CHAPTER 5

VARIATIONS OF HODGE STRUCTURE ON CURVES

Summary. We consider polarizable variations of $\mathbb{C}$-Hodge structure on a punctured smooth projective curve, and we indicate the proof of the corresponding Hodge theorem, as well as the semi-simplicity theorem proved in Chapter 4 for smooth projective varieties. This is the first occurrence of polarizable variations of $\mathbb{C}$-Hodge structure with singularities. It is essential to understand their local behaviour in the neighbourhood of a singular point. This is provided by the theorems of Schmid, that we also explain in this chapter.

5.1. Introduction

A Hodge structure, as explained in Section 2.4, can be considered as a Hodge structure on a vector bundle supported by a point, that is, a vector space. The case where the underlying space is a complex manifold is called a *variation of Hodge structure*. It has been explained in Section 4.1 from a local point of view. The global properties have been considered in Section 4.2.

The question we address in this chapter is the definition and properties of Hodge structures on a vector bundle on a punctured complex projective curve (punctured compact Riemann surface) in the neighbourhood of the punctures (also called the *singularities* of the variation). The notion of a variation of polarized Hodge structure on a non-compact curve is analytic in nature, and a control near the punctures is needed in order to obtain interesting global results.

Our aim in this chapter is to sketch the proof of the *Hodge-Zucker theorem* 5.1.1 on a punctured compact Riemann surface, which is a Hodge theorem "with singularities". We mix the setting of Sections 4.2.b and 4.2.c, that is, we consider a variation of polarized Hodge structure (H, Q) of weight w on a punctured compact Riemann surface $X^* \overset{j}{\hookrightarrow} X$.

***Theorem 5.1.1* (Hodge-Zucker, [Zuc79]).** *In such a case, the cohomology $H^k(X, j_*\underline{\mathcal{H}})$ carries a natural polarized Hodge structure of weight $w+k$ ($k = 0, 1, 2$).*

The way of using L^2 cohomology is the exactly the same as in Section 4.2.c, provided that we replace D' and D'' with $\mathcal{D}'$ and $\mathcal{D}''$. Then we are left with the corresponding L^2 Poincaré and Dolbeault lemmas.

In any case, it is important to extend in some way the variation to the projective curve in order to apply algebraic techniques. What kind of an object should we expect on the projective curve? On the one hand, the theorems of Schmid enable us to extend each step of the Hodge filtration as an algebraic bundle over the curve. On the other hand, Zucker selects the interesting extension among all possible extensions in order to obtain the Hodge-Zucker theorem. This is the *middle extension* $(\mathcal{V}_{\mathrm{mid}}, \nabla)$ of the variation of polarized Hodge structure. This selection is suggested by the L^2 approach to the Hodge theorem. Let us emphasize that this approach is mainly local (except for the L^2 Dolbeault lemma, however), and we will mainly restrict the study to a local setting, where the base manifold is a disc Δ centered at the origin in $\mathbb{C}$ (or simply the germ of Δ at the origin), and we will denote by t its coordinate.

5.2. Variation of Hodge structure on a punctured disc

We now consider the behaviour of a variation of $\mathbb{C}$-Hodge structure near a singular point. From now on, we will work on a disc Δ with coordinate t, as indicated in the introduction of this chapter and we will denote by Δ^* the punctured disc $\Delta \smallsetminus \{0\}$. Assume that $\mathcal{H}$ is a variation of Hodge structure on Δ^*. Our goal is to define a suitable restriction of these data to the origin. As for the case of a point in Δ^*, the underlying vector space of the restricted object should have a dimension equal to the rank of the bundle on Δ^*.

5.2.a. The holomorphic vector bundle with connection. If we are given a holomorphic bundle with connection $(\mathcal{V}, \nabla)$ on Δ^*, there exists a canonical meromorphic extension, called the *Deligne meromorphic extension*, of the bundle $\mathcal{V}$ to a meromorphic bundle $\mathcal{V}_*$ (that is, a free sheaf of $\mathscr{O}_\Delta[1/t]$-modules) equipped with a connection ∇. It consists of all local sections of $j_*\mathcal{V}$ (where $j : \Delta^* \hookrightarrow \Delta$ is the inclusion) whose coefficients in some (or any) basis of multivalued ∇-horizontal sections have moderate growth in any sector with bounded arguments. Equivalently, it is characterized by the property that the coefficients of any multivalued horizontal section expressed in some basis of $\mathcal{V}_*$ are multivalued functions on Δ^* with moderate growth in any sector with bounded arguments.

Similarly, there exists a canonical free $\mathscr{O}_\Delta$-submodule $\mathcal{V}_*^0$ of $\mathcal{V}_*$, called the *Deligne canonical lattice*, consisting of *all* local sections of $j_*\mathcal{V}$ whose coefficients in any basis of horizontal sections on any bounded sector are holomorphic functions on this sector *with at most logarithmic growth*. On this bundle $\mathcal{V}_*^0$, the connection ∇ has a pole of order one. The residue $\mathcal{R}$ of the connection on $\mathcal{V}_*^0$ is an endomorphism of the vector space $\mathcal{V}_*^0/t\mathcal{V}_*^0$. The real part of its eigenvalues belong to $[0, 1)$. The latter two properties also characterize $\mathcal{V}_*^0$ among all lattices of $\mathcal{V}_*$ (i.e., free $\mathscr{O}_\Delta$-submodules of $\mathcal{V}_*$ which generate $\mathcal{V}_*$ as a $\mathscr{O}_\Delta[t^{-1}]$-module).

The existence of a free $\mathscr{O}_\Delta$-submodule $\mathcal{V}_*^0$ of $\mathcal{V}_*$ such that $\mathscr{O}_\Delta[t^{-1}] \otimes \mathcal{V}_*^0 = \mathcal{V}_*$ and on which ∇ has a pole of order one is by definition the condition ensuring that $(\mathcal{V}_*, \nabla)$ has a *regular singularity* at the origin of Δ.

A classical result (see e.g. [**Mal91**, (2.6) p. 24]) asserts that $\mathcal{V}_*^0$ has an $\mathscr{O}_\Delta$-basis with respect to which the matrix of ∇ is constant. More precisely, any $\mathbb{C}$-basis of $\mathcal{V}_*^0/t\mathcal{V}_*^0$ can be lifted to an $\mathscr{O}_\Delta$-basis of $\mathcal{V}_*^0$, and the matrix of ∇ is then equal to the matrix of the residue $\mathcal{R}$ in the given basis of $\mathcal{V}_*^0/t\mathcal{V}_*^0$. These results can be reformulated as follows.

Theorem 5.2.1. *The construction $(\mathcal{V},\nabla)\mapsto(\mathcal{V}_*,\nabla)$ induces an equivalence between the category of vector bundles with connection on the punctured disc Δ^* and that of free $\mathscr{O}_\Delta[1/t]$-modules with a connection ∇ having a regular singularity at the origin.*

Of course, an inverse functor is the restriction of $(\mathcal{V}_*,\nabla)$ to Δ^*. Notice also that this result implies that any morphism $\varphi:(\mathcal{V}_1,\nabla)\to(\mathcal{V}_2,\nabla)$ can be extended in a unique way as a morphism $(\mathcal{V}_{1*},\nabla)\to(\mathcal{V}_{2*},\nabla)$. The proof is obtained by interpreting φ as a horizontal section of $\mathscr{H}om_{\mathscr{O}_{\Delta^*}}(\mathcal{V}_1,\mathcal{V}_2)$ and by using the property that, for a connection with regular singularity (as ∇ on $\mathscr{H}om_{\mathscr{O}_\Delta[1/t]}(\mathcal{V}_{1*},\mathcal{V}_{2*})$), any horizontal section on Δ^* extends in a unique way as a ∇-horizontal section on Δ (see Exercise 5.2.2(4) below).

Exercise 5.2.2 **(The structure of $(\mathcal{V}_*,\nabla)$).**

(1) Show that $(\mathcal{V}_*,\nabla)$ is a successive extension of rank-one meromorphic connections. [*Hint*: Use a Jordan basis for $\mathcal{R}$ of $\mathcal{V}_*^0/t\mathcal{V}_*^0$.]

(2) Assume that $\mathcal{V}$ has rank one. Let v_γ be an $\mathscr{O}_\Delta$-basis of $\mathcal{V}_*^0$ in which the matrix of $t\nabla_{\partial_t}$ is constant. Show that $t\nabla_{\partial_t}v_\gamma=\gamma v_\gamma$ with $\operatorname{Re}\gamma\in[0,1)$. Identify $\mathcal{V}^\nabla$ with the subsheaf of $\rho_*\mathscr{O}_{\widetilde{\Delta}^*}$ consisting of multiples of some (or any) branch of the multivalued function $t^{-\gamma}$, by sending $ct^{-\gamma}$ to $ct^{-\gamma}v_\gamma$.

(3) For $\operatorname{Re}\gamma\in[0,1)$ and $p\geqslant 0$, set $\mathcal{J}_{\gamma,p}=(\mathscr{O}_\Delta[1/t]^{p+1},\nabla)$, where the matrix of ∇_{∂_t} in the canonical basis $\boldsymbol{v}_{\gamma,p}=(v_{\gamma,0},\dots,v_{\gamma,p})$ is given by $t\nabla_{\partial_t}v_{\gamma,k}=\gamma v_{\gamma,k}-v_{\gamma,k-1}$ (so that $v_{\gamma,p}$ is a generating section with respect to $t\nabla_{\partial_t}$). Show that $(\mathcal{V}_*,\nabla)$ has a decomposition

$$(\mathcal{V}_*,\nabla)\simeq\bigoplus_{\gamma\in[0,1)}\Big[\bigoplus_p(\mathcal{J}_{\gamma,p},\nabla)\Big]. \tag{5.2.2 $*$}$$

[*Hint*: Use a Jordan decomposition for $\mathcal{R}$.]

(4) Compute $\operatorname{Ker}\nabla$ on $\mathcal{V}_*$ in terms of this decomposition.

