is strict as such (see Definition A.2.7).

Proof. Recall that, for a graded module, strictness is equivalent to absence of z-torsion (see Exercise A.2.5(1)). Therefore, the second point follows from the first one and from Lemma 7.3.18.

Let us consider the first point. We first claim that a local section m of \mathcal{M} is a local section of $V_{\alpha}\mathcal{M}$ if and only if it satisfies a relation

$$m \cdot b(\mathbf{E}) \in V_{\alpha}\mathcal{M}$$

for some b with z-roots $\leq \alpha$. Indeed, if m is a local section of $V_{\beta}\mathcal{M}$ with $\beta > \alpha$ and satisfying such a relation, the Bézout argument already used and the absence of z-torsion on each $\operatorname{gr}_{\gamma}^{V}\mathcal{M}$ (Lemma 7.3.18) implies that m is a local section of $V_{\leq\beta}\mathcal{M}$. Property 7.3.25(1) implies that there is only a finite set of jumps of the V-filtration between α and β , so by induction we conclude that $m \in V_{\alpha}\mathcal{M}$. The converse is clear.

The grading on \mathscr{M} induces a natural (right) action of $-\partial_z z$ on \mathscr{M} : for a local section $m = \bigoplus_p m_p$ of $\mathscr{M} = \bigoplus_p \mathscr{M}^p$, we set $m(-\partial_z z) := \bigoplus_p pm_p$. This action is natural in the sense that it satisfies the usual commutation relations with the right action of $\widetilde{\mathscr{D}}_X$ (it would be more standard to use the natural left action of $z\partial_z$ on $\mathscr{M}^{\text{left}}$). We claim that, for every $\alpha \in \mathbb{R}$, we have $V_\alpha \mathscr{M}(-\partial_z z) \subset V_\alpha \mathscr{M}$. Let m be a local section of $V_\alpha \mathscr{M}$, which satisfies a relation $mb_m(\mathbf{E}) = m \cdot P$ with $P \in V_{-1} \widetilde{\mathscr{D}}_X$. Then one checks that

$$m(-\partial_z z)b_m(\mathbf{E}) = mb_m(\mathbf{E})(-\partial_z z) + mQ, \quad Q \in V_0 \widehat{\mathscr{D}}_X$$
$$= mP(-\partial_z z) + mQ, \quad P \in V_{-1} \widehat{\mathscr{D}}_X$$
$$= m(-\partial_z z)P + mR, \quad R \in V_0 \widehat{\mathscr{D}}_X.$$

We conclude that $m(-\partial_z z) \in V_{\alpha} \mathscr{M}$ by applying the first claim above.

Since the eigenvalues of $(-\partial_z z)$ on \mathscr{M} are integers and are simple, the same property holds for $V_{\alpha}\mathscr{M}$, showing that $V_{\alpha}\mathscr{M}$ decomposes as the direct sum of its $(-\partial_z z)$ -eigenspaces, which are its graded components of various degrees.

Remark 7.3.27 (The need of a shift). If we regard the actions of t and \mathfrak{d}_t as morphisms in $\mathsf{Mod}(\widetilde{\mathscr{D}}_H)$ -modules, that is, graded morphisms of degree zero, we have to introduce a shift by -1 (see Remark A.2.4) for the action of \mathfrak{d}_t , which sends $F_p z^p$ to $F_{p+1} z^{p+1}$. The same shift has to be introduced for the action of E, as well as for that of $N = (E - \alpha z)$.

Exercise 7.3.28. Check that if (2) and (3) hold for some local decomposition $X \simeq H \times \Delta_t$ at $x_o \in H$, then they hold for any such decomposition.

Remark 7.3.29 (The case of left $\widehat{\mathscr{D}}_X$ -modules). For left $\widehat{\mathscr{D}}_X$ -modules, we take $\beta > -1$ in 7.3.25(2) and $\beta < 0$ in 7.3.25(3) for $\operatorname{gr}_V^\beta \mathscr{M}$. The nilpotent endomorphism N of $\operatorname{gr}_V^\beta \mathscr{M}$ is induced by the action of $-(E - \beta z)$.

Remark 7.3.30 (Side-changing). Let \mathscr{M} be a left $\widetilde{\mathscr{D}}_X$ -module and let $\mathscr{M}^{\text{right}} = \widetilde{\omega}_X \otimes \mathscr{M}$ denote the associated right $\widetilde{\mathscr{D}}_X$ -module. Let us assume that H is defined by one equation g = 0, so that $\text{gr}_V^{\beta} \mathscr{M}$ and $\text{gr}_{\alpha}^V \mathscr{M}^{\text{right}}$ are respectively left and right $\widetilde{\mathscr{D}}_H$ -modules endowed with an action of E.

Assume first that $\mathscr{M} = \widetilde{\mathscr{O}}_X$ and $\mathscr{M}^{\mathrm{right}} = \widetilde{\omega}_X$. We have

$$V^{k}\widetilde{\mathscr{O}}_{X} = \begin{cases} \widetilde{\mathscr{O}}_{X} & \text{if } k \leq 0, \\ g^{k}\widetilde{\mathscr{O}}_{X} & \text{if } k \geq 0, \end{cases} \quad \text{and} \quad V_{k}\widetilde{\omega}_{X} = \begin{cases} \widetilde{\omega}_{X} & \text{if } k \geq -1, \\ g^{-(k+1)}\widetilde{\omega}_{X} & \text{if } k \leq -1. \end{cases}$$

We have $\operatorname{gr}_{-1}^V \widetilde{\omega}_X = \widetilde{\omega}_H \otimes \widetilde{\operatorname{d}} g/z$, so that $\widetilde{\operatorname{d}} g/z$ induces an isomorphism (see Remark A.2.4)

$$\widetilde{\omega}_H(-1) \xrightarrow{\sim} \operatorname{gr}_{-1}^V \widetilde{\omega}_X$$
, that is, $\operatorname{gr}_{-1}^V (\widetilde{\mathscr{O}}_X^{\operatorname{right}}) \simeq (\operatorname{gr}_V^0 \widetilde{\mathscr{O}}_X)^{\operatorname{right}}(-1).$

Arguing similarly for \mathcal{M} and $\mathcal{M}^{\text{right}}$ gives an identification

$$\operatorname{gr}_{\alpha}^{V}(\mathscr{M}^{\operatorname{right}}) \simeq (\operatorname{gr}_{V}^{\beta}\mathscr{M})^{\operatorname{right}}(-1), \quad \beta = -\alpha - 1.$$

With this identification, the actions of E (resp. N) on both sides coincide.

Proposition 7.3.31. Assume that \mathscr{M} is strictly \mathbb{R} -specializable along H. Then, in any local decomposition $X \simeq H \times \Delta_t$ we have

(a)
$$\forall \alpha < 0, t : V_{\alpha} \mathscr{M} \longrightarrow V_{\alpha-1} \mathscr{M}$$
 is an isomorphism;

(b)
$$\forall \alpha \ge 0, \ V_{\alpha} \mathscr{M} = V_{<\alpha} \mathscr{M} + (V_{\alpha-1} \mathscr{M}) \eth_t$$

(c)
$$t: \operatorname{gr}_{\alpha}^{V} \mathcal{M} \longrightarrow \operatorname{gr}_{\alpha-1}^{V} \mathcal{M}$$
 is $\begin{cases} an \text{ isomorphism } if \alpha < 0, \\ injective & if \alpha > 0; \end{cases}$

(d)
$$\tilde{\mathfrak{d}}_t : \operatorname{gr}^V_{\alpha} \mathscr{M} \longrightarrow \operatorname{gr}^V_{\alpha+1} \mathscr{M}(-1)$$
 is $\begin{cases} an \text{ isomorphism } if \alpha > -1, \\ injective & if \alpha < -1; \end{cases}$

In particular (from (b)), \mathscr{M} is generated as a $\widetilde{\mathscr{D}}_X$ -module by $V_0\mathscr{M}$.

Proof. Because $V_{\alpha+\bullet}\mathcal{M}$ is a coherent V-filtration, (a) holds for $\alpha \ll 0$ locally and (b) for $\alpha \gg 0$ locally. Therefore, (a) follows from (c) and (b) follows from (d). By 7.3.25(2) (resp. (3)), the map in (c) (resp. (d)) is onto. The composition $t\mathfrak{d}_t = (\mathbf{E} - \alpha z) + \alpha z$ is injective on $\operatorname{gr}_{\alpha}^V \mathcal{M}$ for $\alpha \neq 0$ since $(\mathbf{E} - \alpha z)$ is nilpotent and $\operatorname{gr}_{\alpha}^V \mathcal{M}$ is strict, hence (c) holds. The argument for (d) is similar.

In the next exercises, we explain which set of data is needed to recover coherent $V_0 \widetilde{\mathscr{D}}_X$ -modules and morphisms between them. This will be used from a more general point of view in Chapter 9.

Exercise 7.3.32 (Recovering morphisms from their restriction to V_0)

Assume that $X = H \times \Delta_t$ and that $\mathcal{M}_1, \mathcal{M}_2$ are strictly \mathbb{R} -specializable along H. Let $\varphi_{\leq 0} : V_0 \mathcal{M}_1 \to V_0 \mathcal{M}_2$ be a morphism in $\mathsf{Mod}(V_0 \widetilde{\mathscr{D}}_X)$ such that the diagram (D_0) commutes:

Show that $\varphi_{\leq 0}$ extends in a unique way as a morphism $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$. [*Hint*: For the uniqueness, use 7.3.31(b); show inductively the existence of $\varphi_{\leq k} : V_k \mathcal{M}_1 \to V_k \mathcal{M}_2$ $(k \geq 1)$; for example, if k = 1, use 7.3.31(d) to show that, for $m, m', n, n' \in V_0 \mathcal{M}_1$, if $m - m' = (n' - n)\partial_t$, then $n' - n \in V_{-1}\mathcal{M}_2$ and deduce that setting $\varphi_{\leq 1}(m + n\partial_t) = \varphi_{\leq 0}(m) + \varphi_{\leq 0}(n)\partial_t$ well defines a $V_0 \widetilde{\mathcal{D}}_X$ -linear morphism $\varphi_{\leq 1} : V_1 \mathcal{M}_1 \to V_1 \mathcal{M}_2$ for which (D₁) commutes.]

Exercise 7.3.33 (Recovering $V_0 \mathscr{M}$). Assume that $X = H \times \Delta_t$ and that \mathscr{M} is strictly \mathbb{R} -specializable along H. We have a natural exact sequence of $V_0 \widetilde{\mathscr{D}}_X$ -modules

$$0 \longrightarrow V_{<0}\mathcal{M} \longrightarrow V_0\mathcal{M} \longrightarrow \operatorname{gr}_0^V \mathcal{M} \longrightarrow 0.$$

We wish to recover explicitly the middle term in terms of the extreme ones and of the morphisms (c) and (d) in Proposition 7.3.31 above, for the most interesting value $\alpha = 0$.

(1) Consider the morphisms

$$gr_{-1}^{V}\mathscr{M}(1) \xrightarrow{A} V_{-1}\mathscr{M} \oplus gr_{-1}^{V}\mathscr{M}(1) \oplus gr_{0}^{V}\mathscr{M} \xrightarrow{B} gr_{-1}^{V}\mathscr{M}$$
$$e \longmapsto (0, e, e\vec{\mathfrak{d}}_{t})$$
$$(m, e, \varepsilon) \longmapsto [m] + e \cdot \vec{\mathfrak{d}}_{t}t - \varepsilon \cdot t$$

where, for $m \in V_{-1}\mathcal{M}$, [m] denotes its class in $\operatorname{gr}_{-1}^V\mathcal{M}$. Show that the composition is zero, hence they define a complex C^{\bullet} of $V_0 \widetilde{\mathscr{D}}_X$ -modules (by regarding each $\operatorname{gr}_{\alpha}^V \mathcal{M}$ as a $V_0 \widetilde{\mathscr{D}}_X$ -module). Show that $H^j(C^{\bullet}) = 0$ for $j \neq 1$.

(2) Consider the morphism from $V_0 \mathscr{M}$ to the middle term given by $\mu \mapsto (\mu \cdot t, 0, [\mu])$, where $[\mu]$ denotes the class of μ in $\operatorname{gr}_0^V \mathscr{M}$. Show that it injects into Ker *B* and that its intersection with Im *A* is zero. [*Hint*: Use 7.3.31(a).]

(3) Show that the induced morphism $V_0 \mathscr{M} \to H^1(C^{\bullet})$ is an isomorphism. [*Hint*: Injectivity follows from (2) above; modulo Im A, any element of Ker B can be represented in a unique way as $(m, 0, \delta)$ with $[m] = \delta \cdot t$; choose any lifting $\tilde{\delta} \in V_0 \mathscr{M}$ of δ and show that there exists $\eta \in V_{<0} \mathscr{M}$ such that $m - \tilde{\delta} \cdot t = \eta \cdot t$ by using 7.3.31(a); conclude by setting $\mu = \tilde{\delta} + \eta$.]

(4) Show that, for any $V_0 \widetilde{\mathscr{D}}_X$ -linear morphism $\varphi_{\leq -1} : V_{-1} \mathscr{M}_1 \to V_{-1} \mathscr{M}_2$, the diagram (D_{-1}) commutes, and conclude that giving a morphism $\varphi_{\leq 0} V_0 \mathscr{M}_1 \to V_0 \mathscr{M}_2$

such that (D₀) commutes is equivalent to giving a pair ($\varphi_{\leq -1}, \varphi_0$) such that, with respect to the morphisms



and setting $\varphi_{-1} = \operatorname{gr}_{-1}^V \varphi_{\leqslant -1}$, we have

$$\eth_t \circ \varphi_{-1} = \varphi_0 \circ \eth_t, \quad \varphi_{-1} \circ t = t \circ \varphi_0.$$

Assume that $X = H \times \Delta_t$. Consider the category whose objects consist of the data $(\mathcal{M}_{\leq -1}, \mathcal{M}_0, \mathbf{c}, \mathbf{v})$, where

• $\mathcal{M}_{\leq -1}$ is a coherent $V_0 \widetilde{\mathscr{D}}_X$ -module on which t is torsion-free and such that $\mathcal{M}_{-1} := \mathcal{M}_{\leq -1}/\mathcal{M}_{\leq -1}t$ is strict and the induced action of $\eth_t t$ on it is nilpotent with index of nilpotence locally bounded on H,

• \mathscr{M}_0 is a strict coherent $\operatorname{gr}_0^V \widetilde{\mathscr{D}}_X$ -module on which the action of $t \eth_t$ is nilpotent with index of nilpotence locally bounded on H,

• the data c, v are $\operatorname{gr}_0^V \widetilde{\mathscr{D}}_X$ -linear morphisms



such that $\mathbf{c} \circ \mathbf{v} = \eth_t t$ on \mathscr{M}_{-1} and $\mathbf{v} \circ \mathbf{c} = t\eth_t$ on \mathscr{M}_0 .

Morphisms in this category consist of pairs $(\varphi_{\leq -1}, \varphi_0)$, where $\varphi_{\leq -1} : \mathscr{M}_{\leq -1} \to \mathscr{N}_{\leq -1}$ is $V_0 \widetilde{\mathscr{D}}_X$ -linear, $\varphi_0 : \mathscr{M}_0 \to \mathscr{N}_0$ is $\operatorname{gr}_0^V \widetilde{\mathscr{D}}_X$ -linear, and the restriction φ_{-1} of $\varphi_{\leq -1}$ to \mathscr{M}_{-1} satisfies

$$\mathbf{c} \circ \varphi_{-1} = \varphi_0 \circ \mathbf{c}, \quad \varphi_{-1} \circ \mathbf{v} = \mathbf{v} \circ \varphi_0.$$

We have a functor from the category of coherent $\widehat{\mathscr{D}}_X$ -modules which are strictly \mathbb{R} -specializable along H to the above category:

$$\mathscr{M} \longmapsto (V_{-1}\mathscr{M}, \operatorname{gr}_0^V \mathscr{M}, \eth_t, t).$$

Corollary 7.3.34 (Recovering morphisms from their restriction to V_{-1} and gr_0^V)

This functor is fully faithful, i.e., any morphism $(\varphi_{\leq -1}, \varphi_0)$ can be lifted in a unique way as a morphism φ .

Proof. Consider the category whose objects are coherent $V_0 \mathscr{D}_X$ -modules $\mathscr{M}_{\leq 0}$ such that

• $\mathcal{M}_{\leq 0}/\mathcal{M}_{\leq 0}t$ is strict,

• $t\mathfrak{d}_t$ acting on $\mathcal{M}_{\leq 0}/\mathcal{M}_{\leq 0}t$ has a minimal polynomial with roots αz satisfying $\alpha \in (-1, 0]$,

• defining $V_{\alpha}\mathcal{M}$ for $\alpha < 0$ as in Exercise 7.3.24, every $\operatorname{gr}_{\alpha}^{V}\mathcal{M}_{\leq 0}$ is strict and 7.3.31(a) holds,

and whose morphisms are $V_0 \widetilde{\mathscr{D}}_X$ -linear morphisms such that (D_0) commutes.

According to Exercise 7.3.32, the functor $\mathscr{M} \mapsto \mathscr{M}_{\leq 0} := V_0 \mathscr{M}$ is fully faithful. Now, the functor $\mathscr{M}_{\leq 0} \mapsto (V_{-1} \mathscr{M}_{\leq 0}, \operatorname{gr}_0^V \mathscr{M}_{\leq 0}, \eth_t, t)$ is an equivalence of categories. Indeed, Exercise 7.3.33 shows that it is essentially surjective and, since the reconstruction is functorial in an obvious way, it enables one to lift in a unique way a pair $(\varphi_{\leq -1}, \varphi_0)$ as a $V_0 \widetilde{\mathscr{D}}_X$ -linear morphism $\varphi_{\leq 0}$ such that (D_0) commutes, showing the full faithfulness.

Remark 7.3.35 (Restriction to z = 1). Let us keep the notation of Exercise 7.3.21. For a coherent \mathscr{D}_X -module \mathscr{M} which is \mathbb{R} -specializable, 7.3.25(2) and (3) are automatically satisfied. Moreover, the morphisms in 7.3.31(c) and (d) are isomorphisms for the given values of α . In other words, for coherent \mathscr{D}_X -modules, being \mathbb{R} -specializable is equivalent to being strictly \mathbb{R} -specializable. In particular, Exercise 7.3.21 applies to coherent $R_F \mathscr{D}_X$ -modules which are strictly \mathbb{R} -specializable along H.

Exercise 7.3.36 (Structure of $\mathcal{M}/V_{<\alpha_o}\mathcal{M}$). Let \mathcal{M} be a coherent right $\widehat{\mathcal{D}}_X$ -module which is strictly \mathbb{R} -specializable along H. Let us fix $\alpha_o \in \mathbb{R}$. Then $\mathcal{M}/V_{<\alpha_o}\mathcal{M}$ is a $V_0 \widetilde{\mathcal{D}}_X$ -module.

(1) Show that $\mathcal{M}/V_{<\alpha_o}\mathcal{M}$ is strict.

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- (2) Show that $\mathcal{M}/V_{<\alpha_o}\mathcal{M}$ decomposes as $\bigoplus_{\alpha \ge \alpha_o} \operatorname{Ker}(\mathbf{E} \alpha z)^N$ with $N \gg 0$.
- (3) Show that the α -summand can be identified with $\operatorname{gr}_{\alpha}^{V} \mathcal{M}$.

(4) Show that $\mathscr{M}/V_{<\alpha_o}\mathscr{M}$ can be identified with $\bigoplus_{\alpha \ge \alpha_o} \operatorname{gr}_{\alpha}^V \mathscr{M}$ as a $V_0 \widetilde{\mathscr{D}}_X$ -module. Does the $V_0 \widetilde{\mathscr{D}}_X$ -module structure of $\mathscr{M}/V_{<\alpha_o} \mathscr{M}$ extend to a $\widetilde{\mathscr{D}}_X$ -module structure? [*Hint*: in local coordinates, what about the relation $[\eth_t, t] = z$ applied to a nonzero section of $\operatorname{gr}_{\alpha_o}^V \mathscr{M}$?]

(5) Assume now that $X \simeq H \times \Delta_t$. Let *s* be a new variable and let us equip $\operatorname{gr}_{\alpha}^V \mathscr{M}[s] := \operatorname{gr}_{\alpha}^V \mathscr{M} \otimes_{\mathbb{C}} \mathbb{C}[s]$ with the following right $V_0 \widetilde{\mathscr{D}}_X$ -structure defined by

$$m_j^{\alpha} s^j \cdot t = \begin{cases} 0 & \text{if } j = 0, \\ \left(m_j^{\alpha} (\mathbf{E} + jz)\right) s^{j-1} & \text{if } j \ge 1, \end{cases}$$
$$m_j^{\alpha} s^j) t \eth_t = \left(m_j^{\alpha} (\mathbf{E} + jz)\right) s^j.$$

Check that this is indeed a $V_0 \widetilde{\mathscr{D}}_X$ -module structure (i.e., $[t \eth_t, t]$ acts as zt). Show that $\mathscr{M}/V_{-1}\mathscr{M}$ can be identified with $\bigoplus_{\alpha \in (-1,0]} \operatorname{gr}_{\alpha}^V \mathscr{M}[s]$. With this structure, show that $\operatorname{gr}_{\alpha}^V \mathscr{M}s^j = \operatorname{Ker}(t\eth_t - (\alpha + j)z)^N$ (with $N \gg 0$ locally).

[*Hint*: use that $\mathfrak{d}_t : \operatorname{gr}^V_{\alpha} \mathscr{M} \to \operatorname{gr}^V_{\alpha+1} \mathscr{M}$ is an isomorphism for $\alpha > -1$ to identify $\bigoplus_{\alpha > -1} \operatorname{gr}^V_{\alpha} \mathscr{M}$ with $\bigoplus_{\alpha \in (-1,0]} \bigoplus_{j \ge 0} \operatorname{gr}^V_{\alpha} \mathscr{M} \mathfrak{d}^j_t$.]

(6) Equip $\operatorname{gr}_{\alpha}^{V}\mathscr{M}[s]$ with the action of \eth_{t} defined by $(m_{j}^{\alpha}s^{j})\eth_{t} = m_{j}^{\alpha}s^{j+1}$. Show that the relation $[\eth_{t}, t] = z$ holds on $\operatorname{sgr}_{\alpha}^{V}\mathscr{M}[s]$, but that $[\eth_{t}, t] = z + (E + z)$ on $\operatorname{gr}_{\alpha}^{V}\mathscr{M}$. Conclude that this action does not define a $\widetilde{\mathscr{D}}_{X}$ -module structure on $\operatorname{gr}_{\alpha}^{V}\mathscr{M}[s]$.

Exercise 7.3.37 (First properties of strictly \mathbb{R} -specializable coherent $\widetilde{\mathscr{D}}_X$ -modules) Show the following properties.

Show the following properties. \sim

(1) Let \mathscr{M} be a coherent $\widehat{\mathscr{D}}_X$ -module which is strictly \mathbb{R} -specializable along H. If $\mathscr{M} = \mathscr{M}_1 \oplus \mathscr{M}_2$ with $\mathscr{M}_1, \mathscr{M}_2 \ \widetilde{\mathscr{D}}_X$ -coherent, then $\mathscr{M}_1, \mathscr{M}_2$ are strictly \mathbb{R} -specializable along H.

(2) In an exact sequence $0 \to \mathcal{M}_1 \to \mathcal{M} \to \mathcal{M}_2 \to 0$ of coherent $\widetilde{\mathcal{D}}_X$ -modules, if \mathcal{M} is strictly \mathbb{R} -specializable along H, set

$$U_{\alpha}\mathcal{M}_{1} = V_{\alpha}\mathcal{M} \cap \mathcal{M}_{1}, \quad U_{\alpha}\mathcal{M}_{2} = \operatorname{image}(V_{\alpha}\mathcal{M}).$$

• Show that these V-filtrations are coherent (see Exercise 7.3.7(1)) and that, for every α , the sequence

$$0 \longrightarrow \operatorname{gr}_{\alpha}^{U} \mathscr{M}_{1} \longrightarrow \operatorname{gr}_{\alpha}^{V} \mathscr{M} \longrightarrow \operatorname{gr}_{\alpha}^{U} \mathscr{M}_{2} \longrightarrow 0$$

is exact.

