## CHAPTER 4

# VARIATIONS OF HODGE STRUCTURE ON A SMOOTH PROJECTIVE VARIETY 


#### Abstract

Summary. The notion of a variation of $\mathbb{C}$-Hodge structure on a complex manifold is the first possible generalization of a $\mathbb{C}$-Hodge structure. It naturally occurs when considering holomorphic families of smooth projective varieties. Later, we will identify this notion with the right notion of a smooth $\mathbb{C}$-Hodge module. We consider global properties of polarizable variations of $\mathbb{C}$-Hodge structure on a smooth projective variety. On the one hand, the Hodge theorem asserts that the de Rham cohomology of a polarizable variation of $\mathbb{C}$-Hodge structure on a smooth projective variety is itself a polarizable graded (-)Hodge-Lefschetz structure. On the other hand, we show that the local system underlying a a polarizable variation of $\mathbb{C}$-Hodge structure on a smooth projective variety is semi-simple, and we classify all such variations with a given underlying semi-simple local system.


### 4.1. Variations of $\mathbb{C}$-Hodge structure

The definition of a variation of $\mathbb{C}$-Hodge structure is modeled on the behaviour of the cohomology of a family of smooth projective varieties parametrized by a smooth algebraic variety, that is, a smooth projective morphism $f: Y \rightarrow X$, that we call below the "geometric setting".

Let us first motivate the definition. Let $X$ be a connected (possibly non compact) complex manifold. In such a setting, the generalization of a vector space $\mathcal{H}^{o}$ is a locally constant sheaf of vector spaces $\underline{\mathcal{H}}$ on $X$. Let us choose a universal covering $\widetilde{X} \rightarrow X$ of $X$ and let us denote by $\Pi$ its group of deck-transformations, which is isomorphic to $\pi_{1}(X, \star)$ for any choice of a base-point $\star \in X$. Let us denote by $\widetilde{\mathcal{H}}$ the space of global sections of the pullback $\underline{\widetilde{\mathcal{H}}}$ of $\underline{\mathcal{H}}$ to $\widetilde{X}$. Then, giving $\underline{\mathcal{H}}$ is equivalent to giving the monodromy representation $\Pi \rightarrow \mathrm{GL}(\widetilde{\mathcal{H}})$. However, it is known that, in the geometric setting, the Hodge decomposition in each fibre of the family does not give rise to locally constant sheaves, but to $C^{\infty}$-bundles.

In the geometric setting, to the locally constant sheaf $R^{k} f_{*} \mathbb{C}_{X}(k \in \mathbb{N})$ is associated the Gauss-Manin connection, which is a holomorphic vector bundle on $Y$ endowed with a holomorphic flat connection. In such a case, the Hodge filtration can be
naturally defined and it is known to produce holomorphic bundles. Therefore, in the general setting of a variation of $\mathbb{C}$-Hodge structure that we intend to define, a better analogue of the complex vector space $\mathcal{H}^{o}$ is a holomorphic vector bundle $\mathcal{H}^{\prime}$ equipped with a flat holomorphic connection $\nabla: \mathcal{H}^{\prime} \rightarrow \Omega_{X}^{1} \otimes_{\mathscr{O}_{X}} \mathcal{H}^{\prime}$, so that the locally constant sheaf $\underline{\mathcal{H}^{\prime}}=\operatorname{Ker} \nabla$, that we also denote by $\mathcal{H}^{\prime \nabla}$, is the desired local system. Note that we can recover $\left(\mathcal{H}^{\prime}, \nabla\right)$ from $\underline{\mathcal{H}}^{\prime}$ since the natural morphism of flat bundles

$$
\left(\mathscr{O}_{X} \otimes_{\mathbb{C}} \underline{\mathcal{H}}^{\prime}, \mathrm{d} \otimes \mathrm{Id}\right) \longrightarrow\left(\mathcal{H}^{\prime}, \nabla\right)
$$