(5) Show that there is no nonzero morphism $\mathcal{J}_{\gamma_1,p}\to\mathcal{J}_{\gamma_2,q}$ if $\gamma_1\neq\gamma_2\in[0,1)$, and conclude that the decomposition indexed by γ above is unique.

We will assume from now on that *the eigenvalues of $\mathcal{R}$ are real.* We can then more generally consider a whole family of Deligne canonical lattices: for every $\beta\in\mathbb{R}$, we denote by $\mathcal{V}_*^\beta$ the lattice defined by the property that the eigenvalues of the residue of the connection belong to $[\beta,\beta+1)$. If we set $\mathcal{V}_*^{>\beta}=\bigcup_{\beta'>\beta}\mathcal{V}_*^{\beta'}$, then $\mathcal{V}_*^{>\beta}$ is the Deligne canonical lattice for which the eigenvalues of the residue of the connection belong to $(\beta,\beta+1]$. We use the notation

$$\operatorname{gr}^\beta\mathcal{V}_*:=\mathcal{V}_*^\beta/\mathcal{V}_*^{>\beta}. \tag{5.2.3}$$

Exercise 5.2.4. Show the following properties.

(1) $\mathcal{V}_*^{\beta+k} = t^k\mathcal{V}_*^\beta$ for every $k\in\mathbb{Z}$.

(2) $\mathrm{gr}^\beta\mathcal{V}_*$ can be identified with the generalized β-eigenspace of the residue of ∇ on $\mathcal{V}_*^{[\beta]}/t\mathcal{V}_*^{[\beta]}$.

(3) The map induced by ∇_{∂_t} sends $\mathrm{gr}^\beta\mathcal{V}_*$ to $\mathrm{gr}^{\beta-1}\mathcal{V}_*$ and, if $\beta\neq 0$, it is an isomorphism. [*Hint*: use that the composition $t\nabla_{\partial_t}:\mathrm{gr}^\beta\mathcal{V}_*\to\mathrm{gr}^\beta\mathcal{V}_*$ is identified with the restriction of the residue of ∇ on $\mathcal{V}_*^{[\beta]}/t\mathcal{V}_*^{[\beta]}$ to its generalized β-eigenspace.]

(4) The map $\nabla_{\partial_t}:\mathcal{V}_*^\beta\to\mathcal{V}_*^{\beta-1}$ is onto [equivalently, $t\nabla_{\partial_t}:\mathcal{V}_*^\beta\to\mathcal{V}_*^\beta$ is onto] provided that $\beta>0$. [*Hint*: Reduce to the case where $\mathcal{V}_*$ has rank one by using Exercise 5.2.2 and has a basis v_γ which satisfies $t\nabla_{\partial_t}v_\gamma=\gamma v_\gamma$ for some $\gamma\in[0,1)$, and show that $\mathcal{V}_*^{\gamma+k}=t^k\mathscr{O}_\Delta v_\gamma$ for $k\in\mathbb{Z}$.]

(5) With respect to a decomposition of $(\mathcal{V}_*,\nabla)$ as in Exercise 5.2.2(3), show that, for $\gamma\in[0,1)$, we have, for $k\in\mathbb{Z}$,

$$\mathcal{V}_*^{\gamma+k}=\bigoplus_{i,\,\gamma_i\geqslant\gamma}t^k\mathscr{O}_\Delta\cdot\boldsymbol{v}_{\gamma_i,p_i}\oplus\bigoplus_{i,\,\gamma_i<\gamma}t^{k+1}\mathscr{O}_\Delta\cdot\boldsymbol{v}_{\gamma_i,p_i}.$$

(6) The subsheaf $\sum_{p\geqslant 0}(\nabla_{\partial_t})^p\mathcal{V}_*^\beta$ of $\mathcal{V}_*$ is an $\mathscr{O}_\Delta$-module and

- does not depend on β provided $\beta>-1$, or provided $\beta\leqslant -1$,
- in the latter case, it is equal to $\mathcal{V}_*$,
- in the former case, we call it the *middle extension* of $(\mathcal{V},\nabla)$ and denote it by $\mathcal{V}_{\mathrm{mid}}$; then $\nabla_{\partial_t}:\mathcal{V}_{\mathrm{mid}}\to\mathcal{V}_{\mathrm{mid}}$ is *onto* and has kernel equal to the sheaf $j_*(\mathcal{V}^\nabla)$.

If we denote by $\mathcal{V}_*^{>-1}$ the lattice on which $\operatorname{Res}\nabla$ has eigenvalues in $(-1,0]$, and if $\beta\in(-1,0]$, $\mathrm{gr}^\beta\mathcal{V}_*$ is identified with the generalized eigenspace of $\operatorname{Res}\nabla$ on $\mathcal{V}_*^{>-1}/t\mathcal{V}_*^{>-1}$ corresponding with the eigenvalue β. We set $\mathrm{N}=-2\pi\mathrm{i}(\operatorname{Res}\nabla)^{\mathrm{nilp}}$ (nilpotent part). This is the endomorphism induced by $-2\pi\mathrm{i}(t\partial_t-\beta)$ on $\mathrm{gr}^\beta\mathcal{V}_*$. [This choice is suggested by the property that the unipotent part of the monodromy operator on $\mathcal{V}^\nabla$ can be identified with $\exp\mathrm{N}$.]

5.2.b. Reminder on Hermitian bundles on the punctured disc. Let $\mathcal{V}$ be a holomorphic vector bundle on Δ^* and let h be a Hermitian metric on the associated C^∞-bundle $\mathcal{H}:=\mathscr{C}^\infty_{\Delta^*}\otimes_{\mathscr{O}_{\Delta^*}}\mathcal{V}$. We denote by $\mathcal{V}_{\mathrm{mod}}$ the subsheaf of $j_*\mathcal{V}$ consisting of local sections whose h-norms have moderate growth in the neighbourhood of the origin. This is an $\mathscr{O}_\Delta[1/t]$-module, which coincides with $\mathcal{V}$ when restricted to Δ^*.

The *parabolic filtration* $\mathcal{V}_{\mathrm{mod}}^\bullet$ is the decreasing filtration, indexed by $\mathbb{R}$, consisting of local sections such that: for any compact neighbourhood K of the origin, in the punctured neighbourhood of which the local section is defined, and for every $\varepsilon>0$, there exists $C=C(K,\varepsilon)>0$ such that the h-norm on $K^*:=K\smallsetminus\{0\}$ of the local section is locally bounded by $C|t|^{\bullet-\varepsilon}$. By definition, we have $\mathcal{V}_{\mathrm{mod}}^\beta=\bigcap_{\beta'<\beta}\mathcal{V}_{\mathrm{mod}}^{\beta'}$.

Clearly, each $\mathcal{V}^{\beta}_{\mathrm{mod}}$ is an $\mathscr{O}_{\Delta}$-submodule of $\mathcal{V}_{\mathrm{mod}}$, which coincides with $\mathcal{V}$ when restricted to Δ^*, and we have

$$\mathcal{V}_{\mathrm{mod}} = \bigcup_{\beta} \mathcal{V}^{\beta}_{\mathrm{mod}}, \qquad \text{and} \qquad \forall\, k \in \mathbb{Z},\quad t^k\mathcal{V}^{\bullet}_{\mathrm{mod}} = \mathcal{V}^{\bullet+k}_{\mathrm{mod}}.$$

A *jump* (or, more correctly, jumping index) of the parabolic filtration is a real number β such that the quotient $\mathcal{V}^{\beta}_{\mathrm{mod}}/\mathcal{V}^{>\beta}_{\mathrm{mod}} \neq 0$, where $\mathcal{V}^{>\beta}_{\mathrm{mod}} := \bigcup_{\beta'>\beta} \mathcal{V}^{\beta'}_{\mathrm{mod}}$. Clearly, if β is a jump, then $\beta + k$ is a jump for every $k \in \mathbb{Z}$. We denote by $J(\beta)$ the set of jumping indices which belong to $[\beta, \beta+1)$. We have $J(\beta+k) = J(\beta)$ for every $k \in \mathbb{Z}$.

Definition 5.2.5. We say that the metric is *moderate* if $\mathcal{V}_{\mathrm{mod}}$ is $\mathscr{O}_{\Delta}[1/t]$-locally free and each $\mathcal{V}^{\beta}_{\mathrm{mod}}$ is $\mathscr{O}_{\Delta}$-locally free.

When the metric is moderate, $J(\beta)$ is finite for every $\beta \in \mathbb{R}$ and we have

$$\mathcal{V}^{\beta}_{\mathrm{mod}}/t\mathcal{V}^{\beta}_{\mathrm{mod}} = \bigoplus_{\beta' \in J(\beta)} \mathrm{gr}^{\beta'}(\mathcal{V}_{\mathrm{mod}}).$$

5.2.c. The theorems of Schmid. Let us consider a variation of polarized $\mathbb{C}$-Hodge structure (H, Q) of weight w on the punctured disc Δ^*. We set $H = (\mathcal{H}, D, F'^{\bullet}\mathcal{H}, F''^{\bullet}\mathcal{H})$. We thus have a positive definite Hermitian metric h on $\mathcal{H}$. On the other hand, we set $\mathcal{V} = \operatorname{Ker} D''$, on which the filtration $F'^{\bullet}\mathcal{H}$ induces a filtration $F^{\bullet}\mathcal{V}$ by holomorphic sub-bundles.

Theorem 5.2.6 **(Schmid).** *The metric* h *on* $\mathcal{H}$ *is moderate and the meromorphic extension* $\mathcal{V}_{\mathrm{mod}}$ *of* $\mathcal{V}$ *with respect to the metric* h *is equal to the canonical Deligne meromorphic extension* $\mathcal{V}_*$ *of* $(\mathcal{V}, \nabla)$.

Remark 5.2.7. In particular, the connection on the meromorphic extension $\mathcal{V}_{\mathrm{mod}}$ must have a regular singularity at the origin.

Example 5.2.8 **(The unitary case).** Let us consider the simple case where the connection is compatible with the Hermitian metric h. This corresponds to a variation of Hodge structure of pure type $(0, 0)$. Then the norm of any horizontal section of $\mathcal{V}$ is constant, hence bounded. By definition of the Deligne meromorphic extension, the norm of any section of $\mathcal{V}_*$ has thus moderate growth. Hence $\mathcal{V}_* \subset \mathcal{V}_{\mathrm{mod}}$.

