CHAPTER 6

PURE HODGE MODULES ON CURVES

Summary. The aim of this chapter is to introduce the general notion of polarized pure Hodge module on a Riemann surface, as the right notion of a singular analogue of a variation of polarized Hodge structure. We will define it by *local* properties, as we do for variations of polarized Hodge structure. For that purpose, we first recall basics on $\mathscr{D}$-modules, which are much more developed in Appendix A and in Chapters 7–10. While the notion of a variation of $\mathbb{C}$-Hodge structure on a punctured compact Riemann surface is purely analytic, that of a pure Hodge module on the corresponding smooth projective curve is partly algebraic.

6.1. Introduction

Let $j : X^* \hookrightarrow X$ be the inclusion of a finite set of points D in a compact Riemann surface X, and let (H, Q) be a variation of polarized Hodge structure on X^*, with associated local system $\underline{\mathcal{H}}$ and filtered holomorphic bundle $(\mathcal{V}, \nabla, F^\bullet\mathcal{V})$. The Hodge-Zucker theorem gives importance to the differential object $(\mathcal{V}_{\mathrm{mid}}, \nabla)$ (see Exercise 5.2.4(6)). However it is, in general, *not* a coherent $\mathscr{O}_X$-module with connection. It is neither a meromorphic bundle with connection in general, i.e., it is not a $\mathscr{O}_X(*D)$-module (where $\mathscr{O}_\Delta(*D)$ denotes the sheaf of meromorphic functions on X with poles on D at most). We have to consider it as a coherent $\mathscr{D}_X$-module, where $\mathscr{D}_X$ denotes the sheaf of holomorphic differential operators. In order to do so, we recall in Section 6.2 the basic notions on $\mathscr{D}$-modules in one complex variable, the general case being treated in Appendix A.

The punctured Riemann surface will then be a punctured disc Δ^* in the remaining part of this introduction. The object analogue to $(\mathcal{V}, \nabla, F^\bullet\mathcal{V})$ on Δ is a holonomic $\mathscr{D}_\Delta$-module $\mathcal{M}$ endowed with a F-filtration $F^\bullet\mathcal{M}$ (this encodes the Griffiths transversality property). Here, the language of triples introduced in Section 2.4.c becomes useful, since we can consider a pair of such objects. A sesquilinear pairing should then take values in a sheaf containing C^∞ functions on Δ and allowing functions like $|t|^{2\beta}\mathrm{L}(t)^k/k!$, as in our discussion of Schmid's theorem. The sheaf of distributions on Δ is a possible candidate, since it is acted on by holomorphic and anti-holomorphic differential operators.

The idea of M. Saito to define the Hodge property is to impose it on the restriction of the data at each point of Δ, so that to apply the definitions of Chapter 2. While this does not cause any trouble at points of $\Delta^* := \Delta \setminus \{0\}$, this leads to problems at the origin for the following reason: the restriction of $\mathcal{M}$ in the sense of $\mathscr{D}$-modules is a complex, which has two cohomology vector spaces in general. The right way to consider the restriction consists in introducing *nearby cycles*. Therefore, the compatibility of the data with the nearby and vanishing cycle functors will be the main tool in the theory of Hodge modules.

However, not all $\mathscr{D}_\Delta$-modules underlie a Hodge module. On the one hand, one has to restrict to holonomic $\mathscr{D}_\Delta$-modules having a regular singularity at the origin. This is "forced" by the theorem of Griffiths-Schmid on the regularity of the connection on the extended Hodge bundles. Moreover, the Hodge-Zucker theorem leads us to focus on regular holonomic $\mathscr{D}_\Delta$-modules which are middle extensions of their restriction to Δ^*. Now, a new phenomenon appears when dealing with $\mathscr{D}_\Delta$-modules, when compared to the case of vector bundles with connection, namely, there do exist $\mathscr{D}_\Delta$-modules supported at the origin, like Dirac distributions, and their Hodge variants are easy to define.

There are thus two kinds of $\mathscr{D}_\Delta$-modules that should underlie a Hodge module. Which extensions between these two kinds can we allow? Since our goal is to define the category of polarized Hodge modules as an analogue over Δ of the category of polarized Hodge structures, we expect to obtain a semi-simple category. In order to achieve this goal, we are therefore led to restrict the kind of extensions between these two families: we only allow direct sums. This is why we introduce the notion of *Support-decomposability* of holonomic $\mathscr{D}_\Delta$-modules.

As for polarized Hodge structures, we will introduce $\mathbb{C}$-Hodge module and polarized $\mathbb{C}$-Hodge modules. In this chapter, we will only consider *left* $\mathscr{D}$-modules in order to keep the analogy with vector bundles with connections and variations of Hodge structures considered in Chapter 5.

6.2. Basics on holonomic $\mathscr{D}$-modules in one variable

We denote by t a coordinate on the disc Δ, by $\mathbb{C}\{t\}$ the ring of convergent power series in the variable t. Let us denote by $\mathscr{D} = \mathbb{C}\{t\}\langle\partial_t\rangle$ the ring of germs at $t = 0$ of holomorphic differential operators. There is a natural increasing filtration $F_\bullet\mathscr{D}$ indexed by $\mathbb{Z}$ defined by

$$F_k\mathscr{D} = \begin{cases} 0 & \text{if } k \leqslant -1, \\ \sum_{j=0}^k \mathbb{C}\{t\}\cdot\partial_t^j & \text{if } k \geqslant 0. \end{cases}$$

This filtration is compatible with the ring structure (i.e., $F_k \cdot F_\ell \subset F_{k+\ell}$ for every $k, \ell \in \mathbb{Z}$). The graded ring $\operatorname{gr}^F \mathscr{D} := \bigoplus_k \operatorname{gr}_k^F \mathscr{D} = \bigoplus_k F_k/F_{k-1}$ is isomorphic to the polynomial ring $\mathbb{C}\{t\}[\tau]$ (graded with respect to the degree in τ).

We also denote by $\mathscr{D}_\Delta$ the *sheaf* of differential operators with holomorphic coefficients on Δ. This is a coherent sheaf, similarly equipped with an increasing filtration $F_\bullet \mathscr{D}_\Delta$ by free $\mathscr{O}_\Delta$-modules of finite rank. The graded sheaf $\operatorname{gr}^F \mathscr{D}_\Delta$ is identified with the sheaf on Δ of functions on the cotangent bundle $T^*\Delta$ which are polynomial in the fibres of the fibration $T^*\Delta \to \Delta$.

6.2.a. Coherent F-filtrations, holonomic modules. Let M be a finitely generated $\mathscr{D}$-module (we basically use left $\mathscr{D}$-modules, but similar properties can be applied to right ones). By an F-filtration of M we mean increasing filtration $F_\bullet M$ by $\mathscr{O} = \mathbb{C}\{t\}$-submodules, indexed by $\mathbb{Z}$, such that, for every $k, \ell \in \mathbb{Z}$, $F_k\mathscr{D} \cdot F_\ell M \subset F_{k+\ell}M$. Such a filtration is said to be *coherent* if it satisfies the following properties:

(1) $F_k M = 0$ for $k \ll 0$,

(2) each $F_k M$ is finitely generated over $\mathscr{O}$,

(3) for every $k, \ell \in \mathbb{Z}$, $F_k\mathscr{D} \cdot F_\ell M \subset F_{k+\ell}M$,

(4) there exists $\ell_0 \in \mathbb{Z}$ such that, for every $k \geqslant 0$ and any $\ell \geqslant \ell_0$, $F_k\mathscr{D} \cdot F_\ell M = F_{k+\ell}M$.

***Remark 6.2.1* (Increasing or decreasing?)** In Hodge theory, one usually uses decreasing filtrations. The trick to go from increasing to decreasing filtrations is to set, for every $p \in \mathbb{Z}$,

$$F^p M := F_{-p} M.$$

The notion of shift is compatible with this convention:

$$F[k]^p M = F^{p+k} M, \quad F[k]_p M = F_{p-k} M.$$

***Exercise 6.2.2* (The Rees module).** The previous properties can be expressed in a simpler way by adding a dummy variable. Let M be a left $\mathscr{D}$-module and let $F_\bullet M$ be an F-filtration of M. Let z be such a variable and let us set $R_F\mathscr{D} = \bigoplus_{k\in\mathbb{Z}} F_k\mathscr{D} \cdot z^k$ and $R_F M = \bigoplus_{k\in\mathbb{Z}} F_k M \cdot z^k$.

(1) Prove that $R_F\mathscr{D}$ is a Noetherian ring.

(2) Prove that $R_F M$ has no $\mathbb{C}[z]$-torsion.

(3) Prove that the F-filtration condition is equivalent to: *$R_F M$ is a left graded $R_F\mathscr{D}$-module.*

(4) Prove that $R_F M/zR_F M = \operatorname{gr}^F M$ and $R_F M/(z-1)R_F M = M$.

(5) Prove that the coherence of $F_\bullet M$ is equivalent to: *$R_F M$ is a finitely generated left $R_F\mathscr{D}$-module.*

(6) Prove that M has a coherent F-filtration if and only if it is finitely generated.

Definition 6.2.3. We say that M is *holonomic* if it is finitely generated and any element of M is annihilated by some nonzero $P \in \mathscr{D}$.

One can prove that any holonomic $\mathscr{D}$-module can be generated by one element (i.e., it is cyclic), hence of the form $\mathscr{D}/I$ where I is a left ideal in $\mathscr{D}$, and that this ideal can be generated by two elements (see [**BM84**]).

6.2.b. The V-filtration. In order to analyze the behaviour of a holonomic module near the origin, we will use another kind of filtration, called the Kashiwara-Malgrange filtration. It is an extension to holonomic modules of the notion of Deligne lattice for meromorphic bundle with connection.

We first define the increasing filtration $V_\bullet\mathscr{D}$ indexed by $\mathbb{Z}$, by giving to any monomial $t^{a_1}\partial_t^{b_1}\cdots t^{a_n}\partial_t^{b_n}$ the V-degree $\sum_i b_i - \sum_i a_i$, and by defining the V-order of an operator $P\in\mathscr{D}$ as the biggest V-degree of its monomials.

Exercise 6.2.4.

(1) Check that the V-order of P does not depend on the way we write its monomials (due to the non-commutativity of $\mathscr{D}$).

(2) Check that each $V_k\mathscr{D}$ is a free $\mathscr{O}$-module, and that, for $k\leqslant 0$, $V_k\mathscr{D}=t^{-k}V_0\mathscr{D}$.

