## CHAPTER 7

## NEARBY AND VANISHING CYCLES OF $\widetilde{\mathscr{D}}$-MODULES


#### Abstract

Summary. We introduce the Kashiwara-Malgrange filtration for a $\widetilde{\mathscr{D}}_{X}$-module, and the notion of strict $\mathbb{R}$-specializability. This leads to the construction of the nearby and vanishing cycle functors. One of the main results is a criterion for the compatibility of this functor with the proper pushforward functor of $\widetilde{\mathscr{D}}$-modules.


Throughout this chapter we use the following notation.

## Notation 7.0.1.

- $X$ denotes a complex manifold.
- $H$ denotes a smooth hypersurface in $X$.
- Locally on $H$, we choose a decomposition $X=H \times \Delta_{t}$, where $\Delta_{t}$ is a small disc in $\mathbb{C}$ with coordinate $t$. We have the corresponding $z$-vector field $\partial_{t}$.
- $D$ denotes an effective divisor on $X$. Locally on $D$, we choose a holomorphic function $g: X \rightarrow \mathbb{C}$ such that $D=(g)$. We then set $X_{0}=g^{-1}(0)$ (this is the support of $D$ in the local setting).
- Recall that $\widetilde{\mathscr{D}}_{X}$ means $\mathscr{D}_{X}$ or $R_{F} \mathscr{D}_{X}$ and, in the latter case, $\widetilde{\mathscr{D}}_{X}$-modules mean graded $\widetilde{\mathscr{D}}_{X}$-modules (see Appendix A). We then use $(k)$ for the shift by $k$ of the grading (see Section A.2.a). When the information on the grading is not essential, we just omit to indicate the corresponding shift. We use the convention that, whenever $\widetilde{\mathscr{D}}_{X}$ means $\mathscr{D}_{X}$, all conditions and statements relying on gradedness or strictness are understood to be empty or tautological.

Remark 7.0.2 (Left and right $\widetilde{\mathscr{D}}$-modules). For various purposes, it is more convenient to work with right $\widetilde{\mathscr{D}}$-modules. However, left $\widetilde{\mathscr{D}}$-modules are more commonly used in applications. We will therefore mainly treat right $\widetilde{\mathscr{D}}$-modules and give the corresponding formulas for left $\widetilde{\mathscr{D}}$-modules in various remarks.

Remark 7.0.3 (Restriction to $z=1$ ). Throughout this chapter we keep the Convention A.2.19. All the constructions can be done either for $\mathscr{D}_{X}$-modules or for graded $R_{F} \mathscr{D}_{X}$-modules, in which case a strictness assumption (strict $\mathbb{R}$-specializability) is most often needed. By "good behaviour with respect to the restriction $z=1$ ", we
mean that the restriction functor $\mathscr{M} \mapsto \mathcal{M}:=\mathscr{M} /(z-1) \mathscr{M}$ is compatible with the constructions. We will see that many, but not all, of the constructions in this chapter have good behaviour with respect to setting $z=1$. We will make this precise for each such construction.

### 7.1. Introduction

This chapter has one main purpose: Given a coherent $\widetilde{\mathscr{D}}_{X}$-module, to give a sufficient condition such that the restriction functor to a divisor $D$, producing a complex of $\widetilde{\mathscr{D}}_{X}$-modules supported on the divisor $D$ which corresponds to the functor ${ }_{\mathrm{D}} \iota_{H * \mathrm{D}} \iota_{H}^{*}$ when $\iota_{H}: H \hookrightarrow X$ is the inclusion of a smooth hypersurface, gives rise to a complex of $\widetilde{\mathscr{D}}_{X}$-modules with coherent cohomology.

The property of being specializable along $D$ will answer this first requirement. However, in the case where $\widetilde{\mathscr{D}}_{X}=R_{F} \mathscr{D}_{X}$, strictness comes into play in a fundamental way in order to ensure a good behaviour. This leads to the notion of strict specializability along $D$. When forgetting the $F$-filtration, i.e., when considering $\mathscr{D}_{X}$-modules, the strictness condition is empty.

Given any holomorphic function $g$ on $X$ with associated divisor $D$ and for every strictly $\mathbb{R}$-specializable $\widetilde{\mathscr{D}}_{X}$-module $\mathscr{M}$ along $D$, we introduce the nearby cycle $\widetilde{\mathscr{D}}_{X}$-modules $\psi_{g, \lambda} \mathscr{M}\left(\lambda \in \mathbb{C}^{*}\right.$ with $\left.|\lambda|=1\right)$ and the vanishing cycle functor $\phi_{g, 1} \mathscr{M}$. They are the "generalized restriction functors", which the usual restriction functors can be deduced from.

The construction is possible when the Kashiwara-Malgrange $V$-filtration exists on a given $\widetilde{\mathscr{D}}_{X}$-module. More precisely, the notion of $V$-filtration is well-defined in the case when $D$ is a smooth divisor. We reduce to this case by considering, when more generally $D=(g)$, the graph inclusion $\iota_{g}: X \hookrightarrow X \times \mathbb{C}$. The $V$-filtration can exist on the pushforward ${ }_{\mathrm{D}} \iota_{*} \mathscr{M}$. We then say that $\mathscr{M}$ is strictly specializable along $D$.

Kashiwara's equivalence is an equivalence (via the pushforward functor $\iota_{Y}: Y \hookrightarrow X$ ) between the category of coherent $\mathscr{D}_{Y}$-modules and that of coherent $\mathscr{D}_{X}$-modules supported on the submanifold $Y$. When $Y$ has codimension one in $X$, this equivalence can be extended as an equivalence between strict coherent $\widetilde{\mathscr{D}}_{Y}$-modules and coherent $\widetilde{\mathscr{D}}_{X}$-modules which are strictly $\mathbb{R}$-specializable along $Y$.

Complex Hodge modules will satisfy a property of semi-simplicity with respect to their support that we introduce in this chapter under the name of strict $S$ decomposability (" S " is for "support"). The support of a coherent $\widetilde{\mathscr{D}}_{X}$-module $\mathscr{M}$ is a closed analytic subspace in $X$. It may have various irreducible components. We introduce a condition which ensures first that $\mathscr{M}$ decomposes as the direct sum of $\widetilde{\mathscr{D}}_{X}$-modules, each of which supported by a single component. However, we wish that each such summand decomposes itself as the direct sum of $\widetilde{\mathscr{D}}_{X}$-modules, each of which supported on an irreducible closed analytic subset of the support of the given summand, in order to satisfy a "geometric simplicity property", namely each such
new summand has no coherent sub-module supported on a strictly smaller closed analytic subset. We then say that such a summand has pure support.

In Section 7.8, we give a criterion in order that the functors $\psi_{g, \lambda}$ and $\phi_{g, 1}$ commute with proper pushforward. This will be an essential step in the theory of complex Hodge modules (see Chapter 13), where we need to prove that the property of strict S-decomposability (i.e., geometric semi-simplicity) is preserved by proper pushforward.

### 7.2. The filtration $V . \widetilde{\mathscr{D}}_{X}$ relative to a smooth hypersurface

Let $H \subset X$ be a smooth hypersurface ${ }^{(1)}$ of $X$ with defining ideal $\mathscr{I}_{H} \subset \mathscr{O}_{X}$. We first define a canonical increasing filtration of $\widetilde{\mathscr{D}}_{X}$ indexed by $\mathbb{Z}$. Let us set $\widetilde{\mathscr{I}}_{H}^{\ell}=\widetilde{\mathscr{O}}_{X}$ for $\ell<0$ and $\widetilde{\mathscr{I}}_{H}^{\ell}=\mathscr{I}_{H}^{\ell} \widetilde{\mathscr{O}}_{X}$ for $\ell \geqslant 0$. For every $k \in \mathbb{Z}$, the subsheaf $V_{k} \widetilde{\mathscr{D}}_{X} \subset \widetilde{\mathscr{D}}_{X}$ $(k \in \mathbb{Z})$ consists of operators $P$ such that $\widetilde{\mathscr{I}_{H}^{j}} P \subset \widetilde{\mathscr{I}_{H}^{j}-k}$ for every $j \in \mathbb{Z}$. For every open set $U$ of $X$ we thus have

$$
\begin{equation*}
V_{k} \widetilde{\mathscr{D}}_{X}(U)=\left\{P \in \widetilde{\mathscr{D}}_{X}(U) \mid \widetilde{\mathscr{I}}_{H}^{\jmath}(U) \cdot P \subset \widetilde{\mathscr{I}}_{H}^{j-k}(U), \forall j \in \mathbb{Z}\right\} \tag{7.2.1}
\end{equation*}
$$

This defines an increasing filtration $V_{\bullet} \widetilde{\mathscr{D}}_{X}$ of $\widetilde{\mathscr{D}}_{X}$ indexed by $\mathbb{Z}$. Note that one can also define $V_{k} \widetilde{\mathscr{D}}_{X}(U)$ as the set of $Q \in \widetilde{\mathscr{D}}_{X}(U)$ such that $Q \cdot \widetilde{\mathscr{I}_{H}^{j}}(U) \subset \widetilde{\mathscr{I}_{H}^{j-k}}(U)$, $\forall j \in \mathbb{Z}$.
Exercise 7.2.2. Show the following properties.
(1) Let us fix a local decomposition $X \simeq H \times \Delta_{t}$ (where $\Delta_{t} \subset \mathbb{C}$ is a disc with coordinate $t)$. With respect to this decomposition,

$$
V_{0} \widetilde{\mathscr{D}}_{X}=\widetilde{\mathscr{O}}_{X}\left\langle\check{\partial}_{x}, t \check{\partial}_{t}\right\rangle, \quad V_{-j} \widetilde{\mathscr{D}}_{X}=V_{0} \widetilde{\mathscr{D}}_{X} \cdot t^{j}, \quad V_{j} \widetilde{\mathscr{D}}_{X}=\sum_{k=0}^{j} V_{0} \widetilde{\mathscr{D}}_{X} \cdot \widetilde{\partial}_{t}^{k} \quad(j \geqslant 0)
$$

(2) For every $k, V_{k} \widetilde{\mathscr{D}}_{X}$ is a locally free $V_{0} \widetilde{\mathscr{D}}_{X}$-module.
(3) $\widetilde{\mathscr{D}}_{X}=\bigcup_{k} V_{k} \widetilde{\mathscr{D}}_{X}$ (the filtration is exhaustive).
(4) $V_{k} \widetilde{\mathscr{D}}_{X} \cdot V_{\ell} \widetilde{\mathscr{D}}_{X} \subset V_{k+\ell} \widetilde{\mathscr{D}}_{X}$ with equality for $k, \ell \leqslant 0$ or $k, \ell \geqslant 0$.
(5) $V_{0} \widetilde{\mathscr{D}}_{X}$ is a sheaf of subalgebras of $\widetilde{\mathscr{D}}_{X}$.
(6) $V_{k} \widetilde{\mathscr{D}}_{X} \mid X \backslash H=\widetilde{\mathscr{D}}_{X \mid X \backslash H}$ for all $k \in \mathbb{Z}$.
(7) $\operatorname{gr}_{k}^{V} \widetilde{\mathscr{D}}_{X}$ is supported on $H$ for all $k \in \mathbb{Z}$,
(8) The induced filtration $V_{k} \widetilde{\mathscr{D}}_{X} \cap \widetilde{\mathscr{O}}_{X}=\widetilde{\mathscr{I}}_{H}^{-k} \widetilde{\mathscr{O}}_{X}$ is the $\widetilde{\mathscr{\mathscr { I }}}_{H}$-adic filtration of $\widetilde{\mathscr{O}}_{X}$ made increasing.
(9) $\left(\bigcap_{k} V_{k} \widetilde{\mathscr{D}}_{X}\right)_{\mid H}=\{0\}$.

## Exercise 7.2.3 (Euler vector field).

(1) Show that the class E of $t \widetilde{\partial}_{t}$ in $\operatorname{gr}_{0}^{V} \widetilde{\mathscr{D}}_{X}$ in some local product decomposition as above does not depend on the choice of such a local product decomposition.

[^0](2) Show that if $H$ has a global equation $g$, then $\operatorname{gr}_{0}^{V} \widetilde{\mathscr{D}}_{X} \simeq \widetilde{\mathscr{D}}_{H}[\mathrm{E}]$.
(3) Conclude that $\operatorname{gr}_{0}^{V} \widetilde{\mathscr{D}}_{X}$ is a sheaf of rings and that E belongs to its center.

Remark 7.2.4 (Structure of $\operatorname{gr}_{0}^{V} \widetilde{\mathscr{D}}_{X}$ ). While $\widetilde{\mathscr{D}}_{H}$ can be identified to the quotient $\operatorname{gr}_{0}^{V} \widetilde{\mathscr{D}}_{X} / \operatorname{Egr}_{0}^{V} \widetilde{\mathscr{D}}_{X}=\operatorname{gr}_{0}^{V} \widetilde{\mathscr{D}}_{X} / \operatorname{gr}_{0}^{V} \widetilde{\mathscr{D}}_{X}$ E, it is not identified with a subsheaf of $\operatorname{gr}_{0}^{V} \widetilde{\mathscr{D}}_{X}$, except when $N_{H} X$ is trivial. When $H$ is globally defined by a holomorphic function $g$, or more generally for any holomorphic function $g: X \rightarrow \mathbb{C}$, we will often use the trick of the graph inclusion $\iota_{g}: X \hookrightarrow X \times \mathbb{C}$ and we will then consider the filtration $V . \widetilde{\mathscr{D}}_{X \times \mathbb{C}}{\underset{\sim}{\sim}}_{x}$ ith respect to $X \times\{0\}$, so that we will be able to identify $\operatorname{gr}_{0}^{V} \widetilde{\mathscr{D}}_{X \times \mathbb{C}}$ with the ring $\widetilde{\mathscr{D}}_{X}[\mathrm{E}]$.
Exercise 7.2.5. Show the equivalence between the category of $\widetilde{\mathscr{O}}_{X}$-modules with integrable logarithmic connection $\widetilde{\nabla}: \mathscr{M} \rightarrow \widetilde{\Omega}_{X}^{1}(\log H) \otimes \mathscr{M}$ and the category of left $V_{0} \widetilde{\mathscr{D}}_{X}$-modules. Show that the residue $\operatorname{Res} \widetilde{\nabla}$ corresponds to the induced action of E on $\mathscr{M} / \widetilde{\mathscr{I}_{H}} \mathscr{M}$.

Let $\nu: N_{H} X \rightarrow H$ denote the normal bundle of $H$ in $X$ and set $\widetilde{\mathscr{D}}_{\left[N_{H} X\right]}:=\nu_{*} \widetilde{\mathscr{D}}_{N_{H} X}$ (where $\nu_{*}$ is taken in the algebraic sense) with its filtration $V_{\bullet} \widetilde{\mathscr{D}}_{\left[N_{H} X\right]}$. Then there is a canonical isomorphism (as graded objects) $\operatorname{gr}^{V} \widetilde{\mathscr{D}}_{X} \simeq \operatorname{gr}^{V} \widetilde{\mathscr{D}}_{\left[N_{H} X\right]}$, and the latter sheaf is isomorphic (forgetting the grading) to $\widetilde{\mathscr{D}}_{\left[N_{H} X\right]}$.
Exercise 7.2.6 (The Rees sheaf of rings $R_{V} \widetilde{\mathscr{D}}_{X}$ ). Introduce the Rees sheaf of rings $R_{V} \widetilde{\mathscr{D}}_{X}:=\bigoplus_{k} V_{k} \widetilde{\mathscr{D}}_{X} \cdot v^{k} \subset \widetilde{\mathscr{D}}_{X}\left[v, v^{-1}\right]$ associated to the filtered sheaf $\left(\widetilde{\mathscr{D}}_{X}, V_{0} \widetilde{\mathscr{D}}_{X}\right)$ (see Definition A.2.3), and similarly $R_{V} \widetilde{\mathscr{O}}_{X}=\bigoplus_{k} V_{k} \widetilde{\mathscr{O}}_{X} \cdot v^{k} \subset \widetilde{\mathscr{O}}_{X}\left[v, v^{-1}\right]$, which is the Rees ring associated to the $\widetilde{\mathscr{I}}_{H}$-adic filtration of $\widetilde{\mathscr{O}}_{X}$.
(1) Show that $R_{V} \widetilde{\mathscr{O}}_{X}=\widetilde{\mathscr{O}}_{X}\left[v, t v^{-1}\right]$, where $t=0$ is a local equation of $H$.
(2) Show that $R_{V} \widetilde{\mathscr{D}}_{X}=\widetilde{\mathscr{O}}_{X}\left[v, t v^{-1}\right]\left\langle v \check{\partial}_{t}, \check{\partial}_{x_{2}}, \ldots, \check{\partial}_{x_{n}}\right\rangle$.
(3) Conclude that $R_{V} \widetilde{\mathscr{D}}_{X}$ is locally free over $R_{V} \widetilde{\mathscr{O}}_{X}$.

Remark 7.2.7 ( $V$-filtration indexed by $A+\mathbb{Z}$ ). The following construction of extending the set of indices will prove useful. Let $A \subset[0,1)$ be a finite subset containing 0 . Let us fix the numbering of $A+\mathbb{Z}=\left\{\ldots, \alpha_{-1}, \alpha_{o}, \alpha_{1}, \ldots\right\}$ which respect the order and such that $\alpha_{o}=0$. We thus have $1=\alpha_{\# A}$. We denote by ${ }^{A} V_{\bullet} \widetilde{\mathscr{D}}_{X}$ the filtration indexed by $A+\mathbb{Z}$ defined by ${ }^{A} V_{\alpha} \widetilde{\mathscr{D}}_{X}:=V_{[\alpha]} \widetilde{D}_{X}$. We consider it as a filtration indexed by $\mathbb{Z}$ by using the previous order-preserving bijection. Since $[\alpha]+[\beta] \leqslant[\alpha+\beta]$, we have ${ }^{A} V_{\alpha} \widetilde{\mathscr{D}}_{X} \cdot{ }^{A} V_{\beta} \widetilde{\mathscr{D}}_{X} \subset{ }^{A} V_{\alpha+\beta} \widetilde{\mathscr{D}}_{X}$, and on the other hand, ${ }^{A} V_{\alpha_{o}} \widetilde{\mathscr{D}}_{X}=V_{0} \widetilde{\mathscr{D}}_{X}$. The Rees ring is $R_{A_{V}} \widetilde{\mathscr{D}}_{X}:=\bigoplus_{k \in \mathbb{Z}}{ }^{A} V_{\alpha_{k}} \widetilde{\mathscr{D}}_{X} v^{k}$. Note also that

$$
\operatorname{gr}^{A} V \widetilde{\mathscr{D}}_{X}=\bigoplus_{k \in \mathbb{Z}} \operatorname{gr}_{\alpha_{k}}^{A_{V}} \widetilde{\mathscr{D}}_{X}=\bigoplus_{k \in \# A \cdot \mathbb{Z}} \operatorname{gr}_{(k / \# A)}^{V} \widetilde{\mathscr{D}}_{X}
$$

It will sometimes be convenient to write, for short, $R_{A_{V}} \widetilde{\mathscr{D}}_{X}:=\bigoplus_{\alpha \in A+\mathbb{Z}}{ }^{A} V_{\alpha} \widetilde{\mathscr{D}}_{X} v^{\alpha}$.
Exercise 7.2.8. Define similarly ${ }^{A} V_{\alpha} \widetilde{\mathscr{O}}_{X}$ and show that $R_{A_{V}} \widetilde{\mathscr{D}}_{X}$ is locally free over $R_{A_{V}} \widetilde{\mathscr{O}}_{X}$.

Remark 7.2.9 (Restriction to $z=1$ ). The $V$-filtration restricts well when setting $z=1$, that is, $V_{k} \mathscr{D}_{X}=V_{k} \widetilde{\mathscr{D}}_{X} /(z-1) V_{k} \widetilde{\mathscr{D}}_{X}=V_{k} \widetilde{\mathscr{D}}_{X} /(z-1) \widetilde{\mathscr{D}}_{X} \cap V_{k} \widetilde{\mathscr{D}}_{X}$.

### 7.3. Specialization of coherent $\widetilde{\mathscr{D}}_{X}$-modules

In this section, $H$ denotes a smooth hypersurface of a complex manifold $X$ and we denote by $t$ a local generator of $\mathscr{I}_{H}$. We use the definitions and notation of Section 7.2.

Caveat 7.3.1. In Subsections 7.3.a-7.3.c, when $\widetilde{\mathscr{D}}_{X}=R_{F} \mathscr{D}_{X}$, we will forget about the grading of the $\widetilde{\mathscr{D}}_{X}$-modules and morphisms involved, in order to keep the notation similar to the case of $\mathscr{D}_{X}$-modules. From Section 7.4 , we will remember the shift of grading for various morphisms, in the case of $R_{F} \mathscr{D}_{X}$-modules (this shift has no influence in the case of $\mathscr{D}_{X}$-modules).

## 7.3.a. Coherent $V$-filtrations

Exercise 7.3.2 (Coherence of $R_{V} \widetilde{\mathscr{D}}_{X}$ ). We consider the Rees sheaf of rings $R_{V} \widetilde{\mathscr{D}}_{X}:=$ $\bigoplus_{k} V_{k} \widetilde{\mathscr{D}}_{X} \cdot v^{k}$ as in Exercise 7.2.6. The aim of this exercise is to show the coherence of the sheaf of rings $R_{V} \widetilde{\mathscr{D}}_{X}$. Since the problem is local, we can assume that there are coordinates $\left(t, x_{2}, \ldots, x_{n}\right)$ such that $H=\{t=0\}$.
(1) Let $K$ be a compact polycylinder in $X$. Show that $R_{V} \widetilde{\mathscr{O}}_{X}(K)=R_{V}\left(\widetilde{\mathscr{O}}_{X}(K)\right)$ is Noetherian, being the Rees ring of the $\widetilde{\mathscr{I}}_{H}$-adic filtration on the ring $\widetilde{\mathscr{O}}_{X}(K)$ (which is Noetherian, by a theorem of Frisch). Similarly, as $\widetilde{\mathscr{O}}_{X, x}$ is flat on $\widetilde{\mathscr{O}}_{X}(K)$ for every $x \in K$, show that the ring $\left(R_{V} \widetilde{\mathscr{O}}_{X}\right)_{x}=R_{V} \widetilde{\mathscr{O}}_{X}(K) \otimes_{\widetilde{\mathscr{O}}_{X}(K)} \widetilde{\mathscr{O}}_{X, x}$ is flat on $R_{V} \widetilde{\mathscr{O}}_{X}(K)$.
(2) Show that $R_{V} \widetilde{\mathscr{O}}_{X}$ is coherent on $X$ by following the strategy developed in [GM93]. [Hint: Let $\widetilde{\Omega}$ be any open set in $X$ and let $\varphi:\left(R_{V} \widetilde{\mathscr{O}}_{X}\right)_{\mid \widetilde{\Omega}}^{q} \rightarrow\left(R_{V} \widetilde{\mathscr{O}}_{X}\right)_{\mid \widetilde{\Omega}}^{p}$ be any morphism. Let $K$ be a polycylinder contained in $\widetilde{\Omega}$. Show that $\operatorname{Ker} \varphi(K)$ is finitely generated over $R_{V} \widetilde{\mathscr{O}}_{X}(K)$ and, if $K^{\circ}$ is the interior of $K$, show that $\operatorname{Ker} \varphi_{\mid K^{\circ}}=\operatorname{Ker} \varphi(K) \otimes_{R_{V} \widetilde{\mathscr{O}}_{X}(K)}\left(R_{V} \widetilde{\mathscr{O}}_{X}\right)_{\mid K^{\circ}}$. Conclude that $\operatorname{Ker} \varphi_{\mid K^{\circ}}$ is finitely generated, whence the coherence of $R_{V} \widetilde{\mathscr{O}}_{X}$.]
(3) Consider the sheaf $\widetilde{\mathscr{O}}_{X}\left[\tau, \xi_{2}, \ldots, \xi_{n}\right]$ equipped with the $V$-filtration for which $\tau$ has degree 1 , the variables $\xi_{2}, \ldots, \xi_{n}$ have degree 0 , and inducing the $V$-filtration (i.e., $t$-adic in the reverse order) on $\widetilde{\mathscr{O}}_{X}$. Firstly, forgetting $\tau$, Show that $R_{V}\left(\widetilde{\mathscr{O}}_{X}\left[\xi_{2}, \ldots, \xi_{n}\right]\right)=\left(R_{V} \widetilde{\mathscr{O}}_{X}\right)\left[\xi_{2}, \ldots, \xi_{n}\right]$. Secondly, using $V_{k}\left(\widetilde{\mathscr{O}}_{X}\left[\tau, \xi_{2}, \ldots, \xi_{n}\right]\right)=$ $\sum_{j \geqslant 0} V_{k-j}\left(\widetilde{\mathscr{O}}_{X}\left[\xi_{2}, \ldots, \xi_{n}\right]\right) \tau^{j}$ for every $k \in \mathbb{Z}$, show that we have a surjective morphism

$$
\begin{aligned}
R_{V} \widetilde{\mathscr{O}}_{X}\left[\xi_{2}, \ldots, \xi_{n}\right] \otimes_{\widetilde{\mathbb{C}}} \widetilde{\mathbb{C}}\left[\tau^{\prime}\right] & \longrightarrow R_{V}\left(\widetilde{\mathscr{O}}_{X}\left[\tau, \xi_{2}, \ldots, \xi_{n}\right]\right) \\
V_{\ell} \widetilde{\mathscr{O}}_{X}\left[\xi_{2}, \ldots, \xi_{n}\right] q^{\ell} \tau^{\prime j} & \longmapsto V_{\ell} \widetilde{\mathscr{O}}_{X}\left[\xi_{2}, \ldots, \xi_{n}\right] \tau^{j} q^{\ell+j}
\end{aligned}
$$

If $K \subset X$ is any polycylinder show that $R_{V}\left(\widetilde{\mathscr{O}}_{X}\left[\tau, \xi_{2}, \ldots, \xi_{n}\right]\right)(K)$ is Noetherian, by using that $\left(R_{V} \widetilde{\mathscr{O}}_{X}(K)\right)\left[\tau^{\prime}, \xi_{2}, \ldots, \xi_{n}\right]$ is Noetherian.
(4) As $R_{V} \widetilde{\mathscr{D}}_{X}$ can be filtered (by the degree of the operators) in such a way that, locally on $X, \operatorname{gr} R_{V} \widetilde{\mathscr{D}}_{X}$ is isomorphic to $R_{V}\left(\widetilde{\mathscr{O}}_{X}\left[\tau, \xi_{2}, \ldots, \xi_{n}\right]\right)$, conclude that, if $K$ is any sufficiently small polycylinder, then $R_{V} \widetilde{\mathscr{D}}_{X}(K)$ is Noetherian.
(5) Use now arguments similar to that of [GM93] to concludes that $R_{V} \widetilde{\mathscr{D}}_{X}$ is coherent.

Definition 7.3.3 (Coherent $V$-filtrations). Let $\mathscr{M}$ be a coherent right $\widetilde{\mathscr{D}}_{X}$-module. A $V$-filtration indexed by $\mathbb{Z}$ is an increasing filtration $U . \mathscr{M}$ which satisfies $U_{\ell} \mathscr{M} \cdot V_{k} \widetilde{\mathscr{D}}_{X} \subset U_{\ell+k} \mathscr{M}$ for every $k, \ell \in \mathbb{Z}$. In particular, each $U_{\ell} \mathscr{M}$ is a right $V_{0} \widetilde{\mathscr{D}}_{X}$-module. We say that it is a coherent $V$-filtration if each $U_{\ell} \mathscr{M}$ is $V_{0} \widetilde{\mathscr{D}}_{X^{-}}$ coherent, locally on $X$, there exists $\ell_{o} \geqslant 0$ such that, for all $k \geqslant 0$,

$$
U_{-k-\ell_{0}} \mathscr{M}=U_{-\ell_{0}} \mathscr{M} \cdot t^{k} \quad \text { and } \quad U_{k+\ell_{0}} \mathscr{M}=\sum_{j=0}^{k} U_{\ell_{o}} \mathscr{M} \widetilde{\partial}_{t}^{j}
$$

Remark 7.3.4 (The case of left $\widetilde{\mathscr{D}}_{X}$-modules). For left $\widetilde{\mathscr{D}}$-modules, it is more usual to consider a decreasing filtration $U^{\bullet} \mathscr{M}$ which satisfies $V_{k} \widetilde{\mathscr{D}}_{X} \cdot U^{\ell} \mathscr{M} \subset U^{\ell-k} \mathscr{M}$ for every $k, \ell \in \mathbb{Z}$. We say that such a filtration is a coherent $V$-filtration if each $U^{\ell} \mathscr{M}$ is $V_{0} \widetilde{\mathscr{D}}_{X}$-coherent, locally on $X$, there exists $\ell_{o} \geqslant 0$ such that, for all $k \geqslant 0$,

$$
U^{k+\ell_{0}} \mathscr{M}=t^{k} U^{\ell_{o}} \mathscr{M} \quad \text { and } \quad U^{-\left(k+\ell_{0}\right)} \mathscr{M}=\sum_{j=0}^{k} \partial_{t}^{j} U^{-\ell_{o}} \mathscr{M}
$$

Exercise 7.3.5 (Characterization of coherent $V$-filtrations). Let $\mathscr{M}$ be a coherent right $\widetilde{\mathscr{D}}_{X}$-module. Show that the following properties are equivalent for a $V$-filtration $U \bullet \mathscr{M}$.
(1) $U \cdot \mathscr{M}$ is a coherent filtration.
(2) The Rees module $R_{U} \mathscr{M}:=\bigoplus_{\ell} U_{\ell} \mathscr{M} v^{\ell}$ is $R_{V} \widetilde{\mathscr{D}}_{X}$-coherent.
(3) For every $x \in X$, replacing $X$ with a small neighbourhood of $x$, there exist integers $\lambda_{j=1, \ldots, q}, \mu_{i=1, \ldots, p}, k_{i=1, \ldots, p}$ and a presentation (recall that $[\bullet]$ means a shift of the grading)

$$
\bigoplus_{j=1}^{q} \widetilde{\mathscr{D}}_{X}\left[\lambda_{j}\right] \longrightarrow \bigoplus_{i=1}^{p} \widetilde{\mathscr{D}}_{X}\left[\mu_{i}\right] \longrightarrow \mathscr{M} \longrightarrow 0
$$

such that $U_{\ell} \mathscr{M}=\operatorname{image}\left(\bigoplus_{i=1}^{p} V_{k_{i}+\ell} \widetilde{\mathscr{D}}_{X}\left[\mu_{i}\right]\right)$.
Note that, as for $\widetilde{\mathscr{I}}_{H}$-adic filtrations on coherent $\widetilde{\mathscr{O}}_{X}$-modules, it is not enough to check the coherence of $\operatorname{gr}_{U} \mathscr{M}$ as a $\operatorname{gr}^{V} \widetilde{\mathscr{D}}_{X}$-module in order to deduce that $U \cdot \mathscr{M}$ is a coherent $V$-filtration.

## Exercise 7.3.6 (From coherent $R_{V} \widetilde{\mathscr{D}}_{X}$-modules to $\widetilde{\mathscr{D}}_{X}$-modules with a coherent $V$-filtration)

(1) Show that a graded $R_{V} \widetilde{\mathscr{D}}_{X}$-module $\mathcal{M}$ can be written as $R_{U} \mathscr{M}$ for some $V$-filtration on some $\widetilde{\mathscr{D}}_{X}$-module $\mathscr{M}$ if and only if it has no $v$-torsion.
(2) Show that, if $\mathcal{M}$ is a graded coherent $R_{V} \widetilde{\mathscr{D}}_{X}$-module, then its $v$-torsion is a graded coherent $R_{V} \widetilde{\mathscr{D}}_{X}$-module.
(3) Conclude that, for any graded coherent $R_{V} \widetilde{\mathscr{D}}_{X}$-module $\mathcal{M}$, there exists a unique coherent $\widetilde{\mathscr{D}}_{X}$-module and a unique coherent $V$-filtration $U^{\bullet} \mathscr{M}$ such that $\mathcal{M} / v$-torsion $=R_{U} \mathscr{M}$.

## Exercise 7.3.7 (Some basic properties of coherent $V$-filtrations)

(1) Show that the filtration naturally induced by a coherent $V$-filtration on a coherent $\widetilde{\mathscr{D}}_{X}$-module on a coherent sub or quotient $\widetilde{\mathscr{D}}_{X}$-modules is a coherent $V$-filtration.
(2) Deduce that, locally on $X$, there exist integers $\lambda_{j=1, \ldots, q}, \ell_{j=1, \ldots, q}, \mu_{i=1, \ldots, p}$, $k_{i=1, \ldots, p}$ and a presentation $\bigoplus_{j=1}^{q} \widetilde{\mathscr{D}}_{X}\left[\lambda_{j}\right] \rightarrow \bigoplus_{i=1}^{p} \widetilde{\mathscr{D}}_{X}\left[\mu_{i}\right] \rightarrow \mathscr{M} \rightarrow 0$ inducing for every $\ell$ a presentation

$$
\bigoplus_{j=1}^{q} V_{\ell j}+\ell \widetilde{\mathscr{D}}_{X}\left[\lambda_{j}\right] \longrightarrow \bigoplus_{i=1}^{p} V_{k_{i}+\ell} \widetilde{\mathscr{D}}_{X}\left[\mu_{i}\right] \longrightarrow U_{\ell} \mathscr{M} \longrightarrow 0
$$

(3) Show that two coherent $V$-filtrations $U \cdot \mathscr{M}$ and $U^{\prime} \cdot \mathscr{M}$ are locally comparable, that is, locally on $X$ there exists $\ell_{o} \geqslant 0$ such that, for every $\ell \in \mathbb{Z}$,

$$
U_{\ell-\ell_{0}} \mathscr{M} \subset U_{\ell}^{\prime} \mathscr{M} \subset U_{\ell+\ell_{0}} \mathscr{M}
$$

(4) If $U \mathscr{M}$ is a coherent $V$-filtration, then for every $\ell_{o} \in \mathbb{Z}$, the filtration $U_{\bullet}+\ell_{o} \mathscr{M}$ is also coherent.
(5) If $U \cdot \mathscr{M}$ and $U^{\prime} \cdot \mathscr{M}$ are two coherent $V$-filtrations, then the filtration $U_{\ell}^{\prime \prime} \mathscr{M}:=$ $U_{\ell} \mathscr{M}+U_{\ell}^{\prime} \mathscr{M}$ is also coherent.
(6) Assume that $H$ is defined by an equation $t=0$. Prove that, locally on $X$, there exists $k_{0}$ such that, for every $k \leqslant k_{0}, t: U_{k} \rightarrow U_{k-1}$ is bijective. [Hint: Use (2) above.]