• Conclude that $U_{\bullet}\mathcal{M}_1$ satisfies the Bernstein property 7.3.23(1) and the strictness property 7.3.23(2) (with index set \mathbb{R}), and thus injectivity in 7.3.31(a) and (d), but possibly not 7.3.25(2) and (3). Deduce that $U_{\alpha}\mathcal{M}_1 = V_{\alpha}\mathcal{M}_1$. [Hint: use the uniqueness property of Lemma 7.3.23.]

• If each $\operatorname{gr}_{\alpha}^{U} \mathscr{M}_{2}$ is also strict, show that $U_{\alpha} \mathscr{M}_{2} = V_{\alpha} \mathscr{M}_{2}$.

 \bullet If moreover one of both $\mathscr{M}_1, \mathscr{M}_2$ is strictly $\mathbb{R}\text{-specializable},$ then so is the other one.

(3) Let $\varphi : \mathscr{M}_1 \to \mathscr{M}_2$ be any morphism between coherent $\widehat{\mathscr{D}}_X$ -modules which are strictly \mathbb{R} -specializable along H. Apply the previous result to $\operatorname{Im} \varphi$.

(4) Let $\iota: X \hookrightarrow X_1$ be a closed inclusion of complex manifolds, and let $H_1 \subset X_1$ be a smooth hypersurface such that $H := X \cap H_1$ is a smooth hypersurface of X. Then a coherent $\widetilde{\mathscr{D}}_X$ -module \mathscr{M} is strictly \mathbb{R} -specializable along H if and only if $\mathscr{M}_1 := {}_{\mathsf{D}}\iota_*\mathscr{M}$ is so along H_1 , and we have, for every α ,

$$(\operatorname{gr}_{\alpha}^{V}\mathscr{M}_{1}, \operatorname{N}) = ({}_{\scriptscriptstyle \mathrm{D}}\iota_{*}\operatorname{gr}_{\alpha}^{V}\mathscr{M}, \operatorname{N}).$$

[*Hint*: assume that $X_1 = H \times \Delta_t \times \Delta_x$ and $X = H \times \Delta_t \times \{0\}$, so that $\mathcal{M}_1 = \iota_* \mathcal{M}[\eth_x]$; show that the filtration $U_\alpha \mathcal{M}_1 := \iota_* V_\alpha \mathcal{M}[\eth_x]$ satisfies all the characteristic properties of the V-filtration of \mathcal{M}_1 along H_1 .]

Example 7.3.38 (Morphisms inducing an isomorphism on $V_{<0}$)

Assume that $X = H \times \Delta_t$. Let \mathscr{M}, \mathscr{N} be strictly \mathbb{R} -specializable along H and let $\varphi : \mathscr{M} \to \mathscr{N}$ be a $\widetilde{\mathscr{D}}_X$ -linear morphism. Since φ is also $V_0 \widetilde{\mathscr{D}}_X$ -linear, it induces a morphism $\mathscr{M}/V_{\alpha_o} \mathscr{M} \to \mathscr{N}/V_{<\alpha_o} \mathscr{N}$ for each α_o , which decomposes with respect to the decomposition 7.3.36(2). Each summand is then identified with $\operatorname{gr}_{\alpha}^V \varphi$. We will consider more specifically the case where φ induces an isomorphism on $V_{<0}$.

We first claim that this condition implies that $\operatorname{Ker} \varphi$ and $\operatorname{Coker} \varphi$ are supported on H, that is, every local section of $\operatorname{Ker} \varphi$, $\operatorname{Coker} \varphi$ is annihilated by some power of t (due to the $\widetilde{\mathscr{D}}_H$ -coherence of these modules). For $\operatorname{Ker} \varphi$, this follows from $\operatorname{Ker} \varphi \cap V_{<0} \mathscr{M} = 0$ together with the property that t is nilpotent on $\mathscr{M}/V_{<0} \mathscr{M}$. For Coker φ , we note that any local section n of \mathscr{N} satisfies $t^k n \in V_{\leq 0} \mathscr{N} = \varphi(V_{\leq 0} \mathscr{M})$ for some k, hence t^k is nilpotent on Coker φ .

The decomposition 7.3.36(2) induces decompositions $\operatorname{Ker} \varphi = \bigoplus_{k \geq 0} \operatorname{Ker} \operatorname{gr}_k^V \varphi$ and $\operatorname{Coker} \varphi = \bigoplus_{k \geq 0} \operatorname{Coker} \operatorname{gr}_k^V \varphi$ as $V_0 \widetilde{\mathscr{D}}_X$ -modules. Moreover, since E acts as 0 on $\operatorname{Ker} \operatorname{gr}_0^V \varphi$, $\operatorname{Coker} \operatorname{gr}_0^V \varphi$, the obstruction in 7.3.36(6) (adapted to the present setting) to extending the $V_0 \widetilde{\mathscr{D}}_X$ -structure to a $\widetilde{\mathscr{D}}_X$ -structure vanishes, and we conclude that the $\widetilde{\mathscr{D}}_X$ -module $\operatorname{Ker} \varphi$, resp. $\operatorname{Coker} \varphi$, is identified with the $\widetilde{\mathscr{D}}_X$ -module $_{\mathrm{D}} i_{H*} \operatorname{Ker} \operatorname{gr}_0^V \varphi$, resp. $_{\mathrm{D}} i_{H*} \operatorname{Coker} \operatorname{gr}_0^V \varphi$.

Definition 7.3.39. A morphism φ between strictly \mathbb{R} -specializable coherent left $\widetilde{\mathscr{D}}_X$ -modules is said to be *strictly* \mathbb{R} -specializable if for every $\alpha \in [-1, 0]$, the induced morphism $\operatorname{gr}_{\alpha}^V \varphi$ is *strict* (i.e., its cokernel is strict), and a similar property for right modules.

Proposition 7.3.40. If φ is strictly \mathbb{R} -specializable, then $\operatorname{gr}_{\alpha}^{V}\varphi$ is strict for every $\alpha \in \mathbb{R}$, and $\operatorname{Ker} \varphi$, $\operatorname{Im} \varphi$ and $\operatorname{Coker} \varphi$ are strictly \mathbb{R} -specializable along H and their V-filtrations are given by

 $V_{\alpha}\operatorname{Ker} \varphi = V_{\alpha}\mathscr{M} \cap \operatorname{Ker} \varphi, \quad V_{\alpha}\operatorname{Coker} \varphi = \operatorname{Coker}(\varphi_{|V_{\alpha}\mathscr{M}}),$ $V_{\alpha}\operatorname{Im} \varphi = \operatorname{Im}(\varphi_{|V_{\alpha}\mathscr{M}}) = V_{\alpha}\mathscr{N} \cap \operatorname{Im} \varphi.$

Proof. Let us endow $\operatorname{Ker} \varphi$ and $\operatorname{Coker} \varphi$ with the filtration U_{\bullet} naturally induced by $V_{\bullet}\mathscr{M}, V_{\bullet}\mathscr{N}$. By using 7.3.31(c) and (d) for \mathscr{M} and \mathscr{N} , we find that $\operatorname{gr}_{\alpha}^{U}\operatorname{Ker} \varphi$ and $\operatorname{gr}_{\alpha}^{U}\operatorname{Coker} \varphi$ are strict for every $\alpha \in \mathbb{R}$. By the uniqueness of the V-filtration, the first line in (7.3.40) holds, and therefore all properties of Definition 7.3.25 hold for $\operatorname{Ker} \varphi$ and $\operatorname{Coker} \varphi$. Now, $\operatorname{Im} \varphi$ has two possible coherent V-filtrations, one induced by $V_{\bullet}\mathscr{N}$ and the other one being the image of $V_{\bullet}\mathscr{M}$. For the first one, strictness of $\operatorname{gr}_{\alpha}\operatorname{Im} \varphi$ holds, hence $\operatorname{Im} \varphi$ is strictly \mathbb{R} -specializable and $V_{\alpha}\operatorname{Im} \varphi = \operatorname{Im} \varphi \cap V_{\alpha}\mathscr{N}$. For the second one $U_{\alpha}\operatorname{Im} \varphi$, $\operatorname{gr}_{\alpha}^{U}\operatorname{Im} \varphi$ is identified with the image of $\operatorname{gr}_{\alpha}^{V}\varphi$, hence is also strict, so $U_{\bullet}\operatorname{Im} \varphi$ is also equal to $V_{\bullet}\operatorname{Im} \varphi$.

Corollary 7.3.41. Let $\mathscr{M}^{\bullet} = \{\cdots \xrightarrow{d_i} \mathscr{M}^i \xrightarrow{d_{i+1}} \cdots\}$ be a complex bounded above whose terms are $\widetilde{\mathscr{D}}_X$ -coherent and strictly \mathbb{R} -specializable along H. Assume that, for every $\alpha \in [-1,0]$, the graded complex $\operatorname{gr}_{\alpha}^V \mathscr{M}^{\bullet}$ is strict, i.e., its cohomology is strict. Then each differential d_i and each $\mathscr{H}^i \mathscr{M}^{\bullet}$ is strictly \mathbb{R} -specializable along H and $\operatorname{gr}_{\alpha}^V$ commutes with taking cohomology.

Proof. By using 7.3.31(c) and (d) for each term of the complex $\operatorname{gr}_{\alpha}^{V} \mathscr{M}^{\bullet}$, we find that strictness of the cohomology holds for every $\alpha \in \mathbb{R}$. We argue by decreasing induction. Assume $\mathscr{M}^{k+1} = 0$. Then the assumption implies that $d_k : \mathscr{M}^{k-1} \to \mathscr{M}^k$ is strictly \mathbb{R} -specializable, so we can apply Proposition 7.3.40 to it. We then replace the complex by $\cdots \mathscr{M}^{k-2} \xrightarrow{d_{k-1}} \operatorname{Ker} d_k \to 0$ and apply the inductive assumption. Moreover, the strict \mathbb{R} -specializability of $\mathscr{M}^k/\operatorname{Ker} d_k \simeq \operatorname{Im} d_{k+1}$ implies that of d_{k-1} . \Box

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Definition 7.3.42 (Strictly \mathbb{R} -specializable *W*-filtered $\widetilde{\mathscr{D}}_X$ -module)

Let $(\mathcal{M}, W_{\bullet}\mathcal{M})$ be a coherent $\widehat{\mathcal{D}}_X$ -module endowed with a locally finite filtration by coherent $\widetilde{\mathcal{D}}_X$ -submodules. We say that $(\mathcal{M}, W_{\bullet}\mathcal{M})$ is a strictly \mathbb{R} -specializable (along H) filtered $\widetilde{\mathcal{D}}_X$ -module if each $W_{\ell}\mathcal{M}$ and each $\operatorname{gr}^W_{\ell}\mathcal{M}$ is strictly \mathbb{R} -specializable.

Lemma 7.3.43. Let $(\mathcal{M}, W_{\bullet}\mathcal{M})$ be a strictly \mathbb{R} -specializable filtered $\tilde{\mathcal{D}}_X$ -module. Then each $W_{\ell}\mathcal{M}/W_k\mathcal{M}$ $(k < \ell)$ is strictly \mathbb{R} -specializable along H.

Proof. By induction on $\ell - k \ge 1$, the case $\ell - k = 1$ holding true by assumption. Let $U_{\bullet}(W_{\ell}\mathscr{M}/W_{k}\mathscr{M})$ be the V-filtration naturally induced by $V_{\bullet}W_{\ell}\mathscr{M}$. It is a coherent filtration. By induction we have $U_{\bullet}(W_{\ell-1}\mathscr{M}/W_{k}\mathscr{M}) = V_{\bullet}(W_{\ell-1}\mathscr{M}/W_{k}\mathscr{M})$ and $U_{\bullet}\mathrm{gr}_{\ell}^{W}\mathscr{M} = V_{\bullet}\mathrm{gr}_{\ell}^{W}\mathscr{M}$. Similarly, $V_{\bullet}W_{\ell}\mathscr{M} \cap W_{\ell-1}\mathscr{M} = V_{\bullet}W_{\ell-1}\mathscr{M}$. We conclude that the sequence

$$0 \longrightarrow \operatorname{gr}^V_{\bullet}(W_{\ell-1}\mathscr{M}/W_k\mathscr{M}) \longrightarrow \operatorname{gr}^U_{\bullet}(W_{\ell}\mathscr{M}/W_k\mathscr{M}) \longrightarrow \operatorname{gr}^V_{\bullet}\operatorname{gr}^W_{\ell}\mathscr{M} \longrightarrow 0$$

is exact, hence the strictness of the middle term.

We will now remember explicitly the grading in the case of $R_F \mathscr{D}_X$ -modules. Recall (see (A.2.3*) and (A.2.4**)) that, given a graded object $M = \bigoplus_p M_p$ (with M_p in degree -p), we set $M(k) = \bigoplus_p M(k)_p$ with $M(k)_p = M_{p-k}$. We have seen that, for strictly \mathbb{R} -specializable $R_F \mathscr{D}$ -modules, the module $\operatorname{gr}_{\alpha}^V \mathscr{M}$ are graded $R_F \mathscr{D}$ -modules in a natural way. Let us emphasize that, in Definition 7.3.25(2) and (3),

- the morphism t is graded of degree zero,
- the morphism ∂_t is graded of degree one; we thus write 7.3.25(3) as

$$\eth_t : \operatorname{gr}^V_{\alpha} \mathscr{M} \xrightarrow{\sim} \operatorname{gr}^V_{\alpha} \mathscr{M}(-1) \quad \text{for } \alpha > -1.$$

Definition 7.4.1 (Nearby and vanishing cycle functors). Let $g: X \to \mathbb{C}$ be a holomorphic function. Let $X \stackrel{\iota_g}{\longrightarrow} X \times \mathbb{C}$ denote the graph inclusion of g. We say that a right $\widetilde{\mathscr{D}}_X$ -module \mathscr{M} is strictly \mathbb{R} -specializable along g = 0 if $\mathscr{H}^0{}_{\mathsf{D}}\iota_{g*}\mathscr{M}$ is strictly \mathbb{R} -specializable along $X \times \{0\}$. We then set

(7.4.2)
$$\begin{cases} \psi_{g,\lambda}\mathscr{M} := \operatorname{gr}_{\alpha}^{V}(\mathscr{H}^{0}{}_{{}_{\mathrm{D}}}\iota_{g*}\mathscr{M})(1), \quad \lambda = \exp(2\pi \mathrm{i}\,\alpha), \ \alpha \in [-1,0), \\ \phi_{g,1}\mathscr{M} := \operatorname{gr}_{0}^{V}(\mathscr{H}^{0}{}_{{}_{\mathrm{D}}}\iota_{g*}\mathscr{M}). \end{cases}$$

Then $\psi_{g,\lambda}\mathcal{M}, \phi_{g,1}\mathcal{M}$ are $\widetilde{\mathcal{D}}_X$ -modules supported on $g^{-1}(0)$, endowed with an endomorphism E induced by $t\mathfrak{d}_t$. We set $\mathbf{N} = (\mathbf{E} - \alpha z)$.

Remark 7.4.3 (Choice of the shift). The choice of a shift (1) for $\psi_{g,\lambda}\mathcal{M}$ and no shift for $\phi_{g,1}\mathcal{M}$ is justified by the following examples.

(1) If $\mathscr{M} = \widetilde{\omega}_{X \times \mathbb{C}}$ we have $\operatorname{gr}_{-1}^V \widetilde{\omega}_{X \times \mathbb{C}}(1) \simeq \widetilde{\omega}_X$ by identifying $\widetilde{\omega}_{X \times \mathbb{C}}$ with $\widetilde{\omega}_X \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{O}}_{X \times \mathbb{C}} \operatorname{d} t/z$.

(2) If \mathscr{M} is a right $\widetilde{\mathscr{D}}_{X\times\mathbb{C}}$ -module of the form ${}_{\mathsf{D}}\iota_*\mathscr{N}$ where \mathscr{N} is a right $\widetilde{\mathscr{D}}_{X\times\{0\}}$ module and $\iota: X \times \{0\} \hookrightarrow X \times \mathbb{C}$ is the inclusion, then $\operatorname{gr}_0^V \mathscr{M} = \mathscr{N}$.

Exercise 7.4.4. Justify that $\psi_{g,\lambda}$ and $\phi_{g,1}$ are functors from the category of \mathbb{R} -specializable right $\widetilde{\mathscr{D}}_X$ -modules to the category of right $\widetilde{\mathscr{D}}_X$ -modules supported on $g^{-1}(0)$. [*Hint*: Use Exercise 7.3.20(2).]

Exercise 7.4.5. Let \mathscr{M} be a right $\widetilde{\mathscr{D}}_X$ -module. When g is smooth and $g^{-1}(0) = H$, show that we have $\psi_{g,\lambda}\mathscr{M} \simeq {}_{\mathsf{D}}\iota_{H*}\mathrm{gr}^V_{\alpha}\mathscr{M}(1)$ and $\phi_{g,1}\mathscr{M} = {}_{\mathsf{D}}\iota_{H*}\mathrm{gr}^V_0\mathscr{M}$, where $i_H: H \hookrightarrow X$ denotes the inclusion.

Remark 7.4.6 (Nearby/vanishing cycle functors for left \mathscr{D}_X -modules)

For left $\hat{\mathscr{D}}_X$ -modules, we also use the graph embedding. However, we now have ${}_{\mathrm{D}}\iota_{g*}\mathscr{M} = \mathscr{H}^{-1}{}_{\mathrm{D}}\iota_{g*}\mathscr{M}$. Therefore, one sets

$$\begin{cases} \psi_{g,\lambda}\mathscr{M}^{\text{left}} := \operatorname{gr}_{V}^{\beta}(\mathscr{H}^{-1}{}_{\scriptscriptstyle{\mathsf{D}}}\iota_{g*}\mathscr{M}^{\text{left}}), & \lambda = \exp(-2\pi \mathrm{i}\,\beta), \ \beta \in (-1,0], \\ \phi_{g,1}\mathscr{M}^{\text{left}} := \operatorname{gr}_{V}^{-1}(\mathscr{H}^{-1}{}_{\scriptscriptstyle{\mathsf{D}}}\iota_{g*}\mathscr{M}^{\text{left}})(-1), \end{cases}$$

with no shift of the grading in the first line, in order that $\operatorname{gr}_V^0 \widetilde{\mathcal{O}}_{X \times \mathbb{C}} = \widetilde{\mathcal{O}}_{X \times \{0\}}$ (with grading). The nilpotent endomorphism N is induced by $-(t\mathfrak{d}_t - \beta z)$.

Lemma 7.4.7 (Side-changing for the nearby/vanishing cycle functors)

The side-changing functor commutes with the nearby/vanishing cycle functors, namely

$$\psi_{g,\lambda}(\mathscr{M}^{\mathrm{right}}) = (\psi_{g,\lambda}\mathscr{M}^{\mathrm{left}})^{\mathrm{right}}, \quad \phi_{g,1}(\mathscr{M}^{\mathrm{right}}) = (\phi_{g,1}\mathscr{M}^{\mathrm{left}})^{\mathrm{right}}.$$

Proof. If \mathscr{N} is a left $\widetilde{\mathscr{D}}_{X\times\mathbb{C}}$ -module which is strictly \mathbb{R} -specializable along $X \times \{0\}$, we have (see Remark 7.3.30)

$$\operatorname{gr}_{\alpha}^{V}(\widetilde{\omega}_{X \times \mathbb{C}} \otimes \mathscr{N}) \simeq \widetilde{\omega}_{X} \otimes \operatorname{gr}_{V}^{\beta}(\mathscr{N})(1) \quad \forall \alpha \in \mathbb{R}, \ \beta = -\alpha - 1.$$

We apply this to $\mathscr{N} = \mathscr{H}^{-1}{}_{\mathsf{D}}\iota_{g*}\mathscr{M}^{\text{left}}$, so that $\mathscr{N}^{\text{right}} = \mathscr{H}^{0}{}_{\mathsf{D}}\iota_{g*}\mathscr{M}^{\text{right}}$.

Proposition 7.4.8. Let $g: X \to \mathbb{C}$ be a holomorphic function and let \mathscr{M} be a coherent $\widetilde{\mathscr{D}}_X$ -module. Assume that \mathscr{M} is strictly \mathbb{R} -specializable along g = 0. Then $\psi_{g,\lambda}\mathscr{M}$ and $\phi_{g,1}\mathscr{M}$ are $\widetilde{\mathscr{D}}_X$ -coherent.

Proof. By assumption, $\psi_{g,\lambda}\mathscr{M}$ and $\phi_{g,1}\mathscr{M}$ are $\operatorname{gr}_0^V \widetilde{\mathscr{D}}_{X \times \mathbb{C}} = \widetilde{\mathscr{D}}_X[E]$ -coherent. Since $E - \alpha z$ is nilpotent on $\psi_{g,\lambda}\mathscr{M}$ ($\lambda = \exp(2\pi i \alpha)$), the $\widetilde{\mathscr{D}}_X$ -coherence follows.

Definition 7.4.9 (Morphisms N, can and var). Assume that \mathscr{M} is strictly \mathbb{R} -specializable along g = 0. The nilpotent operator $N = (t \eth_t - \alpha z)$ is a morphism

$$\psi_{g,\lambda}\mathcal{M} \xrightarrow{\mathbf{N}} \psi_{g,\lambda}\mathcal{M}(-1), \qquad \phi_{g,1}\mathcal{M} \xrightarrow{\mathbf{N}} \phi_{g,1}\mathcal{M}(-1).$$

When $\lambda = 1$, the nilpotent operator N on $\psi_{g,1}\mathcal{M}$ and $\phi_{g,1}\mathcal{M}$ is the operator obtained as the composition var \circ can and can \circ var in the diagram below:

(7.4.9*)
$$\psi_{g,1}\mathcal{M}_{(-1)} \xrightarrow{\operatorname{can} = \cdot \eth_t} \phi_{g,1}\mathcal{M}_{(-1)} \xrightarrow{\operatorname{var} = \cdot t} \phi_{g,1}\mathcal{M}_{(-1)} \xrightarrow{var} \xrightarrow{var} \to \phi_{g,1}\mathcal{M}_{(-1)} \xrightarrow{var} \to \phi_{g,2}\mathcal{M}_{(-1)} \xrightarrow{var} \to \phi_{$$

with the same convention as in (3.2.15).

Remark 7.4.10 (The case of left $\widetilde{\mathscr{D}}_X$ -modules). In this case we have $N = -(t \eth_t - \beta z)$ and the diagram

(7.4.10*)
$$\begin{array}{c} \operatorname{can} = -\eth_t \cdot \\ \psi_{g,1} \mathscr{M} \\ (-1) \quad \operatorname{var} = t \cdot \end{array} \phi_{g,1} \mathscr{M}$$

Exercise 7.4.11. Similarly to Exercise 7.4.5, show that, if $X = H \times \Delta_t$ and g is the projection to Δ_t , so that ι_g is induced by the diagonal embedding $\Delta_t \hookrightarrow \Delta_{t_1} \times \Delta_{t_2}$, then can $= \eth_{t_2}$ and var $= t_2$ for ${}_{\mathsf{D}}\iota_{g*}\mathscr{M}$ are ${}_{\mathsf{D}}\iota_{g*}(\eth_{t_1})$ and ${}_{\mathsf{D}}\iota_{g*}(t_1)$, with $\eth_{t_1} = \eth_t$: $\operatorname{gr}_{-1}^V \mathscr{M} \to \operatorname{gr}_0^V \mathscr{M}(-1)$ and $t_1 = t$: $\operatorname{gr}_0^V \mathscr{M} \to \operatorname{gr}_{-1}^V \mathscr{M}$.

Definition 7.4.12 (Monodromy operator). We work with right \mathscr{D}_X -modules. Assume that \mathcal{M} is \mathbb{R} -specializable along (g). The monodromy operator T on $\psi_{g,\lambda}\mathcal{M}$ is the operator induced by $\exp(2\pi i t\partial_t)$ (for left \mathscr{D}_X -modules $T = \exp(-2\pi i t\partial_t)$), and $T - \lambda \operatorname{Id}$ is nilpotent, and the nilpotent operator N is given by $\frac{1}{2\pi i} \log(T - \lambda \operatorname{Id})$ on $\psi_{g,\lambda}\mathcal{M}$. On $\psi_{g,1}\mathcal{M}, \phi_{g,1}\mathcal{M}$ we have $T = \exp 2\pi i \operatorname{N}$ and $\operatorname{N} = \frac{1}{2\pi i} \log(T - \operatorname{Id})$.