(3) Check that the filtration by the V-order is compatible with the product, and more precisely that

$$V_k\mathscr{D}\cdot V_\ell\mathscr{D}\begin{cases}\subset V_{k+\ell}\mathscr{D} & \text{for every } k,\ell\in\mathbb{Z},\\ = V_{k+\ell}\mathscr{D} & \text{if } k,\ell\leqslant 0 \text{ or if } k,\ell\geqslant 0.\end{cases}$$

Conclude that $V_0\mathscr{D}$ is a ring and that each $V_k\mathscr{D}$ is a left and right $V_0\mathscr{D}$-module.

(4) Check that the Rees object $R_V\mathscr{D}:=\bigoplus_{k\in\mathbb{Z}}V_k\mathscr{D}\cdot v^k$ is a Noetherian ring.

(5) Show that $\mathrm{gr}_0^V\mathscr{D}$ can be identified with the polynomial ring $\mathbb{C}[\mathrm{E}]$, where E is the class of $t\partial_t$ in $\mathrm{gr}_0^V\mathscr{D}$.

(6) Show that E does not depend on the choice of the coordinate t on the disc.

Definition 6.2.5. Let M be a left $\mathscr{D}$-module. By a V-filtration we mean an decreasing filtration $U^\bullet M$ of M, indexed by $\mathbb{Z}$, which satisfies $V_k\mathscr{D}\cdot U^\ell M\subset U^{\ell-k}M$ for every $k,\ell\in\mathbb{Z}$. We say that $U^\bullet M$ is *coherent* if there exists $\ell_0\in\mathbb{N}$ such that the previous inclusion is an equality for every $k\geqslant 0$ and $\ell\leqslant-\ell_0$, and for every $k\leqslant 0$ and $\ell\geqslant\ell_0$.

Exercise 6.2.6.

(1) Show that a filtration $U^\bullet M$ is a V-filtration if and only if the Rees object $R_UM:=\bigoplus_{k\in\mathbb{Z}}U^kMv^{-k}$ is naturally a left graded $R_V\mathscr{D}$-module.

(2) Show that, for every V-filtration $U^\bullet M$ on M, $R_UM/vR_UM=\mathrm{gr}^UM$ and $R_UM/(v-1)R_UM=M$.

(3) Show that any finitely generated $\mathscr{D}$-module has a coherent V-filtration.

(4) Show that a V-filtration is coherent if and only if the Rees module R_UM is finitely generated over $R_V\mathscr{D}$.

(5) Show that, if M is holonomic, then for any coherent V-filtration the graded spaces gr_U^kM are finite dimensional $\mathbb{C}$-vector spaces equipped with a linear action of E.

(6) Show that, if $U^\bullet M$ is a V-filtration of M, then the left multiplication by t induces for every $k \in \mathbb{Z}$ a $\mathbb{C}$-linear homomorphism $\mathrm{gr}_U^k M \to \mathrm{gr}_U^{k+1} M$ and that the action of ∂_t induces $\mathrm{gr}_U^k M \to \mathrm{gr}_U^{k-1} M$. How does E commute with these morphisms?

(7) Show that if a V-filtration is coherent, then $t : U^k M \to U^{k+1} M$ is an isomorphism for every $k \gg 0$ and $\partial_t : \mathrm{gr}_U^k M \to \mathrm{gr}_U^{k-1} M$ is so for every $k \ll 0$.

Proposition 6.2.7 **(The Kashiwara-Malgrange filtration).** *Let M be a holonomic $\mathscr{D}$-module. Then there exists a unique coherent V-filtration denoted by $V^\bullet M$ and called* the Kashiwara-Malgrange filtration of M, *such that the eigenvalues of* E *acting on the finite dimensional vector spaces* $\mathrm{gr}_V^k M$ *have their real part in* $[k, k+1)$.

Proof. Adapt Exercise 7.3.12 to the present setting. □

Exercise 6.2.8 **(V-strictness of morphisms).** Show that any morphism $\varphi : M \to M'$ between holonomic $\mathscr{D}$-modules is V-strict, i.e., satisfies $\varphi(V^k M) = \varphi(M) \cap V^k M'$ for every $k \in \mathbb{Z}$. [*Hint*: show that the right-hand side defines a coherent filtration of $\varphi(M)$ and use the uniqueness of the Kashiwara-Malgrange filtration.]

Exercise 6.2.9. Show that the Kashiwara-Malgrange filtration satisfies the following properties:

(1) for every $k \geqslant 0$, the morphism $V^k M \to V^{k+1} M$ induced by t is an isomorphism;

(2) for every $k \geqslant 0$, the morphism $\mathrm{gr}_V^{-1-k} M \to \mathrm{gr}_V^{-2-k} M$ induced by ∂_t is an isomorphism.

Exercise 6.2.10. Show that, for any holonomic module M, the module $M[t^{-1}] := \mathscr{O}[t^{-1}] \otimes_{\mathscr{O}} M$ is still holonomic and is a finite dimensional vector space over the field of Laurent series $\mathscr{O}[t^{-1}]$, equipped with a connection. Show that its Kashiwara-Malgrange filtration satisfies $V^k M[t^{-1}] = t^k V^0 M[t^{-1}]$ for every $k \in \mathbb{Z}$ (while this only holds for $k \geqslant 0$ for a general holonomic $\mathscr{D}$-module. Conversely, prove that any finite dimensional vector space $(\mathcal{V}_*, \nabla)$ over the field of Laurent series $\mathscr{O}[t^{-1}]$ equipped with a connection is a holonomic $\mathscr{D}$-module.

Exercise 6.2.11 **($\mathscr{D}$-modules with support the origin).** Let M be a finitely generated $\mathscr{D}$-module with support the origin, i.e., each element is annihilated by some power of t (hence M is holonomic). Show that

(1) $V^\beta M = 0$ for $\beta > -1$ and $\mathrm{gr}_V^\beta M = 0$ for $\beta \notin -\mathbb{N}^*$,

(2) $M \simeq (\mathrm{gr}_V^{-1} M)[\partial_t]$, where the action of $\mathscr{D}$ on the right-hand side is given by

$$\begin{aligned} \partial_t \cdot m\partial_t^k &= m\partial_t^{k+1}, \\ t \cdot m\partial_t^k &= -km\partial_t^{k-1}, \end{aligned}$$

(3) M has also the structure of a *right* $\mathscr{D}$-module by setting

$$\begin{aligned} m\partial_t^k \cdot \partial_t &= m\partial_t^{k+1}, \\ m\partial_t^k \cdot t &= km\partial_t^{k-1}. \end{aligned}$$

6.2.c. Nearby and vanishing cycles. For simplicity, in the following we always assume that M is holonomic. We will also assume that the eigenvalues of E acting on $\mathrm{gr}_V^k M$ are real, i.e., belong to $[k, k+1)$. This will be the only case of interest in Hodge theory, according to Lemma 5.2.9. Let $B \subset [0,1)$ be the finite set of eigenvalues of E acting on $\mathrm{gr}_V^0 M$, to which we add 0 if $0 \notin B$. By Exercise 6.2.9, the set B_k of eigenvalues of E acting on $\mathrm{gr}_V^k M$ satisfies $k + (B \smallsetminus \{0\}) \subset B_k \subset k + B$.

For every $\beta \in \mathbb{R}$, we denote by $V^\beta M \subset V^{[\beta]} M$ the pullback by $V^{[\beta]}M \to \mathrm{gr}_V^{[\beta]} M$ of the subspace of $\mathrm{gr}_V^{[\beta]} M$ corresponding to eigenvalues of E which are $\geqslant \beta$, i.e., the subspace $\bigoplus_{\gamma \in [\beta, [\beta]+1)} \mathrm{Ker}(\mathrm{E} - \gamma\,\mathrm{Id})^N$, $N \gg 0$.

In such a way, we obtain a decreasing filtration $V^\bullet M$ indexed by $B + \mathbb{Z} \subset \mathbb{R}$, and we now denote by $\mathrm{gr}_V^\beta M$ the quotient space $V^\beta M / V^{>\beta} M$. We can also consider $V^\bullet M$ as a filtration indexed by $\mathbb{R}$ which jumps at most at $B + \mathbb{Z}$.

Exercise 6.2.12 **(V-strictness of morphisms).** Show the V-strictness of morphisms for the V-filtration indexed by $\mathbb{R}$ (see Exercise 6.2.8).

Exercise 6.2.9 implies:

(1) for every $\beta > -1$, the morphism $V^\beta M \to V^{\beta+1} M$ induced by t is an isomorphism, and so is the morphism $t : \mathrm{gr}_V^\beta M \to \mathrm{gr}_V^{\beta+1} M$; in particular, $V^\beta M$ is $\mathscr{O}$-free if $\beta > -1$;

(2) for every $\beta < 0$, the morphism $\mathrm{gr}_V^\beta M \to \mathrm{gr}_V^{\beta-1} M$ induced by ∂_t is an isomorphism.

In particular, the knowledge of $\mathrm{gr}_V^\beta M$ for $\beta \in [0,1]$ implies that for all β.

Definition 6.2.13 **(The morphisms** $\mathrm{N}, \mathrm{can}, \mathrm{var}$**).** Let M be a holonomic $\mathscr{D}$-module.

(a) We denote by N the nilpotent part of the endomorphism induced by $-2\pi\mathsf{i}\,\mathrm{E}$ on $\mathrm{gr}_V^\beta M$ for every β (we will only consider $\beta \in [0,1]$, according to (1) and (2) above). So we have $\mathrm{N} = -2\pi\mathsf{i}(\mathrm{E} - \beta\,\mathrm{Id})$ on $\mathrm{gr}_V^\beta M$ for $\beta \in [-1, 0]$.

(b) We define $\mathrm{can} : \mathrm{gr}_V^0 M \to \mathrm{gr}_V^{-1} M$ as the homomorphism induced by $-\partial_t$ and $\mathrm{var} : \mathrm{gr}_V^{-1} M \to \mathrm{gr}_V^0 M$ as that induced by $2\pi\mathsf{i}\,t$, so that $\mathrm{var} \circ \mathrm{can} = \mathrm{N} : \mathrm{gr}_V^0 M \to \mathrm{gr}_V^0 M$ and $\mathrm{can} \circ \mathrm{var} = \mathrm{N} : \mathrm{gr}_V^{-1} M \to \mathrm{gr}_V^{-1} M$.

(c) We also denote by $\mathrm{M}_\bullet \mathrm{gr}_V^\beta M$ the monodromy filtration defined by the nilpotent endomorphism N (see Section 3.2.a).