Exercise 7.3.8. Let $\mathscr{U}$ be a coherent left $V_{0} \widetilde{\mathscr{D}}_{X}$-module and let $\mathscr{T}$ be its $t$-torsion subsheaf, i.e., the subsheaf of local sections locally killed by some power of $t$. Show that, locally on $X$, there exists $\ell$ such that $\mathscr{T} \cap \mathscr{U} t^{\ell}=0$. [Hint: Consider the $t$-adic filtration on $V_{0} \widetilde{\mathscr{D}}_{X}$, i.e., the filtration $V_{-j} \widetilde{\mathscr{D}}_{X}$ with $j \geqslant 0$. Show that the filtration $\mathscr{U} t^{j}$ is coherent with respect to it, and locally there is a surjective morphism $\left(V_{0} \widetilde{\mathscr{D}}_{X}\right)^{n} \rightarrow \mathscr{U}$ which is strict with respect to the $V$-filtration. Deduce that its kernel $\mathscr{K}$ is coherent and comes equipped with the induced $V$-filtration, which is coherent. Conclude that, locally on $X$, there exists $j_{0} \geqslant 0$ such that $V_{j_{0}-j} \mathscr{K}=V^{j_{0}} \mathscr{K} \cdot t^{j}$ for every $j \geqslant 0$. Show that, for every $j \geqslant 0$ there is locally an exact sequence (up to shifting the grading on each $V_{\bullet} \widetilde{\mathscr{D}}_{X}$ summand)

$$
\left(V_{-j} \widetilde{\mathscr{D}}_{X}\right)^{m} \longrightarrow\left(V_{-\left(j+j_{0}\right)} \widetilde{\mathscr{D}}_{X}\right)^{n} \longrightarrow \mathscr{U} t^{\left(j+j_{0}\right)} \longrightarrow 0
$$

As $t: V_{k} \widetilde{\mathscr{D}}_{X} \rightarrow V_{k-1} \widetilde{\mathscr{D}}_{X}$ is bijective for $k \leqslant 0$, conclude that $t: \mathscr{U} t^{j_{0}} \rightarrow \mathscr{U} t^{j_{0}+1}$ is so, hence $\mathscr{T} \cap \mathscr{U} t^{j_{0}}=0$.]

Exercise 7.3.9 (Coherent $V$-filtration indexed by $A+\mathbb{Z}$ ). Extend the previous properties to coherent $V$-filtrations indexed by $A+\mathbb{Z}$, where $A \subset[0,1)$ is some finite set (see Remark 7.2.7).
7.3.b. Specializable coherent $\widetilde{\mathscr{D}}_{X}$-modules. Let $H \subset X$ be a smooth hypersurface. Let $\mathscr{M}$ be a left (resp. right) coherent $\widetilde{\mathscr{D}}_{X}$-module and let $m$ be a germ of section of $\mathscr{M}$. In the following, we abuse notation by denoting $\mathrm{E} \in V_{0} \widetilde{\mathscr{D}}_{X}$ any local lifting of the Euler operator $\mathrm{E} \in \operatorname{gr}_{0}^{V} \widetilde{\mathscr{D}}_{X}$, being understood that the corresponding formula does not depend on the choice of such a lifting.

## Definition 7.3.10.

(1) A weak Bernstein equation for $m$ is a relation

$$
\begin{equation*}
m \cdot\left(z^{\ell} b(\mathrm{E})-P\right)=0 \tag{7.3.10*}
\end{equation*}
$$

where

- $\ell$ is some nonnegative integer,
- $b(s)$ is a nonzero polynomial in a variable $s$ with coefficients in $\mathbb{C}$, which takes the form $\prod_{\alpha \in A}(s-\alpha z)^{\nu_{\alpha}}$ for some finite subset $A \in \mathbb{C}$ (depending on $m$ ),
- $P$ is a germ in $V_{-1} \widetilde{\mathscr{D}}_{X}$, i.e., $P=t Q=Q^{\prime} t$ with $Q, Q^{\prime}$ germs in $V_{0} \widetilde{\mathscr{D}}_{X}$.
(2) We say that $\mathscr{M}$ is specializable along $H$ if any germ of section of $\mathscr{M}$ is the solution of some weak Bernstein equation (7.3.10*).
Exercise 7.3.11. Show that a coherent $\widetilde{\mathscr{D}}_{X}$-module $\mathscr{M}$ is specializable along $H$ if and only if one of the following properties holds:
(1) locally on $X$, some coherent $V$-filtration $U \cdot \mathscr{M}$ has a weak Bernstein polynomial, i.e., there exists a nonzero $b(s)$ and a nonnegative integer $\ell$ such that

$$
\begin{equation*}
\forall k \in \mathbb{Z}, \quad \operatorname{gr}_{k}^{U} \mathscr{M} \cdot z^{\ell} b(\mathrm{E}-k z)=0 \tag{7.3.11*}
\end{equation*}
$$

(2) locally on $X$, any coherent $V$-filtration $U \cdot \mathscr{M}$ has a weak Bernstein polynomial. [Hint: in one direction, take the $V$-filtration generated by a finite number of local generators of $\mathscr{M}$; in the other direction, use that two coherent filtrations are locally comparable.]
Exercise 7.3.12. Assume that $\mathscr{M}$ is $\widetilde{\mathscr{D}}_{X}$-coherent and specializable along $H$.
(1) Fix $\ell_{o} \in \mathbb{Z}$ and set $U_{\ell^{\prime}}^{\prime} \mathscr{M}=U_{\ell+\ell_{o}} \mathscr{M}$. Show that $b_{U^{\prime}}(s)$ can be chosen as $b_{U}\left(s-\ell_{o} z\right)$.
(2) Set $b_{U}=b_{1} b_{2}$ where $b_{1}$ and $b_{2}$ have no common root. Show that the filtration $U_{k}^{\prime} \mathscr{M}:=U_{k-1} \mathscr{M}+b_{2}(\mathrm{E}-k z) U_{k} \mathscr{M}$ is a coherent filtration and compute a polynomial $b_{U^{\prime}}$ in terms of $b_{1}, b_{2}$.
(3) Conclude that there exists locally a coherent filtration $U \cdot \mathscr{M}$ for which $b_{U}(s)=$ $\prod_{\alpha \in A}(s-\alpha z)^{\nu_{\alpha}}$ and $\operatorname{Re}(A) \subset(-1,0]$.

Assume that $\mathscr{M}$ is $\widetilde{\mathscr{D}}_{X}$-coherent and specializable along $H$. According to Bézout, for every local section $m$ of $\mathscr{M}$, there exists a minimal polynomial

$$
b_{m}(s)=\prod_{\alpha \in R(m)}(s-\alpha z)^{\nu_{\alpha}}
$$

giving rise to a weak Bernstein equation (7.3.10*). We say that $\mathscr{M}$ is $\mathbb{R}$-specializable along $H$ if for every local section $m$, we have $R(m) \subset \mathbb{R}$. We then set:

$$
\begin{equation*}
\operatorname{ord}_{H}(m)=\max R(m) \tag{7.3.13}
\end{equation*}
$$

Exercise 7.3.14. Assume that $\mathscr{M}$ is an $\mathbb{R}$-specializable coherent $\widetilde{\mathscr{D}}_{X}$-module. Show that, for $m \in \mathscr{M}_{x_{o}}$ and $P \in V_{k} \widetilde{\mathscr{D}}_{X, x_{o}}$, we have

$$
\operatorname{ord}_{H, x_{o}}(m \cdot P) \leqslant \operatorname{ord}_{H, x_{o}}(m)+k
$$

[Hint: use that $\left[\mathrm{E}, V_{-1} \widetilde{\mathscr{D}}_{X}\right] \subset V_{0} \widetilde{\mathscr{D}}_{X}$ and that the coherent $V$-filtrations $\widetilde{\mathscr{D}}_{X}(m P) \cap$ $V_{\bullet} \widetilde{\mathscr{D}}_{X} \cdot m$ and $V_{\bullet} \widetilde{\mathscr{D}}_{X} \cdot m P$ of $\widetilde{\mathscr{D}}_{X} \cdot(m P)$ are locally comparable.]

The filtration by the order along $H$, also called the Kashiwara-Malgrange filtration of $\mathscr{M}$ along $H$, is the increasing filtration $V_{\bullet} \mathscr{M}_{x_{o}}$ indexed by $\mathbb{R}$ defined by

$$
\begin{align*}
V_{\alpha} \mathscr{M}_{x_{o}} & =\left\{m \in \mathscr{M}_{x_{o}} \mid \operatorname{ord}_{H, x_{o}}(m) \leqslant \alpha\right\},  \tag{7.3.15}\\
V_{<\alpha} \mathscr{M}_{x_{o}} & =\left\{m \in \mathscr{M}_{x_{o}} \mid \operatorname{ord}_{H, x_{o}}(m)<\alpha\right\} . \tag{7.3.16}
\end{align*}
$$

We do not claim that it is a coherent $V$-filtration. The order filtration satisfies, $\forall k \in \mathbb{Z}, \forall \alpha, \beta \in \mathbb{R}$

$$
V_{\alpha} \mathscr{M}_{x_{o}} \cdot V_{k} \widetilde{\mathscr{D}}_{X, x_{o}} \subset V_{\alpha+k} \mathscr{M}_{x_{o}} .
$$

It is a filtration of $\mathscr{M}$ by subsheaves $V_{\alpha} \mathscr{M}$ of $V_{0} \widetilde{\mathscr{D}}_{X}$-modules. We set

$$
\begin{equation*}
\operatorname{gr}_{\alpha}^{V} \mathscr{M}:=V_{\alpha} \mathscr{M} / V_{<\alpha} \mathscr{M} \tag{7.3.17}
\end{equation*}
$$

These are $\operatorname{gr}_{0}^{V} \widetilde{\mathscr{D}}_{X}$-modules. In particular, they are endowed with an action of the Euler field E. We already notice, as a preparation to strict $\mathbb{R}$-specializability, that the satisfy part of the strictness condition.

Lemma 7.3.18. The $\operatorname{gr}_{0}^{V} \widetilde{\mathscr{D}}_{X}$-module $\operatorname{gr}_{\alpha}^{V} \mathscr{M}$ has no $z$-torsion.
Proof. It is a matter of proving that, for a section $m$ of $V_{\alpha} \mathscr{M}$, if $m z^{j}$ is a section of $V_{<\alpha} \mathscr{M}$ for some $j \geqslant 0$, then so does $m$. But one checks in a straightforward way that, if $P$ in Exercise 7.3.14 is equal to $z^{j}$, then the inequality there is an equality (with $k=0$ ).

Remark 7.3.19 (The case of left $\widetilde{\mathscr{D}}_{X}$-modules). The order of a local section $m$ is defined
 The filtration by the order along $H$ is the decreasing filtration $V^{\bullet} \mathscr{M}_{x_{o}}$ indexed by $\mathbb{R}$ defined by

$$
\begin{aligned}
V^{\beta} \mathscr{M}_{x_{o}} & =\left\{m \in \mathscr{M}_{x_{o}} \mid \operatorname{ord}_{H, x_{o}}(m) \geqslant \beta\right\}, \\
V^{>\beta} \mathscr{M}_{x_{o}} & =\left\{m \in \mathscr{M}_{x_{o}} \mid \operatorname{ord}_{H, x_{o}}(m)>\beta\right\} .
\end{aligned}
$$

The order filtration satisfies, $\forall k \in \mathbb{Z}, \forall \alpha, \beta \in \mathbb{R}, V_{k} \widetilde{\mathscr{D}}_{X, x_{o}} \cdot V^{\beta} \mathscr{M}_{x_{o}} \subset V^{\beta-k} \mathscr{M}_{x_{o}}$. We set $\operatorname{gr}_{V}^{\beta} \mathscr{M}:=V^{\beta} \mathscr{M} / V^{>\beta} \mathscr{M}$.

## Exercise 7.3.20.

(1) Assume that $\mathscr{M}$ is $\mathbb{R}$-specializable along $H$. Show that any sub- $\widetilde{\mathscr{D}}_{X}$-module $\mathscr{M}^{\prime}$ and any quotient $\widetilde{\mathscr{D}}_{X}$-module $\mathscr{M}^{\prime \prime}$ is also $\mathbb{R}$-specializable along $H$.
(2) Let $\varphi: \mathscr{M}_{1} \rightarrow \mathscr{M}_{2}$ be a morphism between $\mathbb{R}$-specializable modules along $H$. Show that $\varphi$ is compatible with the order filtrations along $H$. Conclude that, on the full sbucategory consisting of $\mathbb{R}$-specializable $\widetilde{\mathscr{D}}_{X}$-modules of the category of $\widetilde{\mathscr{D}}_{X}$-modules (and morphisms consist of all morphisms of $\widetilde{\mathscr{D}}_{X}$-modules), $\operatorname{gr}_{\alpha}^{V}$ is a functor to the category of $\operatorname{gr}_{0}^{V} \widetilde{\mathscr{D}}_{X}$-modules.

Exercise 7.3.21 (Restriction to $z=1$ ). Let $\mathscr{M}$ be a coherent $R_{F} \mathscr{D}_{X}$-module. Assume that $\mathscr{M}$ is $\mathbb{R}$-specializable along $H$.
(1) Show that for every $\alpha$,

$$
(z-1) \mathscr{M} \cap V_{\alpha} \mathscr{M}=(z-1) V_{\alpha} \mathscr{M} .
$$

[Hint: let $m=(z-1) n$ be a local section of $(z-1) \mathscr{M} \cap V_{\alpha} \mathscr{M}$; then $n$ is a local section of $V_{\gamma} \mathscr{M}$ for some $\gamma$; if $\gamma>\alpha$, show that the class of $n \operatorname{in} \operatorname{gr}_{\gamma}^{V} \mathscr{M}$ is a annihilated by $z-1$; conclude with Exercise A.2.5(1).]
(2) Conclude that $\mathcal{M}:=\mathscr{M} /(z-1) \mathscr{M}$ is $\mathbb{R}$-specializable along $H$ and that, for every $\alpha$,

$$
\begin{aligned}
V_{\alpha} \mathcal{M} & =V_{\alpha} \mathscr{M} /(z-1) V_{\alpha} \mathscr{M}=V_{\alpha} \mathscr{M} /\left((z-1) \mathscr{M} \cap V_{\alpha} \mathscr{M}\right), \\
\operatorname{gr}_{\alpha}^{V} \mathcal{M} & =\operatorname{gr}_{\alpha}^{V} \mathscr{M} /(z-1) \operatorname{gr}_{\alpha}^{V} \mathscr{M} .
\end{aligned}
$$

(3) Show that $\left(V_{\alpha} \mathscr{M}\right) \otimes_{\mathbb{C}[z]} \mathbb{C}\left[z, z^{-1}\right]=V_{\alpha} \mathcal{M}\left[z, z^{-1}\right]$.

Exercise 7.3.22 (Side changing). Define the side changing functor for $V_{0} \widetilde{\mathscr{D}}_{X}$-modules by replacing $\widetilde{\mathscr{D}}_{X}$ with $V_{0} \widetilde{\mathscr{D}}_{X}$ in Definition A.3.10. Show that $\mathscr{M}^{\text {left }}$ is $\mathbb{R}$-specializable along $H$ if and only if $\mathscr{M}^{\text {right }}$ is so and, for every $\beta \in \mathbb{R}, V^{\beta}\left(\mathscr{M}^{\text {left }}\right)=$ $\left[V_{-\beta-1}\left(\mathscr{M}^{\text {right }}\right)\right]^{\text {left }}$. [Hint: Use the local computation of Exercise A.3.17.]
7.3.c. Strictly $\mathbb{R}$-specializable coherent $\widetilde{\mathscr{D}}_{X}$-modules. A drawback of the setting of Section 7.3.b is that we cannot ensure that the order filtration is a coherent $V$-filtration.

Lemma 7.3.23 (Kashiwara-Malgrange $V$-filtration). Let $\mathscr{M}$ be an $\mathbb{R}$-specializable coherent $\widetilde{\mathscr{D}}_{X}$-module. Assume that, in the neighbourhood of $x_{o} \in X$ there exists a coherent $V$-filtration $U . \mathscr{M}$ with the following two properties:
(1) its minimal weak Bernstein polynomial $b_{U}(s)=\prod_{\alpha \in A(U)}(s-\alpha z)^{\nu_{\alpha}}$ satisfies $A(U) \subset(-1,0]$,
(2) for every $k, U_{k} \mathscr{M} / U_{k-1} \mathscr{M}$ has no $z$-torsion.

Then such a filtration is unique and equal to the order filtration when considered indexed by integers, which is therefore a coherent $V$-filtration as such. It is called the Kashiwara-Malgrange filtration of $\mathscr{M}$.

Proof. Assume $U . \mathcal{M}$ satisfies (1) and (2). Let $m$ be a local section of $U_{k} \mathscr{M}$ and let $U .\left(m \cdot \widetilde{\mathscr{D}}_{X}\right)$ be the $V$-filtration induced by $U \cdot \mathscr{M}$ on $m \cdot \widetilde{\mathscr{D}}_{X}$. By Exercise 7.3.7(1), it is a coherent $V$-filtration. There exists thus $k_{o} \geqslant 1$ such that $U_{k-k_{o}}\left(m \cdot \widetilde{\mathscr{D}}_{X}\right) \subset$ $m \cdot V_{-1} \widetilde{\mathscr{D}}_{X}$. It follows that $R(m) \subset(A(U)+k) \cup \cdots \cup\left(A(U)+k-k_{o}+1\right)$ and thus $\operatorname{ord}_{H} m=\max R(m) \leqslant k$, so $m \subset V_{k} \mathscr{M}$.

Conversely, assume $m$ is a local section of $V_{k} \mathscr{M}$. It is also a local section of $U_{k+k_{o}} \mathscr{M}$ for some $k_{o} \geqslant 0$. Its class in $\operatorname{gr}_{k+k_{o}}^{U} \mathscr{M}$ is annihilated both by $z^{\ell} b_{m}(\mathrm{E})$ and by $z^{\ell^{\prime}} b_{U}\left(\mathrm{E}-\left(k+k_{o}\right) z\right.$ ) (for some $\ell, \ell^{\prime} \geqslant 0$ ), so if $k_{o}>0$, both polynomials have no common $z$-root, and this class is annihilated by some nonnegative power of $z$, according to Bézout. By Assumtion (2), it is zero, and $m$ is a local section of $U_{k+k_{o}-1} \mathscr{M}$, from which we conclude by induction that $m$ is a local section of $U_{k} \mathscr{M}$, as wanted.

Exercise 7.3.24 (Indexing with $\mathbb{Z}$ or with $\mathbb{R}$ ). The order filtration is naturally indexed by $\mathbb{R}$, while the notion of $V$-filtration considers filtrations indexed by $\mathbb{Z}$. The purpose of this exercise is to show how both notions match when the properties of Lemma 7.3.23 are satisfied. Let $U \cdot \mathscr{M}$ be a filtration for which the properties of Lemma 7.3.23 are satisfied. Then we have seen that $U . \mathscr{M}$ coincides with the "integral part" of the order filtration $V \bullet \mathscr{M}$. Show the following properties.
(1) The weak Bernstein equations $(7.3 .10 *)$ and $(7.3 .11 *)$ hold without any power of $z$, i.e., for every $k$ the operator $\mathrm{E}-k z$ has a minimal polynomial on $U_{k} \mathscr{M} / U_{k-1} \mathscr{M}=$ $V_{k} \mathscr{M} / V_{k-1} \mathscr{M}$ which does not depend on $k$.
(2) The eigen module of $\mathrm{E}-k z$ on this quotient module corresponding to the eigenvalue $\alpha z$ isomorphic to $\operatorname{gr}_{\alpha+k^{\prime}}^{V} \mathscr{M}$ and the corresponding nilpotent endomorphism is

$$
\begin{equation*}
\mathrm{N}:=(\mathrm{E}-(k+\alpha) z) \tag{7.3.24*}
\end{equation*}
$$

In particular, each $\operatorname{gr}_{\alpha+k}^{V} \mathscr{M}$ is strict and we have a canonical identification

$$
V_{k} \mathscr{M} / V_{k-1} \mathscr{M}=\bigoplus_{-1<\alpha \leqslant 0} \operatorname{gr}_{\alpha+k}^{V} \mathscr{M}
$$

(3) For every $\alpha \in(-1,0]$, identify $V_{\alpha+k} \mathscr{M}$ with the pullback of $\bigoplus_{-1<\alpha^{\prime} \leqslant \alpha} \operatorname{gr}_{\alpha^{\prime}+k}{ }^{\mathscr{M}}$ by the projection $V_{k} \mathscr{M} \rightarrow V_{k} \mathscr{M} / V_{k-1} \mathscr{M}$, and show that the shifted order filtration indexed by integers $V_{\alpha+} \cdot \mathscr{M}$ is a coherent $V$-filtration.
(4) Conclude that there exists a finite set $A \subset(-1,0]$ such that the order filtration is indexed by $A+\mathbb{Z}$, and is coherent as such (see Exercise 7.3.9).
Definition 7.3.25 (Strictly $\mathbb{R}$-specializable $\widetilde{\mathscr{D}}_{X}$-modules). Assume that $\mathscr{M}$ is $\mathbb{R}$-specializable along $H$. We say that it is strictly $\mathbb{R}$-specializable along $H$ if
(1) there exists a finite set $A \subset(-1,0]$ such that the filtration by the order along $H$ is a coherent $V$-filtration indexed by $A+\mathbb{Z}$,
and for some (or any) local decomposition $X \simeq H \times \Delta_{t}$,
(2) for every $\alpha<0, t: \operatorname{gr}_{\alpha}^{V} \mathscr{M} \rightarrow \operatorname{gr}_{\alpha-1}^{V} \mathscr{M}$ is onto,
(3) for every $\alpha>-1, \partial_{t}: \operatorname{gr}_{\alpha}^{V} \mathscr{M} \rightarrow \operatorname{gr}_{\alpha+1}^{V} \mathscr{M}(-1)$ is onto.

Proposition 7.3.26. Assume that $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along $H$. Then, every $\operatorname{gr}_{\alpha}^{V} \mathscr{M}$ is a graded $\operatorname{gr}_{0}^{V} \widetilde{\mathscr{D}}_{X}$-module, and is strict as such (see Definition A.2.7).

Proof. Recall that, for a graded module, strictness is equivalent to absence of $z$-torsion (see Exercise A.2.5(1)). Therefore, the second point follows from the first one and from Lemma 7.3.18.

Let us consider the first point. We first claim that a local section $m$ of $\mathscr{M}$ is a local section of $V_{\alpha} \mathscr{M}$ if and only if it satisfies a relation

$$
m \cdot b(\mathrm{E}) \in V_{\alpha} \mathscr{M}
$$

for some $b$ with $z$-roots $\leqslant \alpha$. Indeed, if $m$ is a local section of $V_{\beta} \mathscr{M}$ with $\beta>\alpha$ and satisfying such a relation, the Bézout argument already used and the absence of $z$-torsion on each $\operatorname{gr}_{\gamma}^{V} \mathscr{M}$ (Lemma 7.3.18) implies that $m$ is a local section of $V_{<\beta} \mathscr{M}$. Property $7.3 .25(1)$ implies that there is only a finite set of jumps of the $V$-filtration between $\alpha$ and $\beta$, so by induction we conclude that $m \in V_{\alpha} \mathscr{M}$. The converse is clear.

The grading on $\mathscr{M}$ induces a natural (right) action of $-\partial_{z} z$ on $\mathscr{M}$ : for a local section $m=\bigoplus_{p} m_{p}$ of $\mathscr{M}=\bigoplus_{p} \mathscr{M}^{p}$, we set $m\left(-\partial_{z} z\right):=\bigoplus_{p} p m_{p}$. This action is natural in the sense that it satisfies the usual commutation relations with the right action of $\widetilde{\mathscr{D}}_{X}$ (it would be more standard to use the natural left action of $z \partial_{z}$ on $\left.\mathscr{M}^{\text {left }}\right)$. We claim that, for every $\alpha \in \mathbb{R}$, we have $V_{\alpha} \mathscr{M}\left(-\partial_{z} z\right) \subset V_{\alpha} \mathscr{M}$. Let $m$ be a local section of $V_{\alpha} \mathscr{M}$, which satisfies a relation $m b_{m}(\mathrm{E})=m \cdot P$ with $P \in V_{-1} \widetilde{\mathscr{D}}_{X}$. Then one checks that

$$
\begin{aligned}
m\left(-\partial_{z} z\right) b_{m}(\mathrm{E}) & =m b_{m}(\mathrm{E})\left(-\partial_{z} z\right)+m Q, \quad Q \in V_{0} \widetilde{\mathscr{D}}_{X} \\
& =m P\left(-\partial_{z} z\right)+m Q, \quad P \in V_{-1} \widetilde{\mathscr{D}}_{X} \\
& =m\left(-\partial_{z} z\right) P+m R, \quad R \in V_{0} \widetilde{\mathscr{D}}_{X}
\end{aligned}
$$

We conclude that $m\left(-\partial_{z} z\right) \in V_{\alpha} \mathscr{M}$ by applying the first claim above.
Since the eigenvalues of $\left(-\partial_{z} z\right)$ on $\mathscr{M}$ are integers and are simple, the same property holds for $V_{\alpha} \mathscr{M}$, showing that $V_{\alpha} \mathscr{M}$ decomposes as the direct sum of its $\left(-\partial_{z} z\right)$ eigenspaces, which are its graded components of various degrees.

Remark 7.3.27 (The need of a shift). If we regard the actions of $t$ and $\partial_{t}$ as morphisms in $\operatorname{Mod}\left(\widetilde{\mathscr{D}}_{H}\right)$-modules, that is, graded morphisms of degree zero, we have to introduce a shift by -1 (see Remark A.2.4) for the action of $\partial_{t}$, which sends $F_{p} z^{p}$ to $F_{p+1} z^{p+1}$. The same shift has to be introduced for the action of $E$, as well as for that of $\mathrm{N}=$ $(\mathrm{E}-\alpha z)$.

Exercise 7.3.28. Check that if (2) and (3) hold for some local decomposition $X \simeq$ $H \times \Delta_{t}$ at $x_{o} \in H$, then they hold for any such decomposition.

Remark 7.3.29 (The case of left $\widetilde{\mathscr{D}}_{X}$-modules). For left $\widetilde{\mathscr{D}}_{X}$-modules, we take $\beta>-1$ in 7.3.25(2) and $\beta<0$ in 7.3.25(3) for $\operatorname{gr}_{V}^{\beta} \mathscr{M}$. The nilpotent endomorphism N of $\mathrm{gr}_{V}^{\beta} \mathscr{M}$ is induced by the action of $-(\mathrm{E}-\beta z)$.

Remark 7.3.30 (Side-changing). Let $\mathscr{M}$ be a left $\widetilde{\mathscr{D}}_{X}$-module and let $\mathscr{M}^{\text {right }}=\widetilde{\omega}_{X} \otimes \mathscr{M}$ denote the associated right $\widetilde{\mathscr{D}}_{X}$-module. Let us assume that $H$ is defined by one equation $g=0$, so that $\operatorname{gr}_{V}^{\beta} \mathscr{M}$ and $\operatorname{gr}_{\alpha}^{V} \mathscr{M}^{\text {right }}$ are respectively left and right $\widetilde{\mathscr{D}}_{H^{-}}$ modules endowed with an action of E .

Assume first that $\mathscr{M}=\widetilde{\mathscr{O}}_{X}$ and $\mathscr{M}^{\text {right }}=\widetilde{\omega}_{X}$. We have

$$
V^{k} \widetilde{\mathscr{O}}_{X}=\left\{\begin{array}{ll}
\widetilde{\mathscr{O}}_{X} & \text { if } k \leqslant 0, \\
g^{k} \widetilde{\mathscr{O}}_{X} & \text { if } k \geqslant 0,
\end{array} \quad \text { and } \quad V_{k} \widetilde{\omega}_{X}= \begin{cases}\widetilde{\omega}_{X} & \text { if } k \geqslant-1 \\
g^{-(k+1)} \widetilde{\omega}_{X} & \text { if } k \leqslant-1\end{cases}\right.
$$

We have $\operatorname{gr}{ }_{-1} \widetilde{\omega}_{X}=\widetilde{\omega}_{H} \otimes \widetilde{\mathrm{~d}} g / z$, so that $\widetilde{\mathrm{d}} g / z$ induces an isomorphism (see Remark A.2.4)

$$
\widetilde{\omega}_{H}(-1) \xrightarrow{\sim} \operatorname{gr}_{-1}^{V} \widetilde{\omega}_{X}, \quad \text { that is, } \quad \operatorname{gr}_{-1}^{V}\left(\widetilde{\mathscr{O}}_{X}^{\text {right }}\right) \simeq\left(\operatorname{gr}_{V}^{0} \widetilde{\mathscr{O}}_{X}\right)^{\text {right }}(-1)
$$

Arguing similarly for $\mathscr{M}$ and $\mathscr{M}^{\text {right }}$ gives an identification

$$
\operatorname{gr}_{\alpha}^{V}\left(\mathscr{M}^{\text {right }}\right) \simeq\left(\operatorname{gr}_{V}^{\beta} \mathscr{M}\right)^{\text {right }}(-1), \quad \beta=-\alpha-1
$$

With this identification, the actions of E (resp. N ) on both sides coincide.
Proposition 7.3.31. Assume that $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along $H$. Then, in any local decomposition $X \simeq H \times \Delta_{t}$ we have
(a) $\quad \forall \alpha<0, t: V_{\alpha} \mathscr{M} \longrightarrow V_{\alpha-1} \mathscr{M} \quad$ is an isomorphism;
(b) $\quad \forall \alpha \geqslant 0, V_{\alpha} \mathscr{M}=V_{<\alpha} \mathscr{M}+\left(V_{\alpha-1} \mathscr{M}\right) \partial_{t}$;
(c) $\quad t: \operatorname{gr}_{\alpha}^{V} \mathscr{M} \longrightarrow \operatorname{gr}_{\alpha-1}^{V} \mathscr{M} \quad$ is $\begin{cases}\text { an isomorphism } & \text { if } \alpha<0, \\ \text { injective } & \text { if } \alpha>0 ;\end{cases}$
(d) $\quad \partial_{t}: \operatorname{gr}_{\alpha}^{V} \mathscr{M} \longrightarrow \operatorname{gr}_{\alpha+1}^{V} \mathscr{M}(-1) \quad$ is $\begin{cases}\text { an isomorphism } & \text { if } \alpha>-1, \\ \text { injective } & \text { if } \alpha<-1 ;\end{cases}$

In particular (from (b)), $\mathscr{M}$ is generated as a $\widetilde{\mathscr{D}}_{\text {X }}$-module by $V_{0} \mathscr{M}$.
Proof. Because $V_{\alpha+.} \mathscr{M}$ is a coherent $V$-filtration, (a) holds for $\alpha \ll 0$ locally and (b) for $\alpha \gg 0$ locally. Therefore, (a) follows from (c) and (b) follows from (d). By 7.3.25(2) (resp. (3)), the map in (c) (resp. (d)) is onto. The composition $t \check{\partial}_{t}=$ $(\mathrm{E}-\alpha z)+\alpha z$ is injective on $\operatorname{gr}_{\alpha}^{V} \mathscr{M}$ for $\alpha \neq 0$ since $(\mathrm{E}-\alpha z)$ is nilpotent and $\operatorname{gr}_{\alpha}^{V} \mathscr{M}$ is strict, hence (c) holds. The argument for (d) is similar.

In the next exercises, we explain which set of data is needed to recover coherent $V_{0} \widetilde{\mathscr{D}}_{X}$-modules and morphisms between them. This will be used from a more general point of view in Chapter 9.

## Exercise 7.3.32 (Recovering morphisms from their restriction to $V_{0}$ )

Assume that $X=H \times \Delta_{t}$ and that $\mathscr{M}_{1}, \mathscr{M}_{2}$ are strictly $\mathbb{R}$-specializable along $H$. Let $\varphi_{\leqslant 0}: V_{0} \mathscr{M}_{1} \rightarrow V_{0} \mathscr{M}_{2}$ be a morphism in $\operatorname{Mod}\left(V_{0} \widetilde{\mathscr{D}}_{X}\right)$ such that the diagram $\left(\mathrm{D}_{0}\right)$ commutes:


Show that $\varphi \leqslant 0$ extends in a unique way as a morphism $\varphi: \mathscr{M}_{1} \rightarrow \mathscr{M}_{2}$. [Hint: For the uniqueness, use $7.3 .31(\mathrm{~b})$; show inductively the existence of $\varphi_{\leqslant k}: V_{k} \mathscr{M}_{1} \rightarrow V_{k} \mathscr{M}_{2}$ $(k \geqslant 1)$; for example, if $k=1$, use $7.3 .31(\mathrm{~d})$ to show that, for $m, m^{\prime}, n, n^{\prime} \in V_{0} \mathscr{M}_{1}$, if $m-m^{\prime}=\left(n^{\prime}-n\right) \mathscr{\partial}_{t}$, then $n^{\prime}-n \in V_{-1} \mathscr{M}_{2}$ and deduce that setting $\varphi_{\leqslant 1}\left(m+n \widetilde{\partial}_{t}\right)=$ $\varphi_{\leqslant 0}(m)+\varphi_{\leqslant 0}(n) \check{\partial}_{t}$ well defines a $V_{0} \widetilde{\mathscr{D}}_{X}$-linear morphism $\varphi_{\leqslant 1}: V_{1} \mathscr{M}_{1} \rightarrow V_{1} \mathscr{M}_{2}$ for which $\left(\mathrm{D}_{1}\right)$ commutes.]