Remark 7.4.13 (Monodromy filtration on nearby and vanishing cycles)

The monodromy filtration relative to N on $\psi_{g,\lambda}\mathscr{M}$ and $\phi_{g,1}\mathscr{M}$ (see Exercise 3.1.1 and Remark 3.1.10) is well-defined in the abelian category of graded $\widetilde{\mathscr{D}}_X$ -modules with the automorphism σ induced by the shift (1) of the grading (or in the abelian category of \mathscr{D}_X -modules). The Lefschetz decomposition holds in this category, with respect to the corresponding primitive submodules $P_\ell \psi_{g,\lambda} \mathscr{M}$, $P_\ell \phi_{g,1} \mathscr{M}$ for $\ell \ge 0$.

Nevertheless, strict \mathbb{R} -specializability is not sufficient to ensure that each such primitive submodule (hence each graded piece of the monodromy filtration) is *strict*. The following proposition gives a criterion for the strictness of the primitive parts.

Proposition 7.4.14. Assume \mathscr{M} is strictly \mathbb{R} -specializable along (g) and fix $\lambda \in S^1$. The following properties are equivalent.

- (1) For every $\ell \ge 1$, $N^{\ell} : \psi_{g,\lambda} \mathscr{M} \to \psi_{g,\lambda} \mathscr{M}(-\ell)$ is a strict morphism.
- (2) For every $\ell \in \mathbb{Z}$, $\operatorname{gr}_{\ell}^{\mathrm{M}} \psi_{q,\lambda} \mathscr{M}$ is strict.
- (3) For every $\ell \ge 0$, $P_{\ell}\psi_{q,\lambda}\mathcal{M}$ is strict.

We have a similar assertion for $\phi_{q,1}M$.

Proof. This is Proposition 3.1.11.

Remark 7.4.15 (Restriction to z = 1 of the monodromy filtration)

If \mathscr{M} is a coherent $R_F \mathscr{D}_X$ -module which is strictly \mathbb{R} -specializable along D and setting $\mathfrak{M} = \mathscr{M}/(z-1)\mathscr{M}$, we have $\psi_{g,\lambda}\mathfrak{M} = \psi_{g,\lambda}\mathscr{M}/(z-1)\psi_{g,\lambda}\mathscr{M}$ and $\phi_{g,1}\mathfrak{M} = \phi_{g,1}\mathscr{M}/(z-1)\phi_{g,1}\mathscr{M}$, according to Exercise 7.3.21, and the morphisms can and var for \mathscr{M} obviously restrict to the morphisms can and var for \mathfrak{M} , as well as the nilpotent endomorphism N.

Similarly, the monodromy filtration $M_{\bullet}(N)$ on $\psi_{g,\lambda}\mathcal{M}, \phi_{g,1}\mathcal{M}$ restricts to the monodromy filtration $M_{\bullet}(N)$ on $\psi_{g,\lambda}\mathcal{M}, \phi_{g,1}\mathcal{M}$, since everything behaves $\mathbb{C}[z, z^{-1}]$ -flatly after tensoring with $\mathbb{C}[z, z^{-1}]$.

Exercise 7.4.16 (Strict specializability along $\{t^r = 0\}$). Let t be a smooth function on X, set $X_0 = t^{-1}(0)$ and assume that $X = X_0 \times \mathbb{C}$. Let \mathscr{M} be a coherent $\widetilde{\mathscr{D}}_X$ -module which is strictly \mathbb{R} -specializable along t = 0. The purpose of this exercise is to show that \mathscr{M} is then also strictly \mathbb{R} -specializable along $g = t^r = 0$ for every $r \ge 2$, and to compare nearby cycles of \mathscr{M} with respect to t and to g.

Following the steps below, show that \mathscr{M} is strictly \mathbb{R} -specializable along $\{g = 0\}$ and, denoting by $\iota : X_0 \hookrightarrow X$ the closed inclusion,

(a)
$$(\psi_{a\lambda}\mathcal{M}, \mathbf{N}) = ({}_{\mathbf{D}}\iota_*(\psi_t \lambda^r \mathcal{M}), \mathbf{N}/r)$$
 for every λ ,

- (b) $(\phi_{g,1}\mathcal{M}, \mathbf{N}) = ({}_{\mathrm{D}}\iota_*(\phi_{t,1}\mathcal{M}), \mathbf{N}/r),$
- (c) there is an isomorphism

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$$\left\{ \begin{array}{c} \operatorname{can}_{g} := \operatorname{can}_{t} \circ (t^{r-1})^{-1} \\ \downarrow \\ \psi_{g,1}\mathscr{M} \xrightarrow{(-1) \operatorname{var}_{g}} \phi_{g,1}\mathscr{M} \end{array} \right\} \simeq {}_{\mathrm{D}}\iota_{*} \left\{ \begin{array}{c} \operatorname{can}_{g} := \operatorname{can}_{t} \circ (t^{r-1})^{-1} \\ \downarrow \\ \operatorname{gr}_{-r}^{V}\mathscr{M} \xleftarrow{t^{r-1}} - \psi_{t,1}\mathscr{M} \xrightarrow{(-1) \operatorname{var}_{t}} \phi_{t,1}\mathscr{M} \\ \downarrow \\ (-1) \end{array} \right\}$$

(1) Write ${}_{\mathrm{D}}\iota_{g*}\mathscr{M} = \bigoplus_{k \in \mathbb{N}} \mathscr{M} \otimes \delta \eth_u^k$ as a $\widetilde{\mathscr{D}}_X[u] \langle \eth_u \rangle$ -module, with

$$\begin{split} (m \otimes \delta) \eth_{u}^{k} &= m \otimes \delta \eth_{u}^{k} \quad \forall k \ge 0, \\ (m \otimes \delta) \eth_{t} &= (m \eth_{t}) \otimes \delta - (rt^{r-1}m) \otimes \delta \eth_{u}, \\ (m \otimes \delta) u &= (mt^{r}) \otimes \delta, \\ (m \otimes \delta) \widetilde{\mathscr{O}}_{X} &= (m \widetilde{\mathscr{O}}_{X}) \otimes \delta, \end{split}$$

and with the usual commutation rules. Show the relation

$$r(m \otimes \delta) u \eth_u = [mt \eth_t] \otimes \delta - (mt \otimes \delta) \eth_t.$$

(2) We will denote by V^t the V-filtration with respect to the variable t and by V^u that with respect to the variable u.

For $\alpha \leq 0$, set

$$U_{\alpha}({}_{\mathsf{D}}\iota_{g*}\mathscr{M}) := \left(V_{r\alpha}^{t}\mathscr{M} \otimes \delta\right) \cdot V_{0}^{u}(\widetilde{\mathscr{D}}_{X}[u]\langle \eth_{u}\rangle)$$

and for $\alpha > 0$ define inductively

 $U_{\alpha}({}_{\mathsf{D}}\iota_{g*}\mathscr{M}) := U_{<\alpha}({}_{\mathsf{D}}\iota_{g*}\mathscr{M}) + U_{\alpha-1}({}_{\mathsf{D}}\iota_{g*}\mathscr{M})\eth_{u}.$

Assume that $\alpha \leq 0$. Using the above relation show that, if

$$V_{r\alpha}^{t}\mathscr{M}(t\eth_{t} - r\alpha z)^{\nu_{r\alpha}} \subset V_{< r\alpha}^{t}\mathscr{M},$$
$$U_{\alpha}({}_{\mathsf{D}}\iota_{a*}\mathscr{M})(u\eth_{u} - \alpha z)^{\nu_{r\alpha}} \subset U_{<\alpha}({}_{\mathsf{D}}\iota_{a*}\mathscr{M}),$$

then

and conclude that $u\partial_u - \alpha z$ is nilpotent on $\operatorname{gr}^U_\alpha({}_{\mathsf{D}}\iota_{g*}\mathscr{M})$ for $\alpha \leq 0$.

(3) Show that if m_1, \ldots, m_ℓ generate $V_{r\alpha}^t \mathscr{M}$ over $V_0^t \mathscr{D}_X$, then $m_1 \otimes \delta, \ldots, m_\ell \otimes \delta$ generate $U_{\alpha}({}_{\mathrm{D}}\iota_{g*}\mathscr{M})$ over $V_0^u(\widetilde{\mathscr{D}}_X[u]\langle \eth_u \rangle)$, by using the relation

$$(mt\eth_t)\otimes\delta = (m\otimes\delta)(t\eth_t - ru\eth_u)$$

Conclude that $U_{\alpha}({}_{\mathsf{D}}\iota_{g*}\mathscr{M})$ is $V_0^u(\widetilde{\mathscr{D}}_X[u]\langle \mathfrak{d}_u \rangle)$ -coherent for every $\alpha \leq 0$, hence for every α .

(4) Show that, for every α ,

$$U_{\alpha-1}({}_{\mathsf{D}}\iota_{g*}\mathscr{M}) \subset U_{\alpha}({}_{\mathsf{D}}\iota_{g*}\mathscr{M})u, \text{ resp. } U_{\alpha+1}({}_{\mathsf{D}}\iota_{g*}\mathscr{M}) \subset U_{<\alpha+1}({}_{\mathsf{D}}\iota_{g*}\mathscr{M}) + U_{\alpha}({}_{\mathsf{D}}\iota_{g*}\mathscr{M})\eth_{u},$$

with equality if $\alpha < 0$ (resp. if $\alpha \ge -1$). [*Hint*: Use the analogous property for \mathscr{M} .] Deduce that $U_{\bullet}({}_{\mathsf{D}}\iota_{g*}\mathscr{M})$ is a coherent V-filtration.

(5) Show that, for $\alpha \leq 0$, $V^u_{\alpha}({}_{\mathsf{D}}\iota_{g*}\mathscr{M}) = V^u_{<\alpha}({}_{\mathsf{D}}\iota_{g*}\mathscr{M}) + \sum_{k \geq 0} (V^t_{r\alpha}\mathscr{M} \otimes \delta) \eth^k_t$. Deduce, by considering the degree in \eth_u , that the natural map

$$\bigoplus_{k} \operatorname{gr}_{r\alpha}^{V^{t}} \mathscr{M} \otimes \eth_{t}^{k} \longrightarrow \operatorname{gr}_{\alpha}^{V^{u}}({}_{\mathsf{D}}\iota_{g*}\mathscr{M}) \\
\bigoplus_{k} [m_{k}] \otimes \eth_{t}^{k} \longmapsto \left[\sum_{k} (m_{k} \otimes \delta)\eth_{t}^{k}\right]$$

is an isomorphism of \mathscr{D}_X -modules. Deduce that \mathscr{M} is strictly \mathbb{R} -specializable along g = 0 with (increasing) Kashiwara-Malgrange filtration $V^u({}_{\mathsf{D}}\iota_{g*}\mathscr{M})$ equal to $U_{\bullet}({}_{\mathsf{D}}\iota_{g*}\mathscr{M})$. Conclude the proof of (a) and (b), and then that of (c).

7.5. Strict non-characteristic restrictions

7.5.a. Non-characteristic property. Let $\iota_Y : Y \hookrightarrow X$ denote the inclusion of a closed submanifold with ideal \mathscr{I}_Y (in local coordinates (x_1, \ldots, x_n) , \mathscr{I}_Y is generated by x_1, \ldots, x_p , where $p = \operatorname{codim} Y$). The pullback functor ${}_{\mathsf{D}}\iota_Y^*$ is defined in Section A.7. The case of left $\widetilde{\mathscr{D}}_X$ -modules is easier to treat, so we will consider *left* $\widetilde{\mathscr{D}}_X$ -modules and the corresponding setting for the V-filtration in this section.

Let us make the construction explicit in the case of a closed inclusion. A local section ξ of $\iota_Y^{-1}\widetilde{\Theta}_X$ (vector field on X, considered at points of Y only; we denote by ι_Y^{-1} the sheaf-theoretic pullback) is said to be tangent to Y if, for every local section g of $\widetilde{\mathscr{I}}_Y$, $\xi(g) \in \widetilde{\mathscr{I}}_Y$. This defines a subsheaf $\widetilde{\Theta}_{X|Y}$ of $\iota_Y^{-1}\widetilde{\Theta}_X$. Then $\widetilde{\Theta}_Y = \widetilde{\mathscr{O}}_Y \otimes_{\iota_Y^{-1}\widetilde{\mathscr{O}}_X} \widetilde{\Theta}_{X|Y} = \iota_Y^* \widetilde{\Theta}_{X|Y}$ is a subsheaf of $\iota_Y^* \widetilde{\Theta}_X$.

Given a left $\widetilde{\mathscr{D}}_X$ -module, the action of $\iota_Y^{-1}\widetilde{\Theta}_X$ on $\iota_Y^{-1}\mathscr{M}$ restricts to an action of $\widetilde{\Theta}_Y$ on $\iota_Y^*\mathscr{M} = \widetilde{\mathscr{O}}_Y \otimes_{\iota_Y^{-1}\widetilde{\mathscr{O}}_X} \iota_Y^{-1}\mathscr{M}$. The criterion of Exercise A.3.1 is fulfilled since it is fulfilled for $\widetilde{\Theta}_X$ and \mathscr{M} , defining therefore a left $\widetilde{\mathscr{D}}_Y$ -module structure on $\iota_Y^*\mathscr{M}$: this is ${}_{\mathsf{D}}\iota_Y^*\mathscr{M}$.

Without any other assumption, coherence is not preserved by ${}_{\mathrm{D}}\iota_Y^*$. For example, ${}_{\mathrm{D}}\iota_Y^*\widetilde{\mathscr{D}}_X$ is not $\widetilde{\mathscr{D}}_Y$ -coherent if codim $Y \ge 1$. A criterion for coherence of the pullback is given below in terms of the characteristic variety.

The cotangent map to the inclusion defines a natural bundle morphism $\varpi : T^*X_{|Y} \times \mathbb{C}_z \to T^*Y \times \mathbb{C}_z$, the kernel of which is by definition the conormal bundle $T^*_YX \times \mathbb{C}_z$ of $Y \times \mathbb{C}_z$ in $X \times \mathbb{C}_z$.

Definition 7.5.1 (Non-characteristic property). Let \mathscr{M} be a holonomic $\widehat{\mathscr{D}}_X$ -module with characteristic variety Char \mathscr{M} contained in $\Lambda \times \mathbb{C}_z$, where $\Lambda \subset T^*X$ is Lagrangean (see Section A.10.c). Let $Y \subset X$ be a submanifold of X. We say that Y is non-characteristic with respect to the holonomic $\widetilde{\mathscr{D}}_X$ -module \mathscr{M} , or that \mathscr{M} is non-characteristic along Y, if one of the following equivalent conditions is satisfied:

- $(T_Y^*X \times \mathbb{C}_z) \cap \operatorname{Char} \mathscr{M} \subset T_X^*X \times \mathbb{C}_z,$
- ϖ : Char $\mathscr{M}_{|Y \times \mathbb{C}_z} \to T^*Y \times \mathbb{C}_z$ is finite, i.e., proper with finite fibres.

Exercise 7.5.2. Show that both conditions in Definition 7.5.1 are indeed equivalent. [*Hint*: use the homogeneity property of Char \mathcal{M} .]

Theorem 7.5.3 (Coherence of non-characteristic restrictions)

Assume that \mathscr{M} is $\widetilde{\mathscr{D}}_X$ -coherent and that Y is non-characteristic with respect to \mathscr{M} . Then ${}_{\mathsf{D}}\iota_Y^*\mathscr{M}$ is $\widetilde{\mathscr{D}}_Y$ -coherent and $\operatorname{Char}_{\mathsf{D}}\iota_Y^*\mathscr{M} \subset \varpi(\operatorname{Char}\mathscr{M}_{|Y})$.

Sketch of proof. The question is local near a point $x \in Y$. We may therefore assume that \mathscr{M} has a coherent filtration $F_{\bullet}\mathscr{M}$.

(1) Set $F_{kD}\iota_Y^*\mathscr{M} = \operatorname{image}[\iota_Y^*F_k\mathscr{M} \to \iota_Y^*\mathscr{M}]$. Then, using Exercise A.10.8(2), one shows that $F_{\bullet D}\iota_Y^*\mathscr{M}$ is a coherent filtration with respect to $F_{\bullet D}\iota_Y^*\widetilde{\mathscr{D}}_X$.

(2) The module $\operatorname{gr}^{F}{}_{D}\iota_{Y}^{*}\mathscr{M}$ is a quotient of $\iota_{Y}^{*}\operatorname{gr}^{F}\mathscr{M}$, hence its support is contained in Char $\mathscr{M}_{|Y}$. By Remmert's Theorem, it is a coherent $\operatorname{gr}^{F}\widetilde{\mathscr{D}}_{Y}$ -module.

(3) The filtration $F_{\bullet D} \iota_Y^* \mathscr{M}$ is thus a coherent filtration of the $\widehat{\mathscr{D}}_Y$ -module ${}_D \iota_Y^* \mathscr{M}$. By Exercise A.10.5(1), ${}_D \iota_Y^* \mathscr{M}$ is $\widehat{\mathscr{D}}_Y$ -coherent. Using the coherent filtration above, it is clear that $\operatorname{Char}_D \iota_Y^* \mathscr{M} \subset \varpi(\operatorname{Char} \mathscr{M}_{|Y})$.

Exercise 7.5.4. With the assumptions of Theorem 7.5.3, show similarly that, if Y is defined by $x_1 = \cdots = x_p = 0$ then, considering the map $\boldsymbol{x} : X \to \mathbb{C}^p$ induced by $\boldsymbol{x} := (x_1, \ldots, x_p)$, then \mathscr{M} is $\widetilde{\mathscr{D}}_{X/\mathbb{C}^p}$ -coherent.

Definition 7.5.5 (Strict non-characteristic property). In the setting of Definition 7.5.1, we say that \mathscr{M} is strictly non-characteristic along Y if \mathscr{M} is non-characteristic along Y and moreover $\mathbf{L}_{D}\iota_{Y}^{*}\mathscr{M} = \widetilde{\mathscr{O}}_{Y} \otimes_{\iota_{V}^{-1}\widetilde{\mathscr{O}}_{X}}^{L} \iota_{Y}^{-1}\mathscr{M}$ is strict.

Proposition 7.5.6. If \mathscr{M} is strictly non-characteristic along Y, then $\mathbf{L}_{D}\iota_{Y}^{*}\mathscr{M} = {}_{D}\iota_{Y}^{*}\mathscr{M}$.

Proof. The result holds for \mathscr{D}_X -modules, and therefore it holds after tensoring with $\mathbb{C}[z, z^{-1}]$. As a consequence, $\mathscr{H}^j \mathbf{L}_{\mathsf{D}} \iota^* \mathscr{M}$ is a z-torsion module if $j \neq 0$. It is strict if and only if it is zero.

Proposition 7.5.7. Assume that $\operatorname{codim} Y = 1$ and denote it by H. Then if \mathscr{M} is strictly non-characteristic along H, it is also strictly \mathbb{R} -specializable along H and ${}_{\mathsf{D}}\iota_{H}^{*}\mathscr{M}$ is naturally identified with $\operatorname{gr}_{V}^{0}\mathscr{M}$.

Proof. Since the question is local, we may assume that $X \simeq H \times \Delta_t$. The previous proposition says that $t : \mathcal{M} \to \mathcal{M}$ is injective and the definition amounts to the strictness of $\mathcal{M}/t\mathcal{M}$.

Since \mathscr{M} is $\mathscr{D}_{X/\mathbb{C}}$ -coherent (Exercise 7.5.4), the filtration defined by $U^k \mathscr{M} = t^k \mathscr{M}$ $(k \in \mathbb{N})$ is a coherent V-filtration and $E : \operatorname{gr}_U^0 \mathscr{M} \to \operatorname{gr}_U^0 \mathscr{M}$ acts by 0 since $\eth_t U^0 \mathscr{M} \subset U^0 \mathscr{M} = \mathscr{M}$. It follows that \mathscr{M} is specializable along H and that the Bernstein polynomial of the filtration $U^{\bullet} \mathscr{M}$ has only integral roots. Moreover, $t : \operatorname{gr}_U^k \mathscr{M} \to \operatorname{gr}_U^{k+1} \mathscr{M}$ is onto for $k \ge 0$. We will show by induction on k that each $\operatorname{gr}_U^k \mathscr{M}$ is strict. The assumption is that $\operatorname{gr}_U^0 \mathscr{M}$ is strict. We note that E - kz acts by zero on $\operatorname{gr}_U^k \mathscr{M}$. If $\operatorname{gr}_U^k \mathscr{M}$ is strict, then the composition $\eth_t t$ acts by (k+1)z on $\operatorname{gr}_U^k \mathscr{M}$, hence is injective, so $t : \operatorname{gr}_U^k \mathscr{M} \to \operatorname{gr}_U^{k+1} \mathscr{M}$ is bijective, and $\operatorname{gr}_U^{k+1} \mathscr{M}$ is thus strict. It follows that \mathscr{M} is strictly \mathbb{R} -specializable along H, and the t-adic filtration $U^{\bullet} \mathscr{M}$ is equal to the V-filtration.

Locally, we have an identification ${}_{\mathsf{D}}\iota_{H}^{*}\mathscr{M} = \mathscr{M}/t\mathscr{M} = \operatorname{gr}_{V}^{0}\mathscr{M}$. We note that $\operatorname{gr}_{V}^{0}\mathscr{M}$ is naturally a $\widetilde{\mathscr{D}}_{H}$ -module since E acts by 0, and $\widetilde{\mathscr{D}}_{H} = \operatorname{gr}_{V}^{V}\widetilde{\mathscr{D}}_{X}/\operatorname{Egr}_{0}^{V}\widetilde{\mathscr{D}}_{X}$. Therefore the previous identification is global.

Remark 7.5.8 (The case of right $\widetilde{\mathscr{D}}_X$ -modules). Let \mathscr{M} be a left $\widetilde{\mathscr{D}}_X$ -module and let $\mathscr{M}^{\text{right}} := \widetilde{\omega}_X \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{M}$ be the associated right $\widetilde{\mathscr{D}}_X$ -module (with grading). If \mathscr{M} is strictly non-characteristic along H, then so is $\mathscr{M}^{\text{right}}$. We have

$${}_{\mathrm{D}}\iota_{H}^{*}\mathscr{M}^{\mathrm{right}} := \widetilde{\omega}_{H} \otimes_{\widetilde{\mathscr{O}}_{H}} {}_{\mathrm{D}}\iota_{H}^{*}\mathscr{M} = \widetilde{\omega}_{H} \otimes_{\widetilde{\mathscr{O}}_{H}} \mathrm{gr}_{V}^{0}\mathscr{M} = \mathrm{gr}_{-1}^{V}\mathscr{M}^{\mathrm{right}}(1),$$

according to Remark 7.3.30.

Assume that H is globally defined by the smooth function g. Then

$${}_{\mathrm{D}}\iota_{H*\mathrm{D}}\iota_{H}^{*}\mathscr{M}^{\mathrm{right}} = {}_{\mathrm{D}}\iota_{H*}\mathrm{gr}_{V}^{0}\mathscr{M} = \mathrm{gr}_{-1}^{V}\mathscr{M}^{\mathrm{right}}(1) = \psi_{g,1}\mathscr{M}^{\mathrm{right}},$$

according to Exercise 7.4.5.

7.5.b. Specialization of a strictly non-characteristic divisor with normal crossings. We make explicit an example of computation of nearby cycles along a divisor with normal crossings in a simple situation, anticipating more complicated computations in Chapter 11. Let $D = D_1 \cup D_2$ be a divisor with normal crossings in X and smooth irreducible components D_1, D_2 . We set $D_{1,2} = D_1 \cap D_2$, which is a smooth

manifold of codimension two in X. Let \mathscr{M} be a left $\widetilde{\mathscr{D}}_X$ -module which is strictly noncharacteristic along D_1 , D_2 and $D_{1,2}$. Let us summarize some consequences of the assumption on nearby cycles. In local coordinates we will set $D_i = \{x_i = 0\}$ (i = 1, 2).

(a) \mathscr{M} is strictly \mathbb{R} -specializable along D_1 and D_2 . We denote by $V_{(i)}^{\bullet}\mathscr{M}$ the *V*-filtration of \mathscr{M} along D_i (i = 1, 2).