In fact, both extensions are then equal, and therefore the metric is moderate, as asserted in the general case by Schmid's theorem. Indeed, given any section v of $\mathcal{V}$, we express it on a unitary frame of multivalued horizontal sections, and the norm of the section has moderate growth if and only if the coefficients are multivalued functions with moderate growth in any bounded angular sector. Similarly, we can express an $\mathscr{O}_{\Delta}[t^{-1}]$-basis of $\mathcal{V}_*$ on this unitary frame, and the coefficients have moderate growth. Expressing now v in the chosen $\mathscr{O}_{\Delta}[t^{-1}]$-basis of $\mathcal{V}_*$, we find univalued coefficients with moderate growth, that is, meromorphic functions. In other words, $\mathcal{V}_{\mathrm{mod}} \subset \mathcal{V}_*$.

From now on, we will not distinguish between $\mathcal{V}_{\mathrm{mod}}$ and $\mathcal{V}_*$. One can go further, and analyze the parabolic filtration. But first, we need a result, due to Borel (see [**Sch73**, Lem. 4.5]), which asserts:

Lemma 5.2.9. *For such a variation, the eigenvalues of the monodromy have absolute value equal to one.*

With such a result, we can define the lattices $\mathcal{V}_*^\beta$ for $\beta \in \mathbb{R}$. The next result is:

Theorem 5.2.10 **(Schmid).** *The parabolic filtration $\mathcal{V}_{\mathrm{mod}}^\bullet$ on $\mathcal{V}_*$ induced by the metric* h *is equal to the filtration $\mathcal{V}_*^\bullet$.*

Exercise 5.2.11. Prove the result in the unitary case of Example 5.2.8.

Remark 5.2.12. This result justifies the need of considering the filtration $\mathcal{V}_*^\bullet$ indexed by $\mathbb{R}$ and the graded spaces (5.2.3).

This result characterizes sections of $\mathcal{V}_*^\beta$ in terms of growth of their norm with respect to real powers of $|t|$. In order to analyze the L^2 behaviour of the norm, we will need to refine this result by using a logarithmic scale. Recall that we can lift the monodromy filtration $\mathrm{M}_\bullet \mathrm{gr}^\beta \mathcal{V}_*$ to $\mathrm{M}_\bullet \mathcal{V}_*^\beta$.

Theorem 5.2.13 **(Schmid).** *A germ of section of $\mathrm{M}_\ell \mathcal{V}_*^\beta$ has a non-zero image in $\mathrm{gr}_\ell^{\mathrm{M}} \mathrm{gr}^\beta \mathcal{V}_*$ if and only if its norm has the same order of growth as $|t|^\beta \mathrm{L}(t)^{\ell/2}$.*

Remark 5.2.14. In Section 5.2.a, when extending the vector bundle $\mathcal{V}$ with holomorphic connection ∇ from Δ^* to Δ, we have chosen Deligne's meromorphic extension, that is, we have chosen the (unique) meromorphic extension on which the extended connection is meromorphic and has *regular singularities*. Such a choice, while being canonical and, in some sense, as simple as possible, was not the only possible one. We could have chosen other kinds of meromorphic extensions, on which the extended meromorphic connection has irregular singularities. A posteriori, when considering variations of *polarized* Hodge structures, the theorems of Schmid strongly justify the previous choice.

5.3. The de Rham complexes

5.3.a. The meromorphic de Rham complexes. Let $(\mathcal{V}, \nabla)$ be any holomorphic bundle with connection. Recall that, on Δ^*, the holomorphic de Rham complex $\mathrm{DR}(\mathcal{V}, \nabla)$ is the complex

$$0 \longrightarrow \mathcal{V} \xrightarrow{\nabla} \Omega^1_{\Delta^*} \otimes \mathcal{V} \longrightarrow 0,$$

whose cohomology is nonzero only in degree zero, with $\mathscr{H}^0 \mathrm{DR}(\mathcal{V}, \nabla) = \mathcal{H}^\nabla := \mathrm{Ker}\, \nabla$.

Let us now consider the meromorphic de Rham complex $\mathrm{DR}(\mathcal{V}_*, \nabla)$, defined as the complex

$$0 \longrightarrow \mathcal{V}_* \xrightarrow{\nabla} \Omega^1_{\Delta} \otimes \mathcal{V}_* \longrightarrow 0.$$

Its restriction to Δ^* coincides with $\mathrm{DR}(\mathcal{V},\nabla)$, hence has nonzero cohomology in degree zero only. In other words, $\mathscr{H}^1\,\mathrm{DR}(\mathcal{V}_*,\nabla)$ is a skyscraper sheaf supported at the origin, and $\mathscr{H}^0\,\mathrm{DR}(\mathcal{V}_*,\nabla)$ is some sheaf extension (across the origin) of the locally constant sheaf $\mathcal{V}^\nabla$. We will determine these sheaves.

One can filter the de Rham complex, so that each term of the filtration is a complex whose terms are free $\mathscr{O}_\Delta$-modules of finite rank. For every β, we set

$$V^\beta\,\mathrm{DR}(\mathcal{V}_*,\nabla)=\{0\longrightarrow \mathcal{V}_*^\beta \xrightarrow{\ \nabla\ } \Omega^1_\Delta\otimes\mathcal{V}_*^{\beta-1}\longrightarrow 0\}.$$

Since the action of t is invertible on $\mathcal{V}_*$, the latter complex is quasi-isomorphic to the complex

$$V^\beta\,\mathrm{DR}(\mathcal{V}_*,\nabla)=\{0\longrightarrow \mathcal{V}_*^\beta \xrightarrow{\ t\nabla\ } \Omega^1_\Delta\otimes\mathcal{V}_*^{\beta}\longrightarrow 0\}.$$

***Lemma 5.3.1* (The de Rham complex of the canonical meromorphic extension)**

The inclusion of complexes $V^\beta\,\mathrm{DR}(\mathcal{V}_*,\nabla)\hookrightarrow \mathrm{DR}(\mathcal{V}_*,\nabla)$ *is a quasi-isomorphism provided* $\beta\leqslant 0$. *Moreover, the germs at the origin of these complexes can be computed as the complex of finite dimensional vector spaces*

$$0\longrightarrow \mathrm{gr}^0\mathcal{V}_*\xrightarrow{\ t\partial_t\ }\mathrm{gr}^0\mathcal{V}_*\longrightarrow 0.$$

As a consequence, the natural morphism (in the derived category)

$$\mathrm{DR}(\mathcal{V}_*,\nabla)\longrightarrow \boldsymbol{R}j_*j^{-1}\,\mathrm{DR}(\mathcal{V}_*,\nabla)=\boldsymbol{R}j_*\,\mathrm{DR}(\mathcal{V},\nabla)\xleftarrow{\ \sim\ }\boldsymbol{R}j_*\mathcal{V}^\nabla$$

is an isomorphism.

Proof. For the first statement, we notice that it is enough to check that for every $\beta\leqslant 0$ and any $\gamma<\beta$, the inclusion of complexes $V^\beta\,\mathrm{DR}(\mathcal{V}_*,\nabla)\hookrightarrow V^\gamma\,\mathrm{DR}(\mathcal{V}_*,\nabla)$ is a quasi-isomorphism. This amounts to showing that the quotient complex

$$0\longrightarrow \mathcal{V}_*^\gamma/\mathcal{V}_*^\beta\xrightarrow{\ \partial_t\ }\mathcal{V}_*^{\gamma-1}/\mathcal{V}_*^{\beta-1}\longrightarrow 0$$

is quasi-isomorphic to zero for such pairs (β,γ), and an easy inductive argument reduces to proving that, for every $\gamma<0$, the complex

$$0\longrightarrow \mathrm{gr}^\gamma\mathcal{V}_*\xrightarrow{\ \partial_t\ }\mathrm{gr}^{\gamma-1}\mathcal{V}_*\longrightarrow 0$$

is quasi-isomorphic to zero. The result is now easy since $t\partial_t-\gamma$ is nilpotent on $\mathrm{gr}^\gamma\mathcal{V}_*$.

For the second statement, we are reduced to proving that the germ at the origin of the complex

$$0\longrightarrow \mathcal{V}_*^{>0}\xrightarrow{\ t\partial_t\ }\mathcal{V}_*^{>0}\longrightarrow 0$$

is quasi-isomorphic to zero. [1]

Arguing as in Exercise 5.2.2, one can assume that $\mathcal{V}_*$ has rank one, and has a basis v_γ ($\gamma\in[0,1)$) such that $t\nabla_{\partial_t}v_\gamma=\gamma\cdot v_\gamma$.

1. This is obviously *not true* away from the origin.

(1) If $\gamma \neq 0$, then $\mathcal{V}_*^{>0} = \mathcal{V}_*^0 = \mathscr{O}_\Delta v_\gamma$ and, setting $\mathscr{O} = \mathscr{O}_{\Delta,0}$, the result follows from the property that $(t\partial_t + \gamma) : \mathscr{O} \to \mathscr{O}$ is an isomorphism (easily checked on series expansions).

(2) If $\gamma = 0$, then $\mathcal{V}_*^{>0} = t\mathcal{V}_*^0 = t\mathscr{O}_\Delta v_0$, and the result follows from the property that $(t\partial_t + 1) : \mathscr{O} \to \mathscr{O}$ is an isomorphism, proved as above.