Exercise 7.3.33 (Recovering $V_{0} \mathscr{M}$ ). Assume that $X=H \times \Delta_{t}$ and that $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along $H$. We have a natural exact sequence of $V_{0} \widetilde{\mathscr{D}}_{X}$-modules

$$
0 \longrightarrow V_{<0} \mathscr{M} \longrightarrow V_{0} \mathscr{M} \longrightarrow \operatorname{gr}_{0}^{V} \mathscr{M} \longrightarrow 0
$$

We wish to recover explicitly the middle term in terms of the extreme ones and of the morphisms (c) and (d) in Proposition 7.3.31 above, for the most interesting value $\alpha=0$.
(1) Consider the morphisms

$$
\begin{aligned}
& \operatorname{gr}_{-1}^{V} \mathscr{M}(1) \xrightarrow{A} V_{-1} \mathscr{M} \oplus \operatorname{gr}_{-1}^{V} \mathscr{M}(1) \oplus \operatorname{gr}_{0}^{V} \mathscr{M} \xrightarrow{B} \operatorname{gr}_{-1}^{V} \mathscr{M} \\
& e \longmapsto\left(0, e, e ð_{t}\right) \\
&(m, e, \varepsilon) \longmapsto\longrightarrow m]+e \cdot \check{\partial}_{t} t-\varepsilon \cdot t
\end{aligned}
$$

where, for $m \in V_{-1} \mathscr{M},[m]$ denotes its class in $\operatorname{gr}_{-1}^{V} \mathscr{M}$. Show that the composition is zero, hence they define a complex $C^{\bullet}$ of $V_{0} \widetilde{\mathscr{D}}_{X}$-modules (by regarding each $\operatorname{gr}_{\alpha}^{V} \mathscr{M}$ as a $V_{0} \widetilde{\mathscr{D}}_{X}$-module). Show that $H^{j}\left(C^{\bullet}\right)=0$ for $j \neq 1$.
(2) Consider the morphism from $V_{0} \mathscr{M}$ to the middle term given by $\mu \mapsto(\mu \cdot t, 0,[\mu])$, where $[\mu]$ denotes the class of $\mu$ in $\operatorname{gr}_{0}^{V} \mathscr{M}$. Show that it injects into Ker $B$ and that its intersection with $\operatorname{Im} A$ is zero. [Hint: Use 7.3.31(a).]
(3) Show that the induced morphism $V_{0} \mathscr{M} \rightarrow H^{1}\left(C^{\bullet}\right)$ is an isomorphism. [Hint: Injectivity follows from (2) above; modulo $\operatorname{Im} A$, any element of $\operatorname{Ker} B$ can be represented in a unique way as $(m, 0, \delta)$ with $[m]=\delta \cdot t$; choose any lifting $\widetilde{\delta} \in V_{0} \mathscr{M}$ of $\delta$ and show that there exists $\eta \in V_{<0} \mathscr{M}$ such that $m-\widetilde{\delta} \cdot t=\eta \cdot t$ by using 7.3.31(a); conclude by setting $\mu=\widetilde{\delta}+\eta$.]
(4) Show that, for any $V_{0} \widetilde{\mathscr{D}}_{X}$-linear morphism $\varphi_{\leqslant-1}: V_{-1} \mathscr{M}_{1} \rightarrow V_{-1} \mathscr{M}_{2}$, the diagram $\left(\mathrm{D}_{-1}\right)$ commutes, and conclude that giving a morphism $\varphi \leqslant 0 V_{0} \mathscr{M}_{1} \rightarrow V_{0} \mathscr{M}_{2}$
such that $\left(D_{0}\right)$ commutes is equivalent to giving a pair $\left(\varphi_{\leqslant-1}, \varphi_{0}\right)$ such that, with respect to the morphisms

and setting $\varphi_{-1}=\operatorname{gr}{ }_{-1}^{V} \varphi_{\leqslant-1}$, we have

$$
\partial_{t} \circ \varphi_{-1}=\varphi_{0} \circ \partial_{t}, \quad \varphi_{-1} \circ t=t \circ \varphi_{0}
$$

Assume that $X=H \times \Delta_{t}$. Consider the category whose objects consist of the data $\left(\mathscr{M}_{\leqslant-1}, \mathscr{M}_{0}, \mathrm{c}, \mathrm{v}\right)$, where

- $\mathscr{M}_{\leqslant-1}$ is a coherent $V_{0} \widetilde{\mathscr{D}}_{X}$-module on which $t$ is torsion-free and such that $\mathscr{M}_{-1}:=\mathscr{M}_{\leqslant-1} / \mathscr{M}_{\leqslant-1} t$ is strict and the induced action of $\mathscr{\partial}_{t} t$ on it is nilpotent with index of nilpotence locally bounded on $H$,
- $\mathscr{M}_{0}$ is a strict coherent $\operatorname{gr}_{0}^{V} \widetilde{\mathscr{D}}_{X}$-module on which the action of $t \check{\partial}_{t}$ is nilpotent with index of nilpotence locally bounded on $H$,
- the data c, v are $\operatorname{gr}_{0}^{V} \widetilde{\mathscr{D}}_{X}$-linear morphisms

such that $\mathrm{c} \circ \mathrm{v}=\partial_{t} t$ on $\mathscr{M}_{-1}$ and $\mathrm{v} \circ \mathrm{c}=t \check{\partial}_{t}$ on $\mathscr{M}_{0}$.
Morphisms in this category consist of pairs $\left(\varphi_{\leqslant-1}, \varphi_{0}\right)$, where $\varphi_{\leqslant-1}: \mathscr{M}_{\leqslant-1} \rightarrow \mathscr{N}_{\leqslant-1}$ is $V_{0} \widetilde{\mathscr{D}}_{X}$-linear, $\varphi_{0}: \mathscr{M}_{0} \rightarrow \mathscr{N}_{0}$ is $\operatorname{gr}_{0}^{V} \widetilde{\mathscr{D}}_{X}$-linear, and the restriction $\varphi_{-1}$ of $\varphi_{\leqslant-1}$ to $\mathscr{M}_{-1}$ satisfies

$$
\mathrm{c} \circ \varphi_{-1}=\varphi_{0} \circ \mathrm{c}, \quad \varphi_{-1} \circ \mathrm{v}=\mathrm{v} \circ \varphi_{0} .
$$

We have a functor from the category of coherent $\widetilde{\mathscr{D}}_{X}$-modules which are strictly $\mathbb{R}$-specializable along $H$ to the above category:

$$
\mathscr{M} \longmapsto\left(V_{-1} \mathscr{M}, \operatorname{gr}_{0}^{V} \mathscr{M}, \partial_{t}, t\right) .
$$

## Corollary 7.3.34 (Recovering morphisms from their restriction to $V_{-1}$ and $\operatorname{gr}_{0}^{V}$ )

This functor is fully faithful, i.e., any morphism $\left(\varphi_{\leqslant-1}, \varphi_{0}\right)$ can be lifted in a unique way as a morphism $\varphi$.

Proof. Consider the category whose objects are coherent $V_{0} \widetilde{\mathscr{D}}_{X}$-modules $\mathscr{M}_{\leqslant 0}$ such that

- $\mathscr{M}_{\leqslant 0} / \mathscr{M}_{\leqslant 0} t$ is strict,
- $t \check{\partial}_{t}$ acting on $\mathscr{M}_{\leqslant 0} / \mathscr{M}_{\leqslant 0} t$ has a minimal polynomial with roots $\alpha z$ satisfying $\alpha \in(-1,0]$,
- defining $V_{\alpha} \mathscr{M}$ for $\alpha<0$ as in Exercise 7.3.24, every $\operatorname{gr}_{\alpha}^{V} \mathscr{M}_{\leqslant 0}$ is strict and 7.3.31(a) holds,
and whose morphisms are $V_{0} \widetilde{\mathscr{D}}_{X}$-linear morphisms such that $\left(\mathrm{D}_{0}\right)$ commutes.
According to Exercise 7.3.32, the functor $\mathscr{M} \mapsto \mathscr{M}_{\leqslant 0}:=V_{0} \mathscr{M}$ is fully faithful. Now, the functor $\mathscr{M}_{\leqslant 0} \mapsto\left(V_{-1} \mathscr{M}_{\leqslant 0}, \operatorname{gr}_{0}^{V} \mathscr{M}_{\leqslant 0}, \partial_{t}, t\right)$ is an equivalence of categories. Indeed, Exercise 7.3.33 shows that it is essentially surjective and, since the reconstruction is functorial in an obvious way, it enables one to lift in a unique way a pair $\left(\varphi_{\leqslant-1}, \varphi_{0}\right)$ as a $V_{0} \widetilde{\mathscr{D}}_{X}$-linear morphism $\varphi_{\leqslant 0}$ such that $\left(\mathrm{D}_{0}\right)$ commutes, showing the full faithfulness.
Remark 7.3.35 (Restriction to $z=1$ ). Let us keep the notation of Exercise 7.3.21. For a coherent $\mathscr{D}_{X}$-module $\mathcal{M}$ which is $\mathbb{R}$-specializable, $7.3 .25(2)$ and (3) are automatically satisfied. Moreover, the morphisms in 7.3.31(c) and (d) are isomorphisms for the given values of $\alpha$. In other words, for coherent $\mathscr{D}_{X}$-modules, being $\mathbb{R}$-specializable is equivalent to being strictly $\mathbb{R}$-specializable. In particular, Exercise 7.3.21 applies to coherent $R_{F} \mathscr{D}_{X}$-modules which are strictly $\mathbb{R}$-specializable along $H$.
Exercise 7.3.36 (Structure of $\left.\mathscr{M} / V_{<\alpha_{o}} \mathscr{M}\right)$. Let $\mathscr{M}$ be a coherent right $\widetilde{\mathscr{D}}_{X}$-module which is strictly $\mathbb{R}$-specializable along $H$. Let us fix $\alpha_{o} \in \mathbb{R}$. Then $\mathscr{M} / V_{<\alpha_{o}} \mathscr{M}$ is a $V_{0} \widetilde{\mathscr{D}}_{X}$-module.
(1) Show that $\mathscr{M} / V_{<\alpha_{o}} \mathscr{M}$ is strict.
(2) Show that $\mathscr{M} / V_{<\alpha_{o}} \mathscr{M}$ decomposes as $\bigoplus_{\alpha \geqslant \alpha_{o}} \operatorname{Ker}(\mathrm{E}-\alpha z)^{N}$ with $N \gg 0$.
(3) Show that the $\alpha$-summand can be identified with $\operatorname{gr}_{\alpha}^{V} \mathscr{M}$.
(4) Show that $\mathscr{M} / V_{<\alpha_{o}} \mathscr{M}$ can be identified with $\bigoplus_{\alpha \geqslant \alpha_{o}} \operatorname{gr}_{\alpha}^{V} \mathscr{M}$ as a $V_{0} \widetilde{\mathscr{D}}_{X}$-module. Does the $V_{0} \widetilde{\mathscr{D}}_{X}$-module structure of $\mathscr{M} / V_{<\alpha_{o}} \mathscr{M}$ extend to a $\widetilde{\mathscr{D}}_{X}$-module structure? [Hint: in local coordinates, what about the relation $\left[\partial_{t}, t\right]=z$ applied to a nonzero section of $\operatorname{gr}_{\alpha_{o}}^{V} \mathscr{M}$ ?]
(5) Assume now that $X \simeq H \times \Delta_{t}$. Let $s$ be a new variable and let us equip $\operatorname{gr}_{\alpha}^{V} \mathscr{M}[s]:=\operatorname{gr}_{\alpha}^{V} \mathscr{M} \otimes_{\mathbb{C}} \mathbb{C}[s]$ with the following right $V_{0} \widetilde{\mathscr{D}}_{X}$-structure defined by

$$
\begin{aligned}
m_{j}^{\alpha} s^{j} \cdot t & = \begin{cases}0 & \text { if } j=0, \\
\left(m_{j}^{\alpha}(\mathrm{E}+j z)\right) s^{j-1} & \text { if } j \geqslant 1,\end{cases} \\
\left(m_{j}^{\alpha} s^{j}\right) t \check{\partial}_{t} & =\left(m_{j}^{\alpha}(\mathrm{E}+j z)\right) s^{j} .
\end{aligned}
$$

Check that this is indeed a $V_{0} \widetilde{\mathscr{D}}_{X}$-module structure (i.e., $\left[t \check{\partial}_{t}, t\right]$ acts as $z t$ ). Show that $\mathscr{M} / V_{-1} \mathscr{M}$ can be identified with $\bigoplus_{\alpha \in(-1,0]} \operatorname{gr}_{\alpha}^{V} \mathscr{M}[s]$. With this structure, show that $\operatorname{gr}_{\alpha}^{V} \mathscr{M} s^{j}=\operatorname{Ker}\left(t \partial_{t}-(\alpha+j) z\right)^{N}$ (with $N \gg 0$ locally).
[Hint: use that $\partial_{t}: \operatorname{gr}_{\alpha}^{V} \mathscr{M} \rightarrow \operatorname{gr}_{\alpha+1}^{V} \mathscr{M}$ is an isomorphism for $\alpha>-1$ to identify $\bigoplus_{\alpha>-1} \operatorname{gr}_{\alpha}^{V} \mathscr{M}$ with $\bigoplus_{\alpha \in(-1,0]} \bigoplus_{j \geqslant 0} \operatorname{gr}_{\alpha}^{V} \mathscr{M} \partial_{t}^{j}$.]
(6) Equip $\operatorname{gr}_{\alpha}^{V} \mathscr{M}[s]$ with the action of $\mathscr{\partial}_{t}$ defined by $\left(m_{j}^{\alpha} s^{j}\right) \check{\partial}_{t}=m_{j}^{\alpha} s^{j+1}$. Show that the relation $\left[\partial_{t}, t\right]=z$ holds on $\operatorname{sgr}_{\alpha}^{V} \mathscr{M}[s]$, but that $\left[\partial_{t}, t\right]=z+(\mathrm{E}+z)$ on $\operatorname{gr}_{\alpha}^{V} \mathscr{M}$. Conclude that this action does not define a $\widetilde{\mathscr{D}}_{X}$-module structure on $\operatorname{gr}_{\alpha}^{V} \mathscr{M}[s]$.

## Exercise 7.3.37 (First properties of strictly $\mathbb{R}$-specializable coherent $\widetilde{\mathscr{D}}_{X}$-modules)

Show the following properties.
(1) Let $\mathscr{M}$ be a coherent $\widetilde{\mathscr{D}}_{X}$-module which is strictly $\mathbb{R}$-specializable along $H$. If $\mathscr{M}=\mathscr{M}_{1} \oplus \mathscr{M}_{2}$ with $\mathscr{M}_{1}, \mathscr{M}_{2} \widetilde{\mathscr{D}}_{X}$-coherent, then $\mathscr{M}_{1}, \mathscr{M}_{2}$ are strictly $\mathbb{R}$-specializable along $H$.
(2) In an exact sequence $0 \rightarrow \mathscr{M}_{1} \rightarrow \mathscr{M} \rightarrow \mathscr{M}_{2} \rightarrow 0$ of coherent $\widetilde{\mathscr{D}}_{X}$-modules, if $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along $H$, set

$$
U_{\alpha} \mathscr{M}_{1}=V_{\alpha} \mathscr{M} \cap \mathscr{M}_{1}, \quad U_{\alpha} \mathscr{M}_{2}=\operatorname{image}\left(V_{\alpha} \mathscr{M}\right) .
$$

- Show that these $V$-filtrations are coherent (see Exercise 7.3.7(1)) and that, for every $\alpha$, the sequence

$$
0 \longrightarrow \operatorname{gr}_{\alpha}^{U} \mathscr{M}_{1} \longrightarrow \operatorname{gr}_{\alpha}^{V} \mathscr{M} \longrightarrow \operatorname{gr}_{\alpha}^{U} \mathscr{M}_{2} \longrightarrow 0
$$

is exact.

- Conclude that $U . \mathscr{M}_{1}$ satisfies the Bernstein property $7.3 .23(1)$ and the strictness property $7.3 .23(2)$ (with index set $\mathbb{R}$ ), and thus injectivity in 7.3 .31 (a) and (d), but possibly not 7.3.25(2) and (3). Deduce that $U_{\alpha} \mathscr{M}_{1}=V_{\alpha} \mathscr{M}_{1}$. [Hint: use the uniqueness property of Lemma 7.3.23.]
- If each $\operatorname{gr}_{\alpha}^{U} \mathscr{M}_{2}$ is also strict, show that $U_{\alpha} \mathscr{M}_{2}=V_{\alpha} \mathscr{M}_{2}$.
- If moreover one of both $\mathscr{M}_{1}, \mathscr{M}_{2}$ is strictly $\mathbb{R}$-specializable, then so is the other one.
(3) Let $\varphi: \mathscr{M}_{1} \rightarrow \mathscr{M}_{2}$ be any morphism between coherent $\widetilde{\mathscr{D}}_{X}$-modules which are strictly $\mathbb{R}$-specializable along $H$. Apply the previous result to $\operatorname{Im} \varphi$.
(4) Let $\iota: X \hookrightarrow X_{1}$ be a closed inclusion of complex manifolds, and let $H_{1} \subset X_{1}$ be a smooth hypersurface such that $H:=X \cap H_{1}$ is a smooth hypersurface of $X$. Then a coherent $\widetilde{\mathscr{D}}_{X}$-module $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along $H$ if and only if $\mathscr{M}_{1}:={ }_{\mathrm{D}} \iota_{*} \mathscr{M}$ is so along $H_{1}$, and we have, for every $\alpha$,

$$
\left(\operatorname{gr}_{\alpha}^{V} \mathscr{M}_{1}, \mathrm{~N}\right)=\left({ }_{\mathrm{D}} \iota_{*} \operatorname{gr}_{\alpha}^{V} \mathscr{M}, \mathrm{~N}\right)
$$

[Hint: assume that $X_{1}=H \times \Delta_{t} \times \Delta_{x}$ and $X=H \times \Delta_{t} \times\{0\}$, so that $\mathscr{M}_{1}=\iota_{*} \mathscr{M}\left[\right.$ वे $\left._{x}\right]$; show that the filtration $U_{\alpha} \mathscr{M}_{1}:=\iota_{*} V_{\alpha} \mathscr{M}\left[\tilde{\partial}_{x}\right]$ satisfies all the characteristic properties of the $V$-filtration of $\mathscr{M}_{1}$ along $H_{1}$.]

## Example 7.3.38 (Morphisms inducing an isomorphism on $V_{<0}$ )

Assume that $X=H \times \Delta_{t}$. Let $\mathscr{M}, \mathscr{N}$ be strictly $\mathbb{R}$-specializable along $H$ and let $\varphi: \mathscr{M} \rightarrow \mathscr{N}$ be a $\widetilde{\mathscr{D}}_{X}$-linear morphism. Since $\varphi$ is also $V_{0} \widetilde{\mathscr{D}}_{X}$-linear, it induces a morphism $\mathscr{M} / V_{\alpha_{o}} \mathscr{M} \rightarrow \mathscr{N} / V_{<\alpha_{o}} \mathscr{N}$ for each $\alpha_{o}$, which decomposes with respect to the decomposition 7.3.36(2). Each summand is then identified with $\operatorname{gr}_{\alpha}^{V} \varphi$. We will consider more specifically the case where $\varphi$ induces an isomorphism on $V_{<0}$.

We first claim that this condition implies that $\operatorname{Ker} \varphi$ and $\operatorname{Coker} \varphi$ are supported on $H$, that is, every local section of $\operatorname{Ker} \varphi, \operatorname{Coker} \varphi$ is annihilated by some power of $t$ (due to the $\widetilde{\mathscr{D}}_{H}$-coherence of these modules). For $\operatorname{Ker} \varphi$, this follows from $\operatorname{Ker} \varphi \cap V_{<0} \mathscr{M}=0$ together with the property that $t$ is nilpotent on $\mathscr{M} / V_{<0} \mathscr{M}$.

For Coker $\varphi$, we note that any local section $n$ of $\mathscr{N}$ satisfies $t^{k} n \in V_{<0} \mathscr{N}=\varphi\left(V_{<0} \mathscr{M}\right)$ for some $k$, hence $t^{k}$ is nilpotent on Coker $\varphi$.

The decomposition 7.3.36(2) induces decompositions $\operatorname{Ker} \varphi=\bigoplus_{k \geqslant 0} \operatorname{Kergr}_{k}^{V} \varphi$ and $\operatorname{Coker} \varphi=\bigoplus_{k \geqslant 0} \operatorname{Coker} \operatorname{gr}_{k}^{V} \varphi$ as $V_{0} \widetilde{\mathscr{D}}_{X}$-modules. Moreover, since E acts as 0 on $\operatorname{Ker} \operatorname{gr}_{0}^{V} \varphi$, Coker $\operatorname{gr}_{0}^{V} \varphi$, the obstruction in 7.3.36(6) (adapted to the present setting) to extending the $V_{0} \widetilde{\mathscr{D}}_{X}$-structure to a $\widetilde{\mathscr{D}}_{X}$-structure vanishes, and we conclude that the $\widetilde{\mathscr{D}}_{X}$-module $\operatorname{Ker} \varphi$, resp. Coker $\varphi$, is identified with the $\widetilde{\mathscr{D}}_{X}$-module ${ }_{\mathrm{D}} i_{H *} \operatorname{Ker}^{\operatorname{gr}}{ }_{0}^{V} \varphi$, resp. ${ }_{\mathrm{D}} i_{H *}$ Coker gr ${ }_{0}^{V} \varphi$.

Definition 7.3.39. A morphism $\varphi$ between strictly $\mathbb{R}$-specializable coherent left $\widetilde{\mathscr{D}}_{X}$-modules is said to be strictly $\mathbb{R}$-specializable if for every $\alpha \in[-1,0]$, the induced morphism $\operatorname{gr}_{\alpha}^{V} \varphi$ is strict (i.e., its cokernel is strict), and a similar property for right modules.

Proposition 7.3.40. If $\varphi$ is strictly $\mathbb{R}$-specializable, then $\operatorname{gr}_{\alpha}^{V} \varphi$ is strict for every $\alpha \in \mathbb{R}$, and $\operatorname{Ker} \varphi, \operatorname{Im} \varphi$ and $\operatorname{Coker} \varphi$ are strictly $\mathbb{R}$-specializable along $H$ and their $V$-filtrations are given by

$$
\begin{gathered}
V_{\alpha} \operatorname{Ker} \varphi=V_{\alpha} \mathscr{M} \cap \operatorname{Ker} \varphi, \quad V_{\alpha} \operatorname{Coker} \varphi=\operatorname{Coker}\left(\varphi_{\mid V_{\alpha} \mathscr{M}}\right), \\
V_{\alpha} \operatorname{Im} \varphi=\operatorname{Im}\left(\varphi_{\mid V_{\alpha} \mathscr{M}}\right)=V_{\alpha} \mathscr{N} \cap \operatorname{Im} \varphi .
\end{gathered}
$$

Proof. Let us endow $\operatorname{Ker} \varphi$ and $\operatorname{Coker} \varphi$ with the filtration $U$. naturally induced by $V_{\bullet} \mathscr{M}, V_{\bullet} \mathscr{N}$. By using $7.3 .31(\mathrm{c})$ and (d) for $\mathscr{M}$ and $\mathscr{N}$, we find that $\operatorname{gr}_{\alpha}^{U} \operatorname{Ker} \varphi$ and $\operatorname{gr}_{\alpha}^{U} \operatorname{Coker} \varphi$ are strict for every $\alpha \in \mathbb{R}$. By the uniqueness of the $V$-filtration, the first line in (7.3.40) holds, and therefore all properties of Definition 7.3.25 hold for $\operatorname{Ker} \varphi$ and $\operatorname{Coker} \varphi$. Now, $\operatorname{Im} \varphi$ has two possible coherent $V$-filtrations, one induced by $V . \mathscr{N}$ and the other one being the image of $V \bullet \mathscr{M}$. For the first one, strictness of $\operatorname{gr}_{\alpha} \operatorname{Im} \varphi$ holds, hence $\operatorname{Im} \varphi$ is strictly $\mathbb{R}$-specializable and $V_{\alpha} \operatorname{Im} \varphi=\operatorname{Im} \varphi \cap V_{\alpha} \mathscr{N}$. For the second one $U_{\alpha} \operatorname{Im} \varphi, \operatorname{gr}_{\alpha}^{U} \operatorname{Im} \varphi$ is identified with the image of $\operatorname{gr}_{\alpha}^{V} \varphi$, hence is also strict, so $U_{\bullet} \operatorname{Im} \varphi$ is also equal to $V_{\bullet} \operatorname{Im} \varphi$.

Corollary 7.3.41. Let $\mathscr{M}^{\bullet}=\left\{\cdots \xrightarrow{d_{i}} \mathscr{M}^{i} \xrightarrow{d_{i+1}} \cdots\right\}$ be a complex bounded above whose terms are $\widetilde{\mathscr{D}}_{X}$-coherent and strictly $\mathbb{R}$-specializable along $H$. Assume that, for every $\alpha \in[-1,0]$, the graded complex $\operatorname{gr}_{\alpha}^{V} \mathscr{M}^{\bullet}$ is strict, i.e., its cohomology is strict. Then each differential $d_{i}$ and each $\mathscr{H}^{i} \mathscr{M}^{\bullet}$ is strictly $\mathbb{R}$-specializable along $H$ and $\operatorname{gr}_{\alpha}^{V}$ commutes with taking cohomology.

Proof. By using 7.3.31(c) and (d) for each term of the complex $\operatorname{gr}_{\alpha}^{V} \mathscr{M}^{\bullet}$, we find that strictness of the cohomology holds for every $\alpha \in \mathbb{R}$. We argue by decreasing induction. Assume $\mathscr{M}^{k+1}=0$. Then the assumption implies that $d_{k}: \mathscr{M}^{k-1} \rightarrow \mathscr{M}^{k}$ is strictly $\mathbb{R}$-specializable, so we can apply Proposition 7.3 .40 to it. We then replace the complex by $\cdots \mathscr{M}^{k-2} \xrightarrow{d_{k-1}} \operatorname{Ker} d_{k} \rightarrow 0$ and apply the inductive assumption. Moreover, the strict $\mathbb{R}$-specializability of $\mathscr{M}^{k} / \operatorname{Ker} d_{k} \simeq \operatorname{Im} d_{k+1}$ implies that of $d_{k-1}$.

## Definition 7.3 .42 (Strictly $\mathbb{R}$-specializable $W$-filtered $\widetilde{\mathscr{D}}_{X}$-module)

Let $(\mathscr{M}, W \cdot \mathscr{M})$ be a coherent $\widetilde{\mathscr{D}}_{X}$-module endowed with a locally finite filtration by coherent $\widetilde{\mathscr{D}}_{X}$-submodules. We say that $\left(\mathscr{M}, W_{\bullet} \mathscr{M}\right)$ is a strictly $\mathbb{R}$-specializable (along $H$ ) filtered $\widetilde{\mathscr{D}}_{X}$-module if each $W_{\ell} \mathscr{M}$ and each $\operatorname{gr}_{\ell}^{W} \mathscr{M}$ is strictly $\mathbb{R}$-specializable.

Lemma 7.3.43. Let $(\mathscr{M}, W . \mathscr{M})$ be a strictly $\mathbb{R}$-specializable filtered $\widetilde{\mathscr{D}}_{X}$-module. Then each $W_{\ell} \mathscr{M} / W_{k} \mathscr{M}(k<\ell)$ is strictly $\mathbb{R}$-specializable along $H$.

Proof. By induction on $\ell-k \geqslant 1$, the case $\ell-k=1$ holding true by assumption. Let $U_{\text {. }}\left(W_{\ell} \mathscr{M} / W_{k} \mathscr{M}\right)$ be the $V$-filtration naturally induced by $V_{\bullet} W_{\ell} \mathscr{M}$. It is a coherent filtration. By induction we have $U_{\bullet}\left(W_{\ell-1} \mathscr{M} / W_{k} \mathscr{M}\right)=V_{\bullet}\left(W_{\ell-1} \mathscr{M} / W_{k} \mathscr{M}\right)$ and $U . \mathrm{gr}_{\ell}^{W} \mathscr{M}=V_{\bullet} \operatorname{gr}_{\ell}^{W} \mathscr{M}$. Similarly, $V_{\bullet} W_{\ell} \mathscr{M} \cap W_{\ell-1} \mathscr{M}=V_{\bullet} W_{\ell-1} \mathscr{M}$. We conclude that the sequence

$$
0 \longrightarrow \operatorname{gr}_{\bullet}^{V}\left(W_{\ell-1} \mathscr{M} / W_{k} \mathscr{M}\right) \longrightarrow \operatorname{gr}_{\bullet}^{U}\left(W_{\ell} \mathscr{M} / W_{k} \mathscr{M}\right) \longrightarrow \operatorname{gr}_{\bullet}^{V} \operatorname{gr}_{\ell}^{W} \mathscr{M} \longrightarrow 0
$$

is exact, hence the strictness of the middle term.

### 7.4. Nearby and vanishing cycle functors

We will now remember explicitly the grading in the case of $R_{F} \mathscr{D}_{X}$-modules. Recall (see $(\mathrm{A} .2 .3 *)$ and $(\mathrm{A} .2 .4 * *)$ ) that, given a graded object $M=\bigoplus_{p} M_{p}$ (with $M_{p}$ in degree $-p$ ), we set $M(k)=\bigoplus_{p} M(k)_{p}$ with $M(k)_{p}=M_{p-k}$. We have seen that, for strictly $\mathbb{R}$-specializable $R_{F} \mathscr{D}$-modules, the module $\operatorname{gr}_{\alpha}^{V} \mathscr{M}$ are graded $R_{F} \mathscr{D}$-modules in a natural way. Let us emphasize that, in Definition 7.3.25(2) and (3),

- the morphism $t$ is graded of degree zero,
- the morphism $\partial_{t}$ is graded of degree one; we thus write 7.3.25(3) as

$$
\partial_{t}: \operatorname{gr}_{\alpha}^{V} \mathscr{M} \xrightarrow{\sim} \operatorname{gr}_{\alpha}^{V} \mathscr{M}(-1) \quad \text { for } \alpha>-1
$$

Definition 7.4.1 (Nearby and vanishing cycle functors). Let $g: X \rightarrow \mathbb{C}$ be a holomorphic function. Let $X \xrightarrow{\iota_{g}} X \times \mathbb{C}$ denote the graph inclusion of $g$. We say that a right $\widetilde{\mathscr{D}}_{X}$-module $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along $g=0$ if $\mathscr{H}^{0}{ }_{\mathrm{D}} \iota_{g *} \mathscr{M}$ is strictly $\mathbb{R}$-specializable along $X \times\{0\}$. We then set

$$
\left\{\begin{array}{l}
\psi_{g, \lambda} \mathscr{M}:=\operatorname{gr}_{\alpha}^{V}\left(\mathscr{H}^{0}{ }_{\mathrm{D}} \iota_{g *} \mathscr{M}\right)(1), \quad \lambda=\exp (2 \pi \mathrm{i} \alpha), \alpha \in[-1,0),  \tag{7.4.2}\\
\phi_{g, 1} \mathscr{M}:=\operatorname{gr}_{0}^{V}\left(\mathscr{H}^{0}{ }_{\mathrm{D}} \iota_{g *} \mathscr{M}\right) .
\end{array}\right.
$$

Then $\psi_{g, \lambda} \mathscr{M}, \phi_{g, 1} \mathscr{M}$ are $\widetilde{\mathscr{D}}_{X}$-modules supported on $g^{-1}(0)$, endowed with an endomorphism E induced by $t \check{\partial}_{t}$. We set $\mathrm{N}=(\mathrm{E}-\alpha z)$.
Remark 7.4.3 (Choice of the shift). The choice of a shift (1) for $\psi_{g, \lambda} \mathscr{M}$ and no shift for $\phi_{g, 1} \mathscr{M}$ is justified by the following examples.
(1) If $\mathscr{M}=\widetilde{\omega}_{X \times \mathbb{C}}$ we have $\operatorname{gr}_{-1}^{V} \widetilde{\omega}_{X \times \mathbb{C}}(1) \simeq \widetilde{\omega}_{X}$ by identifying $\widetilde{\omega}_{X \times \mathbb{C}}$ with $\widetilde{\omega}_{X} \otimes_{\widetilde{O}_{X}} \widetilde{\mathscr{O}}_{X \times \mathbb{C}} \widetilde{\mathrm{d}} t / z$.
 module and $\iota: X \times\{0\} \hookrightarrow X \times \mathbb{C}$ is the inclusion, then $\operatorname{gr}_{0}^{V} \mathscr{M}=\mathscr{N}$.

Exercise 7.4.4. Justify that $\psi_{g, \lambda}$ and $\phi_{g, 1}$ are functors from the category of $\mathbb{R}$-specializable right $\widetilde{\mathscr{D}}_{X}$-modules to the category of right $\widetilde{\mathscr{D}}_{X}$-modules supported on $g^{-1}(0)$. [Hint: Use Exercise 7.3.20(2).]