- (b) $\operatorname{gr}_{V_{(i)}}^{\beta} \mathscr{M} = 0$ if $\beta \notin \mathbb{N}$.
- (c) $\operatorname{gr}_{V_{(i)}}^{0} \mathscr{M} = {}_{\mathrm{D}}\iota_{D_{i}}^{*} \mathscr{M} = \iota_{D_{i}}^{*} \mathscr{M}$. In local coordinates, $\operatorname{gr}_{V_{(i)}}^{0} \mathscr{M} = \mathscr{M}/x_{i} \mathscr{M}$.

Lemma 7.5.9. For i = 1, 2, the $\widetilde{\mathscr{D}}_{D_i}$ -module ${}_{\mathsf{D}\iota} {}^*_{D_i}\mathscr{M}$ is strictly non-characteristic, hence strictly \mathbb{R} -specializable, along $D_{1,2}$ and $V^{\bullet}_{(2)} \operatorname{gr}^{0}_{V_{(1)}}\mathscr{M}$ is the filtration induced by $V^{\bullet}_{(2)}\mathscr{M}$, and conversely, so that

$$\mathrm{gr}_{V_{(2)}}^{0}\mathrm{gr}_{V_{(1)}}^{0}\mathscr{M} = \mathrm{gr}_{V_{(1)}}^{0}\mathrm{gr}_{V_{(2)}}^{0}\mathscr{M} = {}_{\mathrm{D}}\iota_{D_{1,2}}^{*}\mathscr{M} = \iota_{D_{1,2}}^{*}\mathscr{M}$$

Proof. The first point is mostly obvious, giving rise to the last formula, according to (c). For the second point, we have to check in local coordinates that $x_2^k(\mathcal{M}/x_1\mathcal{M}) = x_2^k\mathcal{M}/x_1x_2^k\mathcal{M}$ for every $k \ge 1$, that is, the morphism

$$\mathcal{M}/x_1\mathcal{M} \xrightarrow{x_2^k} x_2^k\mathcal{M}/x_1x_2^k\mathcal{M}$$

is an isomorphism. Recall (see Exercise 7.5.4) that \mathscr{M} is $\widehat{\mathscr{D}}_{X/\mathbb{C}^2}$ -coherent, so by taking a local resolution by free $\widehat{\mathscr{D}}_{X/\mathbb{C}^2}$ -modules, we are reduced to proving the assertion for $\mathscr{M} = \widetilde{\mathscr{D}}_{X/\mathbb{C}^2}^{\ell}$, where it is obvious.

Exercise 7.5.10. Conclude from the lemma that (x_1, x_2) is a regular sequence on \mathcal{M} , i.e., $x_1 \mathcal{M} \cap x_2 \mathcal{M} = x_1 x_2 \mathcal{M}$. Show that, for every $k \ge 1$, if we have a relation $\sum_{k_1+k_2=k} x_2^{k_1} x_1^{k_2} m_{k_1,k_2} = 0$ in \mathcal{M} , then there exist $\mu_{i,j} \in \mathcal{M}$ for $i, j \ge 0$ (and the convention that $\mu_{i,j} = 0$ if i or $j \le -1$) such that $m_{k_1,k_2} = x_1 \mu_{k_1-1,k_2} - x_2 \mu_{k_1,k_2-1}$ for every k_1, k_2 .

Our aim is to compute, in the local setting, the nearby cycles of \mathscr{M} along $g = x_1 x_2$ (after having proved that \mathscr{M} is strictly \mathbb{R} -specializable along (g), of course). We consider then the graph inclusion $\iota_g : X \hookrightarrow X \times \mathbb{C}_t$. We will return to the right setting, so we assume $\mathscr{M} = \mathscr{M}^{\text{right}}$, but the following proposition also holds in the left case after side-changing.

Proposition 7.5.11. Under the previous assumptions, the $\widetilde{\mathscr{D}}_{X\times\mathbb{C}}$ -module ${}_{\mathsf{D}}\iota_{g*}\mathscr{M}$ is a minimal extension along (t), we have $\psi_{g,\lambda}\mathscr{M} = 0$ for $\lambda \neq 1$ and there are local isomorphisms

(7.5.11*)
$$P_{\ell}\psi_{g,1}\mathscr{M} \simeq \begin{cases} \psi_{x_{1},1}\mathscr{M} \oplus \psi_{x_{2},1}\mathscr{M} & \text{if } \ell = 0, \\ \psi_{x_{1},1}\psi_{x_{2},1}\mathscr{M}(-1) = \psi_{x_{2},1}\psi_{x_{1},1}\mathscr{M}(-1) & \text{if } \ell = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We set $\mathscr{N} = {}_{\mathsf{D}}\iota_{g*}\mathscr{M}$. We have $\mathscr{N} = \iota_{g*}\mathscr{M}[\eth_t]$ with the usual structure of a right $\widetilde{\mathscr{D}}_{X\times\mathbb{C}}$ -module (see Example A.8.9). We identify $\iota_{g*}\mathscr{M}$ as the component of \eth_t -degree zero in \mathscr{N} . Let $U_{\bullet}\mathscr{N}$ denote the filtration defined by

$$U_{-1}(\mathscr{N}) = \iota_{g*}\mathscr{M} \cdot \widetilde{\mathscr{D}}_X \subset \mathscr{N}, \qquad U^k(\mathscr{N}) = \begin{cases} U_{-1}(\mathscr{N}) \cdot t^k & \text{if } k \ge 0, \\ \sum_{\ell \leqslant -k} U_{-1}(\mathscr{N}) \cdot \eth_t^\ell & \text{if } k \leqslant 0. \end{cases}$$

We wish to prove that $U^{\bullet}\mathcal{N}$ satisfies all the properties of the V-filtration of \mathcal{N} .

Let m be a local section of $\mathscr{M}.$ From the relation

(7.5.12)
$$(m \otimes 1)\eth_{x_1} = (m\eth_{x_1}) \otimes 1 - mx_2 \otimes \eth_t$$

we deduce

(7.5.13)
$$(m \otimes 1)\eth_t t = (m\eth_{x_1} x_1) \otimes 1 - (m \otimes 1) x_1 \eth_{x_1} = (m\eth_{x_2} x_2) \otimes 1 - (m \otimes 1) x_2 \eth_{x_2},$$

showing that $U_{-1}(\mathscr{N})$ is a $V_0 \widetilde{\mathscr{D}}_{X \times \mathbb{C}_t}$ -module. If $(m_i)_{i \in I}$ is a finite set of local $\widetilde{\mathscr{D}}_{X/\mathbb{C}^2}$ generators of \mathscr{M} (see Exercise 7.5.4), we deduce that it is a set of $\widetilde{\mathscr{D}}_X$ -generators, hence
of $V_0 \widetilde{\mathscr{D}}_{X \times \mathbb{C}_t}$ -generators, of $U_{-1}(\mathscr{N})$. It follows that $U^{\bullet}(\mathscr{N})$ is a good V-filtration
of \mathscr{N} . Moreover, the formulas above imply

$$(m \otimes 1)(\mathfrak{d}_t t)^2 = \left((m\mathfrak{d}_{x_1}\mathfrak{d}_{x_2} \otimes 1) + (m \otimes 1)\mathfrak{d}_{x_1}\mathfrak{d}_{x_2} - (m\mathfrak{d}_{x_2} \otimes 1)\mathfrak{d}_{x_1} - (m\mathfrak{d}_{x_1} \otimes 1)\mathfrak{d}_{x_2} \right) \cdot t,$$
giving a Bernstein relation. Since $(\mathfrak{d}_t t)^2$ vanishes on $\operatorname{gr}_{-1}^U(\mathcal{N})$, the monodromy filtration is given by

$$\begin{split} \mathbf{M}_{-2} \mathbf{gr}_{-1}^{U}(\mathscr{N}) &= 0, & \mathbf{M}_{-1} \mathbf{gr}_{-1}^{U}(\mathscr{N}) = \mathbf{gr}_{-1}^{U}(\mathscr{N}) \cdot \eth_{t} t, \\ \mathbf{M}_{0} \mathbf{gr}_{-1}^{U}(\mathscr{N}) &= \mathrm{Ker}[\eth_{t} t : \mathbf{gr}_{-1}^{U}(\mathscr{N}) \to \mathbf{gr}_{-1}^{U}(\mathscr{N})], & \mathbf{M}_{1} \mathbf{gr}_{-1}^{U}(\mathscr{N}) = \mathbf{gr}_{-1}^{U}(\mathscr{N}). \end{split}$$

As a consequence,

$$\begin{split} \mathbf{P}_{0}\mathbf{gr}_{-1}^{U}(\mathscr{N}) &= \mathbf{gr}_{0}^{\mathbf{M}}\mathbf{gr}_{-1}^{U}(\mathscr{N}) = \mathrm{Ker}\,\eth_{t}t/\operatorname{Im}\,\eth_{t}t, \\ \mathbf{P}_{1}\mathbf{gr}_{-1}^{U}(\mathscr{N}) &= \mathbf{gr}_{1}^{\mathbf{M}}\mathbf{gr}_{-1}^{U}(\mathscr{N}) = \mathbf{gr}_{-1}^{U}(\mathscr{N})/\operatorname{Ker}\,\eth_{t}t \xrightarrow{\sim} \mathbf{M}_{-1}\mathbf{gr}_{-1}^{U}(\mathscr{N})(-1). \end{split}$$

We will identify these $\widetilde{\mathscr{D}}_X$ -modules with those given in the statement. This will also prove that $\operatorname{gr}_{-1}^U(\mathscr{N})$ is strict, because $\psi_{x_1,1}\mathscr{M}, \psi_{x_2,1}\mathscr{M}, \psi_{x_1,1}\psi_{x_2,1}\mathscr{M}$ are strict.

Let $G_{\bullet}\mathcal{N}$ denote the filtration by the order with respect to \mathfrak{d}_t . It will be useful to get control on the various objects occurring in the computations, mainly because when working on $\operatorname{gr}^G \mathcal{N}$, the action of \mathfrak{d}_{x_1} amounts to that of $-x_2 \otimes \mathfrak{d}_t$ and similarly for \mathfrak{d}_{x_2} , and the action of x_1, x_2 on \mathcal{M} is well understood, due to Exercise 7.5.10.

Lemma 7.5.14. We have $U_{-1}(\mathscr{N}) \cap G_p(\mathscr{N}) = \sum_{k_1+k_2 \leq p} (\mathscr{M} \otimes 1) \eth_{x_1}^{k_1} \eth_{x_2}^{k_2}$.

Proof. Any local section ν of $U_{-1}(\mathscr{N})$ can be written as $\sum_{k_1,k_2 \ge 0} (m_{k_1,k_2} \otimes 1) \eth_{x_1}^{k_1} \eth_{x_2}^{k_2}$ for some local sections m_{k_1,k_2} of \mathscr{M} and, if $q = \max\{k_1 + k_2 \mid m_{k_1,k_2} \neq 0\}$, the degree of ν with respect to \eth_t is $\leqslant q$ and the coefficient of \eth_t^p is

$$(-1)^q \sum_{k_1+k_2=q} m_{k_1,k_2} x_2^{k_1} x_1^{k_2}.$$

If this coefficient vanishes, Exercise 7.5.10 implies that

$$\nu = \sum_{k_1+k_2 \leqslant q} ((\mu_{k_1-1,k_2} x_1 - \mu_{k_1,k_2-1} x_2) \otimes 1) \eth_{x_1}^{k_1} \eth_{x_2}^{k_2}.$$

The operator against $\mu_{i,j} \otimes 1$ is $(x_1 \eth_{x_1} - x_2 \eth_{x_2}) \eth_{x_1}^i \eth_{x_2}^j$, and (7.5.13) implies

$$(\mu_{i,j}\otimes 1)(x_1\eth_{x_1}-x_2\eth_{x_2})=(\mu_{i,j}(x_1\eth_{x_1}-x_2\eth_{x_2}))\otimes 1$$

so that $\nu \in \sum_{k_1+k_2 \leqslant q-1} (\mathscr{M} \otimes 1) \eth_{x_1}^{k_1} \eth_{x_2}^{k_2}$.

As a consequence, let us prove the equality

(7.5.15)
$$\mathfrak{d}_t^{-1}(U_{-1}(\mathscr{N})) \cap U_{-1}\mathscr{N} = \sum_{k_1,k_2} (\mathscr{M}(x_1,x_2) \otimes 1) \mathfrak{d}_{x_1}^{k_1} \mathfrak{d}_{x_2}^{k_2}$$

and that t acts injectively on $U_{-1}\mathcal{N}$.

Let $\nu = \sum_{q \leqslant p} \nu_q \otimes \eth_t^q$ be a nonzero local section of $U_{-1}(\mathscr{N})$ of *G*-order *p*, so that $\nu_p \neq 0$. We will argue by induction on *p*. By the lemma we have $\nu_p = \sum_{k_1+k_2=p} (m_{k_1,k_2} \otimes 1)\eth_{x_1}^{k_1}\eth_{x_2}^{k_2}$ with $\sum_{k_1+k_2=p} m_{k_1,k_2} x_2^{k_1} x_1^{k_2} \neq 0$ in \mathscr{M} . Assume $\eth_t \nu$ is a local section of $U_{-1}(\mathscr{N})$. Then $\sum_{k_1+k_2=p} m_{k_1,k_2} x_2^{k_1} x_1^{k_2}$ is a local section of $\mathscr{M}(x_1,x_2)^{p+1}$, that is, is equal to $\sum_{k_1+k_2=p} \mu_{k_1,k_2} x_2^{k_1} x_1^{k_2}$ with $\mu_{k_1,k_2} \in \mathscr{M}(x_1,x_2)$, so $\nu - \sum_{k_1+k_2=p} (\mu_{k_1,k_2} \otimes 1)\eth_{x_1}^{k_1}\eth_{x_2}^{k_2}$ a local section of $\eth_t U_{-1}(\mathscr{N}) \cap U_{-1}\mathscr{N}$ and has *G*-order $\leqslant p - 1$. We can conclude by induction.

Assume now that $\nu t = 0$. We have

$$0 = (\nu t)_p = \left[(\nu_p \otimes \eth_t^p) t \right]_p = \nu_p \otimes t \eth_t^p = \nu_p x_1 x_2 \otimes \eth_t^p,$$

so $\nu_p x_1 x_2 = 0$ in \mathcal{M} , and thus $\nu_p = 0$, a contradiction.

Recall that $\mathscr{M} = V_{-1}^{(1)} \mathscr{M}$ (V-filtration relative to x_1), so that $\mathscr{M}/\mathscr{M}x_1 = \operatorname{gr}_{-1}^{V^{(1)}} \mathscr{M}$ and $\mathscr{N}_1 := (\mathscr{M}/\mathscr{M}x_1)[\eth_{x_1}] \simeq \psi_{x_1,1} \mathscr{M}(-1)$, according to Exercise 7.4.5. Similarly, $\mathscr{N}_{12} \simeq \psi_{x_1,1} \psi_{x_2,1} \mathscr{M}(-2)$. The map

(7.5.16)
$$m_{k_1,k_2} \otimes \eth_{x_1}^{k_1} \eth_{x_2}^{k_2} \longmapsto (m_{k_1,k_2} \otimes 1) \eth_{x_1}^{k_1} \eth_{x_2}^{k_2} \cdot \eth_t t$$

sends $\mathcal{M}(x_1, x_2)[\eth_{x_1}, \eth_{x_1}]$ to $U_{-2}\mathcal{N}(-1)$, according to (7.5.12) and defines thus a surjective morphism

$$\psi_{x_1,1}\psi_{x_2,1}\mathscr{M}(-2) = \mathscr{N}_{12} \longrightarrow \mathrm{gr}_{-1}^{\mathrm{M}}\mathrm{gr}_{-1}^{U}\mathscr{N}(-1).$$

Let us prove that it is also injective. Let us denote by $[m_{k_1,k_2}]$ the class of m_{k_1,k_2} in $\mathcal{M}/\mathcal{M}(x_1,x_2)$. Let $\sum [m_{k_1,k_2}] \otimes \eth_{x_1}^{k_1} \eth_{x_2}^{k_2}$ be nonzero and of degree equal to p and set

$$\nu = \sum_{k_1+k_2 \leqslant p} (m_{k_1,k_2} \otimes 1) \eth_{x_1}^{k_1} \eth_{x_2}^{k_2}$$

Assume that $\nu \eth_t t \in U_{-2}\mathscr{N}$, hence, by the injectivity of $t, \nu \eth_t \in U_{-1}\mathscr{N}$. The proof of (7.5.15) above shows that, for $k_1 + k_2 = p$, there exists $\mu_{k_1,k_2} \in \mathscr{M}(x_1, x_2)$ such that $\sum_{k_1+k_2=p} (m_{k_1,k_2} - \mu_{k_1,k_2}) x_2^{k_1} x_1^{k_2} = 0$, and by Exercise 7.5.10 we conclude that $m_{k_1,k_2} \in \mathscr{M}(x_1, x_2)$, so $[m_{k_1,k_2}] = 0$, a contradiction. As a consequence, if $\nu \eth_t t = \sum (m_{k_1,k_2} \otimes 1) \eth_{x_1}^{k_1} \eth_{x_2}^{k_2} \eth_t t$ belongs to $U_{-2} \mathscr{N} = U_{-1} \mathscr{N} \cdot t$, (7.5.15) implies $\nu \in \sum (\mathscr{M}(x_1, x_2) \otimes 1) \eth_{x_1}^{k_1} \eth_{x_2}^{k_2}$. We obtain therefore

(7.5.17)
$$\operatorname{gr}_{1}^{M}\operatorname{gr}_{-1}^{U}\mathscr{N} \xrightarrow{\mathrm{N}} \operatorname{gr}_{-1}^{M}\operatorname{gr}_{-1}^{U}\mathscr{N}(-1) \simeq \psi_{x_{1},1}\psi_{x_{2},1}\mathscr{M}(-2),$$

and these modules are strict. Note that the isomorphism $\mathscr{N}_{12} \xrightarrow{\sim} \operatorname{gr}_1^{\mathrm{M}} \operatorname{gr}_{-1}^U \mathscr{N} = U_{-1} \mathscr{N} / (\mathfrak{d}_t t)^{-1} U_{-1} \mathscr{M})$ is induced by

(7.5.18)
$$m_{k_1,k_2} \otimes \eth_{x_1}^{k_1} \eth_{x_2}^{k_2} \longmapsto (m_{k_1,k_2} \otimes 1) \eth_{x_1}^{k_1} \eth_{x_2}^{k_2}.$$

Let us now consider M_0 . Note that (7.5.15) and the injectivity of t imply

$$\mathcal{M}_{0}\mathrm{gr}_{-1}^{U}\mathscr{N} = \sum_{k_{1},k_{2}} (\mathscr{M}(x_{1},x_{2})\otimes 1)\eth_{x_{1}}^{k_{1}}\eth_{x_{2}}^{k_{2}} \mod U_{-2}\mathscr{N},$$

and clearly $\sum_{k_1,k_2} (\mathscr{M} x_1 x_2 \otimes 1) \eth_{x_1}^{k_1} \eth_{x_2}^{k_2} \subset U_{-2} \mathscr{N}$. Note also that $(mx_1 \otimes 1) \eth_{x_1}^{k_1} \equiv (m\eth_{x_1}^{k_1} x_1) \otimes 1 \mod \operatorname{Im} \eth_t t$, according to (7.5.13). As a consequence,

$$\mathcal{M}_{0}\mathrm{gr}_{-1}^{U}\mathscr{N} = \sum_{k_{1}} (\mathscr{M}x_{2} \otimes 1)\eth_{x_{1}}^{k_{1}} + \sum_{k_{2}} (\mathscr{M}x_{1} \otimes 1)\eth_{x_{2}}^{k_{2}} \mod (U_{-1}\mathscr{N}\eth_{t}t + U_{-2}\mathscr{N}),$$

and we have a surjective morphism

(7.5.19)
$$\psi_{x_1,1}\mathscr{M}(-1) \oplus \psi_{x_2,1}\mathscr{M}(-1) = \mathscr{N}_1 \oplus \mathscr{N}_2 \longrightarrow \operatorname{gr}_0^{\mathrm{M}} \operatorname{gr}_{-1}^U \mathscr{N},$$

sending $m_{k_1,0} \otimes \eth_{x_1}^{k_1}$ to $(m_{k_1,0}x_2 \otimes 1)\eth_{x_1}^{k_1}$ and $m_{0,k_2} \otimes \eth_{x_2}^{k_2}$ to $(m_{0,k_2}x_1 \otimes 1)\eth_{x_2}^{k_2}$. In order to show injectivity, we first check that it is strict with respect to the filtration $G_{\bullet}\mathcal{N}$ and the filtration by the degree in \eth_{x_1}, \eth_{x_2} on $\mathcal{N}_1, \mathcal{N}_2$.

Assume that $(m_{k_1,0}x_2 \otimes 1)\eth_{x_1}^{k_1} + (m_{0,k_2}x_1 \otimes 1)\eth_{x_2}^{k_2} \in G_{p-1}\mathscr{N}$ for $k_1, k_2 \leq p$. Then we find that $m_{p,0} \in \mathscr{M}x_1$ and $m_{0,p}\mathscr{M}x_2$, as wanted. By the same argument we deduce the injectivity.

Due to the strictness of $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_{12}$, we conclude at this point that $\operatorname{gr}_{-1}^U \mathcal{M}$ is strict. If we show that $\operatorname{gr}_k^U \mathcal{M}$ is also strict for any k, then $U_{\bullet} \mathcal{N}$ satisfies all properties characterizing the V-filtration. As a consequence, \mathcal{M} is strictly \mathbb{R} -specializable along $(g), \operatorname{gr}_{-1}^U \mathcal{N} = \psi_{g,1} \mathcal{M}(-1)$, and (7.5.11*) holds.

Clearly, $\mathfrak{d}_t : \operatorname{gr}_{-1}^U \mathscr{N} \to \operatorname{gr}_0^U \mathscr{N}$ is onto. So we are left to proving the following assertions:

(i) $t^k : \operatorname{gr}_{-1}^U \mathcal{N} \to \operatorname{gr}_{-1-k}^U \mathcal{N}$ is an isomorphism (equivalently, injective) for $k \ge 1$, (ii) $t : \operatorname{gr}_0^U \mathcal{N} \to \operatorname{gr}_{-1}^U \mathcal{N}$ is injective (so $\operatorname{gr}_0^U \mathcal{N}$ is strict),

(iii) $\eth_t^k : \operatorname{gr}_0^U \mathscr{N} \to \operatorname{gr}_k^U \mathscr{N}$ is an isomorphism (equivalently, injective) for $k \ge 1$.

Proof of the assertions.

(i) If $\nu \in U_{-1}\mathcal{N}$ satisfies $\nu t^k = \mu t^{k+1}$ for some $\mu \in U_{-1}\mathcal{N}$ then, by injectivity of t on $U_{-1}\mathcal{N}$, $\nu = \mu t$, so $\nu \in U_{-2}\mathcal{N}$.

(ii) If $\nu \in U_{-1}\mathcal{N}$ is such that $\nu \eth_t \cdot t \in U_{-2}\mathcal{N}$, then there exists $\mu \in U_{-1}\mathcal{N}$ such that $(\nu \eth_t - \mu)t = 0$ hence, by t-injectivity, $\nu \eth_t \in U_{-1}\mathcal{N}$.

(iii) We prove the injectivity by induction on $k \ge 1$. Let $\nu \in U_{-1}\mathcal{M}$ and consider $\nu \eth_t \mod U_{-1}\mathcal{N}$ as an element of $\operatorname{gr}_0^U \mathcal{N}$. If $(\nu \eth_t)\eth_t^k \in U_{k-1}\mathcal{N}$, then $(\nu \eth_t^k)\eth_t t = 0$ in $\operatorname{gr}_{k-1}^U \mathcal{N}$. Since $\eth_t t - kz$ is nilpotent on $\operatorname{gr}_{k-1}^U \mathcal{N}$ and since $\operatorname{gr}_{k-1}^U \mathcal{N}$ is strict (by (ii) and the inductive assumption), $\eth_t t$ is injective on $\operatorname{gr}_{k-1}^U \mathcal{N}$, so $(\nu \eth_t)\eth_t^{k-1} = 0$ in $\operatorname{gr}_{k-1}^U \mathcal{N}$, and by induction $\nu \eth_t = 0$ in $\operatorname{gr}_0^U \mathcal{N}$.

This concludes the proof of Proposition 7.5.11.