Exercise 6.2.14. Let M be a holonomic $\mathscr{D}$-module. Prove that

(1) the construction of $\mathrm{gr}_V^\beta M, \mathrm{gr}_V^{-1} M$, can, var, N, is functorial with respect to M, and gr_V^β is an exact functor (i.e., compatible with short exact sequences);

(2) can is onto iff M has no quotient supported at the origin (i.e., there is no surjective morphism $M \to N$ where each element of N is annihilated by some power of t);

(3) var is injective if and only if M has no submodule supported at the origin (i.e., whose elements are annihilated by some power of t);

(4) M is supported at the origin if and only if $\mathrm{gr}_V^\beta M = 0$ for every $\beta > -1$;

(5) $\mathrm{gr}_V^{-1} M = \mathrm{Im}\,\mathrm{can} \oplus \mathrm{Ker}\,\mathrm{var}$ if and only if $M = M' \oplus M''$, where M'' is supported at the origin and M' has neither a quotient nor a submodule supported at the origin (in such a case, we say that M is *S(upport)-decomposable*).

Examples 6.2.15.

(1) If 0 is not a singular point of M, then M is $\mathscr{O}$-free of finite rank and $\mathrm{gr}_V^\beta M = 0$ unless $\beta \in \mathbb{N}$. Then $\mathrm{can} = 0$, $\mathrm{var} = 0$ and $\mathrm{N} = 0$.

(2) If M is *purely irregular*, e.g. $M = (\mathscr{O}[t^{-1}], \nabla)$ with $\nabla = \mathrm{d} + \mathrm{d}t/t^2$, then $\mathrm{gr}_V^\beta M = 0$ for every β. In such a case, the gr_V^β-functors do not bring any interesting information on M.

***Definition 6.2.16* (Middle extension).** We say that a holonomic M is the middle (or minimal) extension of $M[t^{-1}] := \mathscr{O}[t^{-1}] \otimes_{\mathscr{O}} M$ if can is onto and var is injective, that is, if M has neither a quotient nor a submodule supported at the origin.

Exercise 6.2.17.

(1) Show that the Kashiwara-Malgrange filtration of $M[t^{-1}]$ satisfies

  (a) $V^{>-1} M[t^{-1}] = V^{>-1} M$,

  (b) $V^{\beta+k} M[t^{-1}] = t^k V^\beta M[t^{-1}]$ for all $k \in \mathbb{Z}$.

(2) Show that M is a middle extension $\mathscr{D}$-module if and only if M is equal to the $\mathscr{D}$-submodule of $M[t^{-1}]$ generated by $V^{>-1} M$.

***Definition 6.2.18* (Regular singularity).** We say that M has a regular singularity (or is *regular*) at the origin if $V^0 M$ (equivalently, any $V^\beta M$) has finite type over $\mathscr{O}$.

Exercise 6.2.19. Prove that, if M has finite type over $\mathscr{D}$ and is supported at the origin, then M has a regular singularity at the origin.

The following proposition makes the link between the $\mathscr{D}$-module approach and the approach of Section 5.2.a.

Proposition 6.2.20. *Assume that M has a regular singularity at the origin. Then $M[t^{-1}]$ is equal to the germ at 0 of $(\mathcal{V}_*, \nabla)$ (Deligne's canonical meromorphic extension), where $(\mathcal{V}, \nabla)$ is the restriction of M to a punctured small neighbourhood of the origin. Moreover, if M is a middle extension, then M is equal to the germ at 0 of $(\mathcal{V}_{\mathrm{mid}}, \nabla)$. Lastly, the filtration $\mathcal{V}_*^\bullet$ (resp. $\mathcal{V}_{\mathrm{mid}}^\bullet$) is equal to the Kashiwara-Malgrange filtration.*

Proof. Let $\mathcal{M}$ be a coherent $\mathscr{D}_\Delta$-module representative of the germ M on a small disc Δ, having a singularity at 0 only. Set $(\mathcal{V}, \nabla) = \mathcal{M}_{|\Delta^*}$. By the uniqueness of the Deligne lattices with given range of eigenvalues of the residue, we have $\mathcal{V}_*^{>-1} = V^{>-1}\mathcal{M}$. We then have $\mathcal{M}[t^{-1}] = V^{>-1}\mathcal{M}[t^{-1}] = \mathcal{V}_*$, according to Exercise 6.2.17(1). If M is a middle extension, the assertion follows from 6.2.17(2). The last assertion is proved similarly. $\square$

Structure of regular holonomic $\mathscr{D}$-modules. Let M be regular holonomic. For $\beta \in \mathbb{R}$, set

$$M_\beta := \bigcup_k \operatorname{Ker}\bigl[(t\partial_t - \beta)^k : M \longrightarrow M\bigr].$$

Then $M_\beta \cap M_\gamma = 0$ if $\beta \neq \gamma$. Moreover, $M_\beta \cap V^{>\beta}M = 0$: indeed, if $(t\partial_t - \beta)^k m = 0$ and $b(t\partial_t)m = tP(t, t\partial_t)m$ with b having roots $> \beta$, we conclude a relation $m = tQ(t, t\partial_t)$ by Bézout, so the $\mathscr{D}$-module $\mathscr{D} \cdot m$ satisfies $\mathscr{D} \cdot m = V^{-1}(\mathscr{D} \cdot m)$, and its V-filtration is constant; however, the $\mathscr{O}$-finiteness of each term of this V-filtration implies $m = 0$. As a consequence, M_β injects in $\operatorname{gr}_V^\beta M$ and thus has finite dimension. Obviously, multiplication by t sends M_β to $M_{\beta+1}$ and ∂_t goes in the reverse direction. Moreover, $t : M_\beta \to M_{\beta+1}$ is an isomorphism if $\beta > -1$ and $\partial_t : M_{\beta+1} \to M_\beta$ is an isomorphism if $\beta < 0$.

The set consisting of β's such that $M_\beta \neq 0$ is therefore contained in $B + \mathbb{Z}$ (B is the finite set of eigenvalues or E acting on $\operatorname{gr}_V^0 M$), and $M^{\mathrm{alg}} := \bigoplus_\beta M_\beta$ is a regular holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module.

Proposition 6.2.21. *If M is regular holonomic, Then the natural morphism*

$$\mathbb{C}\{t\} \otimes_{\mathbb{C}[t]} M^{\mathrm{alg}} \longrightarrow M$$

is an isomorphism of $\mathscr{D}$-modules, and induces an $\mathbb{R}$-graded isomorphism

$$M^{\mathrm{alg}} \xrightarrow{\sim} \operatorname{gr}_V M^{\mathrm{alg}} \xrightarrow{\sim} \operatorname{gr}_V M. \qquad \square$$

F-filtration on nearby and vanishing cycles. Let M be holonomic and equipped with a coherent F-filtration $F_\bullet M$. There is a natural way to induce a filtration on each vector space $\operatorname{gr}_V^\beta M$ by setting

$$F_p \operatorname{gr}_V^\beta M := \frac{F_p M \cap V^\beta M}{F_p M \cap V^{>\beta} M}. \tag{6.2.22}$$

Exercise 6.2.23. Show that $\mathrm{N} \cdot F_p \operatorname{gr}_V^\beta M \subset F_{p+1} \operatorname{gr}_V^\beta M$ for every $\beta \in \mathbb{R}$ and that

$$\operatorname{can}(F_p \operatorname{gr}_V^0 M) \subset F_{p+1} \operatorname{gr}_V^{-1} M, \quad \operatorname{var}(F_p \operatorname{gr}_V^{-1} M) \subset F_p \operatorname{gr}_V^0 M.$$

The germic version of the de Rham complex. Let us first consider the de Rham complex of M. The holomorphic de Rham complex $\mathrm{DR}\, M$ is defined as the complex

$$\mathrm{DR}\, M = \{0 \to M \xrightarrow{\nabla} \Omega^1 \otimes_{\mathscr{O}} M \to 0\},$$

where the degree of the terms is as usual, i.e., M is in degree zero and $\Omega^1 \otimes_{\mathscr{O}} M$ in degree one. The de Rham complex can be V-filtered, by setting

$$V^\beta \,\mathrm{DR}\, M = \{0 \to V^\beta M \xrightarrow{\nabla} \Omega^1 \otimes_{\mathscr{O}} V^{\beta-1} M \to 0\},$$

for every $\beta \in \mathbb{R}$. The terms of this complex have finite type as $\mathscr{O}$-modules. As the morphism $\operatorname{gr}_V^\beta M \to \operatorname{gr}_V^{\beta-1} M$ induced by ∂_t is an isomorphism for every $\beta < 0$, it follows that the inclusion of complexes

$$V^0 \,\mathrm{DR}\, M \hookrightarrow \mathrm{DR}\, M \tag{6.2.24}$$

is a quasi-isomorphism.

If M comes equipped with a coherent filtration $F^\bullet M$, we set

$$F^p \operatorname{DR} M = \{0 \to F^p M \xrightarrow{\nabla} \Omega^1 \otimes_{\mathscr{O}} F^{p-1} M \to 0\}.$$

6.2.d. Holonomic $\mathscr{D}_\Delta$-modules. We now sheafify the previous constructions and consider a $\mathscr{D}_\Delta$-module $\mathcal{M}$. We assume it is holonomic, that is, its germ at any point of the open disc $\Delta \subset \mathbb{C}$ centered at 0 is holonomic in the previous sense. Then the $\mathscr{D}_\Delta$-module $\mathcal{M}$ is an $\mathscr{O}_\Delta$-module and is equipped with a connection. Moreover, we always assume that the origin of Δ is the only singularity of $\mathcal{M}$ on Δ, that is, away from the origin $\mathcal{M}$ is locally $\mathscr{O}_{\Delta^*}$-free of finite rank.

All the notions of the previous subsection extend in a straightforward way to the present setting. In particular, for an holonomic $\mathscr{D}_\Delta$-module $\mathcal{M}$ having a regular singularity at the origin, Proposition 6.2.21 reads

$$\mathcal{M} \simeq \mathscr{O}_\Delta \otimes_{\mathbb{C}[t]} M^{\mathrm{alg}}.$$

***Definition 6.2.25* (Strict support).** We say that $\mathcal{M}$ as above has *pure support* the disc Δ if its germ M at the origin is a middle extension, as defined in 6.2.16.