Exercise 7.4.5. Let $\mathscr{M}$ be a right $\widetilde{\mathscr{D}}_{X}$-module. When $g$ is smooth and $g^{-1}(0)=H$, show that we have $\psi_{g, \lambda} \mathscr{M} \simeq{ }_{\mathrm{D}} \iota_{H *} \mathrm{gr}_{\alpha}^{V} \mathscr{M}(1)$ and $\phi_{g, 1} \mathscr{M}={ }_{\mathrm{D}} \iota_{H *} \operatorname{gr}_{0}^{V} \mathscr{M}$, where $i_{H}: H \hookrightarrow X$ denotes the inclusion.

## Remark 7.4.6 (Nearby/vanishing cycle functors for left $\widetilde{\mathscr{D}}_{X}$-modules)

For left $\widetilde{\mathscr{D}}_{X}$-modules, we also use the graph embedding. However, we now have ${ }_{\mathrm{D}} \iota_{g *} \mathscr{M}=\mathscr{H}^{-1}{ }_{\mathrm{D}} \iota_{g *} \mathscr{M}$. Therefore, one sets

$$
\left\{\begin{array}{l}
\psi_{g, \lambda} \mathscr{M}^{\text {left }}:=\operatorname{gr}_{V}^{\beta}\left(\mathscr{H}^{-1}{ }_{\mathrm{D}} \iota_{g *} \mathscr{M}^{\text {left }}\right), \\
\phi_{g, 1} \mathscr{M}^{\text {left }}:=\operatorname{gr}_{V}^{-1}\left(\mathscr{H}_{\mathrm{D}}^{-1} \iota_{g *} \mathscr{M}^{\text {left }}\right)(-1),
\end{array} \quad \lambda=\exp (-2 \pi \mathrm{i} \beta), \beta \in(-1,0]\right.
$$

with no shift of the grading in the first line, in order that $\operatorname{gr}_{V}^{0} \widetilde{\mathscr{O}}_{X \times \mathbb{C}}=\widetilde{\mathscr{O}}_{X \times\{0\}}$ (with grading). The nilpotent endomorphism N is induced by $-\left(t \partial_{t}-\beta z\right)$.

## Lemma 7.4.7 (Side-changing for the nearby/vanishing cycle functors)

The side-changing functor commutes with the nearby/vanishing cycle functors, namely

$$
\psi_{g, \lambda}\left(\mathscr{M}^{\text {right }}\right)=\left(\psi_{g, \lambda} \mathscr{M}^{\text {left }}\right)^{\text {right }}, \quad \phi_{g, 1}\left(\mathscr{M}^{\text {right }}\right)=\left(\phi_{g, 1} \mathscr{M}^{\text {left }}\right)^{\text {right }}
$$

Proof. If $\mathscr{N}$ is a left $\widetilde{\mathscr{D}}_{X \times \mathbb{C}}$-module which is strictly $\mathbb{R}$-specializable along $X \times\{0\}$, we have (see Remark 7.3.30)

$$
\operatorname{gr}_{\alpha}^{V}\left(\widetilde{\omega}_{X \times \mathbb{C}} \otimes \mathscr{N}\right) \simeq \widetilde{\omega}_{X} \otimes \operatorname{gr}_{V}^{\beta}(\mathscr{N})(1) \quad \forall \alpha \in \mathbb{R}, \beta=-\alpha-1
$$

We apply this to $\mathscr{N}=\mathscr{H}^{-1}{ }_{\mathrm{D}} \iota_{g *} \mathscr{M}^{\text {left }}$, so that $\mathscr{N}^{\text {right }}=\mathscr{H}^{0}{ }_{\mathrm{D}} \iota_{g *} \mathscr{M}^{\text {right }}$.
Proposition 7.4.8. Let $g: X \rightarrow \mathbb{C}$ be a holomorphic function and let $\mathscr{M}$ be a coherent $\widetilde{\mathscr{D}}_{X}$-module. Assume that $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along $g=0$. Then $\psi_{g, \lambda} \mathscr{M}$ and $\phi_{g, 1} \mathscr{M}$ are $\widetilde{\mathscr{D}}_{X}$-coherent.

Proof. By assumption, $\psi_{g, \lambda} \mathscr{M}$ and $\phi_{g, 1} \mathscr{M}$ are $\operatorname{gr}_{0}^{V} \widetilde{\mathscr{D}}_{X \times \mathbb{C}}=\widetilde{\mathscr{D}}_{X}[\mathrm{E}]$-coherent. Since $\mathrm{E}-\alpha z$ is nilpotent on $\psi_{g, \lambda} \mathscr{M}(\lambda=\exp (2 \pi \mathrm{i} \alpha))$, the $\widetilde{\mathscr{D}}_{X}$-coherence follows.

Definition 7.4.9 (Morphisms N, can and var). Assume that $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along $g=0$. The nilpotent operator $\mathrm{N}=\left(t \mathrm{\partial}_{t}-\alpha z\right)$ is a morphism

$$
\psi_{g, \lambda} \mathscr{M} \xrightarrow{\mathrm{~N}} \psi_{g, \lambda} \mathscr{M}(-1), \quad \phi_{g, 1} \mathscr{M} \xrightarrow{\mathrm{~N}} \phi_{g, 1} \mathscr{M}(-1)
$$

When $\lambda=1$, the nilpotent operator N on $\psi_{g, 1} \mathscr{M}$ and $\phi_{g, 1} \mathscr{M}$ is the operator obtained as the composition var ocan and can $\circ$ var in the diagram below:

with the same convention as in (3.2.15).
Remark 7.4.10 (The case of left $\widetilde{\mathscr{D}}_{X}$-modules). In this case we have $\mathrm{N}=-\left(t \check{\partial}_{t}-\beta z\right)$ and the diagram


Exercise 7.4.11. Similarly to Exercise 7.4.5, show that, if $X=H \times \Delta_{t}$ and $g$ is the projection to $\Delta_{t}$, so that $\iota_{g}$ is induced by the diagonal embedding $\Delta_{t} \hookrightarrow \Delta_{t_{1}} \times \Delta_{t_{2}}$, then can $=\partial_{t_{2}}$ and var $=t_{2}$ for ${ }_{\mathrm{D}} \iota_{g *} \mathscr{M}$ are ${ }_{\mathrm{D}} \iota_{g *}\left(\partial_{t_{1}}\right)$ and ${ }_{\mathrm{D}} \iota_{g *}\left(t_{1}\right)$, with $\partial_{t_{1}}=\partial_{t}$ : $\operatorname{gr}_{-1}^{V} \mathscr{M} \rightarrow \operatorname{gr}_{0}^{V} \mathscr{M}(-1)$ and $t_{1}=t: \operatorname{gr}_{0}^{V} \mathscr{M} \rightarrow \operatorname{gr}_{-1}^{V} \mathscr{M}$.

Definition 7.4.12 (Monodromy operator). We work with right $\mathscr{D}_{X}$-modules. Assume that $\mathcal{M}$ is $\mathbb{R}$-specializable along $(g)$. The monodromy operator T on $\psi_{g, \lambda} \mathcal{M}$ is the operator induced by $\exp \left(2 \pi \mathrm{i} t \partial_{t}\right)$ (for left $\mathscr{D}_{X}$-modules $\mathrm{T}=\exp \left(-2 \pi \mathrm{i} t \partial_{t}\right)$ ), and $\mathrm{T}-\lambda \mathrm{Id}$ is nilpotent, and the nilpotent operator N is given by $\frac{1}{2 \pi \mathrm{i}} \log (\mathrm{T}-\lambda \mathrm{Id})$ on $\psi_{g, \lambda} \mathcal{M}$. On $\psi_{g, 1} \mathcal{M}, \phi_{g, 1} \mathcal{M}$ we have $T=\exp 2 \pi i N$ and $N=\frac{1}{2 \pi \mathrm{i}} \log (\mathrm{T}-\mathrm{Id})$.

## Remark 7.4.13 (Monodromy filtration on nearby and vanishing cycles)

The monodromy filtration relative to N on $\psi_{g, \lambda} \mathscr{M}$ and $\phi_{g, 1} \mathscr{M}$ (see Exercise 3.1.1 and Remark 3.1.10) is well-defined in the abelian category of graded $\widetilde{\mathscr{D}}_{X}$-modules with the automorphism $\sigma$ induced by the shift (1) of the grading (or in the abelian category of $\mathscr{D}_{X}$-modules). The Lefschetz decomposition holds in this category, with respect to the corresponding primitive submodules $\mathrm{P}_{\ell} \psi_{g, \lambda} \mathscr{M}, \mathrm{P}_{\ell} \phi_{g, 1} \mathscr{M}$ for $\ell \geqslant 0$.

Nevertheless, strict $\mathbb{R}$-specializability is not sufficient to ensure that each such primitive submodule (hence each graded piece of the monodromy filtration) is strict. The following proposition gives a criterion for the strictness of the primitive parts.

Proposition 7.4.14. Assume $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along $(g)$ and fix $\lambda \in S^{1}$. The following properties are equivalent.
(1) For every $\ell \geqslant 1, \mathrm{~N}^{\ell}: \psi_{g, \lambda} \mathscr{M} \rightarrow \psi_{g, \lambda} \mathscr{M}(-\ell)$ is a strict morphism.
(2) For every $\ell \in \mathbb{Z}$, $\operatorname{gr}_{\ell}^{\mathrm{M}} \psi_{g, \lambda} \mathscr{M}$ is strict.
(3) For every $\ell \geqslant 0, \mathrm{P}_{\ell} \psi_{g, \lambda} \mathscr{M}$ is strict.

We have a similar assertion for $\phi_{g, 1} \mathcal{M}$.
Proof. This is Proposition 3.1.11.

## Remark 7.4.15 (Restriction to $z=1$ of the monodromy filtration)

If $\mathscr{M}$ is a coherent $R_{F} \mathscr{D}_{X}$-module which is strictly $\mathbb{R}$-specializable along $D$ and setting $\mathcal{M}=\mathscr{M} /(z-1) \mathscr{M}$, we have $\psi_{g, \lambda} \mathcal{M}=\psi_{g, \lambda} \mathscr{M} /(z-1) \psi_{g, \lambda} \mathscr{M}$ and $\phi_{g, 1} \mathcal{M}=$ $\phi_{g, 1} \mathscr{M} /(z-1) \phi_{g, 1} \mathscr{M}$, according to Exercise 7.3.21, and the morphisms can and var for $\mathscr{M}$ obviously restrict to the morphisms can and var for $\mathcal{M}$, as well as the nilpotent endomorphism N .

Similarly, the monodromy filtration M.(N) on $\psi_{g, \lambda} \mathscr{M}, \phi_{g, 1} \mathscr{M}$ restricts to the monodromy filtration $\mathrm{M} .(\mathrm{N})$ on $\psi_{g, \lambda} \mathcal{M}, \phi_{g, 1} \mathcal{M}$, since everything behaves $\mathbb{C}\left[z, z^{-1}\right]$-flatly after tensoring with $\mathbb{C}\left[z, z^{-1}\right]$.

Exercise 7.4.16 (Strict specializability along $\left\{t^{r}=0\right\}$ ). Let $t$ be a smooth function on $X$, set $X_{0}=t^{-1}(0)$ and assume that $X=X_{0} \times \mathbb{C}$. Let $\mathscr{M}$ be a coherent $\widetilde{\mathscr{D}}_{X}$-module which is strictly $\mathbb{R}$-specializable along $t=0$. The purpose of this exercise is to show that $\mathscr{M}$ is then also strictly $\mathbb{R}$-specializable along $g=t^{r}=0$ for every $r \geqslant 2$, and to compare nearby cycles of $\mathscr{M}$ with respect to $t$ and to $g$.

Following the steps below, show that $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along $\{g=0\}$ and, denoting by $\iota: X_{0} \hookrightarrow X$ the closed inclusion,
(a) $\left(\psi_{g, \lambda} \mathscr{M}, \mathrm{~N}\right)=\left({ }_{\mathrm{D}} \iota_{*}\left(\psi_{t, \lambda^{r}} \mathscr{M}\right), \mathrm{N} / r\right)$ for every $\lambda$,
(b) $\left(\phi_{g, 1} \mathscr{M}, \mathrm{~N}\right)=\left({ }_{\mathrm{D}} \iota_{*}\left(\phi_{t, 1} \mathscr{M}\right), \mathrm{N} / r\right)$,
(c) there is an isomorphism
(1) Write ${ }_{\mathrm{D}} \iota_{g *} \mathscr{M}=\bigoplus_{k \in \mathbb{N}} \mathscr{M} \otimes \delta \widetilde{\mathrm{\partial}}_{u}^{k}$ as a $\widetilde{\mathscr{D}}_{X}[u]\left\langle\partial_{u}\right\rangle$-module, with

$$
\begin{aligned}
& (m \otimes \delta) \partial_{u}^{k}=m \otimes \delta \partial_{u}^{k} \quad \forall k \geqslant 0, \\
& (m \otimes \delta) \mathrm{\partial}_{t}=\left(m \check{\mathrm{\partial}}_{t}\right) \otimes \delta-\left(r t^{r-1} m\right) \otimes \delta \check{\mathrm{\Xi}}_{u}, \\
& (m \otimes \delta) u=\left(m t^{r}\right) \otimes \delta, \\
& (m \otimes \delta) \widetilde{\mathscr{O}}_{X}=\left(m \widetilde{\mathscr{O}}_{X}\right) \otimes \delta,
\end{aligned}
$$

and with the usual commutation rules. Show the relation

$$
r(m \otimes \delta) u \check{\partial}_{u}=\left[m t \check{\partial}_{t}\right] \otimes \delta-(m t \otimes \delta) \partial_{t}
$$

(2) We will denote by $V^{t}$ the $V$-filtration with respect to the variable $t$ and by $V^{u}$ that with respect to the variable $u$.

For $\alpha \leqslant 0$, set

$$
U_{\alpha}\left(\mathrm{D}_{\mathrm{D}} \iota_{g *} \mathscr{M}\right):=\left(V_{r \alpha}^{t} \mathscr{M} \otimes \delta\right) \cdot V_{0}^{u}\left(\widetilde{\mathscr{D}}_{X}[u]\left\langle\widetilde{\partial}_{u}\right\rangle\right)
$$

and for $\alpha>0$ define inductively

$$
U_{\alpha}\left({ }_{\mathrm{D}} \iota_{g *} \mathscr{M}\right):=U_{<\alpha}\left({ }_{\mathrm{D}} \iota_{g *} \mathscr{M}\right)+U_{\alpha-1}\left(\mathrm{D}_{\mathrm{D}} \iota_{g *} \mathscr{M}\right) \text { ஓ}_{u} .
$$

Assume that $\alpha \leqslant 0$. Using the above relation show that, if
then

$$
\begin{aligned}
V_{r \alpha}^{t} \mathscr{M}\left(t \partial_{t}-r \alpha z\right)^{\nu_{r \alpha}} & \subset V_{<r \alpha}^{t} \mathscr{M} \\
U_{\alpha}\left({ }_{\mathrm{D}}^{\iota_{g *}} \mathscr{M}\right)\left(u ฎ_{u}-\alpha z\right)^{\nu_{r \alpha}} & \subset U_{<\alpha}\left(\mathrm{p} \iota_{g *} \mathscr{M}\right),
\end{aligned}
$$

and conclude that $\left.u \check{\partial}_{u}-\alpha z\right)$ is nilpotent on $\operatorname{gr}_{\alpha}^{U}\left({ }_{\mathrm{D}} \iota_{g *} \mathscr{M}\right)$ for $\alpha \leqslant 0$.
(3) Show that if $m_{1}, \ldots, m_{\ell}$ generate $V_{r \alpha}^{t} \mathscr{M}$ over $V_{0}^{t} \widetilde{\mathscr{D}}_{X}$, then $m_{1} \otimes \delta, \ldots, m_{\ell} \otimes \delta$ generate $U_{\alpha}\left({ }_{\mathrm{D}} \iota_{g *} \mathscr{M}\right)$ over $V_{0}^{u}\left(\widetilde{\mathscr{D}}_{X}[u]\left\langle\check{\partial}_{u}\right\rangle\right)$, by using the relation

$$
\left(m t \check{\partial}_{t}\right) \otimes \delta=(m \otimes \delta)\left(t \check{\partial}_{t}-r u ð_{u}\right)
$$

Conclude that $U_{\alpha}\left(\mathrm{D}_{g} \iota_{*} \mathscr{M}\right)$ is $V_{0}^{u}\left(\widetilde{\mathscr{D}}_{X}[u]\left\langle\partial_{u}\right\rangle\right)$-coherent for every $\alpha \leqslant 0$, hence for every $\alpha$.
(4) Show that, for every $\alpha$,
$U_{\alpha-1}\left({ }_{\mathrm{D}} \iota_{g *} \mathscr{M}\right) \subset U_{\alpha}\left({ }_{\mathrm{D}} \iota_{g *} \mathscr{M}\right) u$, resp. $U_{\alpha+1}\left({ }_{\mathrm{D}} \iota_{g *} \mathscr{M}\right) \subset U_{<\alpha+1}\left({ }_{\mathrm{D}} \iota_{g *} \mathscr{M}\right)+U_{\alpha}\left({ }_{\mathrm{D}} \iota_{g *} \mathscr{M}\right)$ д $_{u}$, with equality if $\alpha<0$ (resp. if $\alpha \geqslant-1$ ). [Hint: Use the analogous property for $\mathscr{M}$.] Deduce that $U_{\bullet}\left({ }_{\mathrm{D}} \iota_{g *} \mathscr{M}\right)$ is a coherent $V$-filtration.
(5) Show that, for $\alpha \leqslant 0, V_{\alpha}^{u}\left({ }_{\mathrm{D}} \iota_{g *} \mathscr{M}\right)=V_{<\alpha}^{u}\left({ }_{\mathrm{D}} \iota_{g *} \mathscr{M}\right)+\sum_{k \geqslant 0}\left(V_{r \alpha}^{t} \mathscr{M} \otimes \delta\right){\underset{\partial}{t}}_{k}^{k}$. Deduce, by considering the degree in $\check{\partial}_{u}$, that the natural map

$$
\begin{aligned}
\underset{k}{\bigoplus} \operatorname{gr}_{r \alpha}^{V^{t}} \mathscr{M} \otimes \partial_{t}^{k} & \longrightarrow \operatorname{gr}_{\alpha}^{V^{u}}\left({ }_{\mathrm{D}} \iota_{g *} \mathscr{M}\right) \\
\bigoplus_{k}\left[m_{k}\right] \otimes \partial_{t}^{k} & \longmapsto\left[\sum_{k}\left(m_{k} \otimes \delta\right) \mathrm{\partial}_{t}^{k}\right]
\end{aligned}
$$

is an isomorphism of $\widetilde{\mathscr{D}}_{X}$-modules. Deduce that $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along $g=0$ with (increasing) Kashiwara-Malgrange filtration $V_{\bullet}^{u}\left({ }_{\mathrm{D}} \iota_{g *} \mathscr{M}\right)$ equal to $U .\left(\mathrm{D} \iota_{g *} \mathscr{M}\right)$. Conclude the proof of (a) and (b), and then that of (c).

### 7.5. Strict non-characteristic restrictions

7.5.a. Non-characteristic property. Let $\iota_{Y}: Y \hookrightarrow X$ denote the inclusion of a closed submanifold with ideal $\mathscr{I}_{Y}$ (in local coordinates $\left(x_{1}, \ldots, x_{n}\right), \mathscr{I}_{Y}$ is generated by $x_{1}, \ldots, x_{p}$, where $\left.p=\operatorname{codim} Y\right)$. The pullback functor ${ }_{\mathrm{D}} \iota_{Y}^{*}$ is defined in Section A.7. The case of left $\widetilde{\mathscr{D}}_{X}$-modules is easier to treat, so we will consider left $\widetilde{\mathscr{D}}_{X}$-modules and the corresponding setting for the $V$-filtration in this section.

Let us make the construction explicit in the case of a closed inclusion. A local section $\xi$ of $\iota_{Y}^{-1} \widetilde{\Theta}_{X}$ (vector field on $X$, considered at points of $Y$ only; we denote by $\iota_{Y}^{-1}$ the sheaf-theoretic pullback) is said to be tangent to $Y$ if, for every local section $g$ of $\widetilde{\mathscr{I}_{Y}}, \xi(g) \in \widetilde{\mathscr{I}_{Y}}$. This defines a subsheaf $\widetilde{\Theta}_{X \mid Y}$ of $\iota_{Y}^{-1} \widetilde{\Theta}_{X}$. Then $\widetilde{\Theta}_{Y}=$ $\widetilde{\mathscr{O}}_{Y} \otimes_{\iota_{Y}^{-1}} \widetilde{\mathscr{O}}_{X} \widetilde{\Theta}_{X \mid Y}=\iota_{Y}^{*} \widetilde{\Theta}_{X \mid Y}$ is a subsheaf of $\iota_{Y}^{*} \widetilde{\Theta}_{X}$.

Given a left $\widetilde{\mathscr{D}}_{X}$-module, the action of $\iota_{Y}^{-1} \widetilde{\Theta}_{X}$ on $\iota_{Y}^{-1} \mathscr{M}$ restricts to an action of $\widetilde{\Theta}_{Y}$ on $\iota_{Y}^{*} \mathscr{M}=\widetilde{\mathscr{O}}_{Y} \otimes_{\iota_{Y}^{-1}} \widetilde{\mathscr{O}}_{X} \iota_{Y}^{-1} \mathscr{M}$. The criterion of Exercise A.3.1 is fulfilled since it is fulfilled for $\widetilde{\Theta}_{X}$ and $\mathscr{M}$, defining therefore a left $\widetilde{\mathscr{D}}_{Y}$-module structure on $\iota_{Y}^{*} \mathscr{M}$ : this is ${ }_{\mathrm{D}} \iota_{Y}^{*} \mathscr{M}$.

Without any other assumption, coherence is not preserved by ${ }_{\mathrm{D}} \iota_{Y}^{*}$. For example, ${ }_{\mathrm{D}} \iota_{Y}^{*} \widetilde{\mathscr{D}}_{X}$ is not $\widetilde{\mathscr{D}}_{Y}$-coherent if $\operatorname{codim} Y \geqslant 1$. A criterion for coherence of the pullback is given below in terms of the characteristic variety.

The cotangent map to the inclusion defines a natural bundle morphism $\varpi: T^{*} X_{\mid Y} \times \mathbb{C}_{z} \rightarrow T^{*} Y \times \mathbb{C}_{z}$, the kernel of which is by definition the conormal bundle $T_{Y}^{*} X \times \mathbb{C}_{z}$ of $Y \times \mathbb{C}_{z}$ in $X \times \mathbb{C}_{z}$.

Definition 7.5 .1 (Non-characteristic property). Let $\mathscr{M}$ be a holonomic $\widetilde{\mathscr{D}}_{X}$-module with characteristic variety Char $\mathscr{M}$ contained in $\Lambda \times \mathbb{C}_{z}$, where $\Lambda \subset T^{*} X$ is Lagrangean (see Section A.10.c). Let $Y \subset X$ be a submanifold of $X$. We say that $Y$ is non-characteristic with respect to the holonomic $\widetilde{\mathscr{D}}_{X}$-module $\mathscr{M}$, or that $\mathscr{M}$ is noncharacteristic along $Y$, if one of the following equivalent conditions is satisfied:

- $\left(T_{Y}^{*} X \times \mathbb{C}_{z}\right) \cap \operatorname{Char} \mathscr{M} \subset T_{X}^{*} X \times \mathbb{C}_{z}$,
- $\varpi:$ Char $\mathscr{M}_{\mid Y \times \mathbb{C}_{z}} \rightarrow T^{*} Y \times \mathbb{C}_{z}$ is finite, i.e., proper with finite fibres.

Exercise 7.5.2. Show that both conditions in Definition 7.5.1 are indeed equivalent. [Hint: use the homogeneity property of Char $\mathscr{M}$.]

## Theorem 7.5.3 (Coherence of non-characteristic restrictions)

Assume that $\mathscr{M}$ is $\widetilde{\mathscr{D}}_{X}$-coherent and that $Y$ is non-characteristic with respect to $\mathscr{M}$. Then ${ }_{\mathrm{D}} \iota_{Y}^{*} \mathscr{M}$ is $\widetilde{\mathscr{D}}_{Y}$-coherent and $\operatorname{Char}_{\mathrm{D}} \iota_{Y}^{*} \mathscr{M} \subset \varpi\left(\operatorname{Char} \mathscr{M}_{\mid Y}\right)$.

Sketch of proof. The question is local near a point $x \in Y$. We may therefore assume that $\mathscr{M}$ has a coherent filtration $F_{\bullet} \mathscr{M}$.
(1) Set $F_{k_{\mathrm{D}} \iota_{Y}^{*}} \mathscr{M}=\operatorname{image}\left[\iota_{Y}^{*} F_{k} \mathscr{M} \rightarrow \iota_{Y}^{*} \mathscr{M}\right]$. Then, using Exercise A.10.8(2), one shows that $F_{\bullet} \iota_{Y}^{*} \mathscr{M}$ is a coherent filtration with respect to $F_{\bullet D} \iota_{Y}^{*} \widetilde{\mathscr{D}}_{X}$.
(2) The module $\mathrm{gr}_{\mathrm{D}} \iota_{Y}^{*} \mathscr{M}$ is a quotient of $\iota_{Y}^{*} \mathrm{gr}^{F} \mathscr{M}$, hence its support is contained in Char $\mathscr{M}_{\mid Y}$. By Remmert's Theorem, it is a coherent $\mathrm{gr}^{F} \widetilde{\mathscr{D}}_{Y}$-module.
(3) The filtration $F_{\bullet \mathrm{D}} \iota_{Y}^{*} \mathscr{M}$ is thus a coherent filtration of the $\widetilde{\mathscr{D}}_{Y}$-module ${ }_{\mathrm{D}} \iota_{Y}^{*} \mathscr{M}$. By Exercise A.10.5(1), ${ }_{\mathrm{D}} \iota_{Y}^{*} \mathscr{M}$ is $\widetilde{\mathscr{D}}_{Y}$-coherent. Using the coherent filtration above, it is clear that Char $_{\mathrm{D}} \iota_{Y}^{*} \mathscr{M} \subset \varpi\left(\operatorname{Char} \mathscr{M}_{\mid Y}\right)$.

Exercise 7.5.4. With the assumptions of Theorem 7.5.3, show similarly that, if $Y$ is defined by $x_{1}=\cdots=x_{p}=0$ then, considering the map $\boldsymbol{x}: X \rightarrow \mathbb{C}^{p}$ induced by $\boldsymbol{x}:=\left(x_{1}, \ldots, x_{p}\right)$, then $\mathscr{M}$ is $\widetilde{\mathscr{D}}_{X / \mathbb{C}^{p}}$-coherent.

Definition 7.5.5 (Strict non-characteristic property). In the setting of Definition 7.5.1, we say that $\mathscr{M}$ is strictly non-characteristic along $Y$ if $\mathscr{M}$ is non-characteristic along $Y$ and moreover $L_{\mathrm{D}} \iota_{Y}^{*} \mathscr{M}=\widetilde{\mathscr{O}}_{Y} \otimes_{\iota_{Y}^{-1}}^{L} \widetilde{\mathscr{O}}_{X} \iota_{Y}^{-1} \mathscr{M}$ is strict.

Proposition 7.5.6. If $\mathscr{M}$ is strictly non-characteristic along $Y$, then $\boldsymbol{L}_{\mathrm{D}} \iota_{Y}^{*} \mathscr{M}={ }_{\mathrm{D}} \iota_{Y}^{*} \mathscr{M}$.
Proof. The result holds for $\mathscr{D}_{X}$-modules, and therefore it holds after tensoring with $\mathbb{C}\left[z, z^{-1}\right]$. As a consequence, $\mathscr{H}^{j} \boldsymbol{L}_{\mathrm{D}} \iota^{*} \mathscr{M}$ is a $z$-torsion module if $j \neq 0$. It is strict if and only if it is zero.

Proposition 7.5.7. Assume that codim $Y=1$ and denote it by $H$. Then if $\mathscr{M}$ is strictly non-characteristic along $H$, it is also strictly $\mathbb{R}$-specializable along $H$ and ${ }_{\mathrm{D}} \iota_{H}^{*} \mathscr{M}$ is naturally identified with $\operatorname{gr}_{V}^{0} \mathscr{M}$.

Proof. Since the question is local, we may assume that $X \simeq H \times \Delta_{t}$. The previous proposition says that $t: \mathscr{M} \rightarrow \mathscr{M}$ is injective and the definition amounts to the strictness of $\mathscr{M} / t \mathscr{M}$.

Since $\mathscr{M}$ is $\widetilde{\mathscr{D}}_{X / \mathbb{C}}$-coherent (Exercise 7.5.4), the filtration defined by $U^{k} \mathscr{M}=t^{k} \mathscr{M}$ $(k \in \mathbb{N})$ is a coherent $V$-filtration and $\mathrm{E}: \operatorname{gr}_{U}^{0} \mathscr{M} \rightarrow \operatorname{gr}_{U}^{0} \mathscr{M}$ acts by 0 since $\partial_{t} U^{0} \mathscr{M} \subset$ $U^{0} \mathscr{M}=\mathscr{M}$. It follows that $\mathscr{M}$ is specializable along $H$ and that the Bernstein polynomial of the filtration $U^{\bullet} \mathscr{M}$ has only integral roots. Moreover, $t: \operatorname{gr}_{U}^{k} \mathscr{M} \rightarrow$ $\operatorname{gr}_{U}^{k+1} \mathscr{M}$ is onto for $k \geqslant 0$. We will show by induction on $k$ that each $\operatorname{gr}_{U}^{k} \mathscr{M}$ is strict. The assumption is that $\operatorname{gr}_{U}^{0} \mathscr{M}$ is strict. We note that $\mathrm{E}-k z$ acts by zero on $\operatorname{gr}_{U}^{k} \mathscr{M}$. If $\operatorname{gr}_{U}^{k} \mathscr{M}$ is strict, then the composition $\partial_{t} t$ acts by $(k+1) z$ on $\operatorname{gr}_{U}^{k} \mathscr{M}$, hence is injective, so $t: \operatorname{gr}_{U}^{k} \mathscr{M} \rightarrow \operatorname{gr}_{U}^{k+1} \mathscr{M}$ is bijective, and $\operatorname{gr}_{U}^{k+1} \mathscr{M}$ is thus strict. It follows that $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along $H$, and the $t$-adic filtration $U^{\bullet} \mathscr{M}$ is equal to the $V$-filtration.

Locally, we have an identification ${ }_{\mathrm{D}} \iota_{H}^{*} \mathscr{M}=\mathscr{M} / t \mathscr{M}=\operatorname{gr}_{V}^{0} \mathscr{M}$. We note that $\operatorname{gr}_{V}^{0} \mathscr{M}$ is naturally a $\widetilde{\mathscr{D}}_{H}$-module since E acts by 0 , and $\widetilde{\mathscr{D}}_{H}=\operatorname{gr}_{0}^{V} \widetilde{\mathscr{D}}_{X} / \mathrm{E} \operatorname{gr}_{0}^{V} \widetilde{\mathscr{D}}_{X}$. Therefore the previous identification is global.
Remark 7.5.8 (The case of right $\widetilde{\mathscr{D}}_{X}$-modules). Let $\mathscr{M}$ be a left $\widetilde{\mathscr{D}}_{X}$-module and let $\mathscr{M}^{\text {right }}:=\widetilde{\omega}_{X} \otimes_{\widetilde{\sigma}_{X}} \mathscr{M}$ be the associated right $\widetilde{\mathscr{D}}_{X}$-module (with grading). If $\mathscr{M}$ is strictly non-characteristic along $H$, then so is $\mathscr{M}^{\text {right }}$. We have

$$
{ }_{\mathrm{D}} \iota_{H}^{*} \mathscr{M}^{\mathrm{right}}:=\widetilde{\omega}_{H} \otimes_{\widetilde{\mathscr{O}}_{H} \mathrm{D}} \iota_{H}^{*} \mathscr{M}=\widetilde{\omega}_{H} \otimes_{\widetilde{O}_{H}} \operatorname{gr}_{V}^{0} \mathscr{M}=\operatorname{gr}_{-1}^{V} \mathscr{M}^{\mathrm{right}}(1)
$$

according to Remark 7.3.30.
Assume that $H$ is globally defined by the smooth function $g$. Then

$$
{ }_{\mathrm{D}} \iota_{H * \mathrm{D}} \iota_{H}^{*} \mathscr{M}^{\text {right }}={ }_{\mathrm{D}} \iota_{H *} \operatorname{gr}_{V}^{0} \mathscr{M}=\operatorname{gr}_{-1}^{V} \mathscr{M}^{\text {right }}(1)=\psi_{g, 1} \mathscr{M}^{\text {right }}
$$

according to Exercise 7.4.5.
7.5.b. Specialization of a strictly non-characteristic divisor with normal crossings. We make explicit an example of computation of nearby cycles along a divisor with normal crossings in a simple situation, anticipating more complicated computations in Chapter 11. Let $D=D_{1} \cup D_{2}$ be a divisor with normal crossings in $X$ and smooth irreducible components $D_{1}, D_{2}$. We set $D_{1,2}=D_{1} \cap D_{2}$, which is a smooth
manifold of codimension two in $X$. Let $\mathscr{M}$ be a left $\widetilde{\mathscr{D}}_{X}$-module which is strictly noncharacteristic along $D_{1}, D_{2}$ and $D_{1,2}$. Let us summarize some consequences of the assumption on nearby cycles. In local coordinates we will set $D_{i}=\left\{x_{i}=0\right\}(i=1,2)$.
(a) $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along $D_{1}$ and $D_{2}$. We denote by $V_{(i)} \mathscr{M}$ the $V$-filtration of $\mathscr{M}$ along $D_{i}(i=1,2)$.
(b) $\operatorname{gr}_{V_{(i)}}^{\beta} \mathscr{M}=0$ if $\beta \notin \mathbb{N}$.
(c) $\operatorname{gr}_{V_{(i)}}^{0} \mathscr{M}={ }_{\mathrm{D}} \iota_{D_{i}}^{*} \mathscr{M}=\iota_{D_{i}}^{*} \mathscr{M}$. In local coordinates, $\operatorname{gr}_{V_{(i)}}^{0} \mathscr{M}=\mathscr{M} / x_{i} \mathscr{M}$.