For the last statement, we first note that the morphism is functorial in $(\mathcal{V}, \nabla)$. We can therefore reduce to the case of rank one by the argument of Exercise 5.2.2. If $\gamma \neq 0$, the isomorphism is obvious since both complexes are quasi-isomorphic to zero. If $\gamma = 0$, the isomorphism is checked in a straightforward way. $\square$

We now compute the de Rham complex of the middle extension $(\mathcal{V}_{\mathrm{mid}}, \nabla)$. Recall (Exercise 5.2.4(6)) that $\mathcal{V}_{\mathrm{mid}}$ is the $\mathscr{O}_\Delta$-submodule of $\mathcal{V}_*$ which is ∇_{∂_t}-generated by $\mathcal{V}_*^{>-1}$. We then set

$$\mathcal{V}_{\mathrm{mid}}^\beta = \begin{cases} \mathcal{V}_*^\beta & \text{if } \beta > -1, \\ \sum_{p=0}^k (\nabla_{\partial_t})^p \mathcal{V}_*^\gamma & \text{if } k \geqslant 0,\ \gamma \in (-1, 0],\ \text{and } \beta = \gamma - k, \end{cases}$$

and define similarly $\mathcal{V}_{\mathrm{mid}}^{>\beta}$. Then $\mathcal{V}_{\mathrm{mid}}^{-1} = \partial_t \mathcal{V}_*^0 + \mathcal{V}_*^{>-1}$ and $t : \mathcal{V}_{\mathrm{mid}}^{-1} \to t\mathcal{V}_{\mathrm{mid}}^{-1}$ is an isomorphism (i.e., $t : \mathcal{V}_{\mathrm{mid}}^{-1} \to \mathcal{V}_{\mathrm{mid}}^0$ is injective).

Exercise 5.3.2. Show the following properties.

(1) $\mathcal{V}_{\mathrm{mid}}^\beta$ is an $\mathscr{O}_\Delta$-coherent module, which is free of rank equal to $\operatorname{rk} \mathcal{V}$, since, being included in $\mathcal{V}_*$, it is torsion-free.

(2) $\mathcal{V}_{\mathrm{mid}}^\beta = \mathcal{V}_{\mathrm{mid}} \cap \mathcal{V}_*^\beta$. [*Hint*: use that $\partial_t^k : \mathrm{gr}^\gamma \mathcal{V}_* \to \mathrm{gr}^{\gamma-k} \mathcal{V}_*$ is injective for every $\gamma \leqslant -1$ and $k \geqslant 0$.]

(3) For $\beta < 0$, $\partial_t : \mathrm{gr}^\beta \mathcal{V}_{\mathrm{mid}} \to \mathrm{gr}^{\beta-1} \mathcal{V}_{\mathrm{mid}}$ is bijective. [*Hint*: for the injectivity, use (2) to show that $\mathrm{gr}^\beta \mathcal{V}_{\mathrm{mid}} \subset \mathrm{gr}^\beta \mathcal{V}_*$.]

(4) $\mathrm{gr}^{-1} \mathcal{V}_{\mathrm{mid}} \subset \mathrm{gr}^{-1} \mathcal{V}_* \simeq \mathrm{gr}^0 \mathcal{V}_*$ is identified with the image of $t\partial_t : \mathrm{gr}^0 \mathcal{V}_* \to \mathrm{gr}^0 \mathcal{V}_*$.

***Definition 5.3.3* (The morphisms can and var).** We define $\mathrm{can} : \mathrm{gr}^0 \mathcal{V}_{\mathrm{mid}} \to \mathrm{gr}^{-1} \mathcal{V}_{\mathrm{mid}}$ as the homomorphism induced by $-\partial_t$ and $\mathrm{var} : \mathrm{gr}^{-1} \mathcal{V}_{\mathrm{mid}} \to \mathrm{gr}^0 \mathcal{V}_{\mathrm{mid}}$ as that induced by $2\pi\mathrm{i}\, t$, so that $\mathrm{var} \circ \mathrm{can} = \mathrm{N} : \mathrm{gr}^0 \mathcal{V}_{\mathrm{mid}} \to \mathrm{gr}^0 \mathcal{V}_{\mathrm{mid}}$ and $\mathrm{can} \circ \mathrm{var} = \mathrm{N} : \mathrm{gr}^{-1} \mathcal{V}_{\mathrm{mid}} \to \mathrm{gr}^{-1} \mathcal{V}_{\mathrm{mid}}$. By the definition of $\mathcal{V}_{\mathrm{mid}}$, can is onto and var is injective.

The complex $\mathrm{DR}(\mathcal{V}_{\mathrm{mid}}, \nabla)$ is similarly filtered by the subcomplexes $V^\beta \mathrm{DR}(\mathcal{V}_{\mathrm{mid}}, \nabla)$ whose terms are thus $\mathscr{O}_\Delta$-free of finite rank.

***Lemma 5.3.4* (The de Rham complex of the middle extension)**

The inclusion of complexes $V^\beta \mathrm{DR}(\mathcal{V}_{\mathrm{mid}}, \nabla) \hookrightarrow \mathrm{DR}(\mathcal{V}_{\mathrm{mid}}, \nabla)$ *is a quasi-isomorphism provided* $\beta \leqslant 0$. *Moreover, the germs at the origin of these complexes can be computed as the complex of finite dimensional vector spaces*

$$0 \longrightarrow \mathrm{gr}^0 \mathcal{V}_{\mathrm{mid}} \xrightarrow{\partial_t} \mathrm{gr}^{-1} \mathcal{V}_{\mathrm{mid}} \longrightarrow 0.$$

As a consequence, $\mathscr{H}^1\,\mathrm{DR}(\mathcal{V}_{\mathrm{mid}},\nabla)=0$ *and the natural morphism*

$$\mathscr{H}^0\,\mathrm{DR}(\mathcal{V}_{\mathrm{mid}},\nabla)\longrightarrow j_*\mathcal{V}^\nabla$$

is an isomorphism.

Proof. For the first statement, we argue as in Lemma 5.3.1, together with Exercise 5.3.2(3). The second statement is obtained similarly by using Exercise 5.3.2(4). The last statement follows then from that of Lemma 5.3.1. ☐

In particular,

$$(5.3.5)\qquad V^0\,\mathrm{DR}(\mathcal{V}_{\mathrm{mid}},\nabla):=\{0\to\mathcal{V}^0_*\xrightarrow{t\nabla_{\partial_t}}\Omega^1_\Delta\otimes(t\partial_t\mathcal{V}^0_*+\mathcal{V}^{>0}_*)\to 0\}\\ \xrightarrow{\sim}\mathrm{DR}(\mathcal{V}_{\mathrm{mid}},\nabla).$$

We can refine the presentation (5.3.5) by using the lifted monodromy filtration $\mathrm{M}_\bullet\mathcal{V}^0_*$. Indeed, the finite dimensional vector space $\mathrm{gr}^0\mathcal{V}_*$ is equipped with the nilpotent endomorphism induced by $\mathrm{N}=-2\pi\mathrm{i}\,t\partial_t$, hence is endowed with the corresponding monodromy filtration $\mathrm{M}_\bullet\mathrm{gr}^0\mathcal{V}_*$ (see Lemma 3.1.1). We then define the lifted monodromy filtration $\mathrm{M}_\ell\mathcal{V}^0_*$ as the pullback of $\mathrm{M}_\ell\mathrm{gr}^0\mathcal{V}_*$ by the projection $\mathcal{V}^0_*\to\mathrm{gr}^0\mathcal{V}_*$.

Lemma 5.3.6. *The complex* $\mathrm{DR}(\mathcal{V}_{\mathrm{mid}},\nabla)$ *is quasi-isomorphic to*

$$\{0\longrightarrow\mathrm{M}_0\mathcal{V}^0_*\xrightarrow{t\nabla_{\partial_t}}\Omega^1_\Delta\otimes\mathrm{M}_{-2}\mathcal{V}^0_*\longrightarrow 0\}.$$

Proof. Clearly, the complex in the lemma is a subcomplex of (5.3.5). Let us consider the quotient complex. This is

$$(5.3.7)\qquad 0\longrightarrow(\mathrm{gr}^0\mathcal{V}_*/\mathrm{M}_0\mathrm{gr}^0\mathcal{V}_*)\xrightarrow{t\partial_t}(\mathrm{image}\,t\partial_t/\mathrm{M}_{-2}\mathrm{gr}^0\mathcal{V}_*)\longrightarrow 0.$$

Applying Lemma 3.1.11, we find that this complex is quasi-isomorphic to 0 (i.e., the middle morphism is an isomorphism). ☐

5.3.b. The local L^2 condition. The Hodge-Zucker theorem 5.1.1 relies on the L^2 computation of the hypercohomology of a de Rham complex, since this L^2 approach naturally furnishes a Hermitian form on the hypercohomology spaces. In order to analyze the global L^2 condition on a Riemann surface, it is convenient to introduce it in a local way, in the form of an L^2 de Rham complex. We will find in Theorem 5.3.10 the justification for focusing on the de Rham complex of the middle extension.

Hermitian bundle and volume form. Assume that the holomorphic vector bundle $\mathcal{V}$ is endowed with a metric h (equivalently, the C^∞ bundle $\mathcal{H}=\mathscr{C}^\infty_X\otimes_{\mathscr{O}_X}\mathcal{V}$ is endowed with such a metric). If we fix a metric on the punctured disc, with volume element vol, we can define the L^2-norm of a section v of $\mathcal{V}$ on an open set $U\subset\Delta^*$ by the formula

$$\|v\|_2^2=\int_U\mathrm{h}(v,\overline{v})\,\mathrm{d}\,\mathrm{vol}\,.$$

In order to be able to apply the techniques of Section 4.2.c, we choose a metric on Δ_t^* which is complete in the neighbourhood of the puncture. We will assume that, near the puncture, it takes the form

$$\text{(5.3.8)} \quad \mathrm{d\,vol} = \frac{dx^2 + dy^2}{|t|^2 \mathrm{L}(t)^2}, \quad \text{with } x = \operatorname{Re} t,\ y = \operatorname{Im} t,\ \mathrm{L}(t) := \left|\log |t|^2\right| = -\log t\bar{t}.$$

Let us be more explicit concerning the Poincaré metric. Working in polar coordinates $t = re^{i\theta}$ and volume element $\mathrm{d}\theta\, \mathrm{d}r/r$, we find a characterization of the L^2 behaviour of forms near the puncture:

(0) $f \in L^2(\mathrm{d\,vol}) \Leftrightarrow |\log r|^{-1} f \in L^2(\mathrm{d}\theta\, \mathrm{d}r/r)$;

(1) $\omega = f\, \mathrm{d}r/r + g\, \mathrm{d}\theta \in L^2(\mathrm{d\,vol}) \Leftrightarrow f$ and $g \in L^2(\mathrm{d}\theta\, \mathrm{d}r/r)$;

(2) $\eta = h\, \mathrm{d}\theta\, \mathrm{d}r/r \in L^2(\mathrm{d\,vol}) \Leftrightarrow |\log r|\, h \in L^2(\mathrm{d}\theta\, \mathrm{d}r/r)$.