The holomorphic de Rham complex $\operatorname{DR}\mathcal{M}$ is defined as the complex (degrees as above)

$$\operatorname{DR}\mathcal{M} = \{0 \to \mathcal{M} \xrightarrow{\nabla} \Omega^1_\Delta \otimes_{\mathscr{O}_\Delta} \mathcal{M} \to 0\}.$$

Away from the origin, the de Rham complex has cohomology in degree 0 only, and $\mathscr{H}^0 \operatorname{DR}\mathcal{M}_{|\Delta^*} = \mathcal{V}^\nabla$ is a local system of finite dimensional $\mathbb{C}$-vector spaces on Δ^*. In general, $\operatorname{DR}\mathcal{M}$ is a constructible complex on Δ, that is, its cohomology spaces at the origin are finite dimensional $\mathbb{C}$-vector spaces. The subcomplex $V^0 \operatorname{DR}\mathcal{M}$ is quasi-isomorphic to $\operatorname{DR}\mathcal{M}$ and, if $\mathcal{M}$ has a regular singularity at the origin, $V^0 \operatorname{DR}\mathcal{M}$ is a complex whose terms are $\mathscr{O}_\Delta$-coherent (in fact $V^0\mathcal{M}$ is $\mathscr{O}_\Delta$ free).

If $\mathcal{M}$ has pure support the disc Δ, the de Rham complex $\operatorname{DR}\mathcal{M}$ has cohomology in degree zero only, and $\mathscr{H}^0 \operatorname{DR}\mathcal{M} = j_*\mathcal{V}^\nabla$. In such a case, both terms of $V^0 \operatorname{DR}\mathcal{M}$ are $\mathscr{O}_\Delta$-free. On the other hand, if $\mathcal{M}$ is supported at the origin, then $\operatorname{DR}\mathcal{M} \simeq V^0 \operatorname{DR}\mathcal{M}$ reduces to the complex with the single term $V^{-1}\mathcal{M} = \operatorname{gr}_V^{-1}\mathcal{M}$ in degree one.

Exercise 6.2.26. Let $\mathcal{M}, \mathcal{M}'$ be holonomic $\mathscr{D}_\Delta$-modules with singularity at the origin at most. Let $\varphi : \mathcal{M} \to \mathcal{M}'$ be a $\mathscr{D}_\Delta$-linear morphism. Show that, if $\mathcal{M}$ is a middle extension, then φ is zero as soon as it is zero when restricted to Δ^*. [*Hint*: if $\varphi_{|\Delta^*} = 0$, show first that φ is zero on $V^{>-1}\mathcal{M}$ because $V^{>-1}\mathcal{M}'$ is $\mathscr{O}_\Delta$-free, and then use Exercise 6.2.17(2).]

6.2.e. Pushforward of regular holonomic $\mathscr{D}_X$-modules. We now consider the global setting of a compact Riemann surface and a regular holonomic $\mathscr{D}_X$-module $\mathcal{M}$

with singularities at a finite set $D \subset X$. The pushforward (in the sense of left $\mathscr{D}_X$-modules) of $\mathcal{M}$ by the constant map $X \to \mathrm{pt}$ is the complex

$$\boldsymbol{R}\Gamma(X, \mathrm{DR}\,\mathcal{M}),$$

that we regard as a complex of $\mathscr{D}$-modules on a point, that is, a complex of $\mathbb{C}$-vector spaces. The definition of pushforward that we use is that given in Exercise A.8.24. It follows that $\boldsymbol{R}\Gamma(X, \mathrm{DR}\,\mathcal{M})$ has cohomology in degrees $0, 1, 2$.

For a regular holonomic $\mathscr{D}_X$-module $\mathcal{M}$, it is immediate to check that the space $\boldsymbol{H}^k(X, \mathrm{DR}\,\mathcal{M})$ is finite dimensional for every k. Indeed, denote by $V^\beta\mathcal{M}$ the subsheaf of $\mathcal{M}$ which coincides with $V^\beta(\mathcal{M}_{|\Delta})$ on each disc Δ near a singularity and is equal to $\mathcal{M}$ away from the singularities. Then $V^\beta\mathcal{M}$ is $\mathscr{O}_X$-coherent and 6.2.24 gives $V^0(\mathrm{DR}\,\mathcal{M}) \simeq \mathrm{DR}\,\mathcal{M}$, so $\boldsymbol{H}^k(X, \mathrm{DR}\,\mathcal{M}) = \boldsymbol{H}^k(X, V^0\,\mathrm{DR}\,\mathcal{M})$ is finite dimensional since each term of the complex $V^0\,\mathrm{DR}\,\mathcal{M}$ is $\mathscr{O}_X$-coherent and X is compact.

Examples 6.2.27.

(1) Assume $\mathcal{M}$ is supported at one point in X, and let Δ be a small disc centered at that point, with coordinate t. We can assume that $X = \Delta$. Denoting by $\iota : \{0\} \hookrightarrow \Delta$ the inclusion, there exists a vector space $\mathcal{H}$ (equal to $\mathrm{gr}_V^{-1}\mathcal{M}$) such that $\mathcal{M} = \iota_*\mathcal{H}[\partial_t]$. For $k \geqslant 0$ we have $V^k\mathcal{M} = 0$ and $V^{-k-1}\mathcal{M} = \sum_{j\leqslant k} \iota_*\mathcal{H}\partial_t^j$, so that $V^0(\mathrm{DR}\,\mathcal{M})$ is the complex having the skyscraper sheaf with stalk $\mathcal{H}$ at the origin as its term in degree one, and all other terms of the complex are zero. We can thus write

$$\mathrm{DR}\,\mathcal{M} = \iota_*\mathcal{H}[-1].$$

We then find

$$H^k(X, \mathrm{DR}\,\mathcal{M}) = \begin{cases} \mathcal{H} & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(2) Let us consider the same setting as above, but regarding now $\mathcal{M}$ as a *right* $\mathscr{D}_\Delta$-module (see Exercise 6.2.11(3)). It is natural to consider the de Rham complex

$$\{0 \to \mathcal{M} \xrightarrow{\cdot\,\partial_t} \underset{\bullet}{\mathcal{M}} \to 0\}$$

so that the marked term is in degree zero (see Definition A.5.2). We then have

$$H^k(X, \mathrm{DR}\,\mathcal{M}^{\mathrm{right}}) = \begin{cases} \mathcal{H} & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

(3) Assume now that $\mathcal{M} = \mathcal{V}_{\mathrm{mid}}$ and set $\underline{\mathcal{H}} = \mathcal{V}^\nabla$. Then $\mathrm{DR}\,\mathcal{M} = j_*\underline{\mathcal{H}}$ and $\boldsymbol{H}^k(X, \mathrm{DR}\,\mathcal{M}) = H^k(X, j_*\underline{\mathcal{H}})$. As explained in Remark 5.4.15, the only interesting cohomology is now $\boldsymbol{H}^1(X, \mathrm{DR}\,\mathcal{M})$.

6.3. Sesquilinear parings on $\mathscr{D}$-modules on a disc

We have seen in Section 4.1 that the notion of a sesquilinear pairing is instrumental in order to define the polarization of a variation of $\mathbb{C}$-Hodge structure and even, taking the approach of triples, in defining the notion of variation of $\mathbb{C}$-Hodge structure. It takes values in the space of C^∞ functions. In order to extend this notion to that of pairing on $\mathscr{D}$-modules, we need to extend the target space, as suggested by the formula in Lemma 5.4.5. When working with left (resp. right) $\mathscr{D}$-modules, the target space for sesquilinear pairings will be the spaces of distributions (resp. currents) on X. A general presentation of sesquilinear pairing in arbitrary, either in the left or right case, will be given in Chapter 10. We also refer to Appendix A.4.d for general properties of distributions and currents.

6.3.a. Basic distributions. Let us start by noticing that the C^∞ functions on Δ^* (punctured unit disc) considered in Lemma 5.4.5, and that we denote by

$$u_{\beta,p} := |t|^{2\beta}\,\frac{\mathrm{L}(t)^p}{p!},\quad \beta > -1,\, p \in \mathbb{N},$$

defines a distribution on Δ by the formula

$$\langle \eta, u_{\beta,p}\rangle = \int_\Delta u_{\beta,p}\eta,$$

for any C^∞ $(1,1)$-form η with compact support on Δ. In fact, a direct computation in polar coordinates shows that $u_{\beta,p}$ is locally integrable on Δ. These distributions are related by

$$-(t\partial_t - \beta)u_{\beta,p} = -(\bar t\partial_{\bar t} - \beta)u_{\beta,p} = u_{\beta,p-1}, \tag{6.3.1}$$

as can be seen by using integration by parts ($u_{\beta,-1} := 0$).

Proposition 6.3.2. *Suppose that a distribution $u \in \mathfrak{Db}(\Delta)$ solves the equations*

$$(t\partial_t - \beta)^k u = (\bar t\partial_{\bar t} - \beta')^k u = 0$$

for real numbers $\beta, \beta' > -1$ and an integer $k \geqslant 0$.

(a) *We have $u = 0$ unless $\beta - \beta' \in \mathbb{Z}$.*
(b) *If $\beta = \beta'$, then u is a linear combination of $u_{\beta,p}$ with $0 \leqslant p \leqslant k-1$.*

Proof. Let us first show that if $\operatorname{Supp} u \subseteq \{0\}$, then $u = 0$. By continuity, u is annihilated by some large power of t; let $m \in \mathbb{N}$ be the least integer such that $t^m u = 0$. If $m \geqslant 1$, we have

$$\begin{aligned} 0 = t^{m-1}(t\partial_t - \beta)^k u &= (t\partial_t - \beta - (m-1))^k t^{m-1}u\\ &= (\partial_t t - \beta - m)^k t^{m-1}u = (\beta + m)t^{m-1}u,\end{aligned}$$

hence $t^{m-1}u = 0$, due to the fact that $\beta > -1$. The conclusion is that $m = 0$, and hence that $u = 0$.