Lemma 7.5.9. For $i=1,2$, the $\widetilde{\mathscr{D}}_{D_{i}}$-module ${ }_{\mathrm{D}} \iota_{D_{i}}^{*} \mathscr{M}$ is strictly non-characteristic, hence strictly $\mathbb{R}$-specializable, along $D_{1,2}$ and $V_{(2)}^{\bullet} \operatorname{gr}_{V_{(1)}}^{0} \mathscr{M}$ is the filtration induced by $V_{(2)}^{\bullet} \mathscr{M}$, and conversely, so that

$$
\operatorname{gr}_{V_{(2)}}^{0} \operatorname{gr}_{V_{(1)}}^{0} \mathscr{M}=\operatorname{gr}_{V_{(1)}}^{0} \operatorname{gr}_{V_{(2)}}^{0} \mathscr{M}={ }_{\mathrm{D}} \iota_{D_{1,2}}^{*} \mathscr{M}=\iota_{D_{1,2}}^{*} \mathscr{M}
$$

Proof. The first point is mostly obvious, giving rise to the last formula, according to (c). For the second point, we have to check in local coordinates that $x_{2}^{k}\left(\mathscr{M} / x_{1} \mathscr{M}\right)=$ $x_{2}^{k} \mathscr{M} / x_{1} x_{2}^{k} \mathscr{M}$ for every $k \geqslant 1$, that is, the morphism

$$
\mathscr{M} / x_{1} \mathscr{M} \xrightarrow{x_{2}^{k}} x_{2}^{k} \mathscr{M} / x_{1} x_{2}^{k} \mathscr{M}
$$

is an isomorphism. Recall (see Exercise 7.5.4) that $\mathscr{M}$ is $\widetilde{\mathscr{D}}_{X / \mathbb{C}^{2} \text {-coherent, so by taking }}$ a local resolution by free $\widetilde{\mathscr{D}}_{X / \mathbb{C}^{2}}$-modules, we are reduced to proving the assertion for $\mathscr{M}=\widetilde{\mathscr{D}}_{X / \mathbb{C}^{2}}^{\ell}$, where it is obvious.

Exercise 7.5.10. Conclude from the lemma that $\left(x_{1}, x_{2}\right)$ is a regular sequence on $\mathscr{M}$, i.e., $x_{1} \mathscr{M} \cap x_{2} \mathscr{M}=x_{1} x_{2} \mathscr{M}$. Show that, for every $k \geqslant 1$, if we have a relation $\sum_{k_{1}+k_{2}=k} x_{2}^{k_{1}} x_{1}^{k_{2}} m_{k_{1}, k_{2}}=0$ in $\mathscr{M}$, then there exist $\mu_{i, j} \in \mathscr{M}$ for $i, j \geqslant 0$ (and the convention that $\mu_{i, j}=0$ if $i$ or $j \leqslant-1$ ) such that $m_{k_{1}, k_{2}}=x_{1} \mu_{k_{1}-1, k_{2}}-x_{2} \mu_{k_{1}, k_{2}-1}$ for every $k_{1}, k_{2}$.

Our aim is to compute, in the local setting, the nearby cycles of $\mathscr{M}$ along $g=x_{1} x_{2}$ (after having proved that $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along ( $g$ ), of course). We consider then the graph inclusion $\iota_{g}: X \hookrightarrow X \times \mathbb{C}_{t}$. We will return to the right setting, so we assume $\mathscr{M}=\mathscr{M}^{\text {right }}$, but the following proposition also holds in the left case after side-changing.

Proposition 7.5.11. Under the previous assumptions, the $\widetilde{\mathscr{D}}_{X \times \mathbb{C}}$-module ${ }_{\mathrm{D}} \iota_{g *} \mathscr{M}$ is a minimal extension along $(t)$, we have $\psi_{g, \lambda} \mathscr{M}=0$ for $\lambda \neq 1$ and there are local isomorphisms

$$
\mathrm{P}_{\ell} \psi_{g, 1} \mathscr{M} \simeq \begin{cases}\psi_{x_{1}, 1} \mathscr{M} \oplus \psi_{x_{2}, 1} \mathscr{M} & \text { if } \ell=0  \tag{7.5.11*}\\ \psi_{x_{1}, 1} \psi_{x_{2}, 1} \mathscr{M}(-1)=\psi_{x_{2}, 1} \psi_{x_{1}, 1} \mathscr{M}(-1) & \text { if } \ell=1 \\ 0 & \text { otherwise }\end{cases}
$$

Proof. We set $\mathscr{N}={ }_{\mathrm{D}} \iota_{g *} \mathscr{M}$. We have $\mathscr{N}=\iota_{g *} \mathscr{M}\left[\mathrm{\partial}_{t}\right]$ with the usual structure of a right $\widetilde{\mathscr{D}}_{X \times \mathbb{C}}$-module (see Example A.8.9). We identify $\iota_{g *} \mathscr{M}$ as the component of $\partial_{t}$-degree zero in $\mathscr{N}$. Let $U \cdot \mathscr{N}$ denote the filtration defined by

$$
U_{-1}(\mathscr{N})=\iota_{g *} \mathscr{M} \cdot \widetilde{\mathscr{D}}_{X} \subset \mathscr{N}, \quad U^{k}(\mathscr{N})= \begin{cases}U_{-1}(\mathscr{N}) \cdot t^{k} & \text { if } k \geqslant 0 \\ \sum_{\ell \leqslant-k} U_{-1}(\mathscr{N}) \cdot \check{\partial}_{t}^{\ell} & \text { if } k \leqslant 0\end{cases}
$$

We wish to prove that $U^{\bullet} \mathscr{N}$ satisfies all the properties of the $V$-filtration of $\mathscr{N}$.
Let $m$ be a local section of $\mathscr{M}$. From the relation

$$
\begin{equation*}
(m \otimes 1) \check{\mathrm{\partial}}_{x_{1}}=\left(m \check{\mathrm{\partial}}_{x_{1}}\right) \otimes 1-m x_{2} \otimes \mathrm{\partial}_{t} \tag{7.5.12}
\end{equation*}
$$

we deduce

$$
\begin{align*}
& (m \otimes 1) \partial_{t} t=\left(m \check{\partial}_{x_{1}} x_{1}\right) \otimes 1-(m \otimes 1) x_{1} \check{\partial}_{x_{1}} \\
& =\left(m \partial_{x_{2}} x_{2}\right) \otimes 1-(m \otimes 1) x_{2} \text { ஓ }_{x_{2}}, \tag{7.5.13}
\end{align*}
$$

 generators of $\mathscr{M}$ (see Exercise 7.5.4), we deduce that it is a set of $\widetilde{\mathscr{D}}_{X}$-generators, hence of $V_{0} \widetilde{\mathscr{D}}_{X \times \mathbb{C}_{t}}$-generators, of $U_{-1}(\mathscr{N})$. It follows that $U^{\bullet}(\mathscr{N})$ is a good $V$-filtration of $\mathscr{N}$. Moreover, the formulas above imply

giving a Bernstein relation. Since $\left(\partial_{t} t\right)^{2}$ vanishes on $\operatorname{gr}_{-1}^{U}(\mathscr{N})$, the monodromy filtration is given by

$$
\begin{aligned}
\mathrm{M}_{-2} \operatorname{gr}_{-1}^{U}(\mathscr{N}) & =0, & \mathrm{M}_{-1} \operatorname{gr}_{-1}^{U}(\mathscr{N}) & =\operatorname{gr}_{-1}^{U}(\mathscr{N}) \cdot \partial_{t} t \\
\mathrm{M}_{0} \operatorname{gr}_{-1}^{U}(\mathscr{N}) & =\operatorname{Ker}\left[\check{\partial}_{t} t: \operatorname{gr}_{-1}^{U}(\mathscr{N}) \rightarrow \operatorname{gr}_{-1}^{U}(\mathscr{N})\right], & \mathrm{M}_{1} \operatorname{gr}_{-1}^{U}(\mathscr{N}) & =\operatorname{gr}_{-1}^{U}(\mathscr{N})
\end{aligned}
$$

As a consequence,

$$
\begin{aligned}
& \mathrm{P}_{0} \operatorname{gr}_{-1}^{U}(\mathscr{N})=\operatorname{gr}_{0}^{\mathrm{M}} \operatorname{gr}_{-1}^{U}(\mathscr{N})=\operatorname{Ker}_{t} t / \operatorname{Im} \check{\partial}_{t} t, \\
& \mathrm{P}_{1} \operatorname{gr}_{-1}^{U}(\mathscr{N})=\operatorname{gr}_{1}^{\mathrm{M}} \operatorname{gr}_{-1}^{U}(\mathscr{N})=\operatorname{gr}_{-1}^{U}(\mathscr{N}) / \operatorname{Ker} \check{\partial}_{t} t \xrightarrow{\sim} \mathrm{M}_{-1} \operatorname{gr}_{-1}^{U}(\mathscr{N})(-1) .
\end{aligned}
$$

We will identify these $\widetilde{\mathscr{D}}_{X}$-modules with those given in the statement. This will also prove that $\operatorname{gr}_{-1}^{U}(\mathscr{N})$ is strict, because $\psi_{x_{1}, 1} \mathscr{M}, \psi_{x_{2}, 1} \mathscr{M}, \psi_{x_{1}, 1} \psi_{x_{2}, 1} \mathscr{M}$ are strict.

Let $G . \mathscr{N}$ denote the filtration by the order with respect to $\partial_{t}$. It will be useful to get control on the various objects occurring in the computations, mainly because when working on $\operatorname{gr}^{G} \mathscr{N}$, the action of $\partial_{x_{1}}$ amounts to that of $-x_{2} \otimes \partial_{t}$ and similarly for $\mathscr{\partial}_{x_{2}}$, and the action of $x_{1}, x_{2}$ on $\mathscr{M}$ is well understood, due to Exercise 7.5.10.

Lemma 7.5.14. We have $U_{-1}(\mathscr{N}) \cap G_{p}(\mathscr{N})=\sum_{k_{1}+k_{2} \leqslant p}(\mathscr{M} \otimes 1){\underset{\partial}{x_{1}}}_{k_{1}}^{\partial_{x_{2}}}$.
Proof. Any local section $\nu$ of $U_{-1}(\mathscr{N})$ can be written as $\sum_{k_{1}, k_{2} \geqslant 0}\left(m_{k_{1}, k_{2}} \otimes 1\right) \partial_{x_{1}}^{k_{1}} \partial_{x_{2}}^{k_{2}}$ for some local sections $m_{k_{1}, k_{2}}$ of $\mathscr{M}$ and, if $q=\max \left\{k_{1}+k_{2} \mid m_{k_{1}, k_{2}} \neq 0\right\}$, the degree of $\nu$ with respect to $\partial_{t}$ is $\leqslant q$ and the coefficient of $\mathscr{\partial}_{t}^{p}$ is

$$
(-1)^{q} \sum_{k_{1}+k_{2}=q} m_{k_{1}, k_{2}} x_{2}^{k_{1}} x_{1}^{k_{2}} .
$$

If this coefficient vanishes, Exercise 7.5.10 implies that

$$
\nu=\sum_{k_{1}+k_{2} \leqslant q}\left(\left(\mu_{k_{1}-1, k_{2}} x_{1}-\mu_{k_{1}, k_{2}-1} x_{2}\right) \otimes 1\right) \partial_{x_{1}}^{k_{1}} \partial_{x_{2}}^{k_{2}} .
$$

The operator against $\mu_{i, j} \otimes 1$ is $\left(x_{1} \partial_{x_{1}}-x_{2} \partial_{x_{2}}\right) \partial_{x_{1}}^{i} \partial_{x_{2}}^{j}$, and (7.5.13) implies

$$
\left(\mu_{i, j} \otimes 1\right)\left(x_{1} \check{\partial}_{x_{1}}-x_{2} \check{\partial}_{x_{2}}\right)=\left(\mu_{i, j}\left(x_{1} \check{\partial}_{x_{1}}-x_{2} \check{\partial}_{x_{2}}\right)\right) \otimes 1
$$

so that $\nu \in \sum_{k_{1}+k_{2} \leqslant q-1}(\mathscr{M} \otimes 1) \partial_{x_{1}}^{k_{1}} \partial_{x_{2}}^{k_{2}}$.
As a consequence, let us prove the equality

$$
\begin{equation*}
\check{\partial}_{t}^{-1}\left(U_{-1}(\mathscr{N})\right) \cap U_{-1} \mathscr{N}=\sum_{k_{1}, k_{2}}\left(\mathscr{M}\left(x_{1}, x_{2}\right) \otimes 1\right) \check{\partial}_{x_{1}}^{k_{1}} \partial_{x_{2}}^{k_{2}} \tag{7.5.15}
\end{equation*}
$$

and that $t$ acts injectively on $U_{-1} \mathscr{N}$.
Let $\nu=\sum_{q \leqslant p} \nu_{q} \otimes \mathrm{~d}_{t}^{q}$ be a nonzero local section of $U_{-1}(\mathscr{N})$ of $G$-order $p$, so that $\nu_{p} \neq 0$. We will argue by induction on $p$. By the lemma we have $\nu_{p}=$ $\sum_{k_{1}+k_{2}=p}\left(m_{k_{1}, k_{2}} \otimes 1\right){\underset{\partial}{x_{1}}}_{k_{1}} \partial_{x_{2}}^{k_{2}}$ with $\sum_{k_{1}+k_{2}=p} m_{k_{1}, k_{2}} x_{2}^{k_{1}} x_{1}^{k_{2}} \neq 0$ in $\mathscr{M}$. Assume $\partial_{t} \nu$ is a local section of $U_{-1}(\mathscr{N})$. Then $\sum_{k_{1}+k_{2}=p} m_{k_{1}, k_{2}} x_{2}^{k_{1}} x_{1}^{k_{2}}$ is a local section of $\mathscr{M}\left(x_{1}, x_{2}\right)^{p+1}$, that is, is equal to $\sum_{k_{1}+k_{2}=p} \mu_{k_{1}, k_{2}} x_{2}^{k_{1}} x_{1}^{k_{2}}$ with $\mu_{k_{1}, k_{2}} \in \mathscr{M}\left(x_{1}, x_{2}\right)$, so $\nu-\sum_{k_{1}+k_{2}=p}\left(\mu_{k_{1}, k_{2}} \otimes 1\right) \partial_{x_{1}}^{k_{1}} \partial_{x_{2}}^{k_{2}}$ a local section of $\partial_{t} U_{-1}(\mathscr{N}) \cap U_{-1} \mathscr{N}$ and has $G$-order $\leqslant p-1$. We can conclude by induction.

Assume now that $\nu t=0$. We have

$$
0=(\nu t)_{p}=\left[\left(\nu_{p} \otimes \overparen{\partial}_{t}^{p}\right) t\right]_{p}=\nu_{p} \otimes t \overparen{\mathrm{\partial}}_{t}^{p}=\nu_{p} x_{1} x_{2} \otimes \overparen{\partial}_{t}^{p}
$$

so $\nu_{p} x_{1} x_{2}=0$ in $\mathscr{M}$, and thus $\nu_{p}=0$, a contradiction.
Recall that $\mathscr{M}=V_{-1}^{(1)} \mathscr{M}\left(V\right.$-filtration relative to $\left.x_{1}\right)$, so that $\mathscr{M} / \mathscr{M} x_{1}=\operatorname{gr}_{-1}^{V^{(1)}} \mathscr{M}$ and $\mathscr{N}_{1}:=\left(\mathscr{M} / \mathscr{M} x_{1}\right)\left[\check{\partial}_{x_{1}}\right] \simeq \psi_{x_{1}, 1} \mathscr{M}(-1)$, according to Exercise 7.4.5. Similarly, $\mathscr{N}_{12} \simeq \psi_{x_{1}, 1} \psi_{x_{2}, 1} \mathscr{M}(-2)$. The map

$$
\begin{equation*}
m_{k_{1}, k_{2}} \otimes \partial_{x_{1}}^{k_{1}} \partial_{x_{2}}^{k_{2}} \longmapsto\left(m_{k_{1}, k_{2}} \otimes 1\right){\underset{\partial}{x_{1}}}_{k_{1}}^{\partial_{x_{2}} k_{2}} \cdot \partial_{t} t \tag{7.5.16}
\end{equation*}
$$

sends $\mathscr{M}\left(x_{1}, x_{2}\right)\left[\check{\mathrm{\partial}}_{x_{1}}, \check{\partial}_{x_{1}}\right]$ to $U_{-2} \mathscr{N}(-1)$, according to (7.5.12) and defines thus a surjective morphism

$$
\psi_{x_{1}, 1} \psi_{x_{2}, 1} \mathscr{M}(-2)=\mathscr{N}_{12} \longrightarrow \operatorname{gr}_{-1}^{\mathrm{M}} \mathrm{gr}_{-1}^{U} \mathscr{N}(-1)
$$

Let us prove that it is also injective. Let us denote by $\left[m_{k_{1}, k_{2}}\right]$ the class of $m_{k_{1}, k_{2}}$ in $\mathscr{M} / \mathscr{M}\left(x_{1}, x_{2}\right)$. Let $\sum\left[m_{k_{1}, k_{2}}\right] \otimes \partial_{x_{1}}^{k_{1}} \partial_{x_{2}}^{k_{2}}$ be nonzero and of degree equal to $p$ and set

$$
\nu=\sum_{k_{1}+k_{2} \leqslant p}\left(m_{k_{1}, k_{2}} \otimes 1\right){\underset{\partial}{x_{1}}}_{k_{1}}^{\partial_{x_{2}}^{k_{2}}} .
$$

Assume that $\nu \check{\partial}_{t} t \in U_{-2} \mathscr{N}$, hence, by the injectivity of $t, \nu \check{\partial}_{t} \in U_{-1} \mathscr{N}$. The proof of (7.5.15) above shows that, for $k_{1}+k_{2}=p$, there exists $\mu_{k_{1}, k_{2}} \in \mathscr{M}\left(x_{1}, x_{2}\right)$ such that $\sum_{k_{1}+k_{2}=p}\left(m_{k_{1}, k_{2}}-\mu_{k_{1}, k_{2}}\right) x_{2}^{k_{1}} x_{1}^{k_{2}}=0$, and by Exercise 7.5 .10 we conclude that $m_{k_{1}, k_{2}} \in \mathscr{M}\left(x_{1}, x_{2}\right)$, so $\left[m_{k_{1}, k_{2}}\right]=0$, a contradiction.

As a consequence, if $\nu \partial_{t} t=\sum\left(m_{k_{1}, k_{2}} \otimes 1\right) \partial_{x_{1}}^{k_{1}} \partial_{x_{2}}^{k_{2}} \partial_{t} t$ belongs to $U_{-2} \mathscr{N}=U_{-1} \mathscr{N} \cdot t$, (7.5.15) implies $\nu \in \sum\left(\mathscr{M}\left(x_{1}, x_{2}\right) \otimes 1\right){\underset{\partial}{x_{1}}}_{k_{1}}^{\delta_{x_{2}}} \boldsymbol{\partial}^{k_{2}}$. We obtain therefore

$$
\begin{equation*}
\operatorname{gr}_{1}^{\mathrm{M}} \operatorname{gr}_{-1}^{U} \mathscr{N} \xrightarrow[\sim]{\mathrm{N}} \operatorname{gr}_{-1}^{\mathrm{M}} \operatorname{gr}_{-1}^{U} \mathscr{N}(-1) \simeq \psi_{x_{1}, 1} \psi_{x_{2}, 1} \mathscr{M}(-2) \tag{7.5.17}
\end{equation*}
$$

and these modules are strict. Note that the isomorphism $\mathscr{N}_{12} \xrightarrow{\sim} \operatorname{gr}_{1}^{\mathrm{M}} \operatorname{gr}_{-1}^{U}{ }_{1} \mathscr{N}=$ $\left.U_{-1} \mathscr{N} /\left(\check{\partial}_{t} t\right)^{-1} U_{-1} \mathscr{M}\right)$ is induced by

$$
\begin{equation*}
m_{k_{1}, k_{2}} \otimes \partial_{x_{1}}^{k_{1}} \partial_{x_{2}}^{k_{2}} \longmapsto\left(m_{k_{1}, k_{2}} \otimes 1\right) \grave{\partial}_{x_{1}}^{k_{1}} \partial_{x_{2}}^{k_{2}} \tag{7.5.18}
\end{equation*}
$$

Let us now consider $\mathrm{M}_{0}$. Note that (7.5.15) and the injectivity of $t$ imply

$$
\mathrm{M}_{0} \operatorname{gr}_{-1}^{U} \mathscr{N}=\sum_{k_{1}, k_{2}}\left(\mathscr{M}\left(x_{1}, x_{2}\right) \otimes 1\right) \mathfrak{\partial}_{x_{1}}^{k_{1}} \partial_{x_{2}}^{k_{2}} \quad \bmod U_{-2} \mathscr{N}
$$

and clearly $\sum_{k_{1}, k_{2}}\left(\mathscr{M} x_{1} x_{2} \otimes 1\right) \partial_{x_{1}}^{k_{1}} \partial_{x_{2}}^{k_{2}} \subset U_{-2} \mathscr{N}$. Note also that $\left(m x_{1} \otimes 1\right) \partial_{x_{1}}^{k_{1}} \equiv$ $\left(m \widetilde{x}_{x_{1}}^{k_{1}} x_{1}\right) \otimes 1 \bmod \operatorname{Im} \partial_{t} t$, according to (7.5.13). As a consequence,

$$
\mathrm{M}_{0} \operatorname{gr}_{-1}^{U} \mathscr{N}=\sum_{k_{1}}\left(\mathscr{M} x_{2} \otimes 1\right) \check{\partial}_{x_{1}}^{k_{1}}+\sum_{k_{2}}\left(\mathscr{M} x_{1} \otimes 1\right) \check{\partial}_{x_{2}}^{k_{2}} \quad \bmod \left(U_{-1} \mathscr{N} \partial_{t} t+U_{-2} \mathscr{N}\right),
$$

and we have a surjective morphism

$$
\begin{equation*}
\psi_{x_{1}, 1} \mathscr{M}(-1) \oplus \psi_{x_{2}, 1} \mathscr{M}(-1)=\mathscr{N}_{1} \oplus \mathscr{N}_{2} \longrightarrow \operatorname{gr}_{0}^{\mathrm{M}} \operatorname{gr}_{-1}^{U} \mathscr{N} \tag{7.5.19}
\end{equation*}
$$

sending $m_{k_{1}, 0} \otimes \mathscr{\partial}_{x_{1}}^{k_{1}}$ to $\left(m_{k_{1}, 0} x_{2} \otimes 1\right) 夭_{x_{1}}^{k_{1}}$ and $m_{0, k_{2}} \otimes 夭_{x_{2}}^{k_{2}}$ to $\left(m_{0, k_{2}} x_{1} \otimes 1\right) \check{\partial}_{x_{2}}^{k_{2}}$. In order to show injectivity, we first check that it is strict with respect to the filtration $G . \mathscr{N}$ and the filtration by the degree in $\partial_{x_{1}}, \partial_{x_{2}}$ on $\mathscr{N}_{1}, \mathscr{N}_{2}$.

Assume that $\left(m_{k_{1}, 0} x_{2} \otimes 1\right) \partial_{x_{1}}^{k_{1}}+\left(m_{0, k_{2}} x_{1} \otimes 1\right) \partial_{x_{2}}^{k_{2}} \in G_{p-1} \mathscr{N}$ for $k_{1}, k_{2} \leqslant p$. Then we find that $m_{p, 0} \in \mathscr{M} x_{1}$ and $m_{0, p} \mathscr{M} x_{2}$, as wanted. By the same argument we deduce the injectivity.

Due to the strictness of $\mathscr{N}_{1}, \mathscr{N}_{2}, \mathscr{N}_{12}$, we conclude at this point that $\operatorname{gr}_{-1}^{U} \mathscr{M}$ is strict. If we show that $\operatorname{gr}_{k}^{U} \mathscr{M}$ is also strict for any $k$, then $U \cdot \mathscr{N}$ satisfies all properties characterizing the $V$-filtration. As a consequence, $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along $(g), \operatorname{gr}_{-1}^{U} \mathscr{N}=\psi_{g, 1} \mathscr{M}(-1)$, and (7.5.11*) holds.

Clearly, $\partial_{t}: \operatorname{gr}_{-1}^{U} \mathscr{N} \rightarrow \operatorname{gr}_{0}^{U} \mathscr{N}$ is onto. So we are left to proving the following assertions:
(i) $t^{k}: \operatorname{gr}_{-1}^{U} \mathscr{N} \rightarrow \operatorname{gr}_{-1-k^{U}} \mathscr{N}$ is an isomorphism (equivalently, injective) for $k \geqslant 1$,
(ii) $t: \operatorname{gr}_{0}^{U} \mathscr{N} \rightarrow \operatorname{gr}_{-1}^{U} \mathscr{N}$ is injective (so $\operatorname{gr}_{0}^{U} \mathscr{N}$ is strict),
(iii) $\partial_{t}^{k}: \operatorname{gr}_{0}^{U} \mathscr{N} \rightarrow \operatorname{gr}_{k}^{U} \mathscr{N}$ is an isomorphism (equivalently, injective) for $k \geqslant 1$.

Proof of the assertions.
(i) If $\nu \in U_{-1} \mathscr{N}$ satisfies $\nu t^{k}=\mu t^{k+1}$ for some $\mu \in U_{-1} \mathscr{N}$ then, by injectivity of $t$ on $U_{-1} \mathscr{N}, \nu=\mu t$, so $\nu \in U_{-2} \mathscr{N}$.
(ii) If $\nu \in U_{-1} \mathscr{N}$ is such that $\nu \oiint_{t} \cdot t \in U_{-2} \mathscr{N}$, then there exists $\mu \in U_{-1} \mathscr{N}$ such that $\left(\nu \partial_{t}-\mu\right) t=0$ hence, by $t$-injectivity, $\nu \coprod_{t} \in U_{-1} \mathscr{N}$.
(iii) We prove the injectivity by induction on $k \geqslant 1$. Let $\nu \in U_{-1} \mathscr{M}$ and consider $\nu \check{\partial}_{t} \bmod U_{-1} \mathscr{N}$ as an element of $\operatorname{gr}_{0}^{U} \mathscr{N}$. If $\left(\nu \partial_{t}\right) \partial_{t}^{k} \in U_{k-1} \mathscr{N}$, then $\left(\nu \partial_{t}^{k}\right) \partial_{t} t=0$ in $\operatorname{gr}_{k-1}^{U} \mathscr{N}$. Since $\partial_{t} t-k z$ is nilpotent on $\operatorname{gr}_{k-1}^{U} \mathscr{N}$ and since $\operatorname{gr}_{k-1}^{U} \mathscr{N}$ is strict (by (ii) and the inductive assumption), $\partial_{t} t$ is injective on $\operatorname{gr}_{k-1}^{U} \mathscr{N}$, so $\left(\nu ذ_{t}\right) ذ_{t}^{k-1}=0$ in $\operatorname{gr}_{k-1}^{U} \mathscr{N}$, and by induction $\nu \check{\partial}_{t}=0$ in $\operatorname{gr}_{0}^{U} \mathscr{N}$.

This concludes the proof of Proposition 7.5.11.

### 7.6. Strict Kashiwara's equivalence

We now return to the case of right $\widetilde{\mathscr{D}}_{X}$-module when considering the pushforward functor.

Let $\iota_{Y}: Y \subset X$ be the inclusion of a complex submanifold. The following is known as "Kashiwara's equivalence".

Proposition 7.6.1 (see [Kas03, §4.8]). The pushforward functor ${ }_{\mathrm{D}} \iota_{Y *}$ induces a natural equivalence between coherent $\mathscr{D}_{Y}$-modules and coherent $\mathscr{D}_{X}$-modules supported on $Y$, whose quasi-inverse is the restriction functor ${ }_{\mathrm{D}} \iota_{Y}^{*}$.

Be aware however that this result does not hold for graded coherent $R_{F} \mathscr{D}_{X^{-}}$ modules. For example, if $X=\mathbb{C}$ with coordinate $s$ and $\iota_{Y}: Y=\{0\} \hookrightarrow X$ denotes the inclusion, ${ }_{\mathrm{D}} \iota_{Y *} \mathbb{C}[z]=\delta_{s} \cdot \mathbb{C}\left[z, \partial_{s}\right]$ with $\delta_{s} s=0$. On the other hand, consider the $\mathbb{C}[z, s]\left\langle\partial_{s}\right\rangle$-submodule of $\mathbb{C}[z] \otimes_{\mathbb{C}{ }_{\mathrm{D}} \iota_{Y} *} \mathbb{C}=\delta_{s} \mathbb{C}\left[z, \partial_{s}\right]$ generated by $\delta_{s} \partial_{s}$ (note: $\partial_{s}$ and not $\partial_{s}$ ). This submodule is written $\delta_{s} \mathbb{C}[z] \oplus \bigoplus_{k \geqslant 0} \delta_{s} \partial_{s}^{k} \partial_{s}$. It has finite type over $\mathbb{C}[z, s]\left\langle\partial_{s}\right\rangle$ by construction, each element is annihilated by some power of $s$, and $\mathscr{H}^{-1}{ }_{\mathrm{D}} \iota_{Y}^{*}\left(\delta_{s} \partial_{s} \cdot \mathbb{C}[z, s]\left\langle\partial_{s}\right\rangle\right)=\delta_{s} \mathbb{C}[z]$, but it is not equal to ${ }_{\mathrm{D}} \iota_{Y *} \mathbb{C}[z]$.

Note also that this proposition implies in particular that $\mathscr{H}^{k}{ }_{\mathrm{D}} \iota_{Y \mathrm{D}}^{*} \iota_{Y *} \mathscr{M}=0$ for $k \neq-1$, if $\mathscr{M}$ is $\mathscr{D}_{X}$-coherent. In the example above, we have ${ }_{\mathrm{D}} \iota_{Y *} \mathbb{C}=\mathbb{C}\left[\tilde{\partial}_{s}\right]$ and the complex ${ }_{\mathrm{D}} \iota_{Y \mathrm{D}}^{*} \iota_{Y *} \mathbb{C}$ is the complex $\mathbb{C}\left[\partial_{s}\right] \xrightarrow{\cdot s} \mathbb{C}\left[\partial_{s}\right]$ with terms in degrees -1 and 0 . It has cohomology in degree -1 only.

However, this is not true for graded coherent $R_{F} \mathscr{D}_{X}$-modules. With the similar example, the complex ${ }_{\mathrm{D}} \iota_{Y \mathrm{D}}^{*} \iota_{Y *} \mathbb{C}[z]$ is the complex $\mathbb{C}\left[z, \partial_{s}\right] \xrightarrow{\cdot s} \mathbb{C}\left[z, \mathscr{\partial}_{s}\right]$. We have $\mathfrak{\partial}_{s}^{k} \cdot s=k z{\underset{\mathrm{X}}{s}}_{k-1}$, so the cokernel of $s$ is not equal to zero.

Proposition 7.6.2 (Strict Kashiwara's equivalence). Assume that $Y$ is smooth of codimension one in $X$, and let $\iota_{Y}: Y \hookrightarrow X$ denote the inclusion. The functor ${ }_{\mathrm{D}} \iota_{Y *}$ : $\operatorname{Mod}_{\mathrm{coh}}\left(\widetilde{\mathscr{D}}_{Y}\right) \mapsto \operatorname{Mod}_{\mathrm{coh}}\left(\widetilde{\mathscr{D}}_{X}\right)$ is fully faithful. It induces an equivalence between the full subcategory of $\operatorname{Mod}_{\text {coh }}\left(\widetilde{\mathscr{D}}_{Y}\right)$ whose objects are strict, and the full subcategory of $\operatorname{Mod}_{\mathrm{coh}}\left(\widetilde{\mathscr{D}}_{X}\right)$ whose objects are strictly $\mathbb{R}$-specializable along $Y$ and supported on $Y$. An inverse functor is $\mathscr{M} \mapsto \operatorname{gr}_{0}^{V} \mathscr{M}$.

Proof the full faithfulness. It is enough to prove that each morphism $\varphi:{ }_{\mathrm{D}} \iota_{Y *} \mathscr{N}_{1} \rightarrow$ ${ }_{\mathrm{D}} \iota_{Y *} \mathscr{N}_{2}$ takes the form ${ }_{\mathrm{D}} \iota_{Y *} \psi$ for a unique $\psi: \mathscr{N}_{1} \rightarrow \mathscr{N}_{2}$. Because of uniqueness, the assertion is local with respect to $Y$, hence we can assume that there exists a local
coordinate $s$ defining $Y$. Assume $\mathscr{M}={ }_{D^{\iota}}{ }_{Y} * \mathscr{N}$ for some coherent $\widetilde{\mathscr{D}}_{Y}$-module $\mathscr{N}$. Then one can recover $\mathscr{N}$ from $\mathscr{M}$ as the $\widetilde{\mathscr{D}}_{Y}$-module $\mathscr{M} / \mathscr{M} \cdot \partial_{s}$. As a consequence, $\psi$ must be the morphism induced by $\varphi$ on $\mathscr{M} / \mathscr{M} \cdot \partial_{s}$, hence its uniqueness. On the other hand, since $\mathscr{M}_{1}$ is generated by $\mathscr{N}_{1} \otimes \widetilde{\delta}_{s}$ over $\widetilde{\mathscr{D}}_{X}, \varphi$ is determined by its restriction to $\mathscr{N}_{1} \otimes \widetilde{\delta}_{s}$, that is by $\psi$, and the formula is $\varphi={ }_{\mathrm{D}} \iota_{Y *} \psi$.