On the other hand, for every integer ℓ, we have $|\log r|^{\ell/2} \in L^2(\mathrm{d}\theta\, \mathrm{d}r/r)$ if and only if $\ell \leqslant -2$.

The holomorphic L^2 de Rham complex. We will consider the *holomorphic L^2 de Rham complex*

$$\mathrm{DR}(\mathcal{V}_*, \nabla)_{(2)} = \{0 \to \mathcal{V}_{*(2)} \xrightarrow{\nabla} (\Omega^1_\Delta \otimes \mathcal{V}_*)_{(2)} \to 0\},$$

which is the subcomplex of the meromorphic de Rham complex $\mathrm{DR}(\mathcal{V}_*, \nabla)$ defined in the following way:

- $(\Omega^1_\Delta \otimes \mathcal{V}_*)_{(2)}$ is the subsheaf of $\Omega^1_\Delta \otimes \mathcal{V}_*$ consisting of sections which are L^2 (with respect to the metric h on $\mathcal{V}_*$ and the volume $\mathrm{d\,vol}$ on Δ^*),
- $\mathcal{V}_{*(2)}$ is the subsheaf of $\mathcal{V}_*$ consisting of sections v which are L^2, and such that ∇v belongs to $(\Omega^1_\Delta \otimes \mathcal{V}_*)_{(2)}$ defined above.

Let us note that, by the very definition, we get a complex.

Exercise 5.3.9.

(1) Let $r_0 \in (0, 1)$, let $\beta \in \mathbb{R}$ and $\ell \in \mathbb{Z}$. Show that the integral

$$\int_0^{r_0} r^{2\beta+1} \mathrm{L}(r)^\ell \mathrm{d}r$$

is finite iff $\beta > -1$ or $\beta = -1$ and $\ell \leqslant -2$ (recall that $\mathrm{L}(r) := 2\,|\log r| = -2 \log r$).

(2) Deduce from Schmid's theorem 5.2.13 and the characterization of $L^2(\mathrm{d\,vol})$ given above for 1-forms that, when $(\mathcal{V}, \nabla, \mathrm{h})$ underlies a polarized variation of $\mathbb{C}$-Hodge structure, $(\Omega^1_\Delta \otimes \mathcal{V}_*)_{(2)} = dt \otimes \mathrm{M}_{-2}\mathcal{V}_*^{-1}$.

(3) Similarly, show that the holomorphic sections of $\mathcal{V}$ which are L^2 near the origin are the sections of $\mathrm{M}_0\mathcal{V}_*^0$.

(4) Conclude that (in the Hodge case) $\mathcal{V}_{*(2)} = \mathrm{M}_0\mathcal{V}_*^0$ (use that $t\mathrm{M}_{-2}\mathcal{V}_*^{-1} = \mathrm{M}_{-2}\mathcal{V}_*^0$ and that $t\partial_t(\mathrm{M}_0\mathcal{V}_*^0) \subset \mathrm{M}_{-2}\mathcal{V}_*^0$).

According to Lemma 5.3.6, we get

***Theorem 5.3.10* (Zucker).** *If $(\mathcal{V}, \nabla, \mathrm{h})$ underlies a polarized variation of $\mathbb{C}$-Hodge structure, we have $(\mathrm{DR}\,\mathcal{V}_*)_{(2)} \simeq \mathrm{DR}\,\mathcal{V}_{\mathrm{mid}} = j_*\mathcal{V}^\nabla$.*

This theorem is the first step toward a L^2 computation of $j_*\mathcal{V}^\nabla$.

The L^2 de Rham complex. We now work with the associated C^∞ bundle $\mathcal{H} = \mathscr{C}^\infty_{\Delta^*} \otimes_{\mathscr{O}_{\Delta^*}} \mathcal{V}$. It is equipped with a flat C^∞ connection $D = D' + D''$, with $D'' = d'' \otimes \mathrm{Id}$ and D' induced by ∇. We can similarly define the L^2 de Rham complex

$$0 \longrightarrow \mathscr{L}^0_{(2)}(\mathcal{H}) \xrightarrow{\ D\ } \mathscr{L}^1_{(2)}(\mathcal{H}) \xrightarrow{\ D\ } \mathscr{L}^2_{(2)}(\mathcal{H}) \longrightarrow 0,$$

where the upper index refers to the degree of forms. One should give a precise definition of each term. Let us only say that we consider sections of $\mathcal{H}$ having as coefficients forms of degree k which are L^1_{loc} on Δ^* and the norm of these sections should be locally L^2 on the disc Δ. Moreover, we have to ensure that the differential of theses sections, in the sense of currents (i.e., in the weak sense) are also L^2, in order to get a complex.

Theorem 5.3.11 **(L^2 Poincaré lemma, Zucker).** *If $(\mathcal{V}, \nabla, \mathrm{h})$ underlies a polarized variation of $\mathbb{C}$-Hodge structure, the natural inclusion of complexes $(\mathrm{DR}\,\mathcal{V}_*)_{(2)} \hookrightarrow \mathscr{L}^\bullet_{(2)}(\mathcal{H}, D)$ is a quasi-isomorphism.*

Indication for the proof of Theorem 5.3.11. The following two statements have to be shown:

(1) The L^2 complex $\mathscr{L}^\bullet_{(2)}(\mathcal{H}, D)$ has nonzero cohomology in degree zero at most,
(2) The inclusion of the $\mathscr{H}^0$ of the complexes is an isomorphism.

Away from the origin, the second statement follows from Dolbeault-Grothendieck's lemma, while the first one is obtained by solving the $\overline{\partial}$ Neuman's problem.[1]

[1] vérifier et donner une référence

At the origin of the disc, the main observation is the following lemma.

Lemma 5.3.12 **(see [Zuc79, Prop. 6.4]).** *Let $\mathcal{L}$ be a holomorphic line bundle on Δ^* (equipped with the complete metric* (5.3.8)*) with Hermitian metric* h *and having a frame v such that $\|v\|_h \sim \mathrm{L}(t)^k$ for $k \in \mathbb{Z}$. Then, if $k \neq 1$, any germ $\eta = f d\overline{t} \otimes v$ of section of $\mathscr{L}^{(0,1)}_{(2)}(\mathcal{L}, \mathrm{h})$ at the origin is equal to $\overline{\partial}\psi \otimes v$ for some local section $\psi \otimes v$ of $\mathscr{L}^0_{(2)}(\mathcal{L}, \mathrm{h})$.*

This is a $\overline{\partial}$ equation with logarithmically twisted L^2 conditions. It is proved using the decomposition in Fourier series and Hardy inequalities.

Once this lemma is proved, the proof of Theorem 5.3.11 follows (this is not completely straightforward) from Schmid's theorem 5.2.13. □

Applying the hypercohomology functor to Theorems 5.3.11 and 5.3.10, we obtain:

Theorem 5.3.13 **(Zucker).** *Let $j : X^* \hookrightarrow X$ be the inclusion of the complement of a finite set in a compact Riemann surface X. If $(\mathcal{V}, \nabla, \mathrm{h})$ underlies a polarized variation of $\mathbb{C}$-Hodge structure on X^*, the cohomology $H^\bullet(X, j_*\mathcal{V}^\nabla)$ is equal to the L^2 cohomology of the C^∞-bundle with flat connection $(\mathcal{H}, D)$ associated with the holomorphic bundle $(\mathcal{V}, \nabla)$, the L^2 condition being taken with respect to the Hodge metric* h *on $\mathcal{H}$ and a complete metric on X^*, locally equivalent near each puncture to the Poincaré metric.*

5.4. The Hodge filtration

Our aim in this section is to define a Hodge filtration on the cohomology $H^\bullet(X, j_*\mathcal{V}^\nabla)$, and to prove that it endows this cohomology with a polarized Hodge structure. The method will be of a local nature, in a way similar to the computation of the L^2 cohomology.

5.4.a. The holomorphic Hodge filtration. We wish to extend the filtration $F^\bullet\mathcal{V}$ as a filtration $F^\bullet\mathcal{V}_{\mathrm{mid}}$ by sub-bundles satisfying the Griffiths transversality property with respect to the meromorphic connection ∇. A first natural choice would be to set

$$F^p\mathcal{V}_{\mathrm{mid}} := j_*F^p\mathcal{H} \cap \mathcal{V}_{\mathrm{mid}},$$

where $j : \Delta^* \hookrightarrow \Delta$ denotes the inclusion. This choice can lead to a non-coherent $\mathscr{O}_\Delta$-module: for example, if $p \ll 0$, we have $F^p\mathcal{V} = \mathcal{V}$ and we would get $F^p\mathcal{V}_{\mathrm{mid}} = \mathcal{V}_{\mathrm{mid}}$, which is not $\mathscr{O}_\Delta$-coherent. Being more clever, one first defines

$$F^p\mathcal{V}_{\mathrm{mid}}^{>-1} := j_*F^p\mathcal{H} \cap \mathcal{V}_{\mathrm{mid}}^{>-1}. \tag{5.4.1}$$

If this sheaf is $\mathscr{O}_\Delta$-coherent, it will then be natural to define, for every p, in order to obtain Griffiths transversality,

$$F^p\mathcal{V}_{\mathrm{mid}} = \sum_{j\geqslant 0} (\nabla_{\partial_t})^j F^{p+j}\mathcal{V}_{\mathrm{mid}}^{>-1}. \tag{5.4.2}$$

Indeed, with this definition, the relation $\nabla_{\partial_t} F^p\mathcal{V}_{\mathrm{mid}} \subset F^{p-1}\mathcal{V}_{\mathrm{mid}}$ is clearly satisfied.