7.6. Strict Kashiwara's equivalence

We now return to the case of right $\widetilde{\mathscr{D}}_X$ -module when considering the pushforward functor.

Let $\iota_Y : Y \subset X$ be the inclusion of a complex submanifold. The following is known as "Kashiwara's equivalence".

Proposition 7.6.1 (see [Kas03, §4.8]). The pushforward functor ${}_{\mathsf{D}}\iota_{Y*}$ induces a natural equivalence between coherent \mathscr{D}_Y -modules and coherent \mathscr{D}_X -modules supported on Y, whose quasi-inverse is the restriction functor ${}_{\mathsf{D}}\iota_Y^*$.

Be aware however that this result does not hold for graded coherent $R_F \mathscr{D}_X$ modules. For example, if $X = \mathbb{C}$ with coordinate s and $\iota_Y : Y = \{0\} \hookrightarrow X$ denotes the inclusion, ${}_{\mathsf{D}}\iota_{Y*}\mathbb{C}[z] = \delta_s \cdot \mathbb{C}[z, \eth_s]$ with $\delta_s s = 0$. On the other hand, consider the $\mathbb{C}[z, s]\langle\eth_s\rangle$ -submodule of $\mathbb{C}[z] \otimes_{\mathbb{C}} {}_{\mathsf{D}}\iota_{Y*}\mathbb{C} = \delta_s\mathbb{C}[z, \eth_s]$ generated by $\delta_s \eth_s$ (note: \eth_s and not \eth_s). This submodule is written $\delta_s\mathbb{C}[z] \oplus \bigoplus_{k \ge 0} \delta_s\eth_s^k \eth_s$. It has finite type over $\mathbb{C}[z, s]\langle\eth_s\rangle$ by construction, each element is annihilated by some power of s, and $\mathscr{H}^{-1}{}_{\mathsf{D}}\iota_Y^*(\delta_s \eth_s \cdot \mathbb{C}[z, s]\langle\eth_s\rangle) = \delta_s\mathbb{C}[z]$, but it is not equal to ${}_{\mathsf{D}}\iota_{Y*}\mathbb{C}[z]$.

Note also that this proposition implies in particular that $\mathscr{H}_{D}^{k} \iota_{YD}^{l} \iota_{Y*} \mathscr{M} = 0$ for $k \neq -1$, if \mathscr{M} is \mathscr{D}_X -coherent. In the example above, we have ${}_{D} \iota_{Y*} \mathbb{C} = \mathbb{C}[\eth_s]$ and the complex ${}_{D} \iota_{Y*} \mathbb{C}$ is the complex $\mathbb{C}[\eth_s] \xrightarrow{\cdot s} \mathbb{C}[\eth_s]$ with terms in degrees -1 and 0. It has cohomology in degree -1 only.

However, this is not true for graded coherent $R_F \mathscr{D}_X$ -modules. With the similar example, the complex ${}_{\mathbb{D}}\iota_{Y\mathbb{D}}^* \iota_{Y*}\mathbb{C}[z]$ is the complex $\mathbb{C}[z, \eth_s] \xrightarrow{\cdot s} \mathbb{C}[z, \eth_s]$. We have $\eth_s^k \cdot s = kz \eth_s^{k-1}$, so the cokernel of s is not equal to zero.

Proposition 7.6.2 (Strict Kashiwara's equivalence). Assume that Y is smooth of codimension one in X, and let $\iota_Y : Y \hookrightarrow X$ denote the inclusion. The functor ${}_{\mathsf{D}}\iota_{Y*} :$ $\mathsf{Mod}_{coh}(\widetilde{\mathscr{D}}_Y) \mapsto \mathsf{Mod}_{coh}(\widetilde{\mathscr{D}}_X)$ is fully faithful. It induces an equivalence between the full subcategory of $\mathsf{Mod}_{coh}(\widetilde{\mathscr{D}}_Y)$ whose objects are strict, and the full subcategory of $\mathsf{Mod}_{coh}(\widetilde{\mathscr{D}}_X)$ whose objects are strictly \mathbb{R} -specializable along Y and supported on Y. An inverse functor is $\mathscr{M} \mapsto \operatorname{gr}_0^V \mathscr{M}$.

Proof the full faithfulness. It is enough to prove that each morphism $\varphi : {}_{\mathrm{D}}\iota_{Y*}\mathcal{N}_1 \to {}_{\mathrm{D}}\iota_{Y*}\mathcal{N}_2$ takes the form ${}_{\mathrm{D}}\iota_{Y*}\psi$ for a unique $\psi : \mathcal{N}_1 \to \mathcal{N}_2$. Because of uniqueness, the assertion is local with respect to Y, hence we can assume that there exists a local

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coordinate *s* defining *Y*. Assume $\mathscr{M} = {}_{\mathbb{D}}\iota_{Y*}\mathscr{N}$ for some coherent $\widetilde{\mathscr{D}}_Y$ -module \mathscr{N} . Then one can recover \mathscr{N} from \mathscr{M} as the $\widetilde{\mathscr{D}}_Y$ -module $\mathscr{M}/\mathscr{M} \cdot \eth_s$. As a consequence, ψ must be the morphism induced by φ on $\mathscr{M}/\mathscr{M} \cdot \eth_s$, hence its uniqueness. On the other hand, since \mathscr{M}_1 is generated by $\mathscr{N}_1 \otimes \widetilde{\delta}_s$ over $\widetilde{\mathscr{D}}_X$, φ is determined by its restriction to $\mathscr{N}_1 \otimes \widetilde{\delta}_s$, that is by ψ , and the formula is $\varphi = {}_{\mathbb{D}}\iota_{Y*}\psi$.

Lemma 7.6.3. Assume $X \simeq Y \times \mathbb{C}$ with coordinate s on the second factor. Let \mathscr{M} be a coherent $\widetilde{\mathscr{D}}_X$ -module supported on $Y \times \{0\}$.

- (1) Assume that there exists a strict $\widetilde{\mathscr{D}}_Y$ -module \mathscr{N} such that $\mathscr{M} \simeq {}_{\mathsf{D}}\iota_{Y*}\mathscr{N}$. Then
 - (a) $\mathscr{N} = \operatorname{Ker}[s : \mathscr{M} \to \mathscr{M}],$
 - (b) \mathcal{N} is \mathcal{D}_{Y} -coherent,
 - (c) \mathcal{M} is strict and strictly \mathbb{R} -specializable along Y,
 - (d) $\mathscr{N} = \operatorname{gr}_0^V \mathscr{M}$.

(2) Conversely, if \mathscr{M} is strictly \mathbb{R} -specializable along Y, then such an \mathscr{N} exists. In particular, \mathscr{M} is also strict.

Remark 7.6.4 (Strictness and strict \mathbb{R} -specializability). Let \mathscr{M} be as in Lemma 7.6.3, that is, $\widetilde{\mathscr{D}}_X$ -coherent and supported on $Y \times \{0\}$. Then the filtration $U_0\mathscr{M} = \operatorname{Ker} s \subset U_1\mathscr{M} = \operatorname{Ker} s^2 \subset \cdots$ is a filtration by $V_0 \widetilde{\mathscr{D}}_X$ -submodules and obviously admits a weak Bernstein polynomial. Assume moreover that \mathscr{M} is *strict*. Then every $\operatorname{gr}_k^{\mathscr{M}}\mathscr{M}$ is also strict: if $m \in U_k\mathscr{M}$ and $z^\ell m \in U_{k-1}\mathscr{M}$, that is, if $s^{k+1}m = 0$ and $s^k z^\ell m = 0$, then $s^k m = 0$ by strictness of \mathscr{M} and thus m = 0 in $\operatorname{gr}_k^{\mathscr{M}}$. Therefore, $U_{\bullet}\mathscr{M}$ is the Kashiwara-Malgrange filtration $V_{\bullet}\mathscr{M}$ in the sense of Lemma 7.3.23, and Properties 7.3.25(1) and (2) are satisfied.

However, the condition that \mathscr{M} is strict is not enough to obtain the conclusion of 7.6.3(1), as shown by the following example. The point is that 7.3.25(3) may not hold. Assume that Y is reduced to a point and let \mathscr{M} be the $\widetilde{\mathscr{D}}_X$ -submodule of the $\mathscr{D}_X[z]$ -module $\widetilde{\mathbb{C}}\langle\partial_s\rangle$ generated by 1 and ∂_s (recall that $\widetilde{\mathbb{C}} := \mathbb{C}[z]$), that we denote by [1] and $[\partial_s]$ for the sake of clarity. By definition, we have [1]s = 0 and $[\partial_s]s^2 = 0$. For the Kashiwara-Malgrange filtration $V_{\bullet}\mathscr{M}$ defined above, $\eth_s : \operatorname{gr}_0^{\mathscr{M}} \mathscr{M} =$ $\widetilde{\mathbb{C}} \to \operatorname{gr}_1^{\mathscr{M}} \mathscr{M} = [\partial_s]\widetilde{\mathbb{C}}$ is not onto, for its cokernel is $[\partial_s]\mathbb{C}$. In other words, \mathscr{M} is not strictly \mathbb{R} -specializable at s = 0 and not of the form ${}_{\mathbb{D}}\iota_{Y*}\mathscr{N}$.

Proof of Lemma 7.6.3.

(1) Assume $\mathscr{M} = {}_{\mathsf{D}}\iota_{Y*}\mathscr{N}$ for some strict $\widehat{\mathscr{D}}_{Y}$ -module \mathscr{N} . We have ${}_{\mathsf{D}}\iota_{Y*}\mathscr{N} = \bigoplus_{k \ge 0} \iota_{Y*}\mathscr{N} \otimes \delta_s \eth_s^k$ with $\delta_s s = 0$ (see Exercise A.8.30(2)). The action of s on ${}_{\mathsf{D}}\iota_{Y*}\mathscr{N}$ is the z-shift $n \otimes \delta_s \eth_s^k \mapsto zkn \otimes \delta_s \eth_s^{k-1}$, hence $\mathscr{N} = \operatorname{Ker} s$ because \mathscr{N} is strict. Given a finite family of local $\widehat{\mathscr{D}}_X$ -generators of \mathscr{M} , we produce another such family made of homogeneous elements, by taking the components on the previous decomposition. Therefore, there exists a finite family of local sections n_i of \mathscr{N} such that $n_i \otimes \delta_s$ generate \mathscr{M} . Let $\mathscr{N}' \subset \mathscr{N}$ be the $\widehat{\mathscr{D}}_Y$ -submodule they generate. Then ${}_{\mathsf{D}}\iota_Y * \mathscr{N} \to {}_{\mathsf{D}}\iota_Y * \mathscr{N} = \mathscr{M}$ is onto. On the other hand, since \mathscr{N}' is also strict, this map is injective:

If $\sum_{k=1}^{N} n'_k \otimes \delta_s \eth_s^k \mapsto 0$, then $n'_N \otimes \delta_s \eth_s^N \mapsto 0$, and $s^N n'_N \otimes \delta_s \eth_s^N = \star z^N n'_N \otimes \delta_s \eth_s^N \mapsto 0$, where \star is a nonzero constant; so $z^N n'_N = 0$ in \mathscr{N} , hence $n'_N = 0$. We conclude $\mathscr{N}' = \mathscr{N}$ since both are equal to Kers in ${}_{\mathsf{D}}\iota_{Y*}\mathscr{N}$. Therefore, \mathscr{N} is locally finitely $\widetilde{\mathscr{D}}_Y$ -generated in \mathscr{M} , and then is $\widetilde{\mathscr{D}}_Y$ -coherent. One then checks that the filtration $U_j\mathscr{M} := \bigoplus_{k \geq 0}^j \iota_{Y*}\mathscr{N} \otimes \delta_s \eth_s^k$ is a coherent V-filtration of \mathscr{M} , and $\mathscr{N} = \operatorname{gr}_0^U \mathscr{M}$. We deduce that each $\operatorname{gr}_k^U \mathscr{M}$ is strict, and \mathscr{M} is strictly \mathbb{R} -specializable. Lastly, $n \otimes \delta_s$ satisfies $(n \otimes \delta_s) s \eth_s = 0$, so $V_{\bullet} \mathscr{M} = U_{\bullet} \mathscr{M}$ jumps at nonnegative integers only.

(2) Assume that \mathscr{M} is strictly \mathbb{R} -specializable along Y. Then $V_{<0}\mathscr{M} = 0$, according to 7.3.31(a). Similarly, $\operatorname{gr}_{\alpha}^{V}\mathscr{M} = 0$ for $\alpha \notin \mathbb{Z}$. As $s : \operatorname{gr}_{k}^{V}\mathscr{M} \to \operatorname{gr}_{k-1}^{V}\mathscr{M}$ is injective for $k \neq 0$ (see 7.3.31(c)), we conclude that

$$\operatorname{gr}_0^V \mathscr{M} \simeq V_0 \mathscr{M} = \operatorname{Ker}[s : \mathscr{M} \to \mathscr{M}].$$

Since $\mathfrak{d}_s: \operatorname{gr}_k^V \mathscr{M} \to \operatorname{gr}_{k-1}^V \mathscr{M}$ is an isomorphism for $k \leq 0$, we obtain

$$\mathscr{M} = \bigoplus_{\ell \ge 0} \operatorname{gr}_0^V \mathscr{M} \eth_s^\ell = {}_{\scriptscriptstyle \mathrm{D}} \iota_* \operatorname{gr}_0^V \mathscr{M}$$

Lastly, E + z acts by zero on $\operatorname{gr}_0^V \mathscr{M}$, which is therefore a coherent $\widetilde{\mathscr{D}}_Y$ -module by means of the isomorphism $\operatorname{gr}_0^V \widetilde{\mathscr{D}}_X / (E + z) \operatorname{gr}_0^V \widetilde{\mathscr{D}}_X \simeq \widetilde{\mathscr{D}}_Y$. It is strict since \mathscr{M} is strictly \mathbb{R} -specializable.

End of the proof of Proposition 7.6.2. It remains to prove essential surjectivity. Let $V_{\bullet}\mathscr{M}$ be the V-filtration of \mathscr{M} along Y. By the argument in the second part of the proof of Lemma 7.6.3, we have local isomorphisms $\mathscr{M} \xrightarrow{\sim} {}_{\mathsf{D}}\iota_*\mathrm{gr}_0^V\mathscr{M}$ which induce the identity on $V_0\mathscr{M} = \mathrm{gr}_0^V\mathscr{M}$. By full faithfulness they glue in a unique way as a global isomorphism $\mathscr{M} \simeq {}_{\mathsf{D}}\iota_*\mathrm{gr}_0^V\mathscr{M}$.

Corollary 7.6.5. Assume $\operatorname{codim} Y = 1$. Let \mathscr{N} be $\widetilde{\mathscr{D}}_Y$ -coherent and set $\mathscr{M} = {}_{\mathbb{D}}\iota_{Y*}\mathscr{N}$. If $\mathscr{M} = \mathscr{M}_1 \oplus \mathscr{M}_2$ with $\mathscr{M}_1, \mathscr{M}_2$ being $\widetilde{\mathscr{D}}_X$ -coherent, then there exist coherent $\widetilde{\mathscr{D}}_Y$ -submodules $\mathscr{N}_1, \mathscr{N}_2$ of \mathscr{N} such that $\mathscr{N} = \mathscr{N}_1 \oplus \mathscr{N}_2$ and $\mathscr{M}_j = {}_{\mathbb{D}}\iota_{Y*}\mathscr{N}_j$ for j = 1, 2.

Proof. Each \mathcal{M}_i is coherent and supported on Y. We set $\mathcal{N}_i = \mathcal{M}_i \cap \mathcal{N}$. Locally, choose a coordinate s defining Y and set $\mathcal{N}'_i = \mathcal{M}_i/\mathcal{M}_i \cdot \eth_s$. Since $\mathcal{N} = \mathcal{M}/\mathcal{M} \cdot \eth_s$, we deduce that $\mathcal{N} = \mathcal{N}'_1 \oplus \mathcal{N}'_2$, and we have a (local) isomorphism $\mathcal{M}_i \simeq {}_{\mathrm{D}} \iota_* \mathcal{N}'_i$. Then one checks that $\mathcal{N}'i = \mathcal{N}_i$, so it is globally defined.

We now consider the behaviour of strict \mathbb{R} -specializability along a function $g: X \to \mathbb{C}$ with respect to strict Kashiwara's equivalence along Y.

Proposition 7.6.6. Let \mathcal{N} be a coherent $\widetilde{\mathcal{D}}_Y$ -module and set $\mathcal{M} = {}_{\mathsf{D}}\iota_{Y*}\mathcal{N}$.

(1) Assume that \mathcal{N} is strictly \mathbb{R} -specializable along $(g_{|Y})$. Then \mathcal{M} is strictly \mathbb{R} -specializable along (g).

(2) Assume that \mathscr{M} is strictly \mathbb{R} -specializable along (g). Then \mathscr{N} is strictly \mathbb{R} -specializable along $(g_{|Y})$.

In such a case, we have $\psi_{g,\lambda}\mathcal{M} = {}_{\mathsf{D}}\iota_{Y*}\psi_{g|_Y,\lambda}\mathcal{N}$ and $\phi_{g,1}\mathcal{M} = {}_{\mathsf{D}}\iota_{Y*}\phi_{g|_Y,1}\mathcal{N}$. Moreover, $\operatorname{can}_{\mathscr{M}} = {}_{\mathsf{D}}\iota_{Y*}\operatorname{can}_{\mathscr{N}}$ and $\operatorname{var}_{\mathscr{M}} = {}_{\mathsf{D}}\iota_{Y*}\operatorname{var}_{\mathscr{N}}$.

Proof. The first statement is easy to check. Let us consider the second one. We first consider the case where $X = H_Y \times \mathbb{C}_s \times \mathbb{C}_t$, with $Y = H_Y \times \mathbb{C}_t$ and g is the projection to \mathbb{C}_t . We denote by V the V-filtration along t. We have $\mathscr{M} = {}_{\mathsf{D}}\iota_{Y*}\mathscr{N} = \bigoplus_k \iota_{Y*}\mathscr{N} \otimes \delta_s \eth_s^k$.

Let *n* be a local section of \mathscr{N} . If $b(t\mathfrak{d}_t) - tP(y, s, t, \mathfrak{d}_y, \mathfrak{d}_s, t\mathfrak{d}_t)$ is a Bernstein equation for $n \otimes \delta_s$ in ${}_{\mathsf{D}}t_{Y*}\mathscr{N}$, and if $P = P_0 + sQ$, where P_0 does not depend on *s*, then $b(t\mathfrak{d}_t) - tP_0(y, t, \mathfrak{d}_y, \mathfrak{d}_s, t\mathfrak{d}_t)$ is also a Bernstein equation for $n \otimes \delta_s$. The degreezero part with respect to \mathfrak{d}_s of this equation still gives a Bernstein equation for $n \otimes \delta_s$, and thus a Bernstein equation for *n* in \mathscr{N} . We conclude that \mathscr{N} is \mathbb{R} -specializable along H_Y and that $\operatorname{ord}_V(n) \ge \operatorname{ord}_V(n \otimes \delta_s)$.

Let us now prove that the V-filtration of \mathscr{M} is compatible with the decomposition. Let $\sum_{i=0}^{N} n_i \otimes \delta_s \eth_s^i$ be a section in $V_{\alpha}\mathscr{M}$. We will prove by induction on N that $\operatorname{ord}_V(n_i) \leq \alpha$ for every *i*. It is enough to prove it for i = N. We have $\sum_{i=0}^{N} n_i \otimes \delta_s \eth_s^i \cdot s^N = \star z^N n_N \otimes \delta_s \in V_{\alpha}\mathscr{M}$ for some nonzero constant \star . If $n_N \otimes \delta_s \in V_{\gamma}\mathscr{M}$ for $\gamma > \alpha$, then the class of $n_N \otimes \delta_s$ in $\operatorname{gr}_{\gamma}^V\mathscr{M}$ is annihilated by z^N , hence is zero since $\operatorname{gr}_{\gamma}^V\mathscr{M}$ is strict. Therefore, $n_N \otimes \delta_s \in V_{\alpha}\mathscr{M}$, and by the preliminary remark, $\operatorname{ord}_V(n_N) \leq \alpha$. If we denote by $U_{\bullet}\mathscr{N}$ the (possibly not coherent) V-filtration by the V-order, then one has $V_{\alpha}\mathscr{M} = \bigoplus_i \iota_{Y*} U_{\alpha}\mathscr{N} \otimes \delta_s \eth_s^i$ and $\operatorname{gr}_{\alpha}^V\mathscr{M} = \bigoplus_i \iota_{Y*} \operatorname{gr}_{\alpha}^V\mathscr{N} \otimes \delta_s \eth_s^i$. It follows that $U_{\bullet}\mathscr{N}$ is a coherent V-filtration of \mathscr{N} and that each $\operatorname{gr}_{\alpha}^V\mathscr{N}$ is strict. By uniqueness of the V-filtration, we have $U_{\bullet}\mathscr{N} = V_{\bullet}\mathscr{N}$, and Properties 7.3.25(2) and (3) are clearly satisfied, as they hold for \mathscr{M} .

For the general case, the question is local and we can assume that Y is defined by a smooth function h. By assumption, ${}_{\mathsf{D}}\iota_{g*}({}_{\mathsf{D}}\iota_{Y*}\mathscr{N})$ is strictly \mathbb{R} -specializable along t, and thus so is ${}_{\mathsf{D}}\iota_{(h,g)*}({}_{\mathsf{D}}\iota_{Y*}\mathscr{N}) = {}_{\mathsf{D}}\iota_{s=0*}\iota_{g_{|Y}}\mathscr{N}$, after (1). The previous argument implies that $\iota_{g_{|Y}}\mathscr{N}$ is strictly \mathbb{R} -specializable along t, that is, \mathscr{N} is strictly \mathbb{R} -specializable along $g_{|Y}$.

The last statement is then clear by the computation of the V-filtrations above. \Box

7.7. Strictly support-decomposable $\widetilde{\mathscr{D}}$ -modules

Let $g: X \to \mathbb{C}$ be a holomorphic function. We set $X_0 = g^{-1}(0)$. Let $\iota_g: X \hookrightarrow X \times \mathbb{C}$ denote the graph embedding associated with g. We set $H = X \times \{0\} \subset X \times \mathbb{C}$.

We first interpret the strict Kashiwara's equivalence in this setting.

Corollary 7.7.1. Assume that \mathscr{M} is $\widetilde{\mathscr{D}}_X$ -coherent, strictly \mathbb{R} -specializable along D := (g) and supported on X_0 . Then $\mathscr{M} \simeq \phi_{g,1} \mathscr{M}$.

Proof. By Proposition 7.6.2 we have ${}_{\mathsf{D}}\iota_{g*}\mathscr{M} = {}_{\mathsf{D}}\iota_{t*}\mathrm{gr}_{0}^{\mathsf{V}}{}_{\mathsf{D}}\iota_{g*}\mathscr{M} =: {}_{\mathsf{D}}\iota_{t*}\phi_{g,1}\mathscr{M}$. On the other hand, we recover \mathscr{M} from ${}_{\mathsf{D}}\iota_{g*}\mathscr{M}$ as $\mathscr{M} = {}_{\mathsf{D}}p_{*\mathsf{D}}\iota_{g*}\mathscr{M}$, where $p: X \times \mathbb{C} \to \mathbb{C}$ is the projection. We then use that $p \circ \iota_t = \mathrm{Id}_X$.

Proposition 7.7.2. Let \mathscr{M} be a coherent $\widetilde{\mathscr{D}}_X$ -module which is strictly \mathbb{R} -specializable along (g).

- (1) The following properties are equivalent:
 - (a) var : $\phi_{g,1}\mathcal{M} \to \psi_{g,1}\mathcal{M}(-1)$ is injective,
 - (b) ${}_{\mathrm{D}}\iota_{g*}\mathscr{M}$ has no proper subobject in $\mathrm{Mod}_{\mathrm{coh}}(\widetilde{\mathscr{D}}_{X\times\mathbb{C}})$ supported on H,

(c) There is no strictly \mathbb{R} -specializable inclusion $\mathscr{N} \hookrightarrow {}_{\mathsf{D}}\iota_{g*}\mathscr{M}$ with \mathscr{N} strictly \mathbb{R} -specializable supported on H.

(2) If can : $\psi_{g,1}\mathcal{M} \to \phi_{g,1}\mathcal{M}$ is onto, then ${}_{\mathsf{D}}\iota_{g*}\mathcal{M}$ has no proper quotient satisfying 7.3.25(1) and supported on H.