Now let us prove the general case. After pulling back by the exponential mapping

$$\{\operatorname{Re}\tau < 0\} \longrightarrow \Delta^*,\quad \tau \longmapsto e^\tau,$$

we obtain a distribution $\widetilde u$ on the half-plane $\{\operatorname{Re}\tau < 0\}$, with the property that

$$(\partial_\tau - \beta)^k \widetilde u = (\partial_{\bar\tau} - \beta')^k \widetilde u = 0.$$

The equations imply that the product

$$v = e^{-\beta\tau} e^{-\beta'\overline{\tau}} \cdot \widetilde{u}$$

is annihilated by the k-th power of ∂_τ and $\partial_{\overline{\tau}}$, and in particular by the k-th power $(\partial_\tau\partial_{\overline{\tau}})^k$ of the Laplacian. By the regularity of the Laplacian, v is C^∞, and the above equations imply that v is a polynomial $P(\tau,\overline{\tau})$ of degree $\leqslant k$. Consequently,

$$\widetilde{u} = P(\tau,\overline{\tau}) \cdot e^{\beta\tau} e^{\beta'\overline{\tau}}.$$

By construction, $\widetilde{u}$ is invariant under the translation $\tau \mapsto \tau + 2\pi\mathsf{i}$; if $\widetilde{u} \neq 0$, this forces $P(\tau,\overline{\tau})$ to be a polynomial in $\tau + \overline{\tau}$ and $\beta - \beta' \in \mathbb{Z}$.

Now there are two cases. If $\beta - \beta' \notin \mathbb{Z}$, then $\widetilde{u} = 0$, which means that u is supported on the origin; but then $u = 0$, and so the first assertion is proved. If $\beta = \beta'$, then the restriction of u to Δ^* is a linear combination of the C^∞ functions $u_{\beta,p|\Delta^*}$ with $0 \leqslant p \leqslant k-1$. After adding a suitable linear combination of the distributions $u_{\beta,p}$, we can arrange that u is supported on the origin; but then $u = 0$, which proves the second assertion. □

To include the case $\beta = \beta' = -1$ into the picture, we need the following simple facts about distributions.

Exercise 6.3.3. Let δ_0 be the Dirac distribution, defined by $\langle \eta(t)\mathrm{d}t \wedge \mathrm{d}\overline{t}, \delta_0\rangle = \eta(0)$. Using Cauchy's formula, show the formula:

$$\partial_t\partial_{\overline{t}}\mathrm{L}(t) = 2\pi\mathsf{i}\,\delta_0.$$

Proposition 6.3.4. *Suppose that a distribution $u \in \mathfrak{Db}(\Delta)$ solves the equations*

$$(t\partial_t + 1)^k u = (\overline{t}\partial_{\overline{t}} + 1)^k u = 0$$

for some $k \geqslant 1$. Then u is a linear combination of $\partial_t\partial_{\overline{t}} u_{0,p}$ with $1 \leqslant p \leqslant k$.

Proof. Using the relation $t(t\partial_t + 1) = t\partial_t t$, we find $(t\partial_t)^k|t|^2 u = (\overline{t}\partial_{\overline{t}})^k|t|^2 u = 0$, and by Proposition 6.3.2 we deduce $|t|^2 u = \sum_{p=0}^{k-1} c_{p+2} u_{0,p} = |t|^2\partial_t\partial_{\overline{t}} \sum_{q=2}^{k+1} c_q u_{0,q}$, according to the basic relations (6.3.1). On the other hand, distributions solutions of $|t|^2 v = 0$ are $\mathbb{C}$-linear combinations of δ_0, $\partial_t^j\delta_0$, $\partial_{\overline{t}}^j\delta_0$ ($j \geqslant 1$). As a consequence, and using Exercise 6.3.3, we find an expression

$$u = \partial_t\partial_{\overline{t}} \sum_{q=1}^{k+1} c_q u_{0,q} + \sum_{j\geqslant 1} (a_j \partial_t^j\delta_0 + b_j\partial_{\overline{t}}^j\delta_0),$$

and we are left with showing $c_{k+1} = a_j = b_j = 0$.

For that purpose, we note that, for $p = 1, \dots, k+1$,

$$(\partial_t t)^k \partial_t\partial_{\overline{t}} u_{0,p} = \partial_t\partial_{\overline{t}}(t\partial_t)^k u_{0,p} = (-1)^k \partial_t\partial_{\overline{t}} u_{0,p-k} = \begin{cases} 0 & \text{if } p \leqslant k, \\ (-1)^k\, 2\pi\mathsf{i}\,\delta_0 & \text{if } p = k+1. \end{cases}$$

On the other hand,

$$\begin{aligned}(\partial_t t)^k \sum_{j\geqslant 1}(a_j\partial_t^j\delta_0 + b_j\partial_{\bar t}^j\delta_0) &= \sum_{j\geqslant 1} a_j\delta_0\partial_t t)^k\partial_t^j\delta_0\\ &= \sum_{j\geqslant 1} a_j\partial_t^j(\partial_t t - j)^k\delta_0 = \sum_{j\geqslant 1}(-j)^k a_j\partial_t^j\delta_0,\end{aligned}$$

and similarly

$$(\partial_{\bar t}\bar t)^k \sum_{j\geqslant 1}(a_j\partial_t^j\delta_0 + b_j\partial_{\bar t}^j\delta_0) = \sum_{j\geqslant 1}(-j)^k b_j\partial_{\bar t}^j\delta_0,$$

so the equations satisfied by u imply

$$2\pi\mathsf{i}\, c_{k+1}\delta_0 + \sum_{j\geqslant 1}(-j)^k a_j\partial_t^j\delta_0 = 0 \quad\text{and}\quad 2\pi\mathsf{i}\, c_{k+1}\delta_0 + \sum_{j\geqslant 1}(-j)^k b_j\partial_{\bar t}^j\delta_0 = 0,$$

hence $c_{k+1} = a_j = b_j = 0$, as was to be proved. □

6.3.b. Sesquilinear pairings. Let $\mathcal{M}', \mathcal{M}''$ be regular holonomic $\mathscr{D}_\Delta$-modules, each of which written as $\mathcal{M} \simeq \mathscr{O}_\Delta \otimes_{\mathbb{C}[t]} M^{\mathrm{alg}}$ and let $\mathfrak{c} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{Db}_\Delta$ be a sesquilinear pairing between them. For any local sections m', m'' of $\mathcal{M}', \mathcal{M}''$, we have, by definition

$$\begin{aligned}P(t,\partial_t)\mathfrak{c}(m', \overline{m''}) &= \mathfrak{c}(P(t,\partial_t)m', \overline{m''}),\\ \overline{P(t,\partial_t)}\mathfrak{c}(m', \overline{m''}) &= \mathfrak{c}(m', \overline{P(t,\partial_t)m''}).\end{aligned} \tag{6.3.5}$$

For the sake of simplicity, we will set $u_{-1,p} := \partial_t\partial_{\bar t}u_{0,p+1}$. Note that the basic relations (6.3.1) also hold for $u_{-1,p}$. Propositions 6.3.2 and 6.3.4 imply immediately:

Proposition 6.3.6. *Let $\mathfrak{c}$ be a sesquilinear pairing between $\mathcal{M}'$ and $\mathcal{M}''$.*

(1) *The induced pairing $\mathfrak{c} : M'_{\beta'} \otimes \overline{M''_{\beta''}} \to \mathfrak{Db}_\Delta$ vanishes if $\beta' - \beta'' \notin \mathbb{Z}$.*

(2) *If $m' \in M'_\beta$ and $m'' \in M''_\beta$ with $\beta \geqslant -1$, then the induced pairing $\mathfrak{c}(\beta)(m', \overline{m''})$ is a $\mathbb{C}$-linear combination of the basic distributions $u_{\beta,p}$ ($p \geqslant 0$).* □

As a consequence, the pairing $\mathfrak{c}(\beta)$ has a unique expansion $\sum_{p\geqslant 0}\mathfrak{c}_{\beta,p}u_{\beta,p}$, where $\mathfrak{c}_{\beta,p}$ is a sesquilinear pairing $M'_\beta \otimes \overline{M''_\beta} \to \mathbb{C}$ ($\beta \geqslant -1$). Let us set $\mathfrak{c}_\beta := \mathfrak{c}_{\beta,0}$. Using the relations in (6.3.1) and (6.3.5), we get

$$\begin{aligned}\sum_{p\geqslant 0}\mathfrak{c}_{\beta,p}(-(\mathrm{E}-\beta)m', \overline{m''})u_{\beta,p} = \mathfrak{c}(-(\mathrm{E}-\beta)m', \overline{m''}) &= -(t\partial_t - \beta)\mathfrak{c}(m', \overline{m''})\\ &= \sum_{p\geqslant 0}\mathfrak{c}_{\beta,p+1}(m', \overline{m''})u_{\beta,p},\end{aligned}$$

and therefore $\mathfrak{c}_{\beta,p+1}(m', \overline{m''}) = \mathfrak{c}_{\beta,p}(-(\mathrm{E}-\beta)m', \overline{m''})$. So, if we denote by N the nilpotent operator $-(\mathrm{E}-\beta)$ (not to be confused with $\mathrm{N} = -2\pi\mathsf{i}(\mathrm{E}-\beta)$), we have

$$\mathfrak{c}(\beta)(m', \overline{m''}) = \sum_{p\geqslant 0}\mathfrak{c}_\beta(N^p m', \overline{m''})u_{\beta,p} = \sum_{p\geqslant 0}\mathfrak{c}_\beta(m', \overline{N^p m''})u_{\beta,p}$$

(the latter equality is a consequence of (6.3.1)). The calculation above also shows that our pairing $\mathfrak{c}_\beta : M'_\beta \otimes \overline{M''_\beta} \to \mathbb{C}$ satisfies the relation $\mathfrak{c}_\beta \circ (N \otimes \overline{\mathrm{Id}}) = \mathfrak{c}_\beta \circ (\mathrm{Id} \otimes \overline{N})$.