***Exercise 5.4.3* (Extension of the filtration).** Show that, with Definitions given by (5.4.1) and (5.4.2),

(1) for every $\beta > -1$, we have $F^p\mathcal{V}_{\mathrm{mid}} \cap \mathcal{V}_{\mathrm{mid}}^{\beta} = j_*F^p\mathcal{H} \cap \mathcal{V}_{\mathrm{mid}}^{\beta}$ and for every $\beta \geqslant -1$, $F^p\mathcal{V}_{\mathrm{mid}} \cap \mathcal{V}_{\mathrm{mid}}^{>\beta} = j_*F^p\mathcal{H} \cap \mathcal{V}_{\mathrm{mid}}^{>\beta}$;

(2) if $F^p\mathcal{V}_{\mathrm{mid}}^{>-1}$ is $\mathscr{O}_\Delta$-coherent, it is $\mathscr{O}_\Delta$-locally free, hence free (use that $F^p\mathcal{V}_{\mathrm{mid}}^{>-1} \subset \mathcal{V}_{\mathrm{mid}}^{>-1}$);

(3) $F^p\mathcal{V}_{\mathrm{mid}}$ is an $\mathscr{O}_\Delta$-module;

(4) under the assumption in (2), $F^p\mathcal{V}_{\mathrm{mid}}$ is $\mathscr{O}_\Delta$-coherent, and thus $\mathscr{O}_\Delta$-free, and each $F^p\mathcal{V}_{\mathrm{mid}} \cap \mathcal{V}_{\mathrm{mid}}^{\beta}$ is so;

(5) $\bigcup_p F^p\mathcal{V}_{\mathrm{mid}} = \mathcal{V}_{\mathrm{mid}}$.

[*Hint*: Recall that there exists an integer $p_0 \gg 0$ such that $F^{p_0}\mathcal{V} = 0$ and $F^{-p_0}\mathcal{V} = \mathcal{V}$.]

***Exercise 5.4.4* (Relations between $F^\bullet\mathcal{V}_{\mathrm{mid}}$ and $\mathcal{V}_{\mathrm{mid}}^\bullet$).** We now consider the filtration $\mathcal{V}_{\mathrm{mid}}^\bullet$ indexed by $\mathbb{R}$ (see Section 6.2.c).

(1) Assume that each $F^p\mathcal{V}_{\mathrm{mid}}$ defined by (5.4.2) is $\mathscr{O}_\Delta$-coherent. Prove that the filtration $F^\bullet\mathcal{V}_{\mathrm{mid}}$ satisfies the following properties:

(a) for every $\beta > -1$, $t(F^p\mathcal{V}_{\mathrm{mid}} \cap \mathcal{V}_{\mathrm{mid}}^{\beta}) = F^p\mathcal{V}_{\mathrm{mid}} \cap \mathcal{V}_{\mathrm{mid}}^{\beta+1}$;

(b) for every $\beta < 0$, $\partial_t F^p\mathrm{gr}^{\beta}(\mathcal{V}_{\mathrm{mid}}) = F^{p-1}\mathrm{gr}^{\beta-1}(\mathcal{V}_{\mathrm{mid}})$.

[The inclusions $\subset$ are easy; the remarkable property is the existence of inclusions $\supset$; we will call the conjunction of (1a) and (1b) the property of *strict* $\mathbb{R}$-*specializability*, see Section 7.3.c.] Show moreover that

(c) Property (1b) also holds for $\beta = 0$.

(2) Conversely, show that a filtration $F^\bullet \mathcal{V}_{\mathrm{mid}}$ by coherent $\mathscr{O}_\Delta$-submodules which satisfies (5.4.1), (1b) and (1c) also satisfies (5.4.2).

We will mainly consider $\beta \in [-1, 0]$. The properties of the filtration $F^\bullet \mathcal{V}_{\mathrm{mid}}$ are thus governed by those of the filtration $F^\bullet \mathcal{V}_{\mathrm{mid}}^{>-1}$, and since $\mathcal{V}_{\mathrm{mid}}^{>-1} = \mathcal{V}_*^{>-1}$, they do not need the notion of middle extension to be considered. We also notice that the nilpotent endomorphism N shifts by -1 the Hodge filtration on $\mathrm{gr}^\beta \mathcal{V}_{\mathrm{mid}}$, since so does $-t\partial_t$.

5.4.b. The limiting Hodge-Lefschetz structure. The main result, due to Schmid in the case where the monodromy is unipotent in **[Sch73]**, asserts that the limit Lefschetz structure (at $t = 0$) of a polarizable variation of $\mathbb{C}$-Hodge structure on Δ^* is a polarizable Hodge-Lefschetz structure, called the *limit Hodge-Lefschetz structure*. We will first construct the limit Lefschetz structure. We will only consider polarizable variations, so we can fix a polarization and use the simplified setting as in Proposition 2.4.37.

Let (H, Q) be a variation of polarized $\mathbb{C}$-Hodge structure of weight w on Δ^*, that we now write as $((\mathcal{V}, \nabla, F^\bullet \mathcal{V}), \mathfrak{c})$ (see Definition 4.1.10, and 4.1.8 for $\mathfrak{c}$). For every $\beta \in (-1, 0]$, we will define the object $\mathrm{gr}^\beta(H, \mathrm{Q})$ as follows. We set

$$\mathrm{gr}^\beta H = (\mathrm{gr}^\beta \mathcal{V}_*, F^\bullet \mathrm{gr}^\beta \mathcal{V}_*),$$

which is endowed with the nilpotent endomorphism N induced by the action of $-2\pi\mathrm{i}(t\partial_t - \beta)$:

$$(\mathrm{gr}^\beta \mathcal{V}_*, F^\bullet \mathrm{gr}^\beta \mathcal{V}_*) \xrightarrow{\mathrm{N}} (\mathrm{gr}^\beta \mathcal{V}_*, F[-1]^\bullet \mathrm{gr}^\beta \mathcal{V}_*).$$

In order to obtain a candidate Hodge-Lefschetz structure, we need to define a sesquilinear pairing $\mathrm{gr}^\beta \mathfrak{c} : \mathrm{gr}^\beta \mathcal{V}_* \otimes_{\mathbb{C}} \overline{\mathrm{gr}^\beta \mathcal{V}_*} \to \mathbb{C}$ and to check its compatibility with N.

We keep the notation of Exercise 5.2.2(3), but we choose the indices in $(-1, 0]$ instead of $[0, 1)$. Let $\beta', \beta'' \in (-1, 0]$ and let $\mathfrak{c} : \mathcal{J}_{\beta',p|\Delta^*} \otimes \overline{\mathcal{J}_{\beta'',q|\Delta^*}} \to \mathscr{C}^\infty_{\Delta^*}$ be a sesquilinear pairing as in Definition 4.1.8. We denote by $\boldsymbol{v}'_{\beta',p}$ (resp. $\boldsymbol{v}''_{\beta'',q}$) the basis considered in Exercise 5.2.2. Recall also (see (5.3.8)) that we have set $\mathrm{L}(t) = -\log |t|^2 = -\log t\bar{t}$. Notice that $-t\partial_t \mathrm{L}(t)^k/k! = -\bar{t}\partial_{\bar{t}} \mathrm{L}(t)^k/k! = \mathrm{L}(t)^{k-1}/(k-1)!$.

Lemma 5.4.5. *For $i = 0, \dots, p$ and $j = 0, \dots, q$, there exist complex numbers $c_k(i, j)$ such that*

$$\mathfrak{c}(v'_{\beta',i}, \overline{v''_{\beta'',j}}) = \begin{cases} 0 & \text{if } \beta' \neq \beta'', \\ |t|^{2\beta} \sum_{k=0}^{\min(i,j)} c_k(i,j) \dfrac{\mathrm{L}(t)^k}{k!} & \text{if } \beta' = \beta'' =: \beta. \end{cases}$$

Proof. Let us first assume that $i = j = 0$. If we restrict on an open sector centered at the origin on which $t^{\beta'}$ and $t^{\beta''}$ are univalued holomorphic functions, then $\mathfrak{c}(t^{-\beta'}v'_{\beta',0}, \overline{t^{-\beta''}v''_{\beta'',0}})$ is constant since it is annihilated by ∂_t and $\partial_{\bar t}$. Therefore, $\mathfrak{c}(v'_{\beta',0}, \overline{v''_{\beta'',0}}) = c\bar t^{\beta''}t^{\beta'}$ on such a sector. But $\mathfrak{c}(v'_{\beta',0}, \overline{v''_{\beta'',0}})$ is a C^∞ function on the whole Δ^*, hence $\beta' - \beta'' \in \mathbb{Z}$ unless $\mathfrak{c}(v'_{\beta',0}, \overline{v''_{\beta'',0}}) = 0$. Since we assume $\beta', \beta'' \in (-1, 0]$, we obtain the assertion in this case.

In general, we argue similarly by using that, if $\eta \in C^\infty(\Delta^*)$ satisfies $(t\partial_t)^{i+1}\eta = (\bar t\partial_{\bar t})^{j+1}\eta = 0$, then $\eta = \sum_{k=0}^{\min(i,j)} c_k \mathrm{L}(t)^k/k!$. ∎

We conclude that any sesquilinear pairing $\mathfrak{c} : \mathcal{J}_{\beta',p|\Delta^*} \otimes \overline{\mathcal{J}_{\beta'',q|\Delta^*}} \to \mathscr{C}^\infty_{\Delta^*}$ is zero if $\beta' \neq \beta''$, and we are reduced to considering sesquilinear pairings

$$\mathfrak{c} : \mathcal{J}_{\beta,p|\Delta^*} \otimes \overline{\mathcal{J}_{\beta,q|\Delta^*}} \longrightarrow \mathscr{C}^\infty_{\Delta^*}.$$

Let us notice that, due to the explicit expression of $\mathfrak{c}$, we have

$$\mathfrak{c}(v', \overline{t\partial_t v''}) = \mathfrak{c}(t\partial_t v', \overline{v''}).$$

We still denote by $\boldsymbol{v}'_{\beta,p}$ (resp. $\boldsymbol{v}''_{\beta,q}$) the basis induced on $\mathrm{gr}^\beta \mathcal{J}'_{\beta,p} = \mathscr{O}_\Delta \boldsymbol{v}'_{\beta,p}/t\mathscr{O}_\Delta \boldsymbol{v}'_{\beta,p}$ (resp. $\mathrm{gr}^\beta \mathcal{J}''_{\beta,q}$). We define $\mathrm{gr}^\beta \mathfrak{c}$ by the formula

$$\mathrm{gr}^\beta \mathfrak{c}(v'_{\beta,i}, \overline{v''_{\beta,j}}) = c_0(i,j). \tag{5.4.6}$$

We conclude from the previous remark that $\mathrm{gr}^\beta \mathfrak{c}(v', \overline{\mathrm{N}v''}) = -\mathrm{gr}^\beta \mathfrak{c}(\mathrm{N}v', \overline{v''})$ (with N induced by $-2\pi\mathrm{i}(t\partial_t - \beta)$).