Definition 7.7.3 (Minimal extension along g). Let \mathscr{M} be a coherent \mathscr{D}_X -module which is strictly \mathbb{R} -specializable along (g). We say that \mathscr{M} is a minimal extension along (g) if var : $\phi_{g,1}\mathscr{M} \to \psi_{g,1}\mathscr{M}(-1)$ is injective and can : $\psi_{g,1}\mathscr{M} \to \phi_{g,1}\mathscr{M}$ is onto.

Exercise 7.7.4 (can-var diagram for a minimal extension). Show that the diagram (7.4.9*) or (7.4.10*) is isomorphic to the diagram

(7.7.4*)
$$\psi_{g,1}\mathcal{M} \xrightarrow{\operatorname{can} = \mathbb{N}} \operatorname{Im} \mathbb{N}.$$

Proposition 7.7.5. Let \mathscr{M} be as in Proposition 7.7.2. The following properties are equivalent:

(1) $\phi_{g,1}\mathcal{M} = \operatorname{Im} \operatorname{can} \oplus \operatorname{Ker} \operatorname{var},$

(2) $\mathcal{M} = \mathcal{M}' \oplus \mathcal{M}''$ with $\mathcal{M}', \mathcal{M}''$ strictly \mathbb{R} -specializable along $(g), \mathcal{M}'$ being a minimal extension along (g) and \mathcal{M}'' supported on $g^{-1}(0)$.

Moreover, if \mathcal{M}, \mathcal{N} satisfy these properties, any morphism $\varphi : \mathcal{M} \to \mathcal{N}$ decomposes correspondingly.

Proof of Propositions 7.7.2 and 7.7.5. All along this proof, we set $\mathcal{N} = {}_{\mathrm{D}}\iota_{g*}\mathcal{M}$ for short.

7.7.2(1) (1a) \Leftrightarrow (1b): It is enough to show that the morphisms

$$\operatorname{Ker}[t:\mathcal{N}\to\mathcal{N}] \xrightarrow{\operatorname{Ker}[t:\operatorname{gr}_{0}^{V}\mathcal{N}\to\operatorname{gr}_{-1}^{V}\mathcal{N}]} \operatorname{Ker}[t:\operatorname{gr}_{0}^{V}\mathcal{N}\to\operatorname{gr}_{-1}^{V}\mathcal{N}]$$

are isomorphisms. It is clear for the right one, since $t : V^{<0} \mathcal{N} \to V^{<-1} \mathcal{N}$ is an isomorphism, according to 7.3.31(a). For the left one this follows from the fact that t is injective on $\operatorname{gr}_{\alpha}^{V} \mathcal{N}$ for $\alpha \neq 0$ according to 7.3.31(c).

(1b) \Leftrightarrow (1c): let us check \Leftarrow (the other implication is clear). Let \mathscr{T} denote the *t*-torsion submodule of \mathscr{N} and \mathscr{T}' the $\widetilde{\mathscr{D}}_{X\times\mathbb{C}}$ -submodule generated by

$$\mathscr{T}_0 := \operatorname{Ker}[t : \mathscr{N} \longrightarrow \mathscr{N}].$$

Assertion 7.7.6. \mathscr{T}' is strictly \mathbb{R} -specializable and the inclusion $\mathscr{T}' \hookrightarrow \mathscr{N}$ is strictly \mathbb{R} -specializable.

This assertion gives the implication \Leftarrow because Assumption (1c) implies $\mathscr{T}' = 0$, hence $t : \mathscr{N} \to \mathscr{N}$ is injective, so $\mathscr{T} = 0$.

Proof of the assertion. Let us show first that \mathscr{T}' is $\widetilde{\mathscr{D}}_{X\times\mathbb{C}}$ -coherent. As we remarked above, we have $\mathscr{T}_0 = \operatorname{Ker}[t : \operatorname{gr}_0^V \mathscr{N} \to \operatorname{gr}_{-1}^V \mathscr{N}]$. Now, \mathscr{T}_0 is the kernel of a linear morphism between $\widetilde{\mathscr{D}}_H$ -coherent modules $(H = X \times \{0\})$, hence is also $\widetilde{\mathscr{D}}_H$ -coherent. It follows that \mathscr{T}' is $\widetilde{\mathscr{D}}_{X\times\mathbb{C}}$ -coherent.

Let us now show that \mathscr{T}' is strictly \mathbb{R} -specializable. We note that \mathscr{T}_0 is strict because it is isomorphic to a submodule of $\operatorname{gr}_0^V \mathscr{N}$. Let $U_{\bullet} \mathscr{T}'$ be the filtration induced by $V_{\bullet} \mathscr{N}$ on \mathscr{T}' . Then $U_{<0} \mathscr{T}' = 0$, according to 7.3.31(a), and $\operatorname{gr}_{\alpha}^U \mathscr{T}' = 0$ for $\alpha \notin \mathbb{N}$. Let us show by induction on k that

$$U_k \mathscr{T}' = \mathscr{T}_0 + \mathscr{T}_0 \eth_t + \dots + \mathscr{T}_0 \eth_t^k.$$

Let us denote by $U'_k \mathscr{T}'$ the right-hand term. The inclusion \supset is clear. Let $x_o \in H$, $m \in U_k \mathscr{T}'_{x_o}$ and let $\ell \ge k$ such that $m \in U'_\ell \mathscr{T}'_{x_o}$. If $\ell > k$ one has $m \in \mathscr{T}'_{x_o} \cap V_{\ell-1} \mathscr{N}_{x_o}$ hence $mt^\ell \in \mathscr{T}'_{x_o} \cap V_{-1} \mathscr{N}_{x_o} = 0$. Set

$$m = m_0 + m_1 \eth_t + \dots + m_\ell \eth_t^\ell$$

with $m_j t = 0$ $(j = 0, ..., \ell)$. One then has $m_\ell \partial_t^\ell t^\ell = 0$, and since

$$m_{\ell}\eth_{t}^{\ell}t^{\ell} = m_{\ell} \cdot \prod_{j=1}^{\ell} (t\eth_{t} + jz) = \ell! \, m_{\ell}z^{\ell}$$

and \mathscr{T}_0 is strict, one concludes that $m_\ell = 0$, hence $m \in U'_{\ell-1}\mathscr{T}'_{x_o}$. By induction, this implies the other inclusion.

As $\operatorname{gr}_{\alpha}^{U} \mathscr{T}'$ is contained in $\operatorname{gr}_{\alpha}^{V} \mathscr{N}$, one deduces from 7.3.31(d) that $\eth_{t} : \operatorname{gr}_{k}^{U} \mathscr{T}' \to \operatorname{gr}_{k+1}^{U} \mathscr{T}'$ is injective for $k \ge 0$. The previous computation shows that it is onto, hence \mathscr{T}' is strictly \mathbb{R} -specializable and $U_{\bullet} \mathscr{T}'$ is its Malgrange-Kashiwara filtration.

It is now enough to prove that the injective morphism $\operatorname{gr}_0^U \mathscr{T}' \to \operatorname{gr}_0^V \mathscr{N}$ is strict. But its cokernel is identified with the submodule $\operatorname{Im}[t:\operatorname{gr}_0^V \mathscr{N} \to \operatorname{gr}_{-1}^V \mathscr{N}]$ of $\operatorname{gr}_{-1}^V \mathscr{N}$, which is strict.

7.7.2(2) If can is onto, then $\mathscr{N} = \widetilde{\mathscr{D}}_{X \times \mathbb{C}} \cdot V_{<0} \mathscr{N}$. If \mathscr{N} has a *t*-torsion quotient \mathscr{T} satisfying 7.3.25(1), then $V_{<0} \mathscr{T} = 0$, so $V_{<0} \mathscr{N}$ is contained in $\operatorname{Ker}[\mathscr{N} \to \mathscr{T}]$ and thus $\mathscr{N} = \widetilde{\mathscr{D}}_{X \times \widetilde{\mathbb{C}}} \cdot V_{<0} \mathscr{N}$ is also contained in this kernel, that is, $\mathscr{T} = 0$.

 $7.7.5(1) \Rightarrow 7.7.5(2)$ Set

 $U_0 \mathscr{N}' = V_{<0} \mathscr{N} + \eth_t V_{-1} \mathscr{N} \quad \text{and} \quad \mathscr{T}_0 = \operatorname{Ker}[t: \mathscr{N} \longrightarrow \mathscr{N}].$

The assumption (1) is equivalent to $V_0 \mathscr{N} = U_0 \mathscr{N}' \oplus \mathscr{T}_0$. Define

$$U_k \mathcal{N}' = V_k \widehat{\mathscr{D}}_X \cdot U_0 \mathcal{N}' \quad \text{and} \quad U_k \mathcal{N}'' = V_k \widehat{\mathscr{D}}_X \cdot \mathscr{T}_0$$

for $k \ge 0$. As $V_k \mathscr{N} = V_{k-1} \mathscr{N} + \eth_t V_{k-1} \mathscr{N}$ for $k \ge 1$, one has $V_k \mathscr{N} = U_k \mathscr{N}' + U_k \mathscr{N}''$ for $k \ge 0$. Let us show by induction on $k \ge 0$ that this sum is direct. Fix $k \ge 1$ and let $m \in U_k \mathscr{N}' \cap U_k \mathscr{N}''$. Write

$$m = m'_{k-1} + n'_{k-1} \eth_t = m''_{k-1} + n''_{k-1} \eth_t, \quad \begin{cases} m'_{k-1}, n'_{k-1} \in U_{k-1}\mathcal{N}', \\ m''_{k-1}, n''_{k-1} \in U_{k-1}\mathcal{N}''. \end{cases}$$

One has $[n'_{k-1}] \eth_t = [n''_{k-1}] \eth_t$ in $V_k \mathscr{N} / V_{k-1} \mathscr{N}$, hence, as

$$\eth_t: V_{k-1}\mathscr{N}/V_{k-2}\mathscr{N} \longrightarrow V_k\mathscr{N}/V_{k-1}\mathscr{N}$$

is bijective for $k \ge 1$, one gets $[n'_{k-1}] = [n''_{k-1}]$ in $V_{k-1}\mathcal{N}/V_{k-2}\mathcal{N}$ and by induction one deduces that both terms are zero. One concludes that $m \in U_{k-1}\mathcal{N}' \cap U_{k-1}\mathcal{N}'' = 0$ by induction.

This implies that $\mathscr{N} = \mathscr{N}' \oplus \mathscr{N}''$ with $\mathscr{N}' := \bigcup_k U_k \mathscr{N}'$ and \mathscr{N}'' defined similarly. It follows from Exercise 7.3.37(1) that both \mathscr{N}' and \mathscr{N}'' are strictly \mathbb{R} -specializable along H and the filtrations U_{\bullet} above are their Malgrange-Kashiwara filtrations. In particular \mathscr{N}' satisfies (1) and (2). By Corollary 7.6.5 we also know that $\mathscr{N}' = {}_{\mathsf{D}}\iota_{g*}\mathscr{M}'$ and $\mathscr{N}'' = {}_{\mathsf{D}}\iota_{g*}\mathscr{M}''$ for some coherent $\widetilde{\mathscr{D}}_X$ -modules $\mathscr{M}', \mathscr{M}''$.

 $7.7.5(2) \Rightarrow 7.7.5(1)$: One has $V_{<0}\mathcal{N}'' = 0$. Let us show that $\operatorname{Im} \operatorname{can} = \operatorname{gr}_0^V \mathcal{N}'$ and $\operatorname{Ker} \operatorname{var} = \operatorname{gr}_0^V \mathcal{N}''$. The inclusions $\operatorname{Im} \operatorname{can} \subset \operatorname{gr}_0^V \mathcal{N}'$ and $\operatorname{Ker} \operatorname{var} \supset \operatorname{gr}_0^V \mathcal{N}''$ are clear. Moreover $\operatorname{gr}_0^V \mathcal{N}' \cap \operatorname{Ker} \operatorname{var} = 0$ as \mathcal{N}' satisfies (1). Last, $\operatorname{can} : \operatorname{gr}_{-1}^V \mathcal{N}' \to \operatorname{gr}_0^V \mathcal{N}'$ is onto, as \mathcal{N}' satisfies (2). Hence $\operatorname{gr}_0^V \mathcal{N} = \operatorname{Im} \operatorname{can} \oplus \operatorname{Ker} \operatorname{var}$.

Let us now prove the last assertion. Let us consider a morphism $\varphi : \mathcal{M}' \oplus \mathcal{M}'' \to \mathcal{N}' \oplus \mathcal{N}''$. Firstly, by (1b) in Proposition 7.7.2, the component $\mathcal{M}'' \to \mathcal{N}'$ is zero. For the component $\mathcal{M}' \to \mathcal{N}''$, let us denote by \mathcal{M}'_1 its image. The V-filtration on ${}_{\mathrm{D}}\iota_{g*}\mathcal{M}'_1$ induced by $V_{\bullet\mathrm{D}}\iota_{g*}\mathcal{N}''$ is coherent (Exercise 7.3.7(1)) and satisfies 7.3.25(1), hence by Proposition 7.7.2(2) we have ${}_{\mathrm{D}}\iota_{g*}\mathcal{M}'_1 = 0$.

Definition 7.7.7 (Strictly S(upport)-decomposable $\widetilde{\mathscr{D}}_X$ -modules)

We say that a coherent \mathscr{D}_X -module \mathscr{M} is

• strictly S-decomposable along (g) if it is strictly \mathbb{R} -specializable along (g) and satisfies the equivalent conditions 7.7.5;

• strictly S-decomposable at $x_o \in X$ if for any analytic germ $g: (X, x_o) \to (\mathbb{C}, 0)$, \mathscr{M} is strictly S-decomposable along (g) in some neighbourhood of x_o ;

• strictly S-decomposable if it is strictly S-decomposable at all points $x_o \in X$.

Lemma 7.7.8.

(1) If \mathscr{M} is strictly S-decomposable along $\{t = 0\}$, then it is strictly S-decomposable along $\{t^r = 0\}$ for every $r \ge 1$.

(2) If $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, then \mathcal{M} is strictly S-decomposable of some kind if and only if \mathcal{M}_1 and \mathcal{M}_2 are so.

(3) We assume that \mathscr{M} is strictly S-decomposable and its support Z is a pure dimensional closed analytic subset of X. Then the following conditions are equivalent:

(a) for any analytic germ $g : (X, x_o) \to (\mathbb{C}, 0)$ such that $g^{-1}(0) \cap Z$ has everywhere codimension one in Z, ${}_{\mathsf{D}}\iota_{q*}\mathscr{M}$ is a minimal extension along (g);

(b) near any x_o , there is no $\widehat{\mathscr{D}}_X$ -coherent submodule of \mathscr{M} with support having codimension ≥ 1 in Z;

(c) near any x_o , there is no nonzero morphism $\varphi : \mathcal{M} \to \mathcal{N}$, with \mathcal{N} strictly S-decomposable at x_o , such that $\operatorname{Im} \varphi$ is supported in codimension ≥ 1 in Z.

Proof. Property (1) is an immediate consequence of Exercise 7.4.16, and property (2) follows from the fact that for any germ g, the corresponding can and var decompose with respect to the given decomposition of \mathscr{M} . Let us now prove (3). Both conditions (3a) and (3b) reduce to the property that, for any analytic germ $g: (X, x_o) \to (\mathbb{C}, 0)$ which does not vanish identically on any local irreducible component of Z at x_o , the corresponding decomposition $\mathscr{M} = \mathscr{M}' \oplus \mathscr{M}''$ of 7.7.5(2) reduces to $\mathscr{M} = \mathscr{M}'$, i.e., $\mathscr{M}'' = 0$. For the equivalence with (3c), we note that, according to the last assertion in Proposition 7.7.5, and with respect to the decomposition $\varphi = \varphi' \oplus \varphi''$ along a germ g, we have $\operatorname{Im} \varphi \neq 0$ and supported in $g^{-1}(0)$ if and only if $\operatorname{Im} \varphi'' \neq 0$, and thus $\mathscr{M}'' \neq 0$. Conversely, if $\mathscr{M}'' \neq 0$, the projection $\mathscr{M} \to \mathscr{M}''$ gives a morphism contradicting (3c).

Definition 7.7.9 (Pure support). Let \mathscr{M} be strictly S-decomposable and having support a pure dimensional closed analytic subset Z of X. We say that \mathscr{M} has pure support Z if the equivalent conditions of 7.7.8(3) are satisfied.

Proposition 7.7.10 (Generic structure of a strictly S-decomposable module)

Assume that \mathscr{M} is holonomic and strictly S-decomposable with pure support Z, where Z is smooth. Then there exists a unique holonomic and strictly S-decomposable $\widetilde{\mathscr{D}}_Z$ -module \mathscr{N} such that $\mathscr{M} = {}_{\mathrm{D}}\iota_{Z*}\mathscr{N}$. Moreover, there exists a Zariski dense open subset $Z^{\circ} \subset Z$ such that $\mathscr{N}_{|Z^{\circ}}$ is $\widetilde{\mathscr{O}}_{Z^{\circ}}$ -locally free of finite rank.

Proof. By uniqueness, the question is local on Z. We argue by induction on dim X. Let H be a smooth hypersurface containing Z such that $H = \{t = 0\}$ of some local coordinate system (t, x_2, \ldots, x_d) . Since \mathscr{M} is strictly \mathbb{R} -specializable along t, the strict Kashiwara's equivalence implies that $\mathscr{M} = {}_{\square} \iota_{H*} \mathscr{N}$ for a unique coherent $\widetilde{\mathscr{D}}_H$ -module \mathscr{N} . Moreover, \mathscr{N} is strictly \mathbb{R} -specializable along any function g on H, according to Proposition 7.6.6. If $g = g_{|H}$, then one checks that a decomposition 7.7.5(2) for \mathscr{M} along g comes from a decomposition 7.7.5(2) for \mathscr{N} along g. We conclude that \mathscr{N} is also strictly S-decomposable, and has pure support $Z \subset H$. Continuing this way, we reach a coherent $\widetilde{\mathscr{D}}_Z$ -module which is strictly S-decomposable. It is easy to check that \mathscr{N} is holonomic since, if Char \mathscr{M} is obtain by a straightforward formula from Char \mathscr{N} . By deleting from Z the projection of all components of Char \mathscr{N} except the zero section, we obtain a Zariski-dense open subset Z^o of Z such that Char $\mathscr{N}_{|Z^o} \subset T_Z^*Z \times \mathbb{C}_Z$. We conclude from Exercise A.10.16 that $\mathscr{N}_{|Z^o}$ is $\widetilde{\mathscr{O}}_{Z^o}$ -coherent. Let us now prove the \mathcal{O}_{Z^o} -local freeness of $\mathcal{N}_{|Z^o}$. If t is a local coordinate, notice that each term of the V-filtration $V_{\bullet}\mathcal{N}$ is \mathcal{O}_{Z^o} -coherent. It follows that the filtration is locally stationary, hence $\mathcal{N} = V_0\mathcal{N}$, since $\operatorname{gr}_{\alpha}^V\mathcal{N} = 0$ for $\alpha \gg 0$, hence for all $\alpha > 0$. Let m be a local section of \mathcal{N} killed by z. Then m is zero in $\mathcal{N}/\mathcal{N}t$ by strict \mathbb{R} -specializability. As \mathcal{N} is \mathcal{O}_{Z^o} -coherent, Nakayama's lemma (applied to $\mathcal{N} \otimes_{\mathcal{O}_{Z^o}} \mathcal{O}_{Z^o \times \mathbb{C}_z}$) implies that m = 0.

We will now show that a strictly S-decomposable holonomic \mathscr{D}_X -module (see Definition A.10.18) can indeed be decomposed as the direct sum of holonomic $\widetilde{\mathscr{D}}_X$ -modules having pure support. We first consider the local decomposition and, by uniqueness, we get the global one. It is important for that to be able to define a priori the strict components. They are obtained from the characteristic variety of \mathscr{M} , equivalently of \mathscr{M} , according to Corollary 7.7.15 below.

Proposition 7.7.11. Let \mathscr{M} be holonomic and strictly S-decomposable at x_o , and let $(Z_i, x_o)_{i \in I}$ be the family of closed irreducible analytic germs (Z_i, x_o) such that Char $\mathscr{M} = \bigcup_i T^*_{Z_i} X \times \mathbb{C}_z$ near x_o . There exists a unique decomposition $\mathscr{M}_{x_o} = \bigoplus_{i \in I} \mathscr{M}_{Z_i, x_o}$ of germs at x_o such that $\mathscr{M}_{Z_i, x_o} = 0$ or has pure support (Z_i, x_o) .

Proof. We will argue by induction on dim Supp \mathscr{M} . First, we reduce to the case when the support S of \mathscr{M} (see Proposition A.10.13) is irreducible at x_o . For this purpose, let S' be an irreducible component of S at x_o of maximal dimension, and let S'' be the union of all the other ones. Let $g: (X, x_o) \to (\mathbb{C}, 0)$ be an analytic germ such that $S'' \subset g^{-1}(0)$ and $(S', x_o) \not\subset g^{-1}(0)$. Then, according to 7.7.5(2), near x_o , \mathscr{M} has a decomposition $\mathscr{M} = \mathscr{M}' \oplus \mathscr{M}''$, with \mathscr{M}' supported on S' and being a minimal extension along (g), and \mathscr{M}'' supported on S''.

Conversely, if we have any local decomposition $\mathcal{M} = \oplus \mathcal{M}_{S_i}$, with (S_i, x_o) irreducible and \mathcal{M}_{S_i} (strictly S-decomposable after Lemma 7.7.8(2)) having pure support S_i , then $S_i \subset S'$ or $S_i \subset S''$ and $\mathcal{M}' = \bigoplus_{S_i \not\subset S''} \mathcal{M}_{S_i}$, $\mathcal{M}'' = \bigoplus_{S_i \subset S''} \mathcal{M}_{S_i}$.

By induction on the number of irreducible components, we are reduced to the case when (S, x_o) is irreducible. We can assume that dim S > 0.

Choose now a germ $g: (X, x_o) \to (\mathbb{C}, 0)$ which is nonconstant on S and such that $g^{-1}(0)$ contains all components Z_i defined by Char \mathscr{M} , except S. We have, as above, a unique decomposition $\mathscr{M} = \mathscr{M}' \oplus \mathscr{M}''$ of germs at x_o , where \mathscr{M}' is a minimal extension along (g), and \mathscr{M}'' is supported on $g^{-1}(0)$, by the assumption of strict S-decomposability along (g) at x_o . Moreover, \mathscr{M}' and \mathscr{M}'' are also strictly S-decomposable at x_o . We can apply the inductive assumption to \mathscr{M}'' .

Let us show that \mathscr{M}' has pure support S near x_o : if \mathscr{M}'_1 is a coherent submodule of \mathscr{M}' supported on a strict analytic subset $Z \subset S$, then Z is contained in the union of the components Z_i , hence \mathscr{M}'_1 is supported in $g^{-1}(0)$, so is zero. We conclude by 7.7.8(3b).

For the uniqueness, we note that, given such a local decomposition with components \mathscr{M}_{Z_i,x_o} , the components φ_{ij} of any morphism $\varphi : \mathscr{M}_{x_o} \to \mathscr{M}_{x_o}$ vanishes as soon as

 $i \neq j$. Indeed, we have either $\operatorname{codim}_{Z_i}(Z_i \cap Z_j) \geq 1$, or $\operatorname{codim}_{Z_j}(Z_i \cap Z_j)$. In the first case we apply Lemma 7.7.8(3c) to \mathscr{M}_{Z_i,x_o} . In the second case, we apply Lemma 7.7.8(3b) to \mathscr{M}_{Z_j,x_o} . We apply the same result for the identity $\mathscr{M} \to \mathscr{M}$ with respect to two such local decompositions.

By uniqueness of the local decomposition, we get:

Corollary 7.7.12. Let \mathscr{M} be holonomic and strictly S-decomposable on X and let $(Z_i)_{i\in I}$ be the (locally finite) family of closed irreducible analytic subsets Z_i such that Char $\mathscr{M} \subset \bigcup_i T_{Z_i}^* X \times \mathbb{C}_z$. There exists a unique decomposition $\mathscr{M} = \bigoplus_i \mathscr{M}_{Z_i}$ such that each $\mathscr{M}_{Z_i} = 0$ or has pure support Z_i .