Using the power series expansion of the exponential function, we may write the above formula in a purely symbolic way as

$$\mathfrak{c}(\beta)(m',\overline{m''}) = \begin{cases} \mathfrak{c}_\beta(|t|^{2(\beta\,\mathrm{Id}-N)}m',\overline{m''}) & \text{if } \beta>-1,\\ \partial_t\partial_{\bar t}\mathfrak{c}_{-1}\Big(\dfrac{|t|^{-2N}-1}{N}\,m',\overline{m''}\Big) & \text{if } \beta=-1,\end{cases}\qquad (m'\in M'_\beta,\ m''\in M''_\beta).$$

6.3.c. Sesquilinear pairing on nearby cycles. We have seen in Exercise 5.4.7(3) a way to define the sesquilinear pairing $\mathrm{gr}_V^\beta\mathfrak{c}$ by means of a residue formula, if $\beta>-1$. In the present setting, we can conclude:

Lemma 6.3.7. *For every $\beta>-1$, the sesquilinear pairing on $V^\beta\mathcal{M}'\otimes\overline{V^\beta\mathcal{M}''}$ defined by the formula*

$$(m',\overline{m''})\longmapsto \mathrm{Res}_{s=-\beta-1}\int_\Delta |t|^{2s}\mathfrak{c}(m',\overline{m''})\,\chi(t)\ \tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\bar t$$

(for some, or any $\chi\in C^\infty_{\mathrm{c}}(\Delta)$ such that $\chi\equiv1$ near 0) induces a well-defined sesquilinear pairing

$$\mathrm{gr}_V^\beta\mathfrak{c}:\mathrm{gr}_V^\beta\mathcal{M}'\otimes\overline{\mathrm{gr}_V^\beta\mathcal{M}''}\longrightarrow\mathbb{C}$$

which coincides with $\mathfrak{c}_\beta$ via the identification $M_\beta\simeq\mathrm{gr}_V^\beta\mathcal{M}$ ($\mathcal{M}=\mathcal{M}',\mathcal{M}''$) of Proposition 6.2.21. □

Remark 6.3.8. For $m'\in M'_\beta$ and $m''\in M''_\beta$, we recover the equality $\mathrm{gr}_V^\beta\mathfrak{c}(m',\overline{m''})=\mathfrak{c}_\beta(m',\overline{m''})$ (by using the identification $M_\beta=\mathrm{gr}_V^\beta\mathcal{M}$) as already checked in Exercise 5.4.7(3), by means of the formula above for $\mathfrak{c}(\beta)$. Indeed,

$$\begin{aligned}\mathrm{Res}_{s=-\beta-1}&\int_\Delta |t|^{2s}\mathfrak{c}(\beta)(m',\overline{m''})\,\chi(t)\ \tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\bar t\\ &=\mathrm{Res}_{\sigma=0}\int_\Delta \mathfrak{c}_\beta(|t|^{2(\sigma-1-N)}m',\overline{m''})\,\chi(t)\ \tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\bar t\\ &=\mathfrak{c}_\beta\Big(\mathrm{Res}_{\sigma=0}\int_\Delta |t|^{2(\sigma-1-N)}\,\chi(t)\ \tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\bar t\,m',\overline{m''}\Big),\end{aligned}$$

and from Example 5.4.8 and Exercise 5.4.7(1) we have

$$\mathrm{Res}_{\sigma=0}\int_\Delta |t|^{2(\sigma-1-N)}\,\chi(t)\ \tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\bar t=1.$$

6.4. Hodge $\mathscr{D}$-modules on a Riemann surface and the Hodge-Saito theorem

What kind of an algebraic object do we get by considering $\mathcal{V}_{\mathrm{mid}}$ together with its connection and its filtration? How to describe it axiomatically, as we did for variations of Hodge structure? Is there a wider class of filtered $\mathscr{D}$-modules which would give rise to a Hodge theorem? We give an answer to these questions in this section.

6.4.a. Polarized Hodge modules with pure support the origin. Let us start with the simplest case. By a (polarized) $\mathbb{C}$-Hodge module of weight w strictly supported on a point we simply mean a (polarized) $\mathbb{C}$-Hodge structure (H, Q) of weight w (see Section 2.4.a). It will be convenient here to adopt the simple presentation as a filtered w-Hermitian pair $H = ((\mathcal{H}, F^\bullet\mathcal{H}), \mathfrak{c})$ (see Remark 2.4.20 and Proposition 2.4.37), and we omit $\mathrm{Q} = ((-1)^w \mathrm{Id}, \mathrm{Id})$ in the notation.

Let X be a Riemann surface and let Σ be a discrete set of points. Let $\iota_\Sigma : \Sigma \hookrightarrow X$ denote the inclusion. By a *(polarized) $\mathbb{C}$-Hodge module on X of weight w with pure support* Σ we mean the pushforward by ι_Σ of a (polarized) $\mathbb{C}$-Hodge structure $\bigoplus_{x\in\Sigma} H_x$ of weight w.

Before making explicit this pushforward functor, let us recall that the intrinsic way to define the Dirac distribution attached to Σ is to consider it as a $(1,1)$-current, which pairs with test functions (C^∞ functions with compact support on X), by the formula $\langle\delta_\Sigma, \eta\rangle = \sum_{x\in\Sigma} \eta(x)$, while distributions pair with test $(1,1)$-forms. Recall also (see more details in Appendix A.4.d and Section 10.2.a) that the sheaf $\mathfrak{C}_X$ of $(1,1)$-currents is a *right* $\mathscr{D}_X \otimes_{\mathbb{C}} \mathscr{D}_{\overline{X}}$-module: if u is a local current and η is a test function, and if P is a local holomorphic or anti-holomorphic differential operator, we set

$$\langle u \cdot P, \eta\rangle := \langle u, P\eta\rangle.$$

In order to correctly define the pushforward of the sesquilinear pairing $\mathfrak{c}$ by using the δ_Σ current, it will therefore be convenient to work within the framework of *right* $\mathscr{D}_X$-modules. We will first consider the local case of the inclusion $\iota : \{0\} \hookrightarrow \Delta$. Let H be a polarized Hodge structure of weight w. We set the following.

- $\iota_*\mathcal{H}$ is the skyscraper sheaf with stalk $\mathcal{H}$ at the origin.
- $\mathcal{M} = {}_{\mathrm{D}}\iota_*\mathcal{H}$ is the right $\mathscr{D}_\Delta$-module $\iota_*\mathcal{H}[\partial_t]$ (with the right action of t defined by $v\partial_t^k \cdot t = kv\partial_t^{k-1}$, and the right action of ∂_t is the obvious one).
- The F-filtration on ${}_{\mathrm{D}}\iota_*\mathcal{H}$ is defined by

$$F^p\mathcal{M} = F^p{}_{\mathrm{D}}\iota_*\mathcal{H} = \bigoplus_{k\geqslant 0} \iota_*F^{p+k}\mathcal{H} \cdot \partial_t^k.$$

- The pairing ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*\mathfrak{c} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{C}_\Delta$ (sheaf of $(1,1)$-currents on Δ) is defined by $\mathscr{D}_\Delta \otimes_{\mathbb{C}} \overline{\mathscr{D}_\Delta}$-linearity from its restriction to $\iota_*\mathcal{H}' \otimes_{\mathbb{C}} \iota_*\overline{\mathcal{H}''}$ as follows:

$${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*\mathfrak{c}(v', \overline{v''}) = \mathfrak{c}(v', \overline{v''})\delta_0 \quad (\delta_0 = \text{Dirac current at } 0).$$

We will fix later a rule of signs (10.3.16 $*$) for which it will be natural to replace ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*\mathfrak{c}$ with

$${}_{\mathrm{T}}\iota_*\mathfrak{c} := -{}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*\mathfrak{c}.$$

We then set $M = {}_{\mathrm{H}}\iota_*H := ((\mathcal{M}, F^\bullet\mathcal{M}), {}_{\mathrm{T}}\iota_*\mathfrak{c})$.

- We keep in mind the correspondence with triples as in Proposition 2.4.37.

Our aim is now to recover H from ${}_{\mathrm{H}}\iota_*H$ by the pushforward $X \to \mathrm{pt}$ as considered in Section 6.2.e.

Exercise 6.4.1. Recover H from ${}_{\mathrm{H}}\iota_*H$ by applying the following operations.

(1) $\mathcal{H} = \mathrm{Ker}\big(t : \mathcal{H}[\partial_t] \to \mathcal{H}[\partial_t]\big) = \mathrm{gr}_V^{-1}\mathcal{H}[\partial_t]$.
(2) $F^\bullet\mathcal{H} = F^\bullet(\mathcal{H}[\partial_t]) \cap \mathrm{Ker}\, t$.
(3) $\mathfrak{c}$ is the opposite of the coefficient on δ_0 of the restriction of ${}_{\mathrm{T}}\iota_*\mathfrak{c}$ to $\mathcal{H}\otimes\overline{\mathcal{H}}$.

Lemma 6.4.2. *Let us filter the de Rham complex (see Example* 6.2.27(2)*)*

$$\mathrm{DR}({}_{\mathrm{D}}\iota_*\mathcal{H}) = \{0 \to {}_{\mathrm{D}}\iota_*\mathcal{H} \xrightarrow{\;\cdot\,\partial_t\;} {}_{\mathrm{D}}\underset{\bullet}{\iota_*}\mathcal{H} \to 0\}$$

by setting

$$F^p\,\mathrm{DR}({}_{\mathrm{D}}\iota_*\mathcal{H}) = \{0 \to F^{p+1}{}_{\mathrm{D}}\iota_*\mathcal{H} \xrightarrow{\;\cdot\,\partial_t\;} F^p{}_{\mathrm{D}}\iota_*\mathcal{H} \to 0\}.$$

Then $F^p\,\mathrm{DR}({}_{\mathrm{D}}\iota_*\mathcal{H}) \simeq \iota_*F^p\mathcal{H}$.

Proof. Clearly, ∂_t is injective on $F^{p+1}{}_{\mathrm{D}}\iota_*\mathcal{H}$. On the other hand, for $k \geqslant 1$, we have $F^{p+k}\mathcal{H}\otimes\partial_t^k = (F^{p+1+(k-1)}\mathcal{H}\otimes\partial_t^{k-1})\cdot\partial_t$, hence $\mathcal{H}^0F^p\,\mathrm{DR}({}_{\mathrm{D}}\iota_*\mathcal{H}) = \iota_*F^p\mathcal{H}$. □

Let $M := \bigoplus_{x\in\Sigma} {}_{\mathrm{H}}\iota_{x*}H_x$ be a polarized Hodge module of weight w on a compact Riemann surface X, strictly supported on a finite set Σ. According to Example 6.2.27(1), we thus have $\boldsymbol{H}^k(X, \mathrm{DR}\,\mathcal{M}) = 0$ for $k \neq 0$, and $\boldsymbol{H}^0(X, \mathrm{DR}\,\mathcal{M})$ is filtered by

$$F^p\boldsymbol{H}^0(X, \mathrm{DR}\,\mathcal{M}) := \mathrm{image}\big[\boldsymbol{H}^0(X, F^p\,\mathrm{DR}\,\mathcal{M}) \to \boldsymbol{H}^0(X, \mathrm{DR}\,\mathcal{M})\big].$$

Notice that at each point of $x \in \Sigma$, the map above is nothing but $F^p\mathcal{H}_x \to \mathcal{H}_x$, hence is *injective*. Moreover, the complexes $F^p\,\mathrm{DR}\,\mathcal{M}$ and $\mathrm{DR}\,\mathcal{M}$ are supported on Σ, so that their hypercohomology on X is nothing but the direct sum of the stalks of $\mathscr{H}^0F^p\,\mathrm{DR}\,\mathcal{M}$ or $\mathscr{H}^0\,\mathrm{DR}\,\mathcal{M}$ at the points of Σ.