Lemma 7.6.3. Assume $X \simeq Y \times \mathbb{C}$ with coordinate $s$ on the second factor. Let $\mathscr{M}$ be a coherent $\widetilde{\mathscr{D}}_{X}$-module supported on $Y \times\{0\}$.
(1) Assume that there exists a strict $\widetilde{\mathscr{D}}_{Y}$-module $\mathscr{N}$ such that $\mathscr{M} \simeq{ }_{\mathrm{D}} \iota_{Y *} \mathscr{N}$. Then
(a) $\mathscr{N}=\operatorname{Ker}[s: \mathscr{M} \rightarrow \mathscr{M}]$,
(b) $\mathscr{N}$ is $\widetilde{\mathscr{D}}_{Y^{-} \text {-coherent, }}$
(c) $\mathscr{M}$ is strict and strictly $\mathbb{R}$-specializable along $Y$,
(d) $\mathscr{N}=\operatorname{gr}_{0}^{V} \mathscr{M}$.
(2) Conversely, if $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along $Y$, then such an $\mathscr{N}$ exists. In particular, $\mathscr{M}$ is also strict.

Remark 7.6.4 (Strictness and strict $\mathbb{R}$-specializability). Let $\mathscr{M}$ be as in Lemma 7.6.3, that is, $\widetilde{\mathscr{D}}_{X}$-coherent and supported on $Y \times\{0\}$. Then the filtration $U_{0} \mathscr{M}=\operatorname{Ker} s \subset$ $U_{1} \mathscr{M}=\operatorname{Ker} s^{2} \subset \cdots$ is a filtration by $V_{0} \widetilde{\mathscr{D}}_{X}$-submodules and obviously admits a weak Bernstein polynomial. Assume moreover that $\mathscr{M}$ is strict. Then every $\operatorname{gr}_{k}^{U} \mathscr{M}$ is also strict: if $m \in U_{k} \mathscr{M}$ and $z^{\ell} m \in U_{k-1} \mathscr{M}$, that is, if $s^{k+1} m=0$ and $s^{k} z^{\ell} m=0$, then $s^{k} m=0$ by strictness of $\mathscr{M}$ and thus $m=0$ in $\operatorname{gr}_{k}^{U} \mathscr{M}$. Therefore, $U \cdot \mathscr{M}$ is the Kashiwara-Malgrange filtration $V \cdot \mathscr{M}$ in the sense of Lemma 7.3.23, and Properties $7.3 .25(1)$ and (2) are satisfied.

However, the condition that $\mathscr{M}$ is strict is not enough to obtain the conclusion of $7.6 .3(1)$, as shown by the following example. The point is that $7.3 .25(3)$ may not hold. Assume that $Y$ is reduced to a point and let $\mathscr{M}$ be the $\widetilde{\mathscr{D}}_{X}$-submodule of the $\mathscr{D}_{X}[z]$-module $\widetilde{\mathbb{C}}\left\langle\partial_{s}\right\rangle$ generated by 1 and $\partial_{s}$ (recall that $\widetilde{\mathbb{C}}:=\mathbb{C}[z]$ ), that we denote by [1] and $\left[\partial_{s}\right]$ for the sake of clarity. By definition, we have $[1] s=0$ and $\left[\partial_{s}\right] s^{2}=0$. For the Kashiwara-Malgrange filtration $V \cdot \mathscr{M}$ defined above, $\partial_{s}: \operatorname{gr}_{0}^{V} \mathscr{M}=$ $\widetilde{\mathbb{C}} \rightarrow \operatorname{gr}_{1}^{V} \mathscr{M}=\left[\partial_{s}\right] \widetilde{\mathbb{C}}$ is not onto, for its cokernel is $\left[\partial_{s}\right] \mathbb{C}$. In other words, $\mathscr{M}$ is not strictly $\mathbb{R}$-specializable at $s=0$ and not of the form ${ }_{\mathrm{D}} \iota_{Y *} \mathscr{N}$.

Proof of Lemma 7.6.3.
(1) Assume $\mathscr{M}={ }_{\mathrm{D}} \iota_{Y *} \mathscr{N}$ for some strict $\widetilde{\mathscr{D}}_{Y}$-module $\mathscr{N}$. We have ${ }_{\mathrm{D}} \iota_{Y}{ }^{*} \mathscr{N}=$ $\bigoplus_{k \geqslant 0} \iota_{Y *} \mathscr{N} \otimes \delta_{s} \partial_{s}^{k}$ with $\delta_{s} s=0$ (see Exercise A.8.30(2)). The action of $s$ on ${ }_{\mathrm{D}} \iota_{Y}{ }^{*} \mathscr{N}$ is the $z$-shift $n \otimes \delta_{s} \partial_{s}^{k} \mapsto z k n \otimes \delta_{s} \partial_{s}^{k-1}$, hence $\mathscr{N}=\operatorname{Ker} s$ because $\mathscr{N}$ is strict. Given a finite family of local $\widetilde{\mathscr{D}}_{X}$-generators of $\mathscr{M}$, we produce another such family made of homogeneous elements, by taking the components on the previous decomposition. Therefore, there exists a finite family of local sections $n_{i}$ of $\mathscr{N}$ such that $n_{i} \otimes \delta_{s}$ generate $\mathscr{M}$. Let $\mathscr{N}^{\prime} \subset \mathscr{N}$ be the $\widetilde{\mathscr{D}}_{Y}$-submodule they generate. Then ${ }_{\mathrm{D}} \iota_{Y *} \mathscr{N}^{\prime} \rightarrow$ ${ }_{\mathrm{D}} \iota_{Y *} \mathscr{N}=\mathscr{M}$ is onto. On the other hand, since $\mathscr{N}^{\prime}$ is also strict, this map is injective:

If $\sum_{k=1}^{N} n_{k}^{\prime} \otimes \delta_{s} \partial_{s}^{k} \mapsto 0$, then $n_{N}^{\prime} \otimes \delta_{s} \partial_{s}^{N} \mapsto 0$, and $s^{N} n_{N}^{\prime} \otimes \delta_{s} \partial_{s}^{N}=\star z^{N} n_{N}^{\prime} \otimes \delta_{s} \partial_{s}^{N} \mapsto 0$, where $\star$ is a nonzero constant; so $z^{N} n_{N}^{\prime}=0$ in $\mathscr{N}$, hence $n_{N}^{\prime}=0$. We conclude $\mathscr{N}^{\prime}=\mathscr{N}$ since both are equal to $\operatorname{Ker} s$ in $_{\mathrm{D}^{\iota} \iota_{Y} * \mathscr{N} \text {. Therefore, } \mathscr{N} \text { is locally finitely }}$ $\widetilde{\mathscr{D}}_{Y}$-generated in $\mathscr{M}$, and then is $\widetilde{\mathscr{D}}_{Y}$-coherent. One then checks that the filtration $U_{j} \mathscr{M}:=\bigoplus_{k \geqslant 0}^{j} \iota_{Y *} \mathscr{N} \otimes \delta_{s} \partial_{s}^{k}$ is a coherent $V$-filtration of $\mathscr{M}$, and $\mathscr{N}=\operatorname{gr}_{0}^{U} \mathscr{M}$. We deduce that each $\operatorname{gr}_{k}^{U} \mathscr{M}$ is strict, and $\mathscr{M}$ is strictly $\mathbb{R}$-specializable. Lastly, $n \otimes \delta_{s}$ satisfies $\left(n \otimes \delta_{s}\right) s \partial_{s}=0$, so $V_{\bullet} \mathscr{M}=U \cdot \mathscr{M}$ jumps at nonnegative integers only.
(2) Assume that $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along $Y$. Then $V_{<0} \mathscr{M}=0$, according to 7.3.31(a). Similarly, $\operatorname{gr}_{\alpha}^{V} \mathscr{M}=0$ for $\alpha \notin \mathbb{Z}$. As $s: \operatorname{gr}_{k}^{V} \mathscr{M} \rightarrow \operatorname{gr}_{k-1}^{V} \mathscr{M}$ is injective for $k \neq 0$ (see 7.3.31(c)), we conclude that

$$
\operatorname{gr}_{0}^{V} \mathscr{M} \simeq V_{0} \mathscr{M}=\operatorname{Ker}[s: \mathscr{M} \rightarrow \mathscr{M}] .
$$

Since $\partial_{s}: \operatorname{gr}_{k}^{V} \mathscr{M} \rightarrow \operatorname{gr}_{k-1}^{V} \mathscr{M}$ is an isomorphism for $k \leqslant 0$, we obtain

$$
\mathscr{M}=\bigoplus_{\ell \geqslant 0} \operatorname{gr}_{0}^{V} \mathscr{M} \partial_{s}^{\ell}={ }_{\mathrm{D}} \iota_{*} \operatorname{gr}_{0}^{V} \mathscr{M}
$$

Lastly, $\mathrm{E}+z$ acts by zero on $\operatorname{gr}_{0}^{V} \mathscr{M}$, which is therefore a coherent $\widetilde{\mathscr{D}}_{Y}$-module by means of the isomorphism $\operatorname{gr}_{0}^{V} \widetilde{\mathscr{D}}_{X} /(\mathrm{E}+z) \operatorname{gr}_{0}^{V} \widetilde{\mathscr{D}}_{X} \simeq \widetilde{\mathscr{D}}_{Y}$. It is strict since $\mathscr{M}$ is strictly $\mathbb{R}$-specializable.

End of the proof of Proposition 7.6.2. It remains to prove essential surjectivity. Let $V \cdot \mathscr{M}$ be the $V$-filtration of $\mathscr{M}$ along $Y$. By the argument in the second part of the proof of Lemma 7.6.3, we have local isomorphisms $\mathscr{M} \xrightarrow{\sim}{ }_{\mathrm{D}} \iota_{*} \mathrm{gr}_{0}^{V} \mathscr{M}$ which induce the identity on $V_{0} \mathscr{M}=\operatorname{gr}_{0}^{V} \mathscr{M}$. By full faithfulness they glue in a unique way as a global isomorphism $\mathscr{M} \simeq{ }_{\mathrm{D}} \iota_{*} \operatorname{gr}_{0}^{V} \mathscr{M}$.

Corollary 7.6.5. Assume $\operatorname{codim} Y=1$. Let $\mathscr{N}$ be $\widetilde{\mathscr{D}}_{Y}$-coherent and set $\mathscr{M}={\underset{\mathrm{o}}{ }}^{\iota_{Y}}{ }^{*} \mathscr{N}$. If $\mathscr{M}=\mathscr{M}_{1} \oplus \mathscr{M}_{2}$ with $\mathscr{M}_{1}, \mathscr{M}_{2}$ being $\widetilde{\mathscr{D}}_{X}$-coherent, then there exist coherent $\widetilde{\mathscr{D}}_{Y}$-submodules $\mathscr{N}_{1}, \mathscr{N}_{2}$ of $\mathscr{N}$ such that $\mathscr{N}=\mathscr{N}_{1} \oplus \mathscr{N}_{2}$ and $\mathscr{M}_{j}={ }_{\mathrm{D}} \iota_{Y} * \mathscr{N}_{j}$ for $j=1,2$.

Proof. Each $\mathscr{M}_{i}$ is coherent and supported on $Y$. We set $\mathscr{N}_{i}=\mathscr{M}_{i} \cap \mathscr{N}$. Locally, choose a coordinate $s$ defining $Y$ and set $\mathscr{N}_{i}^{\prime}=\mathscr{M}_{i} / \mathscr{M}_{i} \cdot \mathscr{\partial}_{s}$. Since $\mathscr{N}=\mathscr{M} / \mathscr{M} \cdot \mathscr{\partial}_{s}$, we deduce that $\mathscr{N}=\mathscr{N}_{1}^{\prime} \oplus \mathscr{N}_{2}^{\prime}$, and we have a (local) isomorphism $\mathscr{M}_{i} \simeq{ }_{\mathrm{D}} \iota_{*} \mathscr{N}_{i}{ }^{\prime}$. Then one checks that $\mathscr{N}^{\prime} i=\mathscr{N}_{i}$, so it is globally defined.

We now consider the behaviour of strict $\mathbb{R}$-specializability along a function $g: X \rightarrow \mathbb{C}$ with respect to strict Kashiwara's equivalence along $Y$.
Proposition 7.6.6. Let $\mathscr{N}$ be a coherent $\widetilde{\mathscr{D}}_{Y}$-module and set $\mathscr{M}={ }_{\mathrm{D}} \iota_{Y *} \mathscr{N}$.
(1) Assume that $\mathscr{N}$ is strictly $\mathbb{R}$-specializable along $\left(g_{\mid Y}\right)$. Then $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along ( g ).
(2) Assume that $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along $(g)$. Then $\mathscr{N}$ is strictly $\mathbb{R}$-specializable along $\left(g_{\mid Y}\right)$.
In such a case, we have $\psi_{g, \lambda} \mathscr{M}={ }_{\mathrm{D}} \iota_{Y *} \psi_{g_{\mid Y}, \lambda} \mathscr{N}$ and $\phi_{g, 1} \mathscr{M}={ }_{\mathrm{D}} \iota_{Y *} \phi_{g_{\mid Y}, 1} \mathscr{N}$. Moreover, $\operatorname{can} \mathscr{M}={ }_{\mathrm{D}} \ell_{Y *} \operatorname{can}_{\mathscr{N}}$ and $\operatorname{var}_{\mathscr{M}}={ }_{\mathrm{D}} \ell_{Y *} \operatorname{var}_{\mathscr{N}}$.

Proof. The first statement is easy to check. Let us consider the second one. We first consider the case where $X=H_{Y} \times \mathbb{C}_{s} \times \mathbb{C}_{t}$, with $Y=H_{Y} \times \mathbb{C}_{t}$ and $g$ is the projection to $\mathbb{C}_{t}$. We denote by $V$ the $V$-filtration along $t$. We have $\mathscr{M}={ }_{\mathrm{D}} \iota_{Y *} \mathscr{N}=$ $\bigoplus_{k} \iota_{Y *} \mathscr{N} \otimes \delta_{s} \partial_{s}^{k}$.

Let $n$ be a local section of $\mathscr{N}$. If $b\left(t \check{\partial}_{t}\right)-t P\left(y, s, t, \partial_{y}, \partial_{s}, t \partial_{t}\right)$ is a Bernstein equation for $n \otimes \delta_{s}$ in $_{\mathrm{D}} \iota_{Y *} \mathscr{N}$, and if $P=P_{0}+s Q$, where $P_{0}$ does not depend on $s$, then $b\left(t \partial_{t}\right)-t P_{0}\left(y, t, \partial_{y}, \partial_{s}, t \partial_{t}\right)$ is also a Bernstein equation for $n \otimes \delta_{s}$. The degreezero part with respect to $\partial_{s}$ of this equation still gives a Bernstein equation for $n \otimes \delta_{s}$, and thus a Bernstein equation for $n$ in $\mathscr{N}$. We conclude that $\mathscr{N}$ is $\mathbb{R}$-specializable along $H_{Y}$ and that $\operatorname{ord}_{V}(n) \geqslant \operatorname{ord}_{V}\left(n \otimes \delta_{s}\right)$.

Let us now prove that the $V$-filtration of $\mathscr{M}$ is compatible with the decomposition. Let $\sum_{i=0}^{N} n_{i} \otimes \delta_{s} \partial_{s}^{i}$ be a section in $V_{\alpha} \mathscr{M}$. We will prove by induction on $N$ that $\operatorname{ord}_{V}\left(n_{i}\right) \leqslant \alpha$ for every $i$. It is enough to prove it for $i=N$. We have $\sum_{i=0}^{N} n_{i} \otimes \delta_{s} \partial_{s}^{i} \cdot s^{N}=\star z^{N} n_{N} \otimes \delta_{s} \in V_{\alpha} \mathscr{M}$ for some nonzero constant $\star$. If $n_{N} \otimes \delta_{s} \in V_{\gamma} \mathscr{M}$ for $\gamma>\alpha$, then the class of $n_{N} \otimes \delta_{s}$ in $\operatorname{gr}_{\gamma}^{V} \mathscr{M}$ is annihilated by $z^{N}$, hence is zero since $\operatorname{gr}_{\gamma}^{V} \mathscr{M}$ is strict. Therefore, $n_{N} \otimes \delta_{s} \in V_{\alpha} \mathscr{M}$, and by the preliminary remark, $\operatorname{ord}_{V}\left(n_{N}\right) \leqslant \alpha$. If we denote by $U . \mathscr{N}$ the (possibly not coherent) $V$-filtration by the $V$-order, then one has $V_{\alpha} \mathscr{M}=\bigoplus_{i} \iota_{Y *} U_{\alpha} \mathscr{N} \otimes \delta_{s} \partial_{s}^{i}$ and $\operatorname{gr}_{\alpha}^{V} \mathscr{M}=\bigoplus_{i} \iota_{Y *} \operatorname{gr}_{\alpha}^{V} \mathscr{N} \otimes \delta_{s} \partial_{s}^{i}$. It follows that $U \cdot \mathscr{N}$ is a coherent $V$-filtration of $\mathscr{N}$ and that each $\operatorname{gr}_{\alpha}^{V} \mathscr{N}$ is strict. By uniqueness of the $V$-filtration, we have $U \cdot \mathscr{N}=V \cdot \mathscr{N}$, and Properties $7.3 .25(2)$ and (3) are clearly satisfied, as they hold for $\mathscr{M}$.

For the general case, the question is local and we can assume that $Y$ is defined by a smooth function $h$. By assumption, ${ }_{\mathrm{D}} \iota_{g *}\left({ }_{\mathrm{D}} \iota_{Y} * \mathscr{N}\right)$ is strictly $\mathbb{R}$-specializable along $t$, and thus so is ${ }_{\mathrm{D}} \iota_{(h, g) *}\left({ }_{\mathrm{D}} \iota_{Y} \cdot \mathscr{N}\right)={ }_{\mathrm{D}} \iota_{s=0} \iota_{g_{\mid Y}} \mathscr{N}$, after (1). The previous argument implies that $\iota_{g_{\mid Y}} \mathscr{N}$ is strictly $\mathbb{R}$-specializable along $t$, that is, $\mathscr{N}$ is strictly $\mathbb{R}$-specializable along $g_{\mid Y}$.

The last statement is then clear by the computation of the $V$-filtrations above.

### 7.7. Strictly support-decomposable $\widetilde{\mathscr{D}}$-modules

Let $g: X \rightarrow \mathbb{C}$ be a holomorphic function. We set $X_{0}=g^{-1}(0)$. Let $\iota_{g}: X \hookrightarrow$ $X \times \mathbb{C}$ denote the graph embedding associated with $g$. We set $H=X \times\{0\} \subset X \times \mathbb{C}$.

We first interpret the strict Kashiwara's equivalence in this setting.
Corollary 7.7.1. Assume that $\mathscr{M}$ is $\widetilde{\mathscr{D}}_{X}$-coherent, strictly $\mathbb{R}$-specializable along $D:=(g)$ and supported on $X_{0}$. Then $\mathscr{M} \simeq \phi_{g, 1} \mathscr{M}$.

Proof. By Proposition 7.6 .2 we have ${ }_{\mathrm{D}} \iota_{g *} \mathscr{M}={ }_{\mathrm{D}} \iota_{t *} \operatorname{gr}_{0}^{V}{ }_{\mathrm{D}} \iota_{g *} \mathscr{M}=:{ }_{\mathrm{D}} \iota_{t *} \phi_{g, 1} \mathscr{M}$. On the other hand, we recover $\mathscr{M}$ from ${ }_{\mathrm{D}} \iota_{g *} \mathscr{M}$ as $\mathscr{M}={ }_{\mathrm{D}} p_{* \mathrm{D}} \iota_{g *} \mathscr{M}$, where $p: X \times \mathbb{C} \rightarrow \mathbb{C}$ is the projection. We then use that $p \circ \iota_{t}=\operatorname{Id}_{X}$.

Proposition 7.7.2. Let $\mathscr{M}$ be a coherent $\widetilde{\mathscr{D}}_{X}$-module which is strictly $\mathbb{R}$-specializable along ( $g$ ).
(1) The following properties are equivalent:
(a) var: $\phi_{g, 1} \mathscr{M} \rightarrow \psi_{g, 1} \mathscr{M}(-1)$ is injective,
(b) ${ }_{\mathrm{D}} \iota_{g *} \mathscr{M}$ has no proper subobject in $\operatorname{Mod}_{\mathrm{coh}}\left(\widetilde{\mathscr{D}}_{X \times \mathbb{C}}\right)$ supported on $H$,
(c) There is no strictly $\mathbb{R}$-specializable inclusion $\mathscr{N} \hookrightarrow{ }_{\mathrm{D}} \iota_{g *} \mathscr{M}$ with $\mathscr{N}$ strictly $\mathbb{R}$-specializable supported on $H$.
(2) If can : $\psi_{g, 1} \mathscr{M} \rightarrow \phi_{g, 1} \mathscr{M}$ is onto, then ${ }_{\mathrm{D}} \iota_{g *} \mathscr{M}$ has no proper quotient satisfying 7.3.25(1) and supported on $H$.

Definition 7.7.3 (Minimal extension along $g$ ). Let $\mathscr{M}$ be a coherent $\widetilde{\mathscr{D}}_{X}$-module which is strictly $\mathbb{R}$-specializable along $(g)$. We say that $\mathscr{M}$ is a minimal extension along $(g)$ if var : $\phi_{g, 1} \mathscr{M} \rightarrow \psi_{g, 1} \mathscr{M}(-1)$ is injective and can : $\psi_{g, 1} \mathscr{M} \rightarrow \phi_{g, 1} \mathscr{M}$ is onto.

Exercise 7.7.4 (can-var diagram for a minimal extension). Show that the diagram $(7.4 .9 *)$ or $(7.4 .10 *)$ is isomorphic to the diagram


Proposition 7.7.5. Let $\mathscr{M}$ be as in Proposition 7.7.2. The following properties are equivalent:
(1) $\phi_{g, 1} \mathscr{M}=\operatorname{Im}$ can $\oplus$ Ker var,
(2) $\mathscr{M}=\mathscr{M}^{\prime} \oplus \mathscr{M}^{\prime \prime}$ with $\mathscr{M}^{\prime}, \mathscr{M}^{\prime \prime}$ strictly $\mathbb{R}$-specializable along $(g)$, $\mathscr{M}^{\prime}$ being a minimal extension along $(g)$ and $\mathscr{M}^{\prime \prime}$ supported on $g^{-1}(0)$.
Moreover, if $\mathscr{M}, \mathscr{N}$ satisfy these properties, any morphism $\varphi: \mathscr{M} \rightarrow \mathscr{N}$ decomposes correspondingly.

Proof of Propositions 7.7.2 and 7.7.5. All along this proof, we set $\mathscr{N}={ }_{\mathrm{D}} \iota_{g *} \mathscr{M}$ for short.
7.7.2(1) (1a) $\Leftrightarrow(1 \mathrm{~b})$ : It is enough to show that the morphisms

are isomorphisms. It is clear for the right one, since $t: V^{<0} \mathscr{N} \rightarrow V^{<-1} \mathscr{N}$ is an isomorphism, according to 7.3.31(a). For the left one this follows from the fact that $t$ is injective on $\operatorname{gr}_{\alpha}^{V} \mathscr{N}$ for $\alpha \neq 0$ according to 7.3.31(c).
(1b) $\Leftrightarrow(1 \mathrm{c})$ : let us check $\Leftarrow$ (the other implication is clear). Let $\mathscr{T}$ denote the $t$-torsion submodule of $\mathscr{N}$ and $\mathscr{T}^{\prime}$ the $\widetilde{\mathscr{D}}_{X} \times \mathbb{C}^{\text {-submodule generated by }}$

$$
\mathscr{T}_{0}:=\operatorname{Ker}[t: \mathscr{N} \longrightarrow \mathscr{N}] .
$$

Assertion 7.7.6. $\mathscr{T}^{\prime}$ is strictly $\mathbb{R}$-specializable and the inclusion $\mathscr{T}^{\prime} \hookrightarrow \mathscr{N}$ is strictly $\mathbb{R}$-specializable.

This assertion gives the implication $\Leftarrow$ because Assumption (1c) implies $\mathscr{T}^{\prime}=0$, hence $t: \mathscr{N} \rightarrow \mathscr{N}$ is injective, so $\mathscr{T}=0$.

Proof of the assertion. Let us show first that $\mathscr{T}^{\prime}$ is $\widetilde{\mathscr{D}}_{X \times \mathbb{C}^{-} \text {coherent. As we remarked }}$ above, we have $\mathscr{T}_{0}=\operatorname{Ker}\left[t: \operatorname{gr}_{0}^{V} \mathscr{N} \rightarrow \operatorname{gr}_{-1}^{V} \mathscr{N}\right]$. Now, $\mathscr{T}_{0}$ is the kernel of a linear morphism between $\widetilde{\mathscr{D}}_{H^{-}}$-coherent modules $(H=X \times\{0\})$, hence is also $\widetilde{\mathscr{D}}_{H}$-coherent. It follows that $\mathscr{T}^{\prime}$ is $\widetilde{\mathscr{D}}_{X} \times \mathbb{C}^{\text {-coherent. }}$

Let us now show that $\mathscr{T}^{\prime}$ is strictly $\mathbb{R}$-specializable. We note that $\mathscr{T}_{0}$ is strict because it is isomorphic to a submodule of $\operatorname{gr}_{0}^{V} \mathscr{N}$. Let $U . \mathscr{T}^{\prime}$ be the filtration induced by $V . \mathscr{N}$ on $\mathscr{T}^{\prime}$. Then $U_{<0} \mathscr{T}^{\prime}=0$, according to 7.3.31(a), and $\operatorname{gr}_{\alpha}^{U} \mathscr{T}^{\prime}=0$ for $\alpha \notin \mathbb{N}$. Let us show by induction on $k$ that

$$
U_{k} \mathscr{T}^{\prime}=\mathscr{T}_{0}+\mathscr{T}_{0} \partial_{t}+\cdots+\mathscr{T}_{0} \partial_{t}^{k}
$$

Let us denote by $U_{k}^{\prime} \mathscr{T}^{\prime}$ the right-hand term. The inclusion $\supset$ is clear. Let $x_{o} \in H$, $m \in U_{k} \mathscr{T}_{x_{o}}^{\prime}$ and let $\ell \geqslant k$ such that $m \in U_{\ell}^{\prime} \mathscr{T}_{x_{o}}^{\prime}$. If $\ell>k$ one has $m \in \mathscr{T}_{x_{o}}^{\prime} \cap V_{\ell-1} \mathscr{N}_{x_{o}}$ hence $m t^{\ell} \in \mathscr{T}_{x_{o}}^{\prime} \cap V_{-1} \mathscr{N}_{x_{o}}=0$. Set

$$
m=m_{0}+m_{1} \partial_{t}+\cdots+m_{\ell} \partial_{t}^{\ell}
$$

with $m_{j} t=0(j=0, \ldots, \ell)$. One then has $m_{\ell} \partial_{t}^{\ell} t^{\ell}=0$, and since

$$
m_{\ell} \partial_{t}^{\ell} t^{\ell}=m_{\ell} \cdot \prod_{j=1}^{\ell}\left(t \partial_{t}+j z\right)=\ell!m_{\ell} z^{\ell}
$$

and $\mathscr{T}_{0}$ is strict, one concludes that $m_{\ell}=0$, hence $m \in U_{\ell-1}^{\prime} \mathscr{T}_{x_{o}}^{\prime}$. By induction, this implies the other inclusion.

As $\operatorname{gr}_{\alpha}^{U} \mathscr{T}^{\prime}$ is contained in $\operatorname{gr}_{\alpha}^{V} \mathscr{N}$, one deduces from 7.3.31(d) that $\partial_{t}: \operatorname{gr}_{k}^{U} \mathscr{T}^{\prime} \rightarrow$ $\operatorname{gr}_{k+1}^{U} \mathscr{T}^{\prime}$ is injective for $k \geqslant 0$. The previous computation shows that it is onto, hence $\mathscr{T}^{\prime}$ is strictly $\mathbb{R}$-specializable and $U . \mathscr{T}^{\prime}$ is its Malgrange-Kashiwara filtration.

It is now enough to prove that the injective morphism $\operatorname{gr}_{0}^{U} \mathscr{T}^{\prime} \rightarrow \operatorname{gr}_{0}^{V} \mathscr{N}$ is strict. But its cokernel is identified with the submodule $\operatorname{Im}\left[t: \operatorname{gr}_{0}^{V} \mathscr{N} \rightarrow \operatorname{gr}_{-1}^{V} \mathscr{N}\right]$ of $\operatorname{gr}_{-1}^{V} \mathscr{N}$, which is strict.
7.7.2(2) If can is onto, then $\mathscr{N}=\widetilde{\mathscr{D}}_{X \times \mathbb{C}} \cdot V_{<0} \mathscr{N}$. If $\mathscr{N}$ has a $t$-torsion quotient $\mathscr{T}$ satisfying $7.3 .25(1)$, then $V_{<0} \mathscr{T}=0$, so $V_{<0} \mathscr{N}$ is contained in $\operatorname{Ker}[\mathscr{N} \rightarrow \mathscr{T}]$ and thus $\mathscr{N}=\widetilde{\mathscr{D}}_{X \times \widetilde{\mathbb{C}}} \cdot V_{<0} \mathscr{N}$ is also contained in this kernel, that is, $\mathscr{T}=0$.

$$
\text { 7.7.5 }(1) \Rightarrow 7.7 .5(2) \text { Set }
$$

$$
U_{0} \mathscr{N}^{\prime}=V_{<0} \mathscr{N}+\partial_{t} V_{-1} \mathscr{N} \quad \text { and } \quad \mathscr{T}_{0}=\operatorname{Ker}[t: \mathscr{N} \longrightarrow \mathscr{N}]
$$

The assumption (1) is equivalent to $V_{0} \mathscr{N}=U_{0} \mathscr{N}^{\prime} \oplus \mathscr{T}_{0}$. Define

$$
U_{k} \mathscr{N}^{\prime}=V_{k} \widetilde{\mathscr{D}}_{X} \cdot U_{0} \mathscr{N}^{\prime} \quad \text { and } \quad U_{k} \mathscr{N}^{\prime \prime}=V_{k} \widetilde{\mathscr{D}}_{X} \cdot \mathscr{T}_{0}
$$

for $k \geqslant 0$. As $V_{k} \mathscr{N}=V_{k-1} \mathscr{N}+\partial_{t} V_{k-1} \mathscr{N}$ for $k \geqslant 1$, one has $V_{k} \mathscr{N}=U_{k} \mathscr{N}^{\prime}+U_{k} \mathscr{N}^{\prime \prime}$ for $k \geqslant 0$. Let us show by induction on $k \geqslant 0$ that this sum is direct. Fix $k \geqslant 1$ and let $m \in U_{k} \mathscr{N}^{\prime} \cap U_{k} \mathscr{N}^{\prime \prime}$. Write

$$
m=m_{k-1}^{\prime}+n_{k-1}^{\prime} \partial_{t}=m_{k-1}^{\prime \prime}+n_{k-1}^{\prime \prime} \partial_{t}, \quad\left\{\begin{array}{l}
m_{k-1}^{\prime}, n_{k-1}^{\prime} \in U_{k-1} \mathscr{N}^{\prime} \\
m_{k-1}^{\prime \prime}, n_{k-1}^{\prime \prime} \in U_{k-1} \mathscr{N}^{\prime \prime}
\end{array}\right.
$$

One has $\left[n_{k-1}^{\prime}\right] \partial_{t}=\left[n_{k-1}^{\prime \prime}\right] \partial_{t}$ in $V_{k} \mathscr{N} / V_{k-1} \mathscr{N}$, hence, as

$$
\partial_{t}: V_{k-1} \mathscr{N} / V_{k-2} \mathscr{N} \longrightarrow V_{k} \mathscr{N} / V_{k-1} \mathscr{N}
$$

is bijective for $k \geqslant 1$, one gets $\left[n_{k-1}^{\prime}\right]=\left[n_{k-1}^{\prime \prime}\right]$ in $V_{k-1} \mathscr{N} / V_{k-2} \mathscr{N}$ and by induction one deduces that both terms are zero. One concludes that $m \in U_{k-1} \mathscr{N}^{\prime} \cap U_{k-1} \mathscr{N}^{\prime \prime}=0$ by induction.

This implies that $\mathscr{N}=\mathscr{N}^{\prime} \oplus \mathscr{N}^{\prime \prime}$ with $\mathscr{N}^{\prime}:=\bigcup_{k} U_{k} \mathscr{N}^{\prime}$ and $\mathscr{N}^{\prime \prime}$ defined similarly. It follows from Exercise 7.3.37(1) that both $\mathscr{N}^{\prime}$ and $\mathscr{N}^{\prime \prime}$ are strictly $\mathbb{R}$-specializable along $H$ and the filtrations $U$. above are their Malgrange-Kashiwara filtrations. In particular $\mathscr{N}^{\prime}$ satisfies (1) and (2). By Corollary 7.6 .5 we also know that $\mathscr{N}^{\prime}=$ ${ }_{\mathrm{D}} \iota_{g *} \mathscr{M}^{\prime}$ and $\mathscr{N}^{\prime \prime}={ }_{\mathrm{D}} \iota_{g *} \mathscr{M}^{\prime \prime}$ for some coherent $\widetilde{\mathscr{D}}_{\mathrm{X}}$-modules $\mathscr{M}^{\prime}, \mathscr{M}^{\prime \prime}$.
$7.7 .5(2) \Rightarrow 7.7 .5(1)$ : One has $V_{<0} \mathscr{N}^{\prime \prime}=0$. Let us show that $\operatorname{Im}$ can $=\operatorname{gr}_{0}^{V} \mathscr{N}^{\prime}$ and Ker var $=\operatorname{gr}_{0}^{V} \mathscr{N}^{\prime \prime}$. The inclusions $\operatorname{Im}$ can $\subset \operatorname{gr}_{0}^{V} \mathscr{N}^{\prime}$ and Ker var $\supset \operatorname{gr}_{0}^{V} \mathscr{N}^{\prime \prime}$ are clear. Moreover $\operatorname{gr}_{0}^{V} \mathscr{N}^{\prime} \cap$ Ker var $=0$ as $\mathscr{N}^{\prime}$ satisfies (1). Last, can : $\operatorname{gr}_{-1}^{V} \mathscr{N}^{\prime} \rightarrow \operatorname{gr}_{0}^{V} \mathscr{N}^{\prime}$ is onto, as $\mathscr{N}^{\prime}$ satisfies (2). Hence $\operatorname{gr}_{0}^{V} \mathscr{N}=\operatorname{Im}$ can $\oplus$ Ker var.