A closed analytic irreducible subset Z of X such that $\mathcal{M}_Z \neq 0$ is called a *strict* component of \mathcal{M} .

Proof of Corollary 7.7.12. Given the family $(Z_i)_{i \in I}$ and $x_o \in X$, the germs (Z_i, x_o) are equidimensional, and Proposition 7.7.11 gives a unique decomposition $\mathcal{M}_{x_o} = \bigoplus_{i \in I} \mathcal{M}_{Z_i, x_o}$ by gathering the local irreducible components of (Z_i, x_o) . The uniqueness enables us to glue all along Z_i the various germs \mathcal{M}_{Z_i, x_o} .

Corollary 7.7.13. Let $\mathscr{M}', \mathscr{M}''$ be two holonomic $\widetilde{\mathscr{D}}_X$ -module which are strictly S-decomposable and let $(Z_i)_{i \in I}$ be the family of their strict components. Then any morphism $\varphi : \mathscr{M}'_{Z_i} \to \mathscr{M}''_{Z_i}$ vanishes identically if $Z_i \neq Z_j$.

Proof. The image of φ is supported on $Z_i \cap Z_j$, which has everywhere codimension ≥ 1 in Z_i or Z_j if $Z_i \neq Z_j$. We then apply Lemma 7.7.8.

Corollary 7.7.14. Let *M* be holonomic and strictly S-decomposable. Then *M* is strict.

Proof. The question is local, and we can assume that \mathscr{M} has pure support Z with Z closed irreducible analytic near x_o . Proposition 7.7.10 applied to the smooth part of Z produces a dense open subset Z^o of Z such that $\mathscr{M}_{|Z^o}$ is strict. (In fact, since Z^o was defined in terms of the characteristic variety, one can show that it is Zariski open in Z, but this will not matter here.) Let m be a local section of \mathscr{M} near $x_o \in Z$ killed by z. Then $m \cdot \widetilde{\mathscr{D}}_X$ is supported by a proper analytic subset of Z near x_o by the previous argument. As \mathscr{M} has pure support Z, we conclude that m = 0.

Corollary 7.7.15. Let \mathscr{M} be holonomic and strictly S-decomposable. Then $\operatorname{Char} \mathscr{M} = \operatorname{Char} (\mathscr{M}/(z-1)\mathscr{M}) \times \mathbb{C}_z$.

Proof. Since \mathscr{M} is strict, we can apply Exercise A.10.23(1).

Remark 7.7.16 (Restriction to z = 1). Let us keep the notation of Exercise 7.3.21 and let us assume that \mathscr{M} is $R_F \mathscr{D}_X$ -coherent and strictly \mathbb{R} -specializable. It is obvious that, if can is onto for \mathscr{M} , it is also onto for $\mathcal{M} := \mathscr{M}/\mathscr{M}(z-1)$. On the other hand, it is also true that, if var in injective for \mathscr{M} , it is also injective for \mathscr{M} (see Exercise A.2.5(3)). As a consequence, if \mathscr{M} is a minimal extension along (g), so is \mathscr{M} . Moreover, if \mathscr{M} is strictly S-decomposable along (g) at x_o , so is \mathscr{M} , and the strict decomposition $\mathscr{M} = \mathscr{M}' \oplus \mathscr{M}''$ restricts to the decomposition $\mathscr{M} = \mathscr{M}' \oplus \mathscr{M}''$ given by 7.7.5(2).

We conclude that, if \mathscr{M} is strictly S-decomposable, then \mathscr{M} is S-decomposable, and the strict components, together with the pure support, are in one-to-one correspondence.

7.8. Direct image of strictly \mathbb{R} -specializable coherent $\widetilde{\mathscr{D}}_X$ -modules

Let us consider the setting of Theorem A.10.26. So $f : X \to X'$ is a proper holomorphic map, and \mathscr{M} is a coherent right $\widetilde{\mathscr{D}}_X$ -module. Let $H' \subset X'$ be a smooth hypersurface. We will assume that $H := f^*(H')$ is also a smooth hypersurface of X.

If \mathscr{M} has a coherent V-filtration $U_{\bullet}\mathscr{M}$ along H, the $R_V \widehat{\mathscr{D}}_X$ -module $R_U \mathscr{M}$ is therefore coherent. With the assumptions above it is possible to define a sheaf $R_V \widetilde{\mathscr{D}}_{X \to X'}$ and therefore the pushforward ${}_{\mathrm{D}}f_*R_U \mathscr{M}$ as an $R_V \widetilde{\mathscr{D}}_{X'}$ -module (where $V_{\bullet} \widetilde{\mathscr{D}}_{X'}$ is the V-filtration relative to H').

We will show the $R_V \mathscr{D}_{X'}$ -coherence of the cohomology sheaves $\mathscr{H}^k_{\ D} f_* R_U \mathscr{M}$ of the pushforward ${}_{D} f_* R_U \mathscr{M}$ if \mathscr{M} is endowed with a coherent filtration. By the argument of Exercise 7.3.6, by quotienting by the *v*-torsion, we obtain a coherent *V*-filtration on the cohomology sheaves $\mathscr{H}^k_{\ D} f_* \mathscr{M}$ of the pushforward ${}_{D} f_* \mathscr{M}$.

The v-torsion part contains much information however, and this supplementary operation killing the v-torsion looses it. The main result of this section is that, if \mathscr{M} is strictly \mathbb{R} -specializable along H, then so are the cohomology sheaves $\mathscr{H}^k_{\ D} f_*\mathscr{M}$ along H', and moreover, when considering the filtration by the order, the corresponding Rees modules $\mathscr{H}^k_{\ D} f_* \mathscr{R}_V \mathscr{M}$ have no v-torsion, and can thus be written as $R_U \mathscr{H}^k_{\ D} f_* \mathscr{M}$ for some coherent V-filtration $U_{\bullet} \mathscr{H}^k_{\ D} f_* \mathscr{M}$. This coherent V-filtration is nothing but the Kashiwara-Malgrange filtration of $\mathscr{H}^k_{\ D} f_* \mathscr{M}$. We say that the Kashiwara-Malgrange filtration behaves *strictly* with respect to the pushforward functor ${}_{\mathrm{D}} f_*$.

Another way to present this property is to consider the individual terms $V_{\alpha}\mathcal{M}$ of the Kashiwara-Malgrange filtration as $V_0 \widetilde{\mathscr{D}}_X$ -modules. By introducing the sheaf $V_0 \widetilde{\mathscr{D}}_{X \to X'}$, one can define the pushforward complex ${}_{\mathrm{D}} f_* V_{\alpha} \mathcal{M}$ for every α , and the strictness property amounts to saying that for every k and α , the morphisms $\mathcal{H}^k{}_{\mathrm{D}} f_* V_{\alpha} \mathcal{M} \to \mathcal{H}^k{}_{\mathrm{D}} f_* \mathcal{M}$ are *injective*. In the following, we work with right $\widetilde{\mathscr{D}}_X$ -modules and increasing V-filtrations.

7.8.a. Definition of the pushforward functor and the Coherence Theorem

We first note that our assumption on H, H', f is equivalent to the property that, locally at $x_o \in H$, setting $x'_o = f(x_o)$, there exist local decompositions $(X, x_o) \simeq (H, x_o) \times (\mathbb{C}, 0)$ and $(X', x'_o) \simeq (H', x'_o) \times (\mathbb{C}, 0)$ such that $f(y, t) = (f_{|H}(y), t)$.

Let $U_{\bullet}\mathscr{M}$ be a V-filtration of \mathscr{M} and let $R_U\mathscr{M}$ be the associated graded $R_V \mathscr{D}_X$ module. Our first objective is to apply the same reasoning as in Theorem A.10.26 by replacing the category of $\widetilde{\mathscr{D}}$ -modules with that of graded $R_V \widetilde{\mathscr{D}}_X$ -modules. The sheaf $\widetilde{\mathscr{D}}_{X \to X'}$ has a V-filtration: we set $V_k \widetilde{\mathscr{D}}_{X \to X'} := \widetilde{\mathscr{O}}_X \otimes_{f^{-1} \widetilde{\mathscr{O}}_{X'}} f^{-1} V_k \widetilde{\mathscr{D}}_{X'}$. One checks in local decompositions as above that, with respect to the left $\widetilde{\mathscr{D}}_X$ -structure one has $V_\ell \widetilde{\mathscr{D}}_X \cdot V_k \widetilde{\mathscr{D}}_{X \to X'} \subset V_{k+\ell} \widetilde{\mathscr{D}}_{X \to X'}$. We can write

$$R_V \widetilde{\mathscr{D}}_{X \to X'} := \widetilde{\mathscr{O}}_X \otimes_{f^{-1} \widetilde{\mathscr{O}}_{X'}} f^{-1} R_V \widetilde{\mathscr{D}}_{X'} = R_V \widetilde{\mathscr{O}}_X \otimes_{f^{-1} R_V \widetilde{\mathscr{O}}_{X'}} f^{-1} R_V \widetilde{\mathscr{D}}_{X'}.$$

According to Exercise 7.2.6, $R_V \widetilde{\mathscr{D}}_{X'}$ is $R_V \widetilde{\mathscr{O}}_{X'}$ -locally free, so $R_V \widetilde{\mathscr{D}}_{X \to X'}$ is $R_V \widetilde{\mathscr{O}}_{X}$ -locally free.

We define

(7.8.1)
$${}_{\mathrm{D}}f_*R_U\mathscr{M} := \mathbf{R}f_* \big(R_U\mathscr{M} \otimes^{\mathbf{L}}_{R_V\widetilde{\mathscr{D}}_X} R_V\widetilde{\mathscr{D}}_{X \to X'} \big)$$

as an object of $\mathsf{D}^{\mathrm{b}}(R_V \widetilde{\mathscr{D}}_{X'})$.

Theorem 7.8.2. Let \mathscr{M} be a $\widetilde{\mathscr{D}}_X$ -module endowed with a coherent filtration $F_{\bullet}\mathscr{M}$. Let $U_{\bullet}\mathscr{M}$ be a coherent V-filtration of \mathscr{M} . Then the cohomology modules of ${}_{\mathrm{D}}f_*R_U\mathscr{M}$ have coherent $R_V\widetilde{\mathscr{D}}_{X'}$ -cohomology.

Lemma 7.8.3. Let \mathscr{L} be an $R_V \widetilde{\mathscr{O}}_X$ -module. Then

$$(\mathscr{L} \otimes_{R_V \widetilde{\mathscr{O}}_X} R_V \widetilde{\mathscr{D}}_X) \otimes_{R_V \widetilde{\mathscr{D}}_X}^{\mathbf{L}} R_V \widetilde{\mathscr{D}}_{X \to X'} = \mathscr{L} \otimes_{f^{-1} R_V \widetilde{\mathscr{O}}_{X'}} f^{-1} R_V \widetilde{\mathscr{D}}_{X'}.$$

Proof. It is a matter of proving that the left-hand side has cohomology in degree 0 only, since this cohomology is easily seen to be equal to the right-hand side. This can be checked on germs at $x \in X$. Let \mathscr{L}_x^{\bullet} be a resolution of \mathscr{L}_x by free $R_V \widetilde{\mathscr{O}}_{X,x}$ -modules. We have

$$\begin{split} (\mathscr{L}_{x}\otimes_{R_{V}\widetilde{\mathscr{O}}_{X,x}}R_{V}\widetilde{\mathscr{D}}_{X,x})\otimes_{R_{V}\widetilde{\mathscr{D}}_{X,x}}^{L}R_{V}\widetilde{\mathscr{D}}_{X\to X',x} \\ &= (\mathscr{L}_{x}\otimes_{R_{V}\widetilde{\mathscr{O}}_{X,x}}^{L}R_{V}\widetilde{\mathscr{D}}_{X,x})\otimes_{R_{V}\widetilde{\mathscr{D}}_{X,x}}^{L}R_{V}\widetilde{\mathscr{D}}_{X\to X',x} \quad (\text{Ex. 7.2.6}) \\ &= (\mathscr{L}_{x}^{\bullet}\otimes_{R_{V}\widetilde{\mathscr{O}}_{X,x}}R_{V}\widetilde{\mathscr{D}}_{X,x})\otimes_{R_{V}\widetilde{\mathscr{D}}_{X,x}}^{L}R_{V}\widetilde{\mathscr{D}}_{X\to X',x} \\ &= (\mathscr{L}_{x}^{\bullet}\otimes_{R_{V}\widetilde{\mathscr{O}}_{X,x}}R_{V}\widetilde{\mathscr{D}}_{X,x})\otimes_{R_{V}\widetilde{\mathscr{D}}_{X,x}}R_{V}\widetilde{\mathscr{D}}_{X\to X',x} \\ &= \mathscr{L}_{x}^{\bullet}\otimes_{R_{V}\widetilde{\mathscr{O}}_{X,x}}R_{V}\widetilde{\mathscr{D}}_{X\to X',x} \\ &= \mathscr{L}_{x}\otimes_{R_{V}\widetilde{\mathscr{O}}_{X,x}}R_{V}\widetilde{\mathscr{D}}_{X\to X',x} \\ &= \mathscr{L}_{x}\otimes_{R_{V}\widetilde{\mathscr{O}}_{X,x}}R_{V}\widetilde{\mathscr{D}}_{X\to X',x} \quad (R_{V}\widetilde{\mathscr{D}}_{X\to X',x} \text{ is } R_{V}\widetilde{\mathscr{O}}_{X,x}\text{-free}). \quad \Box \end{split}$$

As a consequence of this lemma, we have

$${}_{\mathrm{D}}f_*(\mathscr{L}\otimes_{R_V\widetilde{\mathscr{O}}_X}R_V\mathscr{D}_X)=(Rf_*\mathscr{L})\otimes_{R_V\widetilde{\mathscr{O}}_{X'}}R_V\mathscr{D}_{X'}$$

and the cohomology of this complex is $R_V \widetilde{\mathscr{D}}_{X'}$ -coherent.

Lemma 7.8.4. Assume that \mathscr{M} is a $\widetilde{\mathscr{D}}_X$ -module having a coherent filtration $F_{\bullet}\mathscr{M}$ and let $U_{\bullet}\mathscr{M}$ be a coherent V-filtration of \mathscr{M} . Then in the neighbourhood of any compact set of X, $R_U\mathscr{M}$ has a coherent $F_{\bullet}R_V\widetilde{\mathscr{D}}_X$ -filtration.

Proof. Fix a compact set $K \subset X$. We can thus assume that \mathscr{M} is generated by a coherent $\widetilde{\mathscr{O}}_X$ -module \mathscr{F} in some neighbourhood of K, i.e., $\mathscr{M} = \widetilde{\mathscr{D}}_X \cdot \mathscr{F}$. Consider the V-filtration $U'_{\bullet}\mathscr{M}$ generated by \mathscr{F} , i.e., $U'_{\bullet}\mathscr{M} = V_{\bullet}\widetilde{\mathscr{D}}_X \cdot \mathscr{F}$. Then, clearly, $R_V \widetilde{\mathscr{O}}_X \cdot \mathscr{F} = \bigoplus_k V_k \widetilde{\mathscr{O}}_X \cdot \mathscr{F} v^k$ is a coherent graded $R_V \widetilde{\mathscr{O}}_X$ -module which generates $R_{U'}\mathscr{M}$ as an $R_V \widetilde{\mathscr{D}}_X$ -module.

If the filtration $U''_{\bullet}\mathcal{M}$ is obtained from $U'_{\bullet}\mathcal{M}$ by a shift by $-\ell \in \mathbb{Z}$, i.e., if $R_{U''}\mathcal{M} = v^{\ell}R_{U'}\mathcal{M} \subset \mathcal{M}[v, v^{-1}]$, then $R_{U''}\mathcal{M}$ is generated by the $R_V \widetilde{\mathcal{O}}_X$ -coherent submodule $v^{\ell}R_V \widetilde{\mathcal{O}}_X \cdot \mathscr{F}$.

On the other hand, let $U''_{\bullet}\mathcal{M}$ be a coherent V-filtration such that $R_{U''}\mathcal{M}$ has a coherent $F_{\bullet}R_V\widetilde{\mathcal{D}}_X$ -filtration. Then any coherent V-filtration $U_{\bullet}\mathcal{M}$ such that $U_k\mathcal{M} \subset U''_k\mathcal{M}$ for every k satisfies the same property, because $R_U\mathcal{M}$ is thus a coherent graded $R_V\widetilde{\mathcal{D}}_X$ -submodule of $R_{U''}\mathcal{M}$, so a coherent filtration on the latter induces a coherent filtration on the former.

As any coherent V-filtration $U_{\bullet}\mathcal{M}$ is contained, in some neighbourhood of K, in the coherent V-filtration $U'_{\bullet}\mathcal{M}$ suitably shifted, we get the lemma.

Proof of Theorem 7.8.2. The proof now ends exactly as for Theorem A.10.26. \Box

7.8.b. Strictness of the Kashiwara-Malgrange filtration by pushforward *Theorem 7.8.5* (Pushforward of strictly \mathbb{R} -specializable $\widetilde{\mathscr{D}}$ -modules)

Let $f: X \to X'$ be a proper morphism of complex manifolds, let H' be a smooth hypersurface of X' and assume that $\mathscr{I}_H := \mathscr{I}_{H'} \mathscr{O}_X$ defines a smooth hypersurface Hof X. Let \mathscr{M} be a coherent right $\widetilde{\mathscr{D}}_X$ -module equipped with a coherent filtration. Assume that \mathscr{M} is strictly \mathbb{R} -specializable along H with Kashiwara-Malgrange filtration $V_{\bullet}\mathscr{M}$ indexed by $A + \mathbb{Z}$ with A finite contained in [0, 1), and that each cohomology module $\mathscr{H}^i{}_{{}_{\mathrm{D}}}f_{|H*}\mathrm{gr}^V_{\alpha}\mathscr{M}$ is strict ($\alpha \in [-1, 0]$).

Then each cohomology module $\mathscr{H}^{i}{}_{D}f_{*}\mathscr{M}$, which is $\mathscr{D}_{X'}$ -coherent according to Theorem A.10.26, is strictly \mathbb{R} -specializable along H' and moreover,

- (1) for every α, i , the natural morphism $\mathscr{H}^i{}_{\mathrm{D}}f_*(V_\alpha\mathscr{M}) \to \mathscr{H}^i{}_{\mathrm{D}}f_*\mathscr{M}$ is injective,
- (2) its image is the Kashiwara-Malgrange filtration of $\mathscr{H}^{i}{}_{{}_{\mathrm{D}}}f_{*}\mathscr{M}$ along H',
- (3) for every $\alpha, i, \operatorname{gr}^{V}_{\alpha}(\mathscr{H}^{i}{}_{\mathrm{D}}f_{*}\mathscr{M}) = \mathscr{H}^{i}{}_{\mathrm{D}}f_{|H*}(\operatorname{gr}^{V}_{\alpha}\mathscr{H}^{i}\mathscr{M}).$

As an important corollary we obtain in a straightforward way:

Corollary 7.8.6. Let $f : X \to X'$ be a proper morphism of complex manifolds. Let $g' : X' \to \mathbb{C}$ be any holomorphic function on X' and let \mathscr{M} be $\widetilde{\mathscr{D}}_X$ -coherent and strictly \mathbb{R} -specializable along (g) with $g = g' \circ f$. Assume that for for all i and λ , $\mathscr{H}^i{}_{\mathrm{D}}f_*(\psi_{q,\lambda}\mathscr{M})$ and $\mathscr{H}^i{}_{\mathrm{D}}f_*(\phi_{q,1}\mathscr{M})$ are strict.

Then $\mathscr{H}^i{}_{\mathrm{D}}f_*\mathscr{M}$ is $\widetilde{\mathscr{D}}_{X'}$ -coherent and strictly \mathbb{R} -specializable along (g'), we have for all i and λ ,

$$\begin{split} (\psi_{g,\lambda}(\mathscr{H}^{i}{}_{\mathrm{D}}f_{*}\mathscr{M}),\mathrm{N}) &= \mathscr{H}^{i}{}_{\mathrm{D}}f_{*}(\psi_{g,\lambda}\mathscr{M},\mathrm{N}),\\ (\phi_{g,1}(\mathscr{H}^{i}{}_{\mathrm{D}}f_{*}\mathscr{M}),\mathrm{N}) &= \mathscr{H}^{i}{}_{\mathrm{D}}f_{*}(\phi_{g,1}\mathscr{M},\mathrm{N}), \end{split}$$

and the morphisms can, var for $\mathscr{H}^{i}{}_{\mathrm{D}}f_{*}\mathscr{M}$ are the morphisms $\mathscr{H}^{i}{}_{\mathrm{D}}f_{*}$ can, $\mathscr{H}^{i}{}_{\mathrm{D}}f_{*}$ var. \square

We first explain the mechanism which leads to the strictness property stated in Theorem 7.8.5(1).

Proposition 7.8.7. Let $H' \subset X'$ be a smooth hypersurface. Let $(\mathcal{N}^{\bullet}, U_{\bullet} \mathcal{N}^{\bullet})$ be a V-filtered complex of $\mathcal{D}_{X'}$ -modules, where U_{\bullet} is indexed by $A + \mathbb{Z}$, $A \subset [0,1)$ finite. Let $N \ge 0$ and assume that

(1) $\mathscr{H}^{i}(\operatorname{gr}_{\alpha}^{U}\mathcal{N}^{\bullet})$ is strict for all $\alpha \in A + \mathbb{Z}$ and all $i \geq -N - 1$;

(2) for every $\alpha \in A + \mathbb{Z}$, there exists $\nu_{\alpha} \ge 0$ such that $(E - \alpha z)^{\nu_{\alpha}}$ acts by zero on $\mathscr{H}^{i}(\operatorname{gr}_{\alpha}^{U}\mathcal{N}^{\bullet})$ for every $i \geq -N-1$;

(3) there exists α_o such that for all $\alpha \leq \alpha_o$ and all $i \geq -N-1$, the right multiplication by some (or any) local reduced equation t of H' induces an isomorphism $t: U_{\alpha} \mathcal{N}^i \xrightarrow{\sim} U_{\alpha-1} \mathcal{N}^i;$

- (4) there exists $i_o \in \mathbb{Z}$ such that, for all $i \ge i_o$ and any α , one has $\mathscr{H}^i(U_\alpha \mathcal{N}^{\bullet}) = 0$;
- (5) $\mathscr{H}^{i}(U_{\alpha}\mathscr{N}^{\bullet})$ is $V_{0}\mathscr{D}_{X'}$ -coherent for all $\alpha \in A + \mathbb{Z}$ and all $i \geq -N-1$.

Then for every α and $i \ge -N$ the morphism $\mathscr{H}^i(U_\alpha \mathscr{N}^{\bullet}) \to \mathscr{H}^i(\mathscr{N}^{\bullet})$ is injective. Moreover, the filtration $U_{\bullet}\mathscr{H}^{i}(\mathscr{N}^{\bullet})$ defined by

$$U_{\alpha}\mathscr{H}^{i}(\mathscr{N}^{\bullet}) = \operatorname{image}\left[\mathscr{H}^{i}(U_{\alpha}\mathscr{N}^{\bullet}) \longrightarrow \mathscr{H}^{i}(\mathscr{N}^{\bullet})\right]$$

satisfies $\operatorname{gr}_{\alpha}^{U} \mathscr{H}^{i}(\mathscr{N}^{\bullet}) = \mathscr{H}^{i}(\operatorname{gr}_{\alpha}^{U} \mathscr{N}^{\bullet})$ for all $\alpha \in A + \mathbb{Z}$.