Let us now express $\mathfrak{c}$ from ${}_{\mathrm{T}}\iota_{\Sigma*}\mathfrak{c} =: \mathfrak{c}_M$. Given $m \in \boldsymbol{H}^0(X, \mathrm{DR}\,\mathcal{M})$, we can write $m = \bigoplus_{x\in\Sigma} m_x$ and we can lift m_x as a section of $\mathcal{M}$, denoted in the same way. Then $m_x = \sum_{k\geqslant 0} m_{x,k}\partial_t^k$ for a local coordinate t at x, with $m_{x,k} \in \mathcal{H}_x$. Note that the class of $m_{x,k}\partial_t^k$ is zero in $\boldsymbol{H}^0(X, \mathrm{DR}\,\mathcal{M})$ if $k \geqslant 1$. Doing this for m', m'', we obtain ${}_{\mathrm{T}}\iota_{\Sigma*}\mathfrak{c}(m', \overline{m''}) = -\sum_{x\in\Sigma}\mathfrak{c}(m'_{x,j}, \overline{m''_{x,k}})\delta_x\partial_t^j\partial_{\bar t}^k$ for a local coordinate t at each x. Given a $(1,1)$-current u on X, its integral on X (or pushforward by $a_X : X \to \mathrm{pt}$) is $\langle u, 1\rangle \in \mathbb{C}$ (since X is compact, 1 is a test function), and the rule of signs of (10.3.16 $*$) gives

$${}_{\mathrm{T}}a^0_{X,*}({}_{\mathrm{T}}\iota_{\Sigma*}\mathfrak{c})(m', m'') = -\langle{}_{\mathrm{T}}\iota_{\Sigma*}\mathfrak{c}(m', \overline{m''}), 1\rangle = \sum_{x\in\Sigma}\mathfrak{c}(m'_{x,0}, \overline{m''_{x,0}}).$$

The following is now mostly tautological, but nevertheless interesting to emphasize.

***Theorem 6.4.3* (Hodge-Saito, dimension zero).** *With the previous assumption, the natural map*

$$\boldsymbol{H}^0(X, F^p\,\mathrm{DR}\,\mathcal{M}) \longrightarrow \boldsymbol{H}^0(X, \mathrm{DR}\,\mathcal{M})$$

is injective and its image is denoted by $F^p\boldsymbol{H}^0(X, \mathrm{DR}\,\mathcal{M})$*. Moreover, the data*

$$\big(\boldsymbol{H}^0(X, \mathrm{DR}\,\mathcal{M}), F^\bullet\boldsymbol{H}^0(X, \mathrm{DR}\,\mathcal{M}), {}_{\mathrm{T}}a^0_{X,*}\mathfrak{c}_M\big)$$

is a polarized Hodge structure of weight w (it is isomorphic to $\bigoplus_{x\in\Sigma} H_x$*).*

6.4.b. Polarized $\mathbb{C}$-Hodge modules with pure support Δ, left version

We now return to left $\mathscr{D}_X$-modules for convenience. Let $\mathcal{M}$ be a holonomic $\mathscr{D}_\Delta$-module with regular singularity at the origin, which is the middle extension of its restriction to Δ^* (see Definition 6.2.16), that we assume to be a vector bundle with connection. It follows from Proposition 6.2.20 that $\mathcal{M} = (\mathcal{V}_{\mathrm{mid}}, \nabla)$. Let $F^\bullet\mathcal{M}$ be a coherent F-filtration of $\mathcal{M}$ (taken decreasing).

Definition 6.4.4 **(Strict $\mathbb{R}$-specializability).** We say that the filtered $\mathscr{D}_\Delta$-module $(\mathcal{M}, F^\bullet\mathcal{M})$ is *strictly* $\mathbb{R}$*-specializable* at the origin if the properties 5.4.4(1a) and (1b) are satisfied. (See also Definition 7.3.24 together with Proposition 8.8.2.)

Definition 6.4.5 **(Hodge module with pure support the disc, left version)**
A left *Hodge module* of weight w with pure support Δ and singularity at the origin at most consists of the data $M := \big((\mathcal{M}', F^\bullet\mathcal{M}'), (\mathcal{M}'', F^\bullet\mathcal{M}''), \mathfrak{c}\big)$, where $\mathcal{M}', \mathcal{M}''$ are regular holonomic $\mathscr{D}_\Delta$-modules with singularity at the origin at most and pure support Δ (see Definition 6.2.25), $F^\bullet\mathcal{M}$ is a coherent filtration ($\mathcal{M} = \mathcal{M}'$ or $\mathcal{M}''$), and $\mathfrak{c}$ is a sesquilinear pairing $\mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{Db}_X$, such that

(a) the residue of the connection on some (or any) logarithmic lattice of $\mathcal{M}$ has real eigenvalues (equivalently, the eigenvalues of the monodromy of $\underline{\mathcal{H}}$ have absolute value equal to one),

(b) these data restrict to a variation of Hodge structure $\big((\mathcal{H}', F^\bullet\mathcal{H}'), (\mathcal{H}'', F^\bullet\mathcal{H}''), \mathfrak{c}\big)$ of weight w on Δ^* (regarded as a variation of $\mathbb{C}$-Hodge triple as in Definition 4.1.10),

(c) $(\mathcal{M}, F^\bullet\mathcal{M})$ is strictly $\mathbb{R}$-specializable at the origin,

(d) for every $\beta \in (-1, 0]$, the object

$$\mathrm{gr}_V^\beta M := \big((\mathrm{gr}_V^\beta\mathcal{M}', F^\bullet\mathrm{gr}_V^\beta\mathcal{M}'), (\mathrm{gr}_V^\beta\mathcal{M}'', F^\bullet\mathrm{gr}_V^\beta\mathcal{M}''), \mathrm{gr}_V^\beta\mathfrak{c}\big),$$

together with $\mathrm{N} := (-\mathrm{N}', \mathrm{N}'')$, is a Hodge-Lefschetz structure of weight w (the rule for the sign follows (10.4.24 $*$)).

In other words, we have taken the result of Schmid's theorem 5.4.10 as the definition for the local behaviour of a Hodge module (however, we do not assume here the existence of a polarization, so we do not restrict to the case of $(-1)^w$-Hermitian pairs for the moment).

Remark 6.4.6 **(Morphisms, adjunction, Tate twist).** The notion of morphism is the obvious one, as in the category of triples. A morphism $\varphi : M_1 \to M_2$ is a pair (φ', φ''), where φ' is a filtered morphism $(\mathcal{M}'_1, F^\bullet\mathcal{M}'_1) \to (\mathcal{M}'_2, F^\bullet\mathcal{M}'_2)$ and φ'' a filtered morphism $(\mathcal{M}''_2, F^\bullet\mathcal{M}''_2) \to (\mathcal{M}''_1, F^\bullet\mathcal{M}''_1)$, both satisfying the compatibility relation (2.4.24 $**$) in $\mathfrak{Db}_X$.

Similarly, the adjunction functor $M \mapsto M^*$ (resp. the Tate twist $M(k, \ell)$) is defined as in Remark 2.4.25(6) (resp. (7)) and the category of Hodge modules with pure support Δ is left invariant by these functors (up to the change of weight), according to Remarks 2.4.29 and 3.2.26.

Exercise 6.4.7. Let M be a left Hodge module of weight w with pure support the disc Δ and let $(\mathrm{gr}_V^0 M, \mathrm{N})$ be the associated Hodge-Lefschetz structure centered at w. Consider the associated Hodge-Lefschetz middle extension quiver (see Definition 3.2.14). Show that $\mathrm{Im}\,\mathrm{N}$ has underlying vector spaces $\mathrm{gr}_V^{-1}\mathcal{M}', \mathrm{gr}_V^{-1}\mathcal{M}''$, endowed with the filtration induced on $\mathrm{gr}_V^{-1}\mathcal{M}$ as in (6.2.22).

Corollary 6.4.8. *If* $(\mathcal{M}, F^\bullet\mathcal{M})$ *(*$\mathcal{M} = \mathcal{M}'$ *or* $\mathcal{M}''$*) underlies a left Hodge module of weight* w *and pure support* Δ*, then* $(\mathcal{M}, F^\bullet\mathcal{M}) \simeq (\mathcal{V}_{\mathrm{mid}}, F^\bullet\mathcal{V}_{\mathrm{mid}})$ *as defined by* (5.4.2)*, with* $\mathcal{V} = \mathcal{M}_{|\Delta^*}$*.*

Proof. That $\mathcal{M} \simeq \mathcal{V}_{\mathrm{mid}}$ follows from Definition 6.2.25. It remains to check that the filtrations coincide. By Exercise 5.4.4, it is enough to check that $F^\bullet\mathcal{M} \cap V^{>-1}\mathcal{M} = F^p\mathcal{V}_{\mathrm{mid}}^{>-1}$ and $F^\bullet\mathcal{M}$ satisfies 5.4.4(1c), since we assume that $(\mathcal{M}, F^\bullet\mathcal{M})$ is strictly $\mathbb{R}$-specializable.

Let us first show that

$$F^\bullet\mathcal{M} \cap V^{>-1}\mathcal{M} = (j_*j^{-1}F^\bullet\mathcal{M}) \cap V^{>-1}\mathcal{M},$$

the latter term being equal to $F^p\mathcal{V}_{\mathrm{mid}}^{>-1}$ by (5.4.1). Let m be a local section of $(j_*j^{-1}F^p\mathcal{M} \cap V^{>-1}\mathcal{M}) \cap (F^q\mathcal{M} \cap V^{>-1}\mathcal{M})$ for $q > p$. Then m defines a section of $(F^q\mathcal{M} \cap V^{>-1}\mathcal{M})/(F^p\mathcal{M} \cap V^{>-1}\mathcal{M})$ supported at the origin. Since the latter quotient is $\mathscr{O}_\Delta$-coherent, it follows that $t^N m$ is a local section of $F^p\mathcal{M} \cap V^{>-1}\mathcal{M}$ for some N, hence a local section of $F^p\mathcal{M} \cap V^{>-1+N}\mathcal{M}$. Now, Property 5.4.4(1a) implies that m is a local section of $F^p\mathcal{M} \cap V^{>-1}\mathcal{M}$, hence the desired assertion.