We can now define the pairing $\mathrm{gr}^\beta \mathfrak{c} : \mathrm{gr}^\beta \mathcal{V}_* \otimes_{\mathbb{C}} \overline{\mathrm{gr}^\beta \mathcal{V}_*} \to \mathbb{C}$ by choosing a decomposition (5.2.2 $*$) for $(\mathcal{V}_*, \nabla)$ and by applying (5.4.6) to each pair of terms corresponding to the same $\beta \in (-1, 0]$. This can also be obtained by a residue formula, without explicitly referring to such a decomposition and showing also the independence with respect to it.

***Exercise 5.4.7* (A residue formula for $\mathrm{gr}^\beta \mathfrak{c}$).** Let $\chi(t)$ be a C^∞ function with compact support on Δ which is $\equiv 1$ near $t = 0$. Assume that $\chi(t)$ only depends on $|t|$ (e.g. $\chi(t) = \mu(|t|^2)$ where μ is C^∞).

(1) Show that

$$s \longmapsto (s-1)\int_{\mathbb{C}} |t|^{2s}\chi(t)\ \tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t \wedge \mathrm{d}\bar t$$

is holomorphic for $\operatorname{Re} s > -1$ and extends as an entire function. Show that

$$\operatorname{Res}_{s=-1} \int_{\mathbb{C}} |t|^{2s}\chi(t)\ \tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t \wedge \mathrm{d}\bar t = 1.$$

[*Hint*: by expressing the integrand with respect to the real variables x, y with $t = x + iy$, check the sign of the left-hand side; then compute with polar coordinates up to sign.]

(2) Differentiating k times for $\operatorname{Re} s > -1$, show that

$$\int_{\mathbb{C}} |t|^{2s}\frac{\mathrm{L}(t)^k}{k!}\,\chi(t)\ \tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t \wedge \mathrm{d}\bar t = \frac{(-1)^k}{(s-1)^{k+1}} + F_k(s),$$

where $F_k(s)$ is holomorphic for $\operatorname{Re} s > -1$ and extends as an entire function. Conclude that, for $k \geqslant 1$,

$$\operatorname{Res}_{s=-1} \int_{\mathbb{C}} |t|^{2s} \frac{\mathrm{L}(t)^k}{k!}\, \chi(t)\ \tfrac{\mathrm{i}}{2\pi}\, \mathrm{d}t \wedge \mathrm{d}\bar{t} = 0.$$

(3) Show the formula

$$\operatorname{gr}^\beta \mathfrak{c}(v', \overline{v''}) = \operatorname{Res}_{s=-\beta-1} \int_{\mathbb{C}} |t|^{2s} \mathfrak{c}(v', \overline{v''})\, \chi(t)\ \tfrac{\mathrm{i}}{2\pi}\, \mathrm{d}t \wedge \mathrm{d}\bar{t}.$$

***Example 5.4.8* (A symbolic identity).** Let $\eta \in C_{\mathrm{c}}^\infty(\Delta)$ be any test function. Exercise 5.4.7(1) shows that the function

$$F(s) = \int_\Delta |t|^{2s-2}\, \eta(t)\, \mathrm{d}t \wedge \mathrm{d}\bar{t}$$

is holomorphic on the half space $\operatorname{Re} s > 0$ and extends as a meromorphic function on the s-plane with a simple pole at $s = 0$. An integration by parts gives

$$F(s) = \frac{1}{s^2} \int_\Delta |t|^{2s}\, \partial_t \partial_{\bar{t}} \eta(t)\, \mathrm{d}t \wedge \mathrm{d}\bar{t}, \tag{5.4.8 $*$}$$

[apply Stokes formula first to $\mathrm{d}(|t|^{2s}\eta(t)\mathrm{d}\bar{t}/\bar{t})$ and then to $\mathrm{d}(|t|^{2s}\partial_t\eta(t)\mathrm{d}t)$] and expanding with respect to s (taking into account that $|t|^{2s} = e^{-s\mathrm{L}(t)}$) gives the residue:

$$\operatorname{Res}_{s=0} F(s) = -\int_\Delta \mathrm{L}(t)\, \partial_t \partial_{\bar{t}} \eta(t)\, \mathrm{d}t \wedge \mathrm{d}\bar{t}.$$

Note that, by Exercise 5.4.7(1) and the residue interpretation, if χ is a cut-off function, we have

$$\int_\Delta \mathrm{L}(t)\, \partial_t \partial_{\bar{t}} \chi(t)\, \mathrm{d}t \wedge \mathrm{d}\bar{t} = 2\pi\mathrm{i}\,.$$

We are interested in rewriting the symbolic expression, where N is a nilpotent element of some $\mathbb{C}$-algebra,

$$\int_\Delta |t|^{2s-2-2N}\, \eta(t)\, \mathrm{d}t \wedge \mathrm{d}\bar{t} = \sum_{n=0}^\infty \Big(\int_\Delta \frac{\mathrm{L}(t)^n}{n!}\, |t|^{2s-2}\, \eta(t)\, \mathrm{d}t \wedge \mathrm{d}\bar{t} \Big) N^n$$

in a way that lets us analyze how it behaves near $s = 0$. Let us already note that, if

As long as $\operatorname{Re} s > 0$, differentiation under the integral sign gives

$$(-1)^n \frac{F^{(n)}(s)}{n!} = \int_\Delta \frac{\mathrm{L}(t)^n}{n!}\, |t|^{2s-2}\, \eta(t)\, \mathrm{d}t \wedge \mathrm{d}\bar{t},$$

and since $F(s)$ has a simple pole at $s = 0$, we have $\operatorname{Res}_{s=0} F^{(n)}(s) = 0$ for $n \geqslant 1$. Consequently,

$$\int_\Delta |t|^{2s-2-2N}\, \eta(t)\, \mathrm{d}t \wedge \mathrm{d}\bar{t} = \sum_{n=0}^\infty (-1)^n N^n\, \frac{F^{(n)}(s)}{n!},$$

and this function of s has residue at $s = 0$ equal to that of $F(s)$.

On the other hand, we can expand the expression (5.4.8 $*$) for $F(s)$ into a power series near $s=0$; the result is that

$$F(s)=\frac{1}{s^2}\sum_{p=0}^{\infty}(-1)^p s^p\int_\Delta \frac{\mathrm{L}(t)^p}{p!}\,\partial_t\partial_{\bar t}\eta(t)\,\mathrm{d}t\wedge\mathrm{d}\bar t.$$

After we insert this into the previous expression and simplify the result, we eventually arrive at the symbolic identity

$$\int_\Delta |t|^{2s-2-2N}\,\eta(t)\,\mathrm{d}t\wedge\mathrm{d}\bar t=\int_\Delta \frac{|t|^{2s-2N}-1}{(N-s)^2}\,\partial_t\partial_{\bar t}\,\eta(t)\,\mathrm{d}t\wedge\mathrm{d}\bar t. \tag{5.4.8 $**$}$$

It should be understood as an identity between two families of holomorphic functions – namely the coefficients at N^p on both sides – on the half space $\mathrm{Re}\,s>0$.

Lemma 5.4.9. *The pairing $\mathrm{gr}^\beta\mathfrak{c}$ induces a pairing $\mathrm{gr}^{\mathrm{M}}_\bullet\mathrm{gr}^\beta\mathcal{V}_*\otimes_{\mathbb{C}}\overline{\mathrm{gr}^{\mathrm{M}}_\bullet\mathrm{gr}^\beta\mathcal{V}_*}\to\mathbb{C}$, which is non-degenerate if and only if $\mathfrak{c}$ is non-degenerate.*

Proof. We can regard $\mathrm{gr}^\beta\mathfrak{c}$ as a morphism of Lefschetz pairs $(\mathrm{gr}^\beta\mathcal{V}_*,\mathrm{N})\to(\mathrm{gr}^\beta\mathcal{V}_*,\mathrm{N})^*$, as N is skew-adjoint with respect to $\mathfrak{c}$. It is therefore compatible with the monodromy filtrations (see Section 3.1.a). For the second assertion, we can assume that only terms $\mathcal{J}_{\beta,p}$ (with the same $\beta\in(-1,0]$) occur in the decomposition (5.2.2 $*$). Note that $\mathrm{gr}^{\mathrm{M}}\mathrm{gr}^\beta\mathfrak{c}$ is an isomorphism if and only if $\mathrm{gr}^\beta\mathfrak{c}$ is so. In order to conclude, we can now interpret Lemma 5.4.5 as giving an asymptotic expansion of $\mathfrak{c}$ when $|t|\to 0$, and (5.4.6) as taking its dominant part. We then clearly obtain that $\mathfrak{c}$ is non-degenerate near the origin if and only if $\mathrm{gr}^\beta\mathfrak{c}$ is non-degenerate. The equivalence with non-degeneracy on the whole disk follows then from Remark 4.1.9. $\square$

***Theorem 5.4.10* (Schmid [Sch73]).** *Let (H,Q) be a variation of polarized $\mathbb{C}$-Hodge structure of weight w on Δ^*. Then for every $\beta\in(-1,0]$, the data*

$$(\mathrm{gr}^\beta H,\mathrm{N},\mathrm{gr}^\beta\mathrm{Q})$$

form a polarized Hodge-Lefschetz structure centered at w.