Let us now prove the last assertion. Let us consider a morphism $\varphi: \mathscr{M}^{\prime} \oplus \mathscr{M}^{\prime \prime} \rightarrow$ $\mathscr{N}^{\prime} \oplus \mathscr{N}^{\prime \prime}$. Firstly, by (1b) in Proposition 7.7.2, the component $\mathscr{M}^{\prime \prime} \rightarrow \mathscr{N}^{\prime}$ is zero. For the component $\mathscr{M}^{\prime} \rightarrow \mathscr{N}^{\prime \prime}$, let us denote by $\mathscr{M}_{1}^{\prime}$ its image. The $V$-filtration on ${ }_{\mathrm{D}} \iota_{g *} \mathscr{M}_{1}^{\prime}$ induced by $V_{\bullet}{ }_{\mathrm{D}} \iota_{g *} \mathscr{N}^{\prime \prime}$ is coherent (Exercise 7.3.7(1)) and satisfies 7.3.25(1), hence by Proposition $7.7 .2(2)$ we have ${ }_{\mathrm{D}} \iota_{g *} \mathscr{M}_{1}^{\prime}=0$.

## Definition 7.7.7 (Strictly S(upport)-decomposable $\widetilde{\mathscr{D}}_{X}$-modules)

We say that a coherent $\widetilde{D}_{X}$-module $\mathscr{M}$ is

- strictly $S$-decomposable along $(g)$ if it is strictly $\mathbb{R}$-specializable along $(g)$ and satisfies the equivalent conditions 7.7.5;
- strictly $S$-decomposable at $x_{o} \in X$ if for any analytic germ $g:\left(X, x_{o}\right) \rightarrow(\mathbb{C}, 0)$, $\mathscr{M}$ is strictly S-decomposable along $(g)$ in some neighbourhood of $x_{o}$;
- strictly $S$-decomposable if it is strictly S-decomposable at all points $x_{o} \in X$.


## Lemma 7.7.8.

(1) If $\mathscr{M}$ is strictly $S$-decomposable along $\{t=0\}$, then it is strictly $S$-decomposable along $\left\{t^{r}=0\right\}$ for every $r \geqslant 1$.
(2) If $\mathscr{M}=\mathscr{M}_{1} \oplus \mathscr{M}_{2}$, then $\mathscr{M}$ is strictly $S$-decomposable of some kind if and only if $\mathscr{M}_{1}$ and $\mathscr{M}_{2}$ are so.
(3) We assume that $\mathscr{M}$ is strictly $S$-decomposable and its support $Z$ is a pure dimensional closed analytic subset of $X$. Then the following conditions are equivalent:
(a) for any analytic germ $g:\left(X, x_{o}\right) \rightarrow(\mathbb{C}, 0)$ such that $g^{-1}(0) \cap Z$ has everywhere codimension one in $Z$, ${ }_{\mathrm{D}} \iota_{g *} \mathscr{M}$ is a minimal extension along $(g)$;
(b) near any $x_{o}$, there is no $\widetilde{\mathscr{D}}_{X}$-coherent submodule of $\mathscr{M}$ with support having codimension $\geqslant 1$ in $Z$;
(c) near any $x_{o}$, there is no nonzero morphism $\varphi: \mathscr{M} \rightarrow \mathscr{N}$, with $\mathscr{N}$ strictly $S$-decomposable at $x_{o}$, such that $\operatorname{Im} \varphi$ is supported in codimension $\geqslant 1$ in $Z$.

Proof. Property (1) is an immediate consequence of Exercise 7.4.16, and property (2) follows from the fact that for any germ $g$, the corresponding can and var decompose with respect to the given decomposition of $\mathscr{M}$. Let us now prove (3). Both conditions (3a) and (3b) reduce to the property that, for any analytic germ $g:\left(X, x_{o}\right) \rightarrow(\mathbb{C}, 0)$ which does not vanish identically on any local irreducible component of $Z$ at $x_{o}$, the corresponding decomposition $\mathscr{M}=\mathscr{M}^{\prime} \oplus \mathscr{M}^{\prime \prime}$ of 7.7.5(2) reduces to $\mathscr{M}=\mathscr{M}^{\prime}$, i.e., $\mathscr{M}^{\prime \prime}=0$. For the equivalence with (3c), we note that, according to the last assertion in Proposition 7.7.5, and with respect to the decomposition $\varphi=\varphi^{\prime} \oplus \varphi^{\prime \prime}$ along a germ $g$, we have $\operatorname{Im} \varphi \neq 0$ and supported in $g^{-1}(0)$ if and only if $\operatorname{Im} \varphi^{\prime \prime} \neq 0$, and thus $\mathscr{M}^{\prime \prime} \neq 0$. Conversely, if $\mathscr{M}^{\prime \prime} \neq 0$, the projection $\mathscr{M} \rightarrow \mathscr{M}^{\prime \prime}$ gives a morphism contradicting (3c).

Definition 7.7.9 (Pure support). Let $\mathscr{M}$ be strictly S-decomposable and having support a pure dimensional closed analytic subset $Z$ of $X$. We say that $\mathscr{M}$ has pure support $Z$ if the equivalent conditions of 7.7.8(3) are satisfied.

## Proposition 7.7.10 (Generic structure of a strictly S-decomposable module)

Assume that $\mathscr{M}$ is holonomic and strictly $S$-decomposable with pure support $Z$, where $Z$ is smooth. Then there exists a unique holonomic and strictly $S$-decomposable $\widetilde{\mathscr{D}}_{Z}$-module $\mathscr{N}$ such that $\mathscr{M}={ }_{\mathrm{D}}{ }^{\iota} \widetilde{Z}_{*} \mathscr{N}$. Moreover, there exists a Zariski dense open subset $Z^{\circ} \subset Z$ such that $\mathscr{N}_{Z^{\circ}}$ is $\mathscr{O}_{Z^{o}}$-locally free of finite rank.

Proof. By uniqueness, the question is local on $Z$. We argue by induction on $\operatorname{dim} X$. Let $H$ be a smooth hypersurface containing $Z$ such that $H=\{t=0\}$ of some local coordinate system $\left(t, x_{2}, \ldots, x_{d}\right)$. Since $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along $t$, the strict Kashiwara's equivalence implies that $\mathscr{M}={ }_{\mathrm{D}} \iota_{H *} \mathscr{N}$ for a unique coherent $\widetilde{\mathscr{D}}_{H}$-module $\mathscr{N}$. Moreover, $\mathscr{N}$ is strictly $\mathbb{R}$-specializable along any function $g$ on $H$, according to Proposition 7.6.6. If $g=g_{\mid H}$, then one checks that a decomposition 7.7.5(2) for $\mathscr{M}$ along $g$ comes from a decomposition 7.7.5(2) for $\mathscr{N}$ along $g$. We conclude that $\mathscr{N}$ is also strictly S-decomposable, and has pure support $Z \subset H$. Continuing this way, we reach a coherent $\widetilde{\mathscr{D}}_{Z}$-module which is strictly S-decomposable. It is easy to check that $\mathscr{N}$ is holonomic since, if Char $\mathscr{M}$ is obtain by a straightforward formula from Char $\mathscr{N}$. By deleting from $Z$ the projection of all components of Char $\mathscr{N}$ except the zero section, we obtain a Zariski-dense open subset $Z^{o}$ of $Z$ such that Char $\mathscr{N}_{\mid Z^{\circ}} \subset T_{Z}^{*} Z \times \mathbb{C}_{z}$. We conclude from Exercise A. 10.16 that $\mathscr{N}_{Z^{\circ}}$ is $\widetilde{\mathscr{O}}_{Z^{o} \text {-coherent. }}$

Let us now prove the $\widetilde{\mathscr{O}}_{Z^{o}}$ local freeness of $\mathscr{N}_{\mid Z^{o}}$. If $t$ is a local coordinate, notice that each term of the $V$-filtration $V_{\bullet} \mathscr{N}$ is $\widetilde{\mathscr{O}}_{Z^{\circ}}$-coherent. It follows that the filtration is locally stationary, hence $\mathscr{N}=V_{0} \mathscr{N}$, since $\operatorname{gr}_{\alpha}^{V} \mathscr{N}=0$ for $\alpha \gg 0$, hence for all $\alpha>0$. Let $m$ be a local section of $\mathscr{N}$ killed by $z$. Then $m$ is zero in $\mathscr{N} / \mathscr{N} t$ by strict $\mathbb{R}$-specializability. As $\mathscr{N}$ is $\widetilde{\mathscr{O}}_{Z^{o}}$-coherent, Nakayama's lemma (applied to $\left.\mathscr{N} \otimes_{\widetilde{\mathscr{O}}_{Z^{\circ}}} \mathscr{O}_{Z^{o} \times \mathbb{C}_{z}}\right)$ implies that $m=0$.

We will now show that a strictly S -decomposable holonomic $\widetilde{\mathscr{D}}_{X}$-module (see Definition A.10.18) can indeed be decomposed as the direct sum of holonomic $\widetilde{\mathscr{D}}_{X}$-modules having pure support. We first consider the local decomposition and, by uniqueness, we get the global one. It is important for that to be able to define a priori the strict components. They are obtained from the characteristic variety of $\mathscr{M}$, equivalently of $\mathcal{M}$, according to Corollary 7.7.15 below.

Proposition 7.7.11. Let $\mathscr{M}$ be holonomic and strictly $S$-decomposable at $x_{o}$, and let $\left(Z_{i}, x_{o}\right)_{i \in I}$ be the family of closed irreducible analytic germs $\left(Z_{i}, x_{o}\right)$ such that Char $\mathscr{M}=\bigcup_{i} T_{Z_{i}}^{*} X \times \mathbb{C}_{z}$ near $x_{o}$. There exists a unique decomposition $\mathscr{M}_{x_{o}}=\oplus_{i \in I} \mathscr{M}_{Z_{i}, x_{o}}$ of germs at $x_{o}$ such that $\mathscr{M}_{Z_{i}, x_{o}}=0$ or has pure support $\left(Z_{i}, x_{o}\right)$.

Proof. We will argue by induction on $\operatorname{dim} \operatorname{Supp} \mathscr{M}$. First, we reduce to the case when the support $S$ of $\mathscr{M}$ (see Proposition A.10.13) is irreducible at $x_{o}$. For this purpose, let $S^{\prime}$ be an irreducible component of $S$ at $x_{o}$ of maximal dimension, and let $S^{\prime \prime}$ be the union of all the other ones. Let $g:\left(X, x_{o}\right) \rightarrow(\mathbb{C}, 0)$ be an analytic germ such that $S^{\prime \prime} \subset g^{-1}(0)$ and $\left(S^{\prime}, x_{o}\right) \not \subset g^{-1}(0)$. Then, according to 7.7.5(2), near $x_{o}, \mathscr{M}$ has a decomposition $\mathscr{M}=\mathscr{M}^{\prime} \oplus \mathscr{M}^{\prime \prime}$, with $\mathscr{M}^{\prime}$ supported on $S^{\prime}$ and being a minimal extension along $(g)$, and $\mathscr{M}^{\prime \prime}$ supported on $S^{\prime \prime}$.

Conversely, if we have any local decomposition $\mathscr{M}=\oplus \mathscr{M}_{S_{i}}$, with $\left(S_{i}, x_{o}\right)$ irreducible and $\mathscr{M}_{S_{i}}$ (strictly S-decomposable after Lemma 7.7.8(2)) having pure support $S_{i}$, then $S_{i} \subset S^{\prime}$ or $S_{i} \subset S^{\prime \prime}$ and $\mathscr{M}^{\prime}=\oplus_{S_{i} \not \subset S^{\prime \prime}} \mathscr{M}_{S_{i}}, \mathscr{M}^{\prime \prime}=\oplus_{S_{i} \subset S^{\prime \prime}} \mathscr{M}_{S_{i}}$.

By induction on the number of irreducible components, we are reduced to the case when $\left(S, x_{o}\right)$ is irreducible. We can assume that $\operatorname{dim} S>0$.

Choose now a germ $g:\left(X, x_{o}\right) \rightarrow(\mathbb{C}, 0)$ which is nonconstant on $S$ and such that $g^{-1}(0)$ contains all components $Z_{i}$ defined by Char $\mathscr{M}$, except $S$. We have, as above, a unique decomposition $\mathscr{M}=\mathscr{M}^{\prime} \oplus \mathscr{M}^{\prime \prime}$ of germs at $x_{o}$, where $\mathscr{M}^{\prime}$ is a minimal extension along $(g)$, and $\mathscr{M}^{\prime \prime}$ is supported on $g^{-1}(0)$, by the assumption of strict S-decomposability along $(g)$ at $x_{o}$. Moreover, $\mathscr{M}^{\prime}$ and $\mathscr{M}^{\prime \prime}$ are also strictly S-decomposable at $x_{o}$. We can apply the inductive assumption to $\mathscr{M}^{\prime \prime}$.

Let us show that $\mathscr{M}^{\prime}$ has pure support $S$ near $x_{o}$ : if $\mathscr{M}_{1}^{\prime}$ is a coherent submodule of $\mathscr{M}^{\prime}$ supported on a strict analytic subset $Z \subset S$, then $Z$ is contained in the union of the components $Z_{i}$, hence $\mathscr{M}_{1}^{\prime}$ is supported in $g^{-1}(0)$, so is zero. We conclude by 7.7.8(3b).

For the uniqueness, we note that, given such a local decomposition with components $\mathscr{M}_{Z_{i}, x_{o}}$, the components $\varphi_{i j}$ of any morphism $\varphi: \mathscr{M}_{x_{o}} \rightarrow \mathscr{M}_{x_{o}}$ vanishes as soon as
$i \neq j$. Indeed, we have either $\operatorname{codim}_{Z_{i}}\left(Z_{i} \cap Z_{j}\right) \geqslant 1$, or $\operatorname{codim}_{Z_{j}}\left(Z_{i} \cap Z_{j}\right)$. In the first case we apply Lemma $7.7 .8(3 \mathrm{c})$ to $\mathscr{M}_{Z_{i}, x_{o}}$. In the second case, we apply Lemma 7.7.8(3b) to $\mathscr{M}_{Z_{j}, x_{o}}$. We apply the same result for the identity $\mathscr{M} \rightarrow \mathscr{M}$ with respect to two such local decompositions.

By uniqueness of the local decomposition, we get:
Corollary 7.7.12. Let $\mathscr{M}$ be holonomic and strictly $S$-decomposable on $X$ and let $\left(Z_{i}\right)_{i \in I}$ be the (locally finite) family of closed irreducible analytic subsets $Z_{i}$ such that Char $\mathscr{M} \subset \cup_{i} T_{Z_{i}}^{*} X \times \mathbb{C}_{z}$. There exists a unique decomposition $\mathscr{M}=\oplus_{i} \mathscr{M}_{Z_{i}}$ such that each $\mathscr{M}_{Z_{i}}=0$ or has pure support $Z_{i}$.

A closed analytic irreducible subset $Z$ of $X$ such that $\mathscr{M}_{Z} \neq 0$ is called a strict component of $\mathscr{M}$.

Proof of Corollary 7.7.12. Given the family $\left(Z_{i}\right)_{i \in I}$ and $x_{o} \in X$, the germs $\left(Z_{i}, x_{o}\right)$ are equidimensional, and Proposition 7.7 .11 gives a unique decomposition $\mathscr{M}_{x_{o}}=$ $\oplus_{i \in I} \mathscr{M}_{Z_{i}, x_{o}}$ by gathering the local irreducible components of $\left(Z_{i}, x_{o}\right)$. The uniqueness enables us to glue all along $Z_{i}$ the various germs $\mathscr{M}_{Z_{i}, x}$.
Corollary 7.7.13. Let $\mathscr{M}^{\prime}, \mathscr{M}^{\prime \prime}$ be two holonomic $\widetilde{\mathscr{D}}_{X}$-module which are strictly $S$-decomposable and let $\left(Z_{i}\right)_{i \in I}$ be the family of their strict components. Then any morphism $\varphi: \mathscr{M}_{Z_{i}}^{\prime} \rightarrow \mathscr{M}_{Z_{j}}^{\prime \prime}$ vanishes identically if $Z_{i} \neq Z_{j}$.

Proof. The image of $\varphi$ is supported on $Z_{i} \cap Z_{j}$, which has everywhere codimension $\geqslant 1$ in $Z_{i}$ or $Z_{j}$ if $Z_{i} \neq Z_{j}$. We then apply Lemma 7.7.8.

Corollary 7.7.14. Let $\mathscr{M}$ be holonomic and strictly $S$-decomposable. Then $\mathscr{M}$ is strict.
Proof. The question is local, and we can assume that $\mathscr{M}$ has pure support $Z$ with $Z$ closed irreducible analytic near $x_{o}$. Proposition 7.7 .10 applied to the smooth part of $Z$ produces a dense open subset $Z^{o}$ of $Z$ such that $\mathscr{M}_{\mid Z^{\circ}}$ is strict. (In fact, since $Z^{o}$ was defined in terms of the characteristic variety, one can show that it is Zariski open in $Z$, but this will not matter here.) Let $m$ be a local section of $\mathscr{M}$ near $x_{o} \in Z$ killed by $z$. Then $m \cdot \widetilde{\mathscr{D}}_{X}$ is supported by a proper analytic subset of $Z$ near $x_{o}$ by the previous argument. As $\mathscr{M}$ has pure support $Z$, we conclude that $m=0$.

Corollary 7.7.15. Let $\mathscr{M}$ be holonomic and strictly $S$-decomposable. Then $\mathrm{Char} \mathscr{M}=$ $\operatorname{Char}(\mathscr{M} /(z-1) \mathscr{M}) \times \mathbb{C}_{z}$.

Proof. Since $\mathscr{M}$ is strict, we can apply Exercise A.10.23(1).
Remark 7.7.16 (Restriction to $z=1$ ). Let us keep the notation of Exercise 7.3.21 and let us assume that $\mathscr{M}$ is $R_{F} \mathscr{D}_{X}$-coherent and strictly $\mathbb{R}$-specializable. It is obvious that, if can is onto for $\mathscr{M}$, it is also onto for $\mathcal{M}:=\mathscr{M} / \mathscr{M}(z-1)$. On the other hand, it is also true that, if var in injective for $\mathscr{M}$, it is also injective for $\mathcal{M}$ (see Exercise A.2.5(3)). As a consequence, if $\mathscr{M}$ is a minimal extension along $(g)$, so is $\mathcal{M}$. Moreover,
if $\mathscr{M}$ is strictly S-decomposable along $(g)$ at $x_{o}$, so is $\mathcal{M}$, and the strict decomposition $\mathscr{M}=\mathscr{M}^{\prime} \oplus \mathscr{M}^{\prime \prime}$ restricts to the decomposition $\mathcal{M}=\mathcal{M}^{\prime} \oplus \mathcal{M}^{\prime \prime}$ given by 7.7.5(2).

We conclude that, if $\mathscr{M}$ is strictly S-decomposable, then $\mathcal{M}$ is S-decomposable, and the strict components, together with the pure support, are in one-to-one correspondence.

### 7.8. Direct image of strictly $\mathbb{R}$-specializable coherent $\widetilde{\mathscr{D}}_{X}$-modules

Let us consider the setting of Theorem A.10.26. So $f: X \rightarrow X^{\prime}$ is a proper holomorphic map, and $\mathscr{M}$ is a coherent right $\widetilde{\mathscr{D}}_{X}$-module. Let $H^{\prime} \subset X^{\prime}$ be a smooth hypersurface. We will assume that $H:=f^{*}\left(H^{\prime}\right)$ is also a smooth hypersurface of $X$.

If $\mathscr{M}$ has a coherent $V$-filtration $U \mathscr{M}$ along $H$, the $R_{V} \widetilde{\mathscr{D}}_{X}$-module $R_{U} \mathscr{M}$ is therefore coherent. With the assumptions above it is possible to define a sheaf $R_{V} \widetilde{\mathscr{D}}_{X \rightarrow X^{\prime}}$ and therefore the pushforward ${ }_{\mathrm{D}} f_{*} R_{U} \mathscr{M}$ as an $R_{V} \widetilde{\mathscr{D}}_{X^{\prime}-\text { module (where } V_{\bullet}} \widetilde{\mathscr{D}}_{X^{\prime}}$ is the $V$-filtration relative to $H^{\prime}$ ).

We will show the $R_{V} \widetilde{\mathscr{D}}_{X^{\prime}}$-coherence of the cohomology sheaves $\mathscr{H}^{k}{ }_{\mathrm{D}} f_{*} R_{U} \mathscr{M}$ of the pushforward ${ }_{\mathrm{D}} f_{*} R_{U} \mathscr{M}$ if $\mathscr{M}$ is endowed with a coherent filtration. By the argument of Exercise 7.3.6, by quotienting by the $v$-torsion, we obtain a coherent $V$-filtration on the cohomology sheaves $\mathscr{H}_{\mathrm{D}}^{k} f_{*} \mathscr{M}$ of the pushforward ${ }_{\mathrm{D}} f_{*} \mathscr{M}$.

The $v$-torsion part contains much information however, and this supplementary operation killing the $v$-torsion looses it. The main result of this section is that, if $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along $H$, then so are the cohomology sheaves $\mathscr{H}^{k}{ }_{\mathrm{D}} f_{*} \mathscr{M}$ along $H^{\prime}$, and moreover, when considering the filtration by the order, the corresponding Rees modules $\mathscr{H}_{\mathrm{D}}^{k} f_{*} R_{V} \mathscr{M}$ have no $v$-torsion, and can thus be written as $R_{U} \mathscr{H}_{\mathrm{D}}^{k} f_{*} \mathscr{M}$ for some coherent $V$-filtration $U . \mathscr{H}_{\mathrm{D}}^{k} f_{*} \mathscr{M}$. This coherent $V$-filtration is nothing but the Kashiwara-Malgrange filtration of $\mathscr{H}_{\mathrm{D}}^{k} f_{*} \mathscr{M}$. We say that the Kashiwara-Malgrange filtration behaves strictly with respect to the pushforward functor ${ }_{\mathrm{D}} f_{*}$.

Another way to present this property is to consider the individual terms $V_{\alpha} \mathscr{M}$ of the Kashiwara-Malgrange filtration as $V_{0} \widetilde{\mathscr{D}}_{X}$-modules. By introducing the sheaf $V_{0} \widetilde{\mathscr{D}}_{X \rightarrow X^{\prime}}$, one can define the pushforward complex ${ }_{\mathrm{D}} f_{*} V_{\alpha} \mathscr{M}$ for every $\alpha$, and the strictness property amounts to saying that for every $k$ and $\alpha$, the morphisms $\mathscr{H}_{\mathrm{D}}^{k} f_{*} V_{\alpha} \mathscr{M} \rightarrow \mathscr{H}_{\mathrm{D}}^{k} f_{*} \mathscr{M}$ are injective. In the following, we work with right $\widetilde{\mathscr{D}}_{X}$-modules and increasing $V$-filtrations.

## 7.8.a. Definition of the pushforward functor and the Coherence Theorem

We first note that our assumption on $H, H^{\prime}, f$ is equivalent to the property that, locally at $x_{o} \in H$, setting $x_{o}^{\prime}=f\left(x_{o}\right)$, there exist local decompositions $\left(X, x_{o}\right) \simeq$ $\left(H, x_{o}\right) \times(\mathbb{C}, 0)$ and $\left(X^{\prime}, x_{o}^{\prime}\right) \simeq\left(H^{\prime}, x_{o}^{\prime}\right) \times(\mathbb{C}, 0)$ such that $f(y, t)=\left(f_{\mid H}(y), t\right)$.

Let $U \cdot \mathscr{M}$ be a $V$-filtration of $\mathscr{M}$ and let $R_{U} \mathscr{M}$ be the associated graded $R_{V} \widetilde{\mathscr{D}}_{X^{-}}$ module. Our first objective is to apply the same reasoning as in Theorem A.10.26 by replacing the category of $\widetilde{\mathscr{D}}$-modules with that of graded $R_{V} \widetilde{\mathscr{D}}_{X}$-modules.

The sheaf $\widetilde{\mathscr{D}}_{X \rightarrow X^{\prime}}$ has a $V$-filtration: we set $V_{k} \widetilde{\mathscr{D}}_{X \rightarrow X^{\prime}}:=\widetilde{\mathscr{O}}_{X} \otimes_{f^{-1} \widetilde{\mathscr{O}}_{X^{\prime}}} f^{-1} V_{k} \widetilde{\mathscr{D}}_{X^{\prime}}$. One checks in local decompositions as above that, with respect to the left $\widetilde{\mathscr{D}}_{X}$-structure one has $V_{\ell} \widetilde{\mathscr{D}}_{X} \cdot V_{k} \widetilde{\mathscr{D}}_{X \rightarrow X^{\prime}} \subset V_{k+\ell} \widetilde{\mathscr{D}}_{X \rightarrow X^{\prime}}$. We can write

$$
R_{V} \widetilde{\mathscr{D}}_{X \rightarrow X^{\prime}}:=\widetilde{\mathscr{O}}_{X} \otimes_{f^{-1} \widetilde{\mathscr{O}}_{X^{\prime}}} f^{-1} R_{V} \widetilde{\mathscr{D}}_{X^{\prime}}=R_{V} \widetilde{\mathscr{O}}_{X} \otimes_{f^{-1} R_{V} \widetilde{\mathscr{O}}_{X^{\prime}}} f^{-1} R_{V} \widetilde{\mathscr{D}}_{X^{\prime}}
$$

According to Exercise 7.2.6, $R_{V} \widetilde{\mathscr{D}}_{X^{\prime}}$ is $R_{V} \widetilde{\mathscr{O}}_{X^{\prime}}$-locally free, so $R_{V} \widetilde{\mathscr{D}}_{X \rightarrow X^{\prime}}$ is $R_{V} \widetilde{\mathscr{O}}_{X^{-}}$ locally free.

We define

$$
\begin{equation*}
{ }_{\mathrm{D}} f_{*} R_{U} \mathscr{M}:=\boldsymbol{R} f_{*}\left(R_{U} \mathscr{M} \otimes_{R_{V} \tilde{\mathscr{D}}_{X}}^{\boldsymbol{L}} R_{V} \widetilde{\mathscr{D}}_{X \rightarrow X^{\prime}}\right) \tag{7.8.1}
\end{equation*}
$$

as an object of $\mathrm{D}^{\mathrm{b}}\left(R_{V} \widetilde{\mathscr{D}}_{X^{\prime}}\right)$.
Theorem 7.8.2. Let $\mathscr{M}$ be a $\widetilde{\mathscr{D}}_{X}$-module endowed with a coherent filtration $F_{.} \mathscr{M}$. Let $U . \mathscr{M}$ be a coherent $V$-filtration of $\mathscr{M}$. Then the cohomology modules of ${ }_{\mathrm{D}} f_{*} R_{U} \mathscr{M}$ have coherent $R_{V} \widetilde{\mathscr{D}}_{X^{\prime}}$-cohomology.

Lemma 7.8.3. Let $\mathscr{L}$ be an $R_{V} \widetilde{\mathscr{O}}_{X}$-module. Then

$$
\left(\mathscr{L} \otimes_{R_{V} \tilde{\mathscr{O}}_{X}} R_{V} \widetilde{\mathscr{D}}_{X}\right) \otimes_{R_{V} \tilde{\mathscr{D}}_{X}}^{\boldsymbol{L}} R_{V} \widetilde{\mathscr{D}}_{X \rightarrow X^{\prime}}=\mathscr{L} \otimes_{f^{-1} R_{V} \tilde{\mathscr{O}}_{X^{\prime}}} f^{-1} R_{V} \widetilde{\mathscr{D}}_{X^{\prime}} .
$$

Proof. It is a matter of proving that the left-hand side has cohomology in degree 0 only, since this cohomology is easily seen to be equal to the right-hand side. This can be checked on germs at $x \in X$. Let $\mathscr{L}_{x}^{\bullet}$ be a resolution of $\mathscr{L}_{x}$ by free $R_{V} \widetilde{\mathscr{O}}_{X, x}$-modules. We have

$$
\begin{aligned}
\left(\mathscr{L}_{x} \otimes_{R_{V}} \widetilde{\mathscr{O}}_{X, x}\right. & \left.R_{V} \widetilde{\mathscr{D}}_{X, x}\right) \otimes_{R_{V}}^{L} \widetilde{\mathscr{D}}_{X, x} \\
& =\left(\mathscr{L}_{x} \otimes_{V}^{L} \widetilde{\mathscr{D}}_{X \rightarrow X^{\prime}, x} \widetilde{\mathscr{O}}_{X, x}\right. \\
& \left.R_{V} \widetilde{\mathscr{D}}_{X, x}\right) \otimes_{R_{V}}^{L} \widetilde{\mathscr{D}}_{X, x} \\
& =\left(\mathscr{L}_{x}^{\bullet} \otimes_{R_{V}} \widetilde{\mathscr{D}}_{X, x} \widetilde{\mathscr{D}}_{X \rightarrow X^{\prime}, x} \quad\right. \text { (Ex. 7.2.6) } \\
& =\left(\widetilde{\mathscr{D}}_{X, x}\right) \otimes_{R_{V}}^{L} \widetilde{\mathscr{D}}_{X, x} R_{V V} \widetilde{\mathscr{D}}_{X \rightarrow X^{\prime}, x} \\
& \left.=\widetilde{\mathscr{L}}_{x, x}^{\bullet} \otimes_{R_{V}} \widetilde{\mathscr{O}}_{X, x} \widetilde{\mathscr{D}}_{X, x}\right) \otimes_{R_{V}} \widetilde{\mathscr{D}}_{X, x} \widetilde{\mathscr{D}}_{X \rightarrow X^{\prime}, x} \\
& R_{V} \widetilde{\mathscr{D}}_{X \rightarrow X^{\prime}, x} \\
& =\mathscr{L}_{x} \otimes_{R_{V} \widetilde{\mathscr{O}}_{X, x}}^{L} R_{V} \widetilde{\mathscr{D}}_{X \rightarrow X^{\prime}, x} \\
& =\mathscr{L}_{x} \otimes_{R_{V} \widetilde{\mathscr{O}}_{X, x}} R_{V} \widetilde{\mathscr{D}}_{X \rightarrow X^{\prime}, x} \quad\left(R_{V} \widetilde{\mathscr{D}}_{X \rightarrow X^{\prime}, x} \text { is } R_{V} \widetilde{\mathscr{O}}_{X, x} \text {-free }\right) .
\end{aligned}
$$

As a consequence of this lemma, we have

$$
{ }_{\mathrm{D}} f_{*}\left(\mathscr{L} \otimes_{R_{V} \widetilde{\sigma}_{X}} R_{V} \widetilde{\mathscr{D}}_{X}\right)=\left(\boldsymbol{R} f_{*} \mathscr{L}\right) \otimes_{R_{V} \widetilde{\mathscr{O}}_{X^{\prime}}} R_{V} \widetilde{\mathscr{D}}_{X^{\prime}}
$$

and the cohomology of this complex is $R_{V} \widetilde{\mathscr{D}}_{X^{\prime}}$-coherent.
Lemma 7.8.4. Assume that $\mathscr{M}$ is a $\widetilde{\mathscr{D}}_{X}$-module having a coherent filtration $F_{.} \mathscr{M}$ and let $U . \mathscr{M}$ be a coherent $V$-filtration of $\mathscr{M}$. Then in the neighbourhood of any compact set of $X, R_{U} \mathscr{M}$ has a coherent $F_{\bullet} R_{V} \widetilde{\mathscr{D}}_{X}$-filtration.

Proof. Fix a compact set $K \subset X$. We can thus assume that $\mathscr{M}$ is generated by a coherent $\widetilde{\mathscr{O}}_{X}$-module $\mathscr{F}$ in some neighbourhood of $K$, i.e., $\mathscr{M}=\widetilde{\mathscr{D}}_{X} \cdot \mathscr{F}$. Consider the $V$-filtration $U_{\bullet}^{\prime} \mathscr{M}$ generated by $\mathscr{F}$, i.e., $U_{\bullet}^{\prime} \mathscr{M}=V_{\bullet} \widetilde{\mathscr{D}}_{X} \cdot \mathscr{F}$. Then, clearly, $R_{V} \widetilde{\mathscr{O}}_{X} \cdot \mathscr{F}=$ $\bigoplus_{k} V_{k} \widetilde{\mathscr{O}}_{X} \cdot \mathscr{F} v^{k}$ is a coherent graded $R_{V} \widetilde{\mathscr{O}}_{X}$-module which generates $R_{U^{\prime}} \mathscr{M}$ as an $R_{V} \widetilde{\mathscr{D}}_{X}$-module.

If the filtration $U_{\bullet}^{\prime \prime} \mathscr{M}$ is obtained from $U_{\bullet}^{\prime} \mathscr{M}$ by a shift by $-\ell \in \mathbb{Z}$, i.e., if $R_{U^{\prime \prime}} \mathscr{M}=$ $v^{\ell} R_{U^{\prime}} \mathscr{M} \subset \mathscr{M}\left[v, v^{-1}\right]$, then $R_{U^{\prime \prime}} \mathscr{M}$ is generated by the $R_{V} \widetilde{\mathscr{O}}_{X}$-coherent submodule $v^{\ell} R_{V} \widetilde{\mathscr{O}}_{X} \cdot \mathscr{F}$.