Proof. It will have three steps. During the proof, the indices α, β, γ will run in $A + \mathbb{Z}$. *First step.* This step proves a formal analogue of the conclusion of the proposition. Put

$$\widehat{U_{\alpha}\mathcal{N}^{\bullet}} = \varprojlim_{\gamma} U_{\alpha}\mathcal{N}^{\bullet} / U_{\gamma}\mathcal{N}^{\bullet} \quad \text{and} \quad \widehat{\mathcal{N}^{\bullet}} = \varinjlim_{\alpha} \widehat{U_{\alpha}\mathcal{N}^{\bullet}}.$$

Under the assumption of Proposition 7.8.7, we will prove the following:

(a) For all $\beta \leq \alpha$, $\widehat{U_{\alpha}\mathcal{N}^{\bullet}} \to \widehat{U_{\alpha}\mathcal{N}^{\bullet}}$ is injective (hence, for all α , $\widehat{U_{\alpha}\mathcal{N}^{\bullet}} \to \widehat{\mathcal{N}^{\bullet}}$ is injective) and $\widehat{U_{\alpha}\mathcal{N}^{\bullet}}/\widehat{U_{<\alpha}\mathcal{N}^{\bullet}} = U_{\alpha}\mathcal{N}^{\bullet}/U_{<\alpha}\mathcal{N}^{\bullet}.$

- (b) For every $\beta \leq \alpha$ and any $i, \mathcal{H}^i(U_\alpha \mathcal{N}^{\bullet}/U_\beta \mathcal{N}^{\bullet})$ is strict.
- (c) $\mathscr{H}^{i}(\widehat{U}_{\alpha}\mathscr{N}^{\bullet}) = \underline{\lim}_{\sim} \mathscr{H}^{i}(U_{\alpha}\mathscr{N}^{\bullet}/U_{\gamma}\mathscr{N}^{\bullet}) \ (i \ge -N).$
- (d) $\mathscr{H}^{i}(\widehat{U_{\alpha}\mathscr{N}^{\bullet}}) \to \mathscr{H}^{i}(\widehat{\mathscr{N}^{\bullet}})$ is injective $(i \ge -N)$. (e) $\mathscr{H}^{i}(\widehat{\mathscr{N}^{\bullet}}) = \varinjlim_{\alpha} \mathscr{H}^{i}(\widehat{U_{\alpha}\mathscr{N}^{\bullet}}) \ (i \ge -N)$.

We note that the statements (b)–(d) imply that $\mathscr{H}^i(\widehat{\mathscr{N}}^{\bullet})$ is strict for $i \ge -N$, although $\mathscr{H}^i(\mathscr{N}^{\bullet})$ need not be strict.

Define $U_{\alpha}\mathscr{H}^{i}(\widehat{\mathscr{N}^{\bullet}}) = \operatorname{image}[\mathscr{H}^{i}(\widehat{U_{\alpha}}\mathscr{N}^{\bullet}) \to \mathscr{H}^{i}(\widehat{\mathscr{N}^{\bullet}})]$. Then the statements (a) and (d) imply that

$$\operatorname{gr}_{\alpha}^{U}\mathscr{H}^{i}(\widehat{\mathscr{N}^{\bullet}}) = \mathscr{H}^{i}(\widehat{U_{\alpha}\mathscr{N}^{\bullet}}/\widehat{U_{<\alpha}\mathscr{N}^{\bullet}}) = \mathscr{H}^{i}(\operatorname{gr}_{\alpha}^{U}\mathscr{N}^{\bullet}) \quad (i \ge -N).$$

For $\gamma < \beta < \alpha$ consider the exact sequence of complexes

$$0 \longrightarrow U_{\beta} \mathscr{N}^{\bullet} / U_{\gamma} \mathscr{N}^{\bullet} \longrightarrow U_{\alpha} \mathscr{N}^{\bullet} / U_{\gamma} \mathscr{N}^{\bullet} \longrightarrow U_{\alpha} \mathscr{N}^{\bullet} / U_{\beta} \mathscr{N}^{\bullet} \longrightarrow 0.$$

As the projective system $(U_{\alpha} \mathcal{N}^{\bullet} / U_{\gamma} \mathcal{N}^{\bullet})_{\gamma}$ trivially satisfies the Mittag-Leffler condition (ML) (see e.g. **[KS90**, Prop. 1.12.4]), the sequence remains exact after passing to the projective limit, so we get an exact sequence of complexes

$$0\longrightarrow \widehat{U_{\beta}\mathcal{N}^{\bullet}}\longrightarrow \widehat{U_{\alpha}\mathcal{N}^{\bullet}}\longrightarrow U_{\alpha}\mathcal{N}^{\bullet}/U_{\beta}\mathcal{N}^{\bullet}\longrightarrow 0,$$

hence (a).

Let us show by induction on $\rho = \alpha - \gamma \in A + \mathbb{N}$ that, for all $\gamma < \alpha$ and $i \ge -N$,

- (i) $\prod_{\gamma < \beta \leq \alpha} (E \beta z)^{\nu_{\beta}}$ annihilates $\mathscr{H}^{i}(U_{\alpha}/U_{\gamma}),$
- (ii) for all β such that $\gamma < \beta < \alpha$, we have an exact sequence,

(7.8.8)
$$0 \to \mathscr{H}^{i}(U_{\beta}\mathscr{N}^{\bullet}/U_{\gamma}\mathscr{N}^{\bullet}) \to \mathscr{H}^{i}(U_{\alpha}\mathscr{N}^{\bullet}/U_{\gamma}\mathscr{N}^{\bullet}) \to \mathscr{H}^{i}(U_{\alpha}\mathscr{N}^{\bullet}/U_{\beta}\mathscr{N}^{\bullet}) \to 0.$$

(iii) $\mathscr{H}^{i}(U_{\alpha}\mathscr{N}^{\bullet}/U_{\beta}\mathscr{N}^{\bullet})$ is strict

(iii) $\mathscr{H}^{i}(U_{\alpha}\mathscr{N}^{\bullet}/U_{\gamma}\mathscr{N}^{\bullet})$ is strict.

If γ is the predecessor of α in $A + \mathbb{Z}$, (i) and (iii) are true by assumption and (ii) is empty. Moreover, (ii)_{ρ} and (iii)_{$<\rho$} imply (iii)_{ρ}. For $\gamma < \beta < \alpha$ and $\alpha - \gamma = \rho$, consider the exact sequence

$$\cdots \xrightarrow{\psi^{i}} \mathscr{H}^{i}(U_{\beta}/U_{\gamma}) \longrightarrow \mathscr{H}^{i}(U_{\alpha}/U_{\gamma}) \longrightarrow \mathscr{H}^{i}(U_{\alpha}/U_{\beta})$$
$$\xrightarrow{\psi^{i+1}} \mathscr{H}^{i+1}(U_{\beta}/U_{\gamma}) \longrightarrow \cdots$$

For any $i \ge -N$, any local section of $\operatorname{Im} \psi^{i+1}$ is then killed by $\prod_{\beta < \delta \le \alpha} (\mathbb{E} - \delta z)$ and by $\prod_{\gamma < \delta \le \beta} (\mathbb{E} - \delta z)$ according to (i)_{< ρ}, hence is zero by (iii)_{< ρ}, and the same property holds for $\operatorname{Im} \psi^i$, so the previous sequence of \mathscr{H}^i is exact. Arguing similarly, we get the exactness of (7.8.8) for $\alpha - \gamma = \rho$, hence (ii)_{ρ}, from which (i)_{ρ} follows.

Consequently, the projective system $(\mathscr{H}^i(U_{\alpha}\mathscr{N}^{\bullet}/U_{\gamma}\mathscr{N}^{\bullet}))_{\gamma}$ satisfies (ML), so we get (c). Moreover, taking the limit on γ in (7.8.8) gives, according to (ML), an exact sequence

$$0 \longrightarrow \mathscr{H}^{i}(\widehat{U_{\beta}\mathcal{N}^{\bullet}}) \longrightarrow \mathscr{H}^{i}(\widehat{U_{\alpha}\mathcal{N}^{\bullet}}) \longrightarrow \mathscr{H}^{i}(U_{\alpha}\mathcal{N}^{\bullet}/U_{\beta}\mathcal{N}^{\bullet}) \longrightarrow 0,$$

hence (d). Now, (e) is clear.

Second step. For every i, α , denote by $\mathscr{T}^i_{\alpha} \subset \mathscr{H}^i(U_{\alpha}\mathscr{N}^{\bullet})$ the $\mathscr{I}_{H'}$ -torsion subsheaf of $\mathscr{H}^i(U_{\alpha}\mathscr{N}^{\bullet})$. We set locally $\mathscr{I}_{H'} = t\mathscr{O}_{X'}$. We will now prove that it is enough to show

(7.8.9)
$$\exists \alpha_o, \quad \alpha \leqslant \alpha_o \Longrightarrow \mathscr{T}^i_\alpha = 0 \quad \forall i \ge -N.$$

We assume that (7.8.9) is proved (step 3). Let $\gamma \leq \alpha_o$ and $i \geq -N$, so that $\mathscr{T}^i_{\gamma} = 0$, and let $\alpha \geq \gamma$. Then, by definition of a V-filtration, $t^{\lceil \alpha - \gamma \rceil}$ acts by 0 on $U_{\alpha} \mathscr{N}^{\bullet} / U_{\gamma} \mathscr{N}^{\bullet}$, so that the image of $\mathscr{H}^{i-1}(U_{\alpha} \mathscr{N}^{\bullet} / U_{\gamma} \mathscr{N}^{\bullet})$ in $\mathscr{H}^i(U_{\gamma} \mathscr{N}^{\bullet})$ is contained in \mathscr{T}^i_{γ} , and thus is zero. We therefore have an exact sequence for every $i \geq -N$:

$$0 \longrightarrow \mathscr{H}^{i}(U_{\gamma}\mathscr{N}^{\bullet}) \longrightarrow \mathscr{H}^{i}(U_{\alpha}\mathscr{N}^{\bullet}) \longrightarrow \mathscr{H}^{i}(U_{\alpha}\mathscr{N}^{\bullet}/U_{\gamma}\mathscr{N}^{\bullet}) \longrightarrow 0$$

Using (7.8.8), we get for every $\beta < \alpha$ the exact sequence

 $0 \longrightarrow \mathscr{H}^{i}(U_{\beta}\mathcal{N}^{\bullet}) \longrightarrow \mathscr{H}^{i}(U_{\alpha}\mathcal{N}^{\bullet}) \longrightarrow \mathscr{H}^{i}(U_{\alpha}\mathcal{N}^{\bullet}/U_{\beta}\mathcal{N}^{\bullet}) \longrightarrow 0.$

This implies that $\mathscr{H}^i(U_\beta \mathscr{N}^{\bullet}) \to \mathscr{H}^i(\mathscr{N}^{\bullet}) = \lim_{\Delta \to \infty} \mathscr{H}^i(U_\alpha \mathscr{N}^{\bullet})$ is injective. For every α , let us set

$$U_{\alpha}\mathscr{H}^{i}(\mathscr{N}^{\bullet}) := \operatorname{image} \big[\mathscr{H}^{i}(U_{\alpha}\mathscr{N}^{\bullet}) \hookrightarrow \mathscr{H}^{i}(\mathscr{N}^{\bullet}) \big].$$

We thus have, for every $\alpha \in A + \mathbb{Z}$ and $i \ge -N$,

$$\operatorname{gr}^U_{\alpha} \mathscr{H}^i(\mathcal{N}^{\bullet}) = \mathscr{H}^i(\operatorname{gr}^U_{\alpha} \mathcal{N}^{\bullet}).$$

Third step: proof of (7.8.9). Let us choose α_o as in 7.8.7(3). We notice that the multiplication by t induces an isomorphism $t: U_{\alpha} \widetilde{\mathcal{N}^i} \xrightarrow{\sim} U_{\alpha-1} \widetilde{\mathcal{N}^i}$ for $\alpha \leq \alpha_o$ and $i \ge -N-1$, hence an isomorphism $t: \mathscr{H}^i(\widehat{U_{\alpha}\mathscr{N}^{\bullet}}) \xrightarrow{\sim} \mathscr{H}^i(\widehat{U_{\alpha-1}\mathscr{N}^{\bullet}})$, and that (d) in Step one implies that, for all $i \ge -N$ and all $\alpha \le \alpha_o$, the multiplication by t on $\mathscr{H}^{i}(\widehat{U_{\alpha}}\mathcal{N}^{\bullet})$ is injective.

The proof of (7.8.9) is done by decreasing induction on *i*. It clearly hods for $i \ge i_o$ (given by 7.8.7(4)). We assume that, for every $\alpha \leq \alpha_o$, we have $\mathscr{T}^{i+1}_{\alpha} = 0$. We have (after 7.8.7(3)) an exact sequence of complexes, for every $k \in \mathbb{N}$ and $\bullet \ge -N-1$,

$$0 \longrightarrow U_{\alpha} \mathscr{N}^{\bullet} \xrightarrow{t^{\kappa}} U_{\alpha} \mathscr{N}^{\bullet} \longrightarrow U_{\alpha} \mathscr{N}^{\bullet} / U_{\alpha-k} \mathscr{N}^{\bullet} \longrightarrow 0.$$

As $\mathscr{T}^{i+1}_{\alpha} = 0$, we have, for every $k \ge 1$ an exact sequence

$$\mathscr{H}^{i}(U_{\alpha}\mathscr{N}^{\bullet}) \xrightarrow{t^{k}} \mathscr{H}^{i}(U_{\alpha}\mathscr{N}^{\bullet}) \longrightarrow \mathscr{H}^{i}(U_{\alpha}\mathscr{N}^{\bullet}/U_{\alpha-k}\mathscr{N}^{\bullet}) \longrightarrow 0,$$

hence, according to Step one,

$$\mathscr{H}^{i}(\widehat{U_{\alpha}\mathscr{N}^{\bullet}})/\mathscr{H}^{i}(\widehat{U_{\alpha-k}\mathscr{N}^{\bullet}}) = \mathscr{H}^{i}(U_{\alpha}\mathscr{N}^{\bullet}/U_{\alpha-k}\mathscr{N}^{\bullet}) = \mathscr{H}^{i}(U_{\alpha}\mathscr{N}^{\bullet})/t^{k}\mathscr{H}^{i}(U_{\alpha}\mathscr{N}^{\bullet}).$$

According to Assumption 7.8.7(5) and Exercise 7.3.8, for k big enough (locally on X'), the map $\mathscr{T}^i_{\alpha} \to \mathscr{H}^i(U_{\alpha}\mathscr{N}^{\bullet})/t^k\mathscr{H}^i(U_{\alpha}\mathscr{N}^{\bullet})$ is injective. It follows that $\mathscr{T}^i_{\alpha} \to \mathscr{H}^i(\widehat{U_{\alpha}}\mathcal{N}^{\bullet})$ is injective too. But we know that t is injective on $\mathscr{H}^i(\widehat{U_{\alpha}}\mathcal{N}^{\bullet})$ for $\alpha \leq \alpha_o$, hence $\mathscr{T}^i_{\alpha} = 0$, thus concluding Step 3.

Proof of Theorem 7.8.5

Lemma 7.8.10. Let $U_{\bullet}\mathscr{M}$ be a V-filtration indexed by $A + \mathbb{Z}$ of a $\widetilde{\mathscr{D}}_X$ -module \mathscr{M} which satisfies the following properties:

- (a) $t: U_{\alpha}\mathcal{M} \to U_{\alpha-1}\mathcal{M}$ is bijective for every $\alpha < 0$, (b) $\eth_t: \operatorname{gr}_{\alpha}^U \mathcal{M} \to \operatorname{gr}_{\alpha+1}^U \mathcal{M}$ is bijective for every $\alpha > -1$.

We define $R_U \mathscr{M}$ as in Remark 7.2.7, which is thus an $R_{AV} \widetilde{\mathscr{D}}_X$ -module. Then $R_U\mathscr{M}$ has a resolution $\mathscr{L}^{\bullet} \otimes_{\widetilde{\mathscr{O}}_Y} R_{AV} \widetilde{\mathscr{D}}_X$, where each \mathscr{L}^i is an $\widetilde{\mathscr{O}}_X$ -module.

Proof. By assumption, the morphism $\varphi : \bigoplus_{\gamma \in [-1,0]} U_{\gamma} \mathscr{M} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X \to \mathscr{M}$ is surjective and induces surjective morphisms $\bigoplus_{\gamma \in [-1,0]} U_{\gamma} \mathscr{M} \otimes_{\widetilde{\mathscr{O}}_X} {}^{A}\!V_{\alpha-\gamma} \widetilde{\mathscr{D}}_X \to U_{\alpha} \mathscr{M}$ for every $\alpha \in A + \mathbb{Z}$, hence a surjective morphism $\bigoplus_{\gamma \in [-1,0]} U_{\gamma} \mathscr{M} v^{\gamma} \otimes_{\widetilde{\mathscr{O}}_X} R_{AV} \widetilde{\mathscr{D}}_X \to R_U \mathscr{M}$, with the convention of Remark 7.2.7. We note that the V-filtered induced $\hat{\mathscr{D}}_X$ -module that we have introduced also satisfies (a) and (b). Set $\mathscr{K} = \operatorname{Ker} \varphi$, that we equip with the induced filtration $U_{\bullet}\mathscr{K}$. We thus have an exact sequence for every α :

$$0 \longrightarrow U_{\alpha} \mathscr{K} \longrightarrow \bigoplus_{\gamma \in [-1,0]} U_{\gamma} \mathscr{M} \otimes_{\widetilde{\mathscr{O}}_{X}} {}^{A}\!V_{\alpha - \gamma} \widetilde{\mathscr{D}}_{X} \longrightarrow U_{\alpha} \mathscr{M} \longrightarrow 0,$$

from which we deduce that $U_{\bullet}\mathcal{K}$ satisfies (a) and (b), enabling us to continue the process.

The assertion of the theorem is local on X', and we will work in the neighbourhood of a point $x'_o \in H'$. The Kashiwara-Malgrange filtration $V_{\bullet}\mathscr{M}$ satisfies the properties 7.8.10(a) and (b), according to Proposition 7.3.31. We can then use a resolution as in Lemma 7.8.10, that we stop at a finite step chosen large enough (due to the cohomological finiteness of f) such that, for the corresponding bounded complex $\mathscr{L}^{\bullet} \otimes_{\widetilde{\mathcal{O}}_X} R_{AV} \widetilde{\mathscr{D}}_X$, one has

$$\mathscr{H}^{i}{}_{{}_{\mathrm{D}}}f_{*}(R_{V}\mathscr{M})\neq 0 \Longrightarrow \mathscr{H}^{i}{}_{{}_{\mathrm{D}}}f_{*}(R_{V}\mathscr{M}) = \mathscr{H}^{i}{}_{{}_{\mathrm{D}}}f_{*}(\mathscr{L}^{\bullet}\otimes_{\widetilde{\mathscr{O}}_{X}}R_{AV}\widetilde{\mathscr{D}}_{X})$$

and similarly for every α ,

$$\mathscr{H}^{i}{}_{\mathrm{D}}f_{|H*}(\mathrm{gr}^{V}_{\alpha}\mathscr{M})\neq 0 \Longrightarrow \mathscr{H}^{i}{}_{\mathrm{D}}f_{|H*}(\mathrm{gr}^{V}_{\alpha}\mathscr{M}) = \mathscr{H}^{i}{}_{\mathrm{D}}f_{|H*}(\mathscr{L}^{\bullet}\otimes_{\widetilde{\mathscr{O}}_{X}}\mathrm{gr}^{A_{V}}_{\alpha}\widetilde{\mathscr{D}}_{X}).$$

In such a case, $\mathscr{H}^{i}{}_{{}_{\mathrm{D}}}f_{*}(R_{V}\mathscr{M}) = \mathscr{H}^{i}(f_{*}\operatorname{God}^{\bullet}(\mathscr{L}^{\bullet}\otimes_{f^{-1}\widetilde{\mathcal{O}}_{X'}}f^{-1}R_{AV}\widetilde{\mathscr{D}}_{X'}))$, according to Lemma 7.8.3. We thus set

$$(\mathscr{N}^{\bullet}, U_{\bullet}\mathscr{N}^{\bullet}) = \left(f_* \operatorname{God}^{\bullet}(\mathscr{L}^{\bullet} \otimes_{f^{-1}\widetilde{\mathscr{O}}_{X'}} f^{-1}\widetilde{\mathscr{D}}_{X'}), f_* \operatorname{God}^{\bullet}(\mathscr{L}^{\bullet} \otimes_{f^{-1}\widetilde{\mathscr{O}}_{X'}} f^{-1A}V_{\bullet}\widetilde{\mathscr{D}}_{X'})\right).$$

Since the sequences

$$0 \longrightarrow {}^{A}V_{\alpha}\widetilde{\mathscr{D}}_{X'} \longrightarrow \widetilde{\mathscr{D}}_{X'} \longrightarrow \widetilde{\mathscr{D}}_{X'}/{}^{A}V_{\alpha}\widetilde{\mathscr{D}}_{X'} \longrightarrow 0$$
$$0 \longrightarrow {}^{A}V_{<\alpha}\widetilde{\mathscr{D}}_{X'} \longrightarrow {}^{A}V_{\alpha}\widetilde{\mathscr{D}}_{X'} \longrightarrow \operatorname{gr}_{\alpha}^{A_{V}}\widetilde{\mathscr{D}}_{X'} \longrightarrow 0$$

and

are exact sequences of locally free $\widehat{\mathscr{O}}_{X'}$ -modules, they remain exact after applying $\mathscr{L}^{\bullet} \otimes_{\widetilde{\mathscr{O}}_{X'}}$, then also after applying the Godement functor (see Exercise A.8.13(1)), and then after applying f_* since the latter complexes consist of flabby sheaves.

This implies that $U_{\alpha}\mathcal{N}^{\bullet}$ is indeed a subcomplex of \mathcal{N}^{\bullet} and $\operatorname{gr}_{\alpha}^{U}\mathcal{N}^{\bullet} = f_* \operatorname{God}^{\bullet}(\mathscr{L}^{\bullet} \otimes_{f^{-1}\widetilde{\mathcal{O}}_{X'}} f^{-1} \operatorname{gr}_{\alpha}^{A_V} \widetilde{\mathscr{D}}_{X'}).$

Property 7.8.7(5) is satisfied, according to Theorem 7.8.2, and Properties 7.8.7(3) and (4) are clear.

We have $\mathscr{H}^{i}(\mathrm{gr}_{\alpha}^{U}\mathscr{N}^{\bullet}) = \mathscr{H}^{i}({}_{\mathrm{D}}f_{|H*}\mathrm{gr}_{\alpha}^{V}\mathscr{M})$ for $i \geq -N$ for some N such that $\mathscr{H}^{i}({}_{\mathrm{D}}f_{|H*}\mathrm{gr}_{\alpha}^{V}\mathscr{M}) = 0$ if i < -N, so that 7.8.7(1) holds by assumption and 7.8.7(2) is satisfied by taking the maximum of the local values ν_{α} along the compact fibre $f^{-1}(x'_{o})$.

From Proposition 7.8.7 we conclude that 7.8.5(1) holds for $\alpha \in A + \mathbb{Z}$ and any *i*. Denoting by $U_{\bullet}\mathscr{H}^{i}{}_{\mathrm{D}}f_{*}\mathscr{M}$ the image filtration in 7.8.5(1), we thus have $R_{U}\mathscr{H}^{i}{}_{\mathrm{D}}f_{*}\mathscr{M} = \mathscr{H}^{i}{}_{\mathrm{D}}f_{*}R_{V}\mathscr{M}$ and therefore

$$\operatorname{gr}_{\alpha}^{U}(\mathscr{H}^{i}{}_{\mathrm{D}}f_{*}\mathscr{M}) = \mathscr{H}^{i}{}_{\mathrm{D}}f_{|H*}\operatorname{gr}_{\alpha}^{V}\mathscr{M}.$$

7.9. COMMENTS

In particular, the left-hand term is strict by assumption on the right-hand term.

By the coherence theorem 7.8.2, we conclude that $U_{\bullet}\mathscr{H}^{i}{}_{{}_{\mathrm{D}}}f_{*}\mathscr{M}$ is a coherent ^AV-filtration of $\mathscr{H}^{i}{}_{{}_{\mathrm{D}}}f_{*}\mathscr{M}$. Therefore, $U_{\bullet}\mathscr{H}^{i}{}_{{}_{\mathrm{D}}}f_{*}\mathscr{M}$ satisfies the assumptions of Lemma 7.3.23 (extended to filtrations indexed by $A + \mathbb{Z}$). Moreover, the properties 7.3.25(2) and (3) are also satisfied since they hold for \mathscr{M} . We conclude that $\mathscr{H}^{i}{}_{{}_{\mathrm{D}}}f_{*}\mathscr{M}$ is strictly \mathbb{R} -specializable along H' and that $U_{\bullet}\mathscr{H}^{i}{}_{{}_{\mathrm{D}}}f_{*}\mathscr{M}$ is its Kashiwara-Malgrange filtration. Now, Properties (1)–(3) in Theorem 7.8.5 are clear.

7.9. Comments

Here come the references to the existing work which has been the source of inspiration for this chapter.