It remains to check 5.4.4(1c). This amounts to proving that can is an epimorphism in the category of filtered vector spaces. This follows from the property that it is a morphism of Hodge-Lefschetz structures (see Exercise 3.2.11(2)). □

Remark 6.4.9. Strict $\mathbb{R}$-specializability, as defined by 6.4.4 and assumed in Definition 6.4.5, would not have been enough to prove Corollary 6.4.8. Hodge theory is used in an essential way here, by means of Exercise 3.2.11, to ensure Property 5.4.4(1c).

Proposition 6.4.10. *Let* X *be a Riemann surface. There is no nonzero morphism* $M_1 \to M_2$ *between left* $\mathbb{C}$*-Hodge modules of weight* w_1, w_2 *with pure support* X *if* $w_1 > w_2$*.*

Proof. Let Σ be the union of the singular sets of M_1 and M_2. The $\mathscr{D}$-module part of $\mathrm{Im}\,\varphi$ has support in Σ, by applying Proposition 2.4.5(2) at points of $X \smallsetminus \Sigma$, but is included in a $\mathscr{D}$-module with pure support of dimension one, hence is zero. □

***Proposition 6.4.11* (Abelianity).** *Let* X *be a Riemann surface. The category* $\mathsf{HM}_X(X, w)$ *of left Hodge modules with pure support equal to* X *is* abelian *and any morphism is strict with respect to the* F*-filtrations.*

Proof. The question is local, so we can assume that $X = \Delta$ and that the only singularity of both source and target of the morphism is the origin. Let $\varphi : M_1 \to M_2$ be a morphism. We will first show that $\mathrm{Ker}\,\varphi$ and $\mathrm{Coker}\,\varphi$ are also strictly $\mathbb{R}$-specializable and have pure support Δ. We consider the following diagram of exact sequences in MHS (see Exercise 6.4.7):

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}(\mathrm{gr}_V^0\varphi)(1) \arrow[r] \arrow[d, "\mathrm{can}"] & \mathrm{gr}_V^0 M_1(1) \arrow[r, "\mathrm{gr}_V^0\varphi"] \arrow[d, two heads, "\mathrm{can}"] \arrow[dd, bend right, "\mathrm{N}_1"'] & \mathrm{gr}_V^0 M_2(1) \arrow[r] \arrow[d, two heads, "\mathrm{can}"] \arrow[dd, bend right, "\mathrm{N}_2"'] & \operatorname{Coker}(\mathrm{gr}_V^0\varphi)(1) \arrow[r] \arrow[d, two heads, "\mathrm{can}"] & 0 \\
0 \arrow[r] & \operatorname{Ker}(\mathrm{gr}_V^{-1}\varphi) \arrow[r] \arrow[d, hook, "\mathrm{var}"] & \mathrm{gr}_V^{-1} M_1 \arrow[r, "\mathrm{gr}_V^{-1}\varphi"] \arrow[d, hook, "\mathrm{var}"] & \mathrm{gr}_V^{-1} M_2 \arrow[r] \arrow[d, hook, "\mathrm{var}"] & \operatorname{Coker}(\mathrm{gr}_V^{-1}\varphi) \arrow[r] \arrow[d, "\mathrm{var}"] & 0 \\
0 \arrow[r] & \operatorname{Ker}(\mathrm{gr}_V^0\varphi) \arrow[r] & \mathrm{gr}_V^0 M_1 \arrow[r, "\mathrm{gr}_V^0\varphi"] & \mathrm{gr}_V^0 M_2 \arrow[r] & \operatorname{Coker}(\mathrm{gr}_V^0\varphi) \arrow[r] & 0
\end{tikzcd}$$

We have to prove that the left up can is an epimorphism and that the right down var is a monomorphism. This amounts to showing that $\operatorname{Im}\mathrm{N}_1 \cap \operatorname{Ker}(\mathrm{gr}_V^0\varphi) = \mathrm{N}_1(\operatorname{Ker}(\mathrm{gr}_V^0\varphi))$ (because this is equivalent to $\operatorname{Im}\mathrm{can} \cap \operatorname{Ker}(\mathrm{gr}_V^{-1}\varphi) = \mathrm{can}(\operatorname{Ker}(\mathrm{gr}_V^{-1}\varphi)))$ and $\operatorname{Im}\mathrm{N}_2 \cap \operatorname{Im}(\mathrm{gr}_V^0\varphi) = \mathrm{N}_2(\operatorname{Im}(\mathrm{gr}_V^0\varphi))$. This follows from Lemma 3.1.7 in the setting of Exercise 3.2.11(7).

If we consider φ on the $\mathscr{D}$-module components of the Hodge module, that we simply denote by $\mathcal{M}$ (so that we do not distinguish between φ' and φ''), we clearly have $\operatorname{Ker}(\mathrm{gr}_V^0\varphi) = \mathrm{gr}_V^0(\operatorname{Ker}\varphi)$ and similarly for Coker, and for gr_V^{-1}, proving thus that $\operatorname{Ker}\varphi$ and $\operatorname{Coker}\varphi$ are middle extensions.

It remains to prove that $\operatorname{Ker}\varphi$ and $\operatorname{Coker}\varphi$, as filtered $\mathscr{D}_\Delta$-modules, are strictly $\mathbb{R}$-specializable at the origin. We note that $\mathrm{gr}_V^\beta\varphi$ is *strict* for every $\beta \in (-1, 0]$, as well as $\mathrm{gr}_V^{-1}\varphi$, since they are morphisms of mixed Hodge structures. It is then straightforward to check that $\operatorname{Ker}\mathrm{gr}_V^0\varphi$ (with filtration induced by that of $\mathrm{gr}_V^0\mathcal{M}$) is equal to $\mathrm{gr}_V^0\operatorname{Ker}\varphi$ (with filtration on $\operatorname{Ker}\varphi$ induced from that of $\mathcal{M}$), and similarly for Coker. $\square$

Corollary 6.4.12. *Let φ be a morphism in $\mathsf{HM}_X(X, w)$. Assume that it is injective on the $\mathscr{D}_X$-module component. Then it is a monomorphism, i.e., the Hodge filtration on the source of φ is the filtration induced by that on its target.* $\square$

Definition 6.4.13 **(Polarized Hodge module with pure support the disc, left version)** Let X be a Riemann surface and let M be a Hodge module of weight w with pure support X and singularity in a discrete set $\Sigma \subset X$. A *polarization* is a morphism $\mathrm{Q} : M \to M^*(-w)$ (see Remark 6.4.6) whose restriction to X^* induces a polarization of the corresponding variation of $\mathbb{C}$-Hodge structure.

Remark 6.4.14. A polarized Hodge module of weight w is isomorphic to one of the form $((\mathcal{M}', F^\bullet\mathcal{M}'), (\mathcal{M}'', F^\bullet\mathcal{M}''), \mathfrak{c})$ with polarization $((-1)^w \operatorname{Id}, \operatorname{Id})$, so that we will in general consider it as a pair $(\mathcal{M}, F^\bullet\mathcal{M}), \mathfrak{c})$.

It follows from Schmid's theorem 5.4.10 that a polarization Q as above induces, for every $\lambda \in S^1$, a polarization of the Hodge-Lefschetz structure $(\mathrm{gr}_V^\beta M, \mathrm{N})$ ($\beta \in (-1, 0]$) centered at w, for each local coordinate t at a point of Σ.

Exercise 6.4.15. Same as Exercise 6.4.7 with polarization.

The definition of a polarized Hodge module consists therefore in taking Schmid's theorem 5.4.10 as a defining property. This leads to the definition of the category $\mathsf{pHM}_X(X,w)$ of left polarizable $\mathbb{C}$-Hodge modules of weight w with pure support X: this is the full sub-category of $\mathsf{HM}_X(X,w)$ consisting of Hodge modules which admit a polarization. However, the morphisms are not supposed to be compatible with a given polarization.

Let $\Sigma \subset X$ be a finite subset and let j denote the inclusion $X^* = X \smallsetminus \Sigma \xhookrightarrow{j} X$. Schmid's theorem implies more precisely:

Corollary 6.4.16 **(of Theorem 5.4.10).** *The restriction functor j^*, from the category of left polarizable $\mathbb{C}$-Hodge modules with pure support X and singularity at Σ at most to the category of polarizable variations of $\mathbb{C}$-Hodge structure on $X^* = X \smallsetminus \Sigma$ is an equivalence of categories.* □

According to Exercise 4.1.14(2), we find:

Corollary 6.4.17. *The category $\mathsf{pHM}_X(X,w)$ is semi-simple.* □

The Hodge-Saito theorem. Let us now start with a polarized Hodge module (M,Q) on a compact Riemann surface X (see Definition 6.4.13). We assume that it has pure support X. Away from a finite set $\Sigma \xhookrightarrow{\iota} X$, it corresponds to a variation of polarized Hodge structure of weight w. For $\mathcal{M} = \mathcal{M}'$ or $\mathcal{M}''$, we have $\mathcal{M} = \mathcal{V}_{\mathrm{mid}}$ and the de Rham complex $\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}$ is naturally filtered (see Formula 2.3.4), so that we get in a natural way a filtration on its hypercohomology. Recall (see §6.2.d) that this de Rham complex is a resolution of $j_*\underline{\mathcal{H}}$. The Hodge-Zucker theorem 5.1.1 (together with Remark 5.4.15(3)) implies the following theorem in a straightforward way.

Theorem 6.4.18 **(Hodge-Saito, dimension one, left version).** *For (M,Q) and $(\mathcal{M},F^\bullet\mathcal{M})$ as above ($\mathcal{M} = \mathcal{M}'$ or $\mathcal{M}''$),*

(1) *the filtered complex $\boldsymbol{R}\Gamma(X,F^\bullet\,\mathrm{DR}\,\mathcal{M})$ is strict, i.e., for every k,p, the natural morphism $\boldsymbol{H}^k(X,F^p\,\mathrm{DR}\,\mathcal{M}) \to \boldsymbol{H}^k(X,\mathrm{DR}\,\mathcal{M})$ is injective,*

(2) *the data $\boldsymbol{H}^1(X,\mathrm{DR}\,\mathcal{M})$ equipped with the filtration induced by $F^p(\mathrm{DR}\,\mathcal{M})$, together with the sesquilinear pairing induced $\mathfrak{c}$ and the morphism induced by Q, form a polarized Hodge structure of weight $w+1$.* [1] □

6.5. Comments

Here come the references to the existing work which has been the source of inspiration for this chapter.

1. We refer to Section 12.2.b for the general definition of the sesquilinear pairing and the polarization "induced" by $\mathfrak{c}$ and Q.