On the other hand, let $U_{\bullet}^{\prime \prime} \mathscr{M}$ be a coherent $V$-filtration such that $R_{U^{\prime \prime}} \mathscr{M}$ has a coherent $F_{\bullet} R_{V} \widetilde{\mathscr{D}}_{X}$-filtration. Then any coherent $V$-filtration $U . \mathscr{M}$ such that $U_{k} \mathscr{M} \subset$ $U_{k}^{\prime \prime} \mathscr{M}$ for every $k$ satisfies the same property, because $R_{U} \mathscr{M}$ is thus a coherent graded $R_{V} \widetilde{\mathscr{D}}_{X}$-submodule of $R_{U^{\prime \prime}} \mathscr{M}$, so a coherent filtration on the latter induces a coherent filtration on the former.

As any coherent $V$-filtration $U \cdot \mathscr{M}$ is contained, in some neighbourhood of $K$, in the coherent $V$-filtration $U^{\prime} \mathscr{M}$ suitably shifted, we get the lemma.
Proof of Theorem 7.8.2. The proof now ends exactly as for Theorem A.10.26.

## 7.8.b. Strictness of the Kashiwara-Malgrange filtration by pushforward

## Theorem 7.8.5 (Pushforward of strictly $\mathbb{R}$-specializable $\widetilde{\mathscr{D}}$-modules)

Let $f: X \rightarrow X^{\prime}$ be a proper morphism of complex manifolds, let $H^{\prime}$ be a smooth hypersurface of $X^{\prime}$ and assume that $\mathscr{I}_{H}:=\mathscr{I}_{H^{\prime}} \mathscr{O}_{X}$ defines a smooth hypersurface $H$ of $X$. Let $\mathscr{M}$ be a coherent right $\widetilde{\mathscr{D}}_{X}$-module equipped with a coherent filtration. Assume that $\mathscr{M}$ is strictly $\mathbb{R}$-specializable along $H$ with Kashiwara-Malgrange filtration $V . \mathscr{M}$ indexed by $A+\mathbb{Z}$ with $A$ finite contained in $[0,1)$, and that each cohomology module $\mathscr{H}^{i}{ }_{\mathrm{D}} f_{\mid H *} \operatorname{gr}_{\alpha}^{V} \mathscr{M}$ is strict $(\alpha \in[-1,0])$.

Then each cohomology module $\mathscr{H}^{i}{ }_{\mathrm{D}} f_{*} \mathscr{M}$, which is $\widetilde{\mathscr{D}}_{X^{\prime}}$-coherent according to Theorem A.10.26, is strictly $\mathbb{R}$-specializable along $H^{\prime}$ and moreover,
(1) for every $\alpha$, $i$, the natural morphism $\mathscr{H}^{i}{ }_{\mathrm{D}} f_{*}\left(V_{\alpha} \mathscr{M}\right) \rightarrow \mathscr{H}^{i}{ }_{\mathrm{D}} f_{*} \mathscr{M}$ is injective,
(2) its image is the Kashiwara-Malgrange filtration of $\mathscr{H}^{i}{ }_{\mathrm{D}} f_{*} \mathscr{M}$ along $H^{\prime}$,
(3) for every $\alpha, i, \operatorname{gr}_{\alpha}^{V}\left(\mathscr{H}^{i}{ }_{\mathrm{D}} f_{*} \mathscr{M}\right)=\mathscr{H}^{i}{ }_{\mathrm{D}} f_{\mid H *}\left(\operatorname{gr}_{\alpha}^{V} \mathscr{H}^{i} \mathscr{M}\right)$.

As an important corollary we obtain in a straightforward way:
Corollary 7.8.6. Let $f: X \rightarrow X^{\prime}$ be a proper morphism of complex manifolds. Let $g^{\prime}: X^{\prime} \rightarrow \mathbb{C}$ be any holomorphic function on $X^{\prime}$ and let $\mathscr{M}$ be $\widetilde{\mathscr{D}}_{X}$-coherent and strictly $\mathbb{R}$-specializable along ( $g$ ) with $g=g^{\prime} \circ f$. Assume that for for all $i$ and $\lambda$, $\mathscr{H}^{i}{ }_{\mathrm{D}} f_{*}\left(\psi_{g, \lambda} \mathscr{M}\right)$ and $\mathscr{H}^{i}{ }_{\mathrm{D}}{ }_{\mathrm{D}} f_{*}\left(\phi_{g, 1} \mathscr{M}\right)$ are strict.

Then $\mathscr{H}^{i}{ }_{\mathrm{D}} f_{*} \mathscr{M}$ is $\widetilde{D}_{X^{\prime}}$-coherent and strictly $\mathbb{R}$-specializable along $\left(g^{\prime}\right)$, we have for all $i$ and $\lambda$,

$$
\begin{aligned}
& \left(\psi_{g, \lambda}\left(\mathscr{H}^{i}{ }_{\mathrm{D}} f_{*} \mathscr{M}\right), \mathrm{N}\right)=\mathscr{H}^{i}{ }_{\mathrm{D}} f_{*}\left(\psi_{g, \lambda} \mathscr{M}, \mathrm{~N}\right), \\
& \left(\phi_{g, 1}\left(\mathscr{H}^{i}{ }_{\mathrm{D}} f_{*} \mathscr{M}\right), \mathrm{N}\right)=\mathscr{H}^{i}{ }_{\mathrm{D}} f_{*}\left(\phi_{g, 1} \mathscr{M}, \mathrm{~N}\right),
\end{aligned}
$$

and the morphisms can, var for $\mathscr{H}^{i}{ }_{\mathrm{D}} f_{*} \mathscr{M}$ are the morphisms $\mathscr{H}^{i}{ }_{\mathrm{D}} f_{*}$ can, $\mathscr{H}^{i}{ }_{\mathrm{D}} f_{*}$ var.

We first explain the mechanism which leads to the strictness property stated in Theorem 7.8.5(1).

Proposition 7.8.7. Let $H^{\prime} \subset X^{\prime}$ be a smooth hypersurface. Let ( $\left.\mathscr{N}^{\bullet}, U . \mathscr{N}^{\bullet}\right)$ be a $V$-filtered complex of $\widetilde{\mathscr{D}}_{X^{\prime}}$-modules, where $U$. is indexed by $A+\mathbb{Z}, A \subset[0,1)$ finite. Let $N \geqslant 0$ and assume that
(1) $\mathscr{H}^{i}\left(\operatorname{gr}_{\alpha}^{U} \mathscr{N}^{\bullet}\right)$ is strict for all $\alpha \in A+\mathbb{Z}$ and all $i \geqslant-N-1$;
(2) for every $\alpha \in A+\mathbb{Z}$, there exists $\nu_{\alpha} \geqslant 0$ such that $(\mathrm{E}-\alpha z)^{\nu_{\alpha}}$ acts by zero on $\mathscr{H}^{i}\left(\operatorname{gr}_{\alpha}^{U} \mathscr{N}^{\bullet}\right)$ for every $i \geqslant-N-1$;
(3) there exists $\alpha_{o}$ such that for all $\alpha \leqslant \alpha_{o}$ and all $i \geqslant-N-1$, the right multiplication by some (or any) local reduced equation $t$ of $H^{\prime}$ induces an isomorphism $t: U_{\alpha} \mathscr{N}^{i} \xrightarrow{\sim} U_{\alpha-1} \mathscr{N}^{i} ;$
(4) there exists $i_{o} \in \mathbb{Z}$ such that, for all $i \geqslant i_{o}$ and any $\alpha$, one has $\mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet}\right)=0$;
(5) $\mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet}\right)$ is $V_{0} \widetilde{\mathscr{D}}_{X^{\prime}}$-coherent for all $\alpha \in A+\mathbb{Z}$ and all $i \geqslant-N-1$.

Then for every $\alpha$ and $i \geqslant-N$ the morphism $\mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet}\right) \rightarrow \mathscr{H}^{i}\left(\mathscr{N}^{\bullet}\right)$ is injective. Moreover, the filtration $U . \mathscr{H}^{i}\left(\mathscr{N}^{\bullet}\right)$ defined by

$$
U_{\alpha} \mathscr{H}^{i}\left(\mathscr{N}^{\bullet}\right)=\operatorname{image}\left[\mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet}\right) \longrightarrow \mathscr{H}^{i}\left(\mathscr{N}^{\bullet}\right)\right]
$$

satisfies $\operatorname{gr}_{\alpha}^{U} \mathscr{H}^{i}\left(\mathscr{N}^{\bullet}\right)=\mathscr{H}^{i}\left(\operatorname{gr}_{\alpha}^{U} \mathscr{N}^{\bullet}\right)$ for all $\alpha \in A+\mathbb{Z}$.
Proof. It will have three steps. During the proof, the indices $\alpha, \beta, \gamma$ will run in $A+\mathbb{Z}$. First step. This step proves a formal analogue of the conclusion of the proposition. Put

$$
\widehat{U_{\alpha} \mathscr{N}^{\bullet}}={\underset{خ}{\gamma}}_{\lim _{\alpha}} U_{\alpha} \mathscr{N}^{\bullet} / U_{\gamma} \mathscr{N}^{\bullet} \quad \text { and } \quad \widehat{\mathcal{N}^{\bullet}}=\underset{\alpha}{\lim } \widehat{U_{\alpha} \mathscr{N}^{\bullet}}
$$

Under the assumption of Proposition 7.8.7, we will prove the following:
(a) For all $\beta \leqslant \alpha, \widehat{U_{\beta \mathscr{N}^{\bullet}}} \rightarrow \widehat{U_{\alpha \mathscr{N}^{\bullet}}}$ is injective (hence, for all $\alpha, \widehat{U_{\alpha \mathscr{N}^{\bullet}}} \rightarrow{\widehat{\mathscr{N}^{\bullet}}}^{\text {is }}$ injective) and $\widehat{U_{\alpha} \mathscr{N}} \bullet / \widehat{U_{<\alpha \mathscr{N}}}=U_{\alpha} \mathscr{N}^{\bullet} / U_{<\alpha \mathscr{N}^{\bullet}}$.
(b) For every $\beta \leqslant \alpha$ and any $i, \mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet} / U_{\beta} \mathscr{N}^{\bullet}\right)$ is strict.
(c) $\mathscr{H}^{i}\left(\widehat{U_{\alpha} \mathscr{N}^{\bullet}}\right)=\lim _{\gamma} \mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet} / U_{\gamma} \mathscr{N}^{\bullet}\right)(i \geqslant-N)$.
(d) $\mathscr{H}^{i}\left(\widehat{U_{\alpha} \mathscr{N}^{\bullet}}\right) \rightarrow \mathscr{H}^{i}\left(\widehat{\mathscr{N}^{\bullet}}\right)$ is injective $(i \geqslant-N)$.
(e) $\mathscr{H}^{i}\left(\widehat{\mathscr{N}^{\bullet}}\right)=\lim _{\alpha} \mathscr{H}^{i}\left(\widehat{U_{\alpha} \mathscr{N}^{\bullet}}\right)(i \geqslant-N)$.

We note that the statements (b)-(d) imply that $\mathscr{H}^{i}\left(\widehat{\mathscr{N}^{\bullet}}\right)$ is strict for $i \geqslant-N$, although $\mathscr{H}^{i}\left(\mathscr{N}^{\bullet}\right)$ need not be strict.

Define $U_{\alpha} \mathscr{H}^{i}\left(\widehat{\mathscr{N}^{\bullet}}\right)=\operatorname{image}\left[\mathscr{H}^{i}\left(\widehat{U_{\alpha} \mathscr{N}^{\bullet}}\right) \rightarrow \mathscr{H}^{i}\left(\widehat{\mathscr{N}^{\bullet}}\right)\right]$. Then the statements (a) and (d) imply that

$$
\operatorname{gr}_{\alpha}^{U} \mathscr{H}^{i}\left(\widehat{\mathscr{N}^{\bullet}}\right)=\mathscr{H}^{i}\left(\widehat{U_{\alpha} \mathscr{N}} \cdot / \widehat{U_{<\alpha} \mathscr{N}^{\bullet}}\right)=\mathscr{H}^{i}\left(\operatorname{gr}_{\alpha}^{U} \mathscr{N}^{\bullet}\right) \quad(i \geqslant-N)
$$

For $\gamma<\beta<\alpha$ consider the exact sequence of complexes

$$
0 \longrightarrow U_{\beta} \mathscr{N}^{\bullet} / U_{\gamma} \mathscr{N}^{\bullet} \longrightarrow U_{\alpha} \mathscr{N}^{\bullet} / U_{\gamma} \mathscr{N}^{\bullet} \longrightarrow U_{\alpha} \mathscr{N}^{\bullet} / U_{\beta} \mathscr{N}^{\bullet} \longrightarrow 0
$$

As the projective system $\left(U_{\alpha} \mathscr{N}^{\bullet} / U_{\gamma} \mathscr{N}^{\bullet}\right)_{\gamma}$ trivially satisfies the Mittag-Leffler condition (ML) (see e.g. [KS90, Prop. 1.12.4]), the sequence remains exact after passing to the projective limit, so we get an exact sequence of complexes

$$
0 \longrightarrow \widehat{U_{\beta} \mathscr{N}^{\bullet}} \longrightarrow \widehat{U_{\alpha} \mathscr{N}^{\bullet}} \longrightarrow U_{\alpha} \mathscr{N}^{\bullet} / U_{\beta} \mathscr{N}^{\bullet} \longrightarrow 0
$$

hence (a).
Let us show by induction on $\rho=\alpha-\gamma \in A+\mathbb{N}$ that, for all $\gamma<\alpha$ and $i \geqslant-N$,
(i) $\prod_{\gamma<\beta \leqslant \alpha}(\mathrm{E}-\beta z)^{\nu_{\beta}}$ annihilates $\mathscr{H}^{i}\left(U_{\alpha} / U_{\gamma}\right)$,
(ii) for all $\beta$ such that $\gamma<\beta<\alpha$, we have an exact sequence,
(7.8.8) $0 \rightarrow \mathscr{H}^{i}\left(U_{\beta} \mathscr{N}^{\bullet} / U_{\gamma} \mathscr{N}^{\bullet}\right) \rightarrow \mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet} / U_{\gamma} \mathscr{N}^{\bullet}\right) \rightarrow \mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet} / U_{\beta} \mathscr{N}^{\bullet}\right) \rightarrow 0$.
(iii) $\mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet} / U_{\gamma} \mathscr{N}^{\bullet}\right)$ is strict.

If $\gamma$ is the predecessor of $\alpha$ in $A+\mathbb{Z}$, (i) and (iii) are true by assumption and (ii) is empty. Moreover, (ii) $\rho_{\rho}$ and (iii) ${ }_{<\rho}$ imply (iii) $)_{\rho}$. For $\gamma<\beta<\alpha$ and $\alpha-\gamma=\rho$, consider the exact sequence

$$
\begin{aligned}
\cdots \stackrel{\psi^{i}}{\longrightarrow} \mathscr{H}^{i}\left(U_{\beta} / U_{\gamma}\right) \longrightarrow \mathscr{H}^{i}\left(U_{\alpha} / U_{\gamma}\right) \longrightarrow \mathscr{H}^{i}\left(U_{\alpha} / U_{\beta}\right) \\
\xrightarrow{\psi^{i+1}} \mathscr{H}^{i+1}\left(U_{\beta} / U_{\gamma}\right) \longrightarrow \cdots
\end{aligned}
$$

For any $i \geqslant-N$, any local section of $\operatorname{Im} \psi^{i+1}$ is then killed by $\prod_{\beta<\delta \leqslant \alpha}(\mathrm{E}-\delta z)$ and by $\prod_{\gamma<\delta \leqslant \beta}(\mathrm{E}-\delta z)$ according to $(\mathrm{i})_{<\rho}$, hence is zero by (iii) $)_{<\rho}$, and the same property holds for $\operatorname{Im} \psi^{i}$, so the previous sequence of $\mathscr{H}^{i}$ is exact. Arguing similarly, we get the exactness of (7.8.8) for $\alpha-\gamma=\rho$, hence (ii) ${ }_{\rho}$, from which (i) ${ }_{\rho}$ follows.

Consequently, the projective system $\left(\mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet} / U_{\gamma} \mathscr{N}^{\bullet}\right)\right)_{\gamma}$ satisfies (ML), so we get (c). Moreover, taking the limit on $\gamma$ in (7.8.8) gives, according to (ML), an exact sequence

$$
0 \longrightarrow \mathscr{H}^{i}\left(\widehat{U_{\beta} \mathscr{N}^{\bullet}}\right) \longrightarrow \mathscr{H}^{i}\left(\widehat{U_{\alpha} \mathscr{N}^{\bullet}}\right) \longrightarrow \mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet} / U_{\beta} \mathscr{N}^{\bullet}\right) \longrightarrow 0
$$

hence (d). Now, (e) is clear.
Second step. For every $i, \alpha$, denote by $\mathscr{T}_{\alpha}^{i} \subset \mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet}\right)$ the $\mathscr{I}_{H^{\prime}}$-torsion subsheaf of $\mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet}\right)$. We set locally $\mathscr{I}_{H^{\prime}}=t \mathscr{O}_{X^{\prime}}$. We will now prove that it is enough to show

$$
\begin{equation*}
\exists \alpha_{o}, \quad \alpha \leqslant \alpha_{o} \Longrightarrow \mathscr{T}_{\alpha}^{i}=0 \forall i \geqslant-N . \tag{7.8.9}
\end{equation*}
$$

We assume that (7.8.9) is proved (step 3). Let $\gamma \leqslant \alpha_{o}$ and $i \geqslant-N$, so that $\mathscr{T}_{\gamma}^{i}=0$, and let $\alpha \geqslant \gamma$. Then, by definition of a $V$-filtration, $t^{\lceil\alpha-\gamma\rceil}$ acts by 0 on $U_{\alpha} \mathscr{N}^{\bullet} / U_{\gamma} \mathscr{N}^{\bullet}$, so that the image of $\mathscr{H}^{i-1}\left(U_{\alpha} \mathscr{N}^{\bullet} / U_{\gamma} \mathscr{N}^{\bullet}\right)$ in $\mathscr{H}^{i}\left(U_{\gamma} \mathscr{N}^{\bullet}\right)$ is contained in $\mathscr{T}_{\gamma}^{i}$, and thus is zero. We therefore have an exact sequence for every $i \geqslant-N$ :

$$
0 \longrightarrow \mathscr{H}^{i}\left(U_{\gamma} \mathscr{N}^{\bullet}\right) \longrightarrow \mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet}\right) \longrightarrow \mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet} / U_{\gamma} \mathscr{N}^{\bullet}\right) \longrightarrow 0
$$

Using (7.8.8), we get for every $\beta<\alpha$ the exact sequence

$$
0 \longrightarrow \mathscr{H}^{i}\left(U_{\beta} \mathscr{N}^{\bullet}\right) \longrightarrow \mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet}\right) \longrightarrow \mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet} / U_{\beta} \mathscr{N}^{\bullet}\right) \longrightarrow 0
$$

This implies that $\mathscr{H}^{i}\left(U_{\beta} \mathscr{N}^{\bullet}\right) \rightarrow \mathscr{H}^{i}\left(\mathscr{N}^{\bullet}\right)=\underline{\lim }_{\rightarrow} \mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet}\right)$ is injective. For every $\alpha$, let us set

$$
U_{\alpha} \mathscr{H}^{i}\left(\mathscr{N}^{\bullet}\right):=\text { image }\left[\mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet}\right) \longleftrightarrow \mathscr{H}^{i}\left(\mathscr{N}^{\bullet}\right)\right] .
$$

We thus have, for every $\alpha \in A+\mathbb{Z}$ and $i \geqslant-N$,

$$
\operatorname{gr}_{\alpha}^{U} \mathscr{H}^{i}\left(\mathscr{N}^{\bullet}\right)=\mathscr{H}^{i}\left(\operatorname{gr}_{\alpha}^{U} \mathscr{N}^{\bullet}\right) .
$$

Third step: proof of (7.8.9). Let us choose $\alpha_{o}$ as in 7.8.7(3). We notice that the multiplication by $t$ induces an isomorphism $t: \widehat{U_{\alpha} \mathscr{N}^{i}} \xrightarrow{\sim} \widehat{U_{\alpha-1} \mathscr{N}^{i}}$ for $\alpha \leqslant \alpha_{o}$ and $i \geqslant-N-1$, hence an isomorphism $t: \mathscr{H}^{i}\left(\widehat{U_{\alpha \mathscr{N}^{\bullet}}}\right) \xrightarrow{\sim} \mathscr{H}^{i}\left(\widehat{U_{\alpha-1} \mathscr{N}} \bullet\right)$, and that (d) in Step one implies that, for all $i \geqslant-N$ and all $\alpha \leqslant \alpha_{o}$, the multiplication by $t$ on $\mathscr{H}^{i}\left(\widehat{U_{\alpha} \mathscr{N}} \cdot\right)$ is injective.

The proof of (7.8.9) is done by decreasing induction on $i$. It clearly hods for $i \geqslant i_{o}$ (given by 7.8.7(4)). We assume that, for every $\alpha \leqslant \alpha_{o}$, we have $\mathscr{T}_{\alpha}^{i+1}=0$. We have (after 7.8.7(3)) an exact sequence of complexes, for every $k \in \mathbb{N}$ and $\bullet \geqslant-N-1$,

$$
0 \longrightarrow U_{\alpha} \mathscr{N}^{\bullet} \xrightarrow{t^{k}} U_{\alpha} \mathscr{N}^{\bullet} \longrightarrow U_{\alpha} \mathscr{N}^{\bullet} / U_{\alpha-k} \mathscr{N}^{\bullet} \longrightarrow 0
$$

As $\mathscr{T}_{\alpha}^{i+1}=0$, we have, for every $k \geqslant 1$ an exact sequence

$$
\mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet}\right) \xrightarrow{t^{k}} \mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet}\right) \longrightarrow \mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet} / U_{\alpha-k} \mathscr{N}^{\bullet}\right) \longrightarrow 0
$$

hence, according to Step one,

$$
\mathscr{H}^{i}\left(\widehat{U_{\alpha} \mathscr{N}^{\bullet}}\right) / \mathscr{H}^{i}\left(\widehat{U_{\alpha-k} \mathscr{N}}\right)=\mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet} / U_{\alpha-k} \mathscr{N}^{\bullet}\right)=\mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet}\right) / t^{k} \mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet}\right) .
$$

According to Assumption 7.8.7(5) and Exercise 7.3.8, for $k$ big enough (locally on $X^{\prime}$ ), the map $\mathscr{T}_{\alpha}^{i} \rightarrow \mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet}\right) / t^{k} \mathscr{H}^{i}\left(U_{\alpha} \mathscr{N}^{\bullet}\right)$ is injective. It follows that $\mathscr{T}_{\alpha}^{i} \rightarrow \mathscr{H}^{i}\left(\widehat{U_{\alpha} \mathscr{N}^{\bullet}}\right)$ is injective too. But we know that $t$ is injective on $\mathscr{H}^{i}\left(\widehat{U_{\alpha} \mathscr{N}^{\bullet}}\right)$ for $\alpha \leqslant \alpha_{o}$, hence $\mathscr{T}_{\alpha}^{i}=0$, thus concluding Step 3 .

Proof of Theorem 7.8.5
Lemma 7.8.10. Let $U . \mathscr{M}$ be a $V$-filtration indexed by $A+\mathbb{Z}$ of a $\widetilde{\mathscr{D}}_{X}$-module $\mathscr{M}$ which satisfies the following properties:
(a) $t: U_{\alpha} \mathscr{M} \rightarrow U_{\alpha-1} \mathscr{M}$ is bijective for every $\alpha<0$,
(b) $\partial_{t}: \operatorname{gr}_{\alpha}^{U} \mathscr{M} \rightarrow \operatorname{gr}_{\alpha+1}^{U} \mathscr{M}$ is bijective for every $\alpha>-1$.

We define $R_{U} \mathscr{M}$ as in Remark 7.2.7, which is thus an $R_{A_{V}} \widetilde{\mathscr{D}}_{X}$-module. Then $R_{U} \mathscr{M}$ has a resolution $\mathscr{L}^{\bullet} \otimes_{\widetilde{\mathscr{O}}_{X}} R_{A_{V}} \widetilde{\mathscr{D}}_{X}$, where each $\mathscr{L}^{i}$ is an $\widetilde{\mathscr{O}}_{X}$-module.

Proof. By assumption, the morphism $\varphi: \bigoplus_{\gamma \in[-1,0]} U_{\gamma} \mathscr{M} \otimes_{\widetilde{\mathscr{O}}_{X}} \widetilde{\mathscr{D}}_{X} \rightarrow \mathscr{M}$ is surjective and induces surjective morphisms $\bigoplus_{\gamma \in[-1,0]} U_{\gamma} \mathscr{M} \otimes_{\widetilde{\sigma}_{X}}{ }^{A} V_{\alpha-\gamma} \widetilde{\mathscr{D}}_{X} \rightarrow U_{\alpha} \mathscr{M}$ for every $\alpha \in A+\mathbb{Z}$, hence a surjective morphism $\bigoplus_{\gamma \in[-1,0]} U_{\gamma} \mathscr{M} v^{\gamma} \otimes_{\widetilde{\mathscr{O}}_{X}} R_{A_{V}} \widetilde{\mathscr{D}}_{X} \rightarrow R_{U} \mathscr{M}$,
with the convention of Remark 7.2.7. We note that the $V$-filtered induced $\widetilde{\mathscr{D}}_{X}$-module that we have introduced also satisfies (a) and (b). Set $\mathscr{K}=\operatorname{Ker} \varphi$, that we equip with the induced filtration $U . \mathscr{K}$. We thus have an exact sequence for every $\alpha$ :

$$
0 \longrightarrow U_{\alpha} \mathscr{K} \longrightarrow \bigoplus_{\gamma \in[-1,0]} U_{\gamma} \mathscr{M} \otimes_{\widetilde{\mathscr{O}}_{X}}{ }^{A} V_{\alpha-\gamma} \widetilde{\mathscr{D}}_{X} \longrightarrow U_{\alpha} \mathscr{M} \longrightarrow 0
$$

from which we deduce that $U . \mathscr{K}$ satisfies (a) and (b), enabling us to continue the process.

The assertion of the theorem is local on $X^{\prime}$, and we will work in the neighbourhood of a point $x_{o}^{\prime} \in H^{\prime}$. The Kashiwara-Malgrange filtration $V_{\bullet} \mathscr{M}$ satisfies the properties 7.8.10(a) and (b), according to Proposition 7.3.31. We can then use a resolution as in Lemma 7.8.10, that we stop at a finite step chosen large enough (due to the cohomological finiteness of $f$ ) such that, for the corresponding bounded complex $\mathscr{L} \bullet \otimes_{\widetilde{D}_{X}} R_{A_{V}} \widetilde{\mathscr{D}}_{X}$, one has

$$
\mathscr{H}_{\mathrm{D}}^{i} f_{*}\left(R_{V} \mathscr{M}\right) \neq 0 \Longrightarrow \mathscr{H}_{\mathrm{D}}^{i} f_{*}\left(R_{V} \mathscr{M}\right)=\mathscr{H}_{\mathrm{D}}^{i} f_{*}\left(\mathscr{L}^{\bullet} \otimes_{\widetilde{\mathscr{O}}_{X}} R_{A_{V}} \widetilde{\mathscr{D}}_{X}\right)
$$

and similarly for every $\alpha$,

$$
\mathscr{H}_{\mathrm{D}}^{i} f_{\mid H *}\left(\operatorname{gr}_{\alpha}^{V} \mathscr{M}\right) \neq 0 \Longrightarrow \mathscr{H}^{i}{ }_{\mathrm{D}} f_{\mid H *}\left(\operatorname{gr}_{\alpha}^{V} \mathscr{M}\right)=\mathscr{H}_{\mathrm{D}}^{i} f_{\mid H *}\left(\mathscr{L}^{\bullet} \otimes_{\widetilde{\mathscr{O}}_{X}} \operatorname{gr}_{\alpha}^{A_{V}} \widetilde{\mathscr{D}}_{X}\right)
$$

In such a case, $\mathscr{H}^{i}{ }_{\mathrm{D}} f_{*}\left(R_{V} \mathscr{M}\right)=\mathscr{H}^{i}\left(f_{*} \operatorname{God}{ }^{\bullet}\left(\mathscr{L}^{\bullet} \otimes_{f^{-1} \widetilde{\mathscr{O}}_{X^{\prime}}} f^{-1} R_{A_{V}} \widetilde{\mathscr{D}}_{X^{\prime}}\right)\right)$, according to Lemma 7.8.3. We thus set

$$
\left(\mathscr{N}^{\bullet}, U \cdot \mathscr{N}^{\bullet}\right)=\left(f_{*} \operatorname{God}{ }^{\bullet}\left(\mathscr{L}^{\bullet} \otimes_{f^{-1} \widetilde{\mathscr{O}}_{X^{\prime}}} f^{-1} \widetilde{\mathscr{D}}_{X^{\prime}}\right), f_{*} \operatorname{God}\left(\mathscr{L}^{\bullet} \otimes_{f^{-1}} \widetilde{\mathscr{O}}_{X^{\prime}} f^{-1 A} V_{\bullet} \widetilde{\mathscr{D}}_{X^{\prime}}\right)\right)
$$

Since the sequences
and

$$
0 \longrightarrow{ }^{A} V_{\alpha} \widetilde{\mathscr{D}}_{X^{\prime}} \longrightarrow \widetilde{\mathscr{D}}_{X^{\prime}} \longrightarrow \widetilde{\mathscr{D}}_{X^{\prime}} /{ }^{A} V_{\alpha} \widetilde{\mathscr{D}}_{X^{\prime}} \longrightarrow 0
$$

$0 \longrightarrow{ }^{A} V_{<\alpha} \mathscr{D}_{X^{\prime}} \longrightarrow{ }^{A} V_{\alpha} \mathscr{D}_{X^{\prime}} \longrightarrow \operatorname{gr}_{\alpha} \mathscr{D}_{X^{\prime}} \longrightarrow 0$
are exact sequences of locally free $\widetilde{\mathscr{O}}_{X^{\prime}}$-modules, they remain exact after applying $\mathscr{L} \cdot \otimes_{\widetilde{\mathscr{O}}_{X}}$, then also after applying the Godement functor (see Exercise A.8.13(1)), and then after applying $f_{*}$ since the latter complexes consist of flabby sheaves.

This implies that $U_{\alpha} \mathscr{N}^{\bullet}$ is indeed a subcomplex of $\mathscr{N}^{\bullet}$ and $\operatorname{gr}_{\alpha}^{U} \mathscr{N}^{\bullet}=$ $f_{*} \operatorname{God}^{\bullet}\left(\mathscr{L} \bullet \otimes_{f-1} \widetilde{\mathscr{O}}_{X^{\prime}} f^{-1} \operatorname{gr}_{\alpha}^{A} V \widetilde{\mathscr{D}}_{X^{\prime}}\right)$.

Property 7.8.7(5) is satisfied, according to Theorem 7.8.2, and Properties 7.8.7(3) and (4) are clear.

We have $\mathscr{H}^{i}\left(\operatorname{gr}_{\alpha}^{U} \mathscr{N}^{\bullet}\right)=\mathscr{H}^{i}\left({ }_{\mathrm{D}} f_{\mid H *} \operatorname{gr}_{\alpha}^{V} \mathscr{M}\right)$ for $i \geqslant-N$ for some $N$ such that $\mathscr{H}^{i}\left({ }_{\mathrm{D}} f_{\mid H * \mathrm{gr}_{\alpha}}{ }^{V} \mathscr{M}\right)=0$ if $i<-N$, so that 7.8.7(1) holds by assumption and 7.8.7(2) is satisfied by taking the maximum of the local values $\nu_{\alpha}$ along the compact fibre $f^{-1}\left(x_{o}^{\prime}\right)$.

From Proposition 7.8 .7 we conclude that 7.8.5(1) holds for $\alpha \in A+\mathbb{Z}$ and any $i$. Denoting by $U . \mathscr{H}^{i}{ }_{\mathrm{D}} f_{*} \mathscr{M}$ the image filtration in 7.8.5(1), we thus have $R_{U} \mathscr{H}^{i}{ }_{\mathrm{D}} f_{*} \mathscr{M}=$ $\mathscr{H}^{i}{ }_{\mathrm{D}}{ }_{*} R_{V} R_{\mathscr{M}}$ and therefore

$$
\operatorname{gr}_{\alpha}^{U}\left(\mathscr{H}_{\mathrm{D}}^{i}{ }_{*} f_{\mathscr{M}}\right)=\mathscr{H}_{\mathrm{D}}^{i} f_{\mid H * \mathrm{gr}_{\alpha}^{V} \mathscr{M} .} .
$$

In particular, the left-hand term is strict by assumption on the right-hand term.
By the coherence theorem 7.8 .2 , we conclude that $U . \mathscr{H}^{i}{ }_{\mathrm{D}} f_{*} \mathscr{M}$ is a coherent ${ }^{A} V$-filtration of $\mathscr{H}^{i}{ }_{\mathrm{D}} f_{*} \mathscr{M}$. Therefore, $U \cdot \mathscr{H}^{i}{ }_{\mathrm{D}} f_{*} \mathscr{M}$ satisfies the assumptions of Lemma 7.3.23 (extended to filtrations indexed by $A+\mathbb{Z}$ ). Moreover, the properties 7.3.25(2) and (3) are also satisfied since they hold for $\mathscr{M}$. We conclude that $\mathscr{H}^{i}{ }_{\mathrm{D}} f_{*} \mathscr{M}$ is strictly $\mathbb{R}$-specializable along $H^{\prime}$ and that $U . \mathscr{H}^{i}{ }_{\mathrm{D}} f_{*} \mathscr{M}$ is its Kashiwara-Malgrange filtration. Now, Properties (1)-(3) in Theorem 7.8.5 are clear.

### 7.9. Comments

Here come the references to the existing work which has been the source of inspiration for this chapter.


[^0]:    1. Other settings can be considered, for example a smooth subvariety, or a finite family of smooth subvarieties, but they will not be needed for our purpose.