5.4.c. The L^2 Dolbeault lemma. One of the important points in order to prove the E_1-degeneration of the Hodge-to-de Rham spectral sequence in the context of the Hodge-Zucker theorem 5.1.1 is the Dolbeault lemma, making the bridge between the holomorphic world and the L^2 world of harmonic sections. We will briefly give indications on its proof.

Recall that the Dolbeault lemma on a complex manifold X says that $H^q(X,\Omega^p_X)\simeq H^{p,q}_{\mathrm{d}''}(X)=H^q\big(\Gamma(X,\mathscr{E}^{p,\bullet}_X),\mathrm{d}''\big)$.

If we now consider a variation of polarized Hodge structure on X, as in Section 4.2.b, the complex $\mathscr{E}^\bullet_X(\mathcal{H})$ is filtered by taking into account the holomorphic degree of the form and the Hodge degree of the section. Moreover, this filtration splits as direct sum of terms $\mathscr{E}^{i,j}_X\otimes\mathcal{H}^{k,\ell}$, and each of these terms is a summand in the p, q term of the decomposition if $p = i + k$ and $q = j + \ell$. The Dolbeault lemma then says that

$$H^q\big(X, \mathrm{gr}_F^p \mathrm{DR}(\mathcal{V}, \nabla)\big) = H^q\big(\Gamma(X, \mathrm{gr}_F^p \mathscr{E}_X^\bullet(\mathcal{H}))\big).$$

Let us note that the differential in the complex $\mathrm{gr}_F^p \mathscr{E}_X^\bullet(\mathcal{H})$ is induced by $\mathcal{D}''$, as introduced in Exercise 4.2.2(4).

Let us now come back to the context of the Hodge-Zucker theorem 5.1.1. The first point to be settled is the freeness of each step of the F-filtration of $\mathcal{V}_{\mathrm{mid}}$. Recall that it is defined with (5.4.1) and (5.4.2).

Let us first consider $F^p\mathcal{V}_{*(2)} := j_*F^p\mathcal{V} \cap \mathcal{V}_{*(2)}$. According to Exercise 5.3.9(4), this is also $j_*F^p\mathcal{V} \cap \mathrm{M}_0\mathcal{V}_*^0$. If we show its coherence, then $F^p\mathcal{V}_*^0$ will be coherent as well, as $\mathcal{V}_*^0/\mathrm{M}_0\mathcal{V}_*^0$ is finite dimensional. In the same way, $F^p\mathcal{V}_*^{>-1}$ will be coherent. It will then also be locally free of rank equal to $\mathrm{rk}\, F^p\mathcal{V}$, and $F^p\mathcal{V}_{\mathrm{mid}}$ defined by (5.4.2) will be a filtration of $\mathcal{V}_{\mathrm{mid}}$ satisfying $F^p\mathcal{V}_{\mathrm{mid}} \cap \mathcal{V}_{*(2)} = F^p\mathcal{V}_{*(2)}$.

Proposition 5.4.11 **([Zuc79, Prop. 5.2]).** *The $\mathscr{O}_\Delta$-module $F^p\mathcal{V}_{*(2)}$ is coherent.*

This is shown using results of Schmid [**Sch73**]. We can thus apply the results of Exercises 5.4.3 and 5.4.4.

From Proposition 3.2.25 we obtain a nearby/vanishing Hodge-Lefschetz quiver on $(\psi_{t,1}\mathcal{V}_{\mathrm{mid}}, \phi_{t,1}\mathcal{V}_{\mathrm{mid}})$.

It is not difficult to filter the complex $\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}$ by the usual procedure as in (2.3.4) from the filtration of $\mathcal{V}_{\mathrm{mid}}$. On the other hand, according to Theorem 5.3.10, the inclusion $(\mathrm{DR}\,\mathcal{V}_*)_{(2)} \hookrightarrow \mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}$ is a quasi-isomorphism. Is it a *filtered* quasi-isomorphism?

Firstly, we have to *define* the filtration $F^\bullet(\mathrm{DR}\,\mathcal{V}_*)_{(2)}$. Using the interpretation of Exercise 5.3.9, we are reduced to defining the filtration on $\mathrm{M}_0\mathcal{V}_*^0$ and $\mathrm{M}_{-2}\mathcal{V}_*^{-1}$. The natural choice is simply to induce the filtration $F^\bullet\mathcal{V}_{\mathrm{mid}}$ on these submodules. Therefore, answering the question above amounts to answering the following ones:

(5.4.12) Is (5.3.5) a filtered quasi-isomorphism, when the terms are equipped with the induced filtration?

(5.4.13) Is the quasi-isomorphism of Lemma 5.3.6 a filtered quasi-isomorphism, when the terms are equipped with the induced filtration?

The answer to both questions is *yes*. For the first question, we have to show that, for every $\beta < 0$ and any p, the complex

$$0 \longrightarrow F^p\mathrm{gr}_V^\beta\mathcal{V}_{\mathrm{mid}} \xrightarrow{\ \partial_t\ } F^{p-1}\mathrm{gr}_V^{\beta-1}\mathcal{V}_{\mathrm{mid}} \longrightarrow 0$$

is quasi-isomorphic to zero. This is Exercise 5.4.4(1b). Using now Exercise 5.4.4(1a), we can replace the filtered complex $F^\bullet\mathcal{H}^0\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}$ with the filtered complex corresponding to (5.3.5).

For the second question, we have to prove the filtered analogue of Lemma 5.3.6. This is done by an argument of strictness: both terms in (5.3.7) are shown to be mixed Hodge structures, and the morphism between them (if one Tate-twists the right-hand term by -1) a morphism of mixed Hodge structures, hence is strictly compatible with the Hodge filtration.

Let us come back to the Dolbeault lemma. On the holomorphic side, we have $H^q(X, \mathrm{gr}_F^p \mathrm{DR}\, \mathcal{V}_{\mathrm{mid}})$, that we now can write as $H^q(X, \mathrm{gr}_F^p(\mathrm{DR}\, \mathcal{V}_{\mathrm{mid}})_{(2)})$, a form which will help us to compare with the L^2 side.

The L^2 Dolbeault complex has to be taken with respect to the differential $\mathcal{D}''$ and the L^2 condition on a section $\eta \otimes v$ concerns the derivative $\mathcal{D}''(\eta \otimes v)$.

By using Lemma 5.3.12, one gets

Theorem 5.4.14 **(L^2 Dolbeault lemma, Zucker).** *With the assumptions of Theorem* 5.3.13*, the natural morphism (induced by the inclusion of complexes)* $H^q(X, \mathrm{gr}_F^p(\mathrm{DR}\, \mathcal{V}_{\mathrm{mid}})_{(2)}) \to H^q(X, \mathrm{gr}_F^p \mathscr{L}_{(2)}(\mathcal{H}, \mathcal{D}''))$ *is an isomorphism.*

Remarks 5.4.15.

(1) As in Remark 4.2.4(4), a consequence of the Hodge-Zucker theorem 5.1.1 is that the maximal constant subsheaf of $\underline{\mathcal{H}}$ has stalk $H^0(X^*, \underline{\mathcal{H}}) = H^0(X, j_*\underline{\mathcal{H}})$, and thus underlies a constant variation of polarizable Hodge structure of weight w which is a direct summand in $\underline{\mathcal{H}}$.

(2) We could also express the Hodge-Zucker theorem 5.1.1 in terms of a polarizable graded $(-)$Hodge-Lefschetz structure of weight $w+1$, but the previous remark shows that the only interesting cohomology is $H^1(X, j_*\underline{\mathcal{H}})$. It is primitive, so the polarization on it can be expressed without referring to an ample line bundle. The positivity property of the polarization is proved exactly as in Theorem 4.2.3 in the case of compact Riemann surfaces, by replacing C^∞ sections with L^2 sections with respect to the complete metric fixed on X^*, and using the pairing (**??**). There is no need here to argue on primitivity of L^2 sections.

(3) (*Degeneration at E_1 of the Hodge-to-de Rham spectral sequence*)

One checks that the filtered complex $\boldsymbol{R}\Gamma\big(X, F^\bullet(\mathrm{DR}\, \mathcal{V}_{\mathrm{mid}})_{(2)}\big)$ is *strict*, exactly as in Remark 4.2.4(2).

5.5. Semi-simplicity

We extend in this section the results of Section 4.3 to the case of a punctured projective curve.

5.5.a. The semi-simplicity theorem. Let X be a smooth projective curve and let X^* be a Zariski open subset of X (i.e., the complement of a finite set of points). Let $H = (\mathcal{H}, F'^\bullet\mathcal{H}, F''^\bullet\mathcal{H}, D, \mathrm{Q})$ be a variation of polarized $\mathbb{C}$-Hodge structure of weight w on X^* (see Definition 4.1.3), and let $\underline{\mathcal{H}} = \mathrm{Ker}\, \nabla$ be the associated complex local system.

Theorem 5.5.1. *Under these assumptions, the complex local system* $\underline{\mathcal{H}}$ *is semi-simple.*

5.5.b. Structure of variations of polarized $\mathbb{C}$-Hodge structure

Let X, X^* be as in Section 5.5.a and let $(\mathcal{H}, F'^{\bullet}\mathcal{H}, F''^{\bullet}\mathcal{H}, D, \mathrm{Q})$ be a variation of polarized $\mathbb{C}$-Hodge structure of weight w on X^*. We now use on X^* the same notation as in Section 4.3.c.

We note that *Proposition* 4.3.7 *and Lemma* 4.3.5 *hold in this setting*. Indeed, the reference to Theorem 4.3.3 is replaced with a reference to Theorem 5.5.1, so the new argument needed only concerns the existence of a pure Hodge structure on $H^0(X^*, \mathscr{E}nd(\underline{\mathcal{H}}))$, which is provided by the Hodge-Zucker theorem 5.1.1, according to Remark 5.4.15(1).

5.6. Comments

Here come the references to the existing work which has been the source of inspiration for this chapter.