is an isomorphism. A filtration is then a finite (exhaustive) decreasing filtration by sub-bundles $F^{\bullet} \mathcal{H}^{\prime}$ (recall that a sub-bundle $F^{p} \mathcal{H}^{\prime}$ of $\mathcal{H}^{\prime}$ is a locally free $\mathscr{O}_{X}$-submodule of $\mathcal{H}^{\prime}$ such that $\mathcal{H}^{\prime} / F^{p} \mathcal{H}^{\prime}$ is also a locally free $\mathscr{O}_{X}$-module; $F^{\bullet} \mathcal{H}^{\prime}$ is a filtration by sub-bundles if each $F^{p} \mathcal{H}^{\prime} / F^{p+1} \mathcal{H}^{\prime}$ is a locally free $\mathscr{O}_{X}$-module). The main property, known as Griffiths transversality property is that the filtration should satisfy

$$
\begin{equation*}
\nabla\left(F^{p} \mathcal{H}^{\prime}\right) \subset \Omega_{X}^{1} \otimes_{\mathscr{O}_{X}} F^{p-1} \mathcal{H}^{\prime} \quad \forall p \in \mathbb{Z} \tag{4.1.1}
\end{equation*}
$$

However, the analogue of a bi-filtered vector space is not a bi-filtered holomorphic flat bundle, since one knows in the geometric setting that one of the filtrations should behave holomorphically, while the other one should behave anti-holomorphically. This leads to a presentation by $C^{\infty}$-bundles.

Let $\mathcal{H}=\mathscr{C}_{X}^{\infty} \otimes_{\mathscr{O}_{X}} \mathcal{H}^{\prime}$ be the associated $C^{\infty}$ bundle and let $D$ be the connection on $\mathcal{H}$ defined by $D(\varphi \otimes v)=\mathrm{d} \varphi \otimes v+\varphi \otimes \nabla v$ (this is a flat connection which decomposes with respect to types as $D=D^{\prime}+D^{\prime \prime}$ and $\left.D^{\prime \prime}=d^{\prime \prime} \otimes \mathrm{Id}\right)$. Then $D^{\prime \prime}$ is a holomorphic structure on $\mathcal{H}$, i.e., $\operatorname{Ker} D^{\prime \prime}$ is a holomorphic bundle with connection $\nabla$ induced by $D^{\prime}$ : this is $\left(\mathcal{H}^{\prime}, \nabla\right)$ by construction. Each bundle $F^{p} \mathcal{H}^{\prime}$ gives rise similarly to a $C^{\infty}$-bundle $F^{\prime p} \mathcal{H}$ which is holomorphic in the sense that $D^{\prime \prime} F^{\prime p} \mathcal{H} \subset \mathscr{E}_{X}^{0,1} \otimes F^{\prime p} \mathcal{H}$ (and thus $\left(D^{\prime \prime}\right)^{2}=0$ on $F^{\prime p} \mathcal{H}$ ).

On the other hand, $D^{\prime}$ defines an anti-holomorphic structure on $\mathcal{H}$, and $\operatorname{Ker} D^{\prime}$ is an anti-holomorphic bundle with a flat anti-holomorphic connection $\bar{\nabla}$ induced by $D^{\prime \prime}$. If we wish to work with holomorphic bundle, we can thus consider the conjugate bundle ${ }^{(1)} \mathcal{H}^{\prime \prime}=\overline{\operatorname{Ker} D^{\prime}}$, that we equip with the holomorphic flat connection $\nabla=$ $\overline{D_{\mid \text {Ker } D^{\prime}}^{\prime \prime}}$. A filtration of $\mathcal{H}$ by anti-holomorphic sub-bundles is by definition a filtration $F^{\prime \prime \bullet} \mathcal{H}$ by $C^{\infty}$-sub-bundles on which $D^{\prime}=0$. It corresponds to a filtration of $\mathcal{H}^{\prime \prime}$ by holomorphic sub-bundles $F^{\bullet} \mathcal{H}^{\prime \prime}$.

Conversely, given a flat $C^{\infty}$ bundle $(\mathcal{H}, D)$, we decompose the flat connection into its $(1,0)$ part $D^{\prime}$ and its $(0,1)$ part $D^{\prime \prime}$.

Exercise 4.1.2. Show that a connection $D$ is flat if and only if it satisfies

$$
D^{\prime 2}=0, \quad D^{\prime \prime 2}=0, \quad D^{\prime} D^{\prime \prime}+D^{\prime \prime} D^{\prime}=0
$$

[^0]Since, by flatness, $\left(D^{\prime \prime}\right)^{2}=0$, the Koszul-Malgrange theorem [KM58] implies that Ker $D^{\prime \prime}$ is a holomorphic bundle $\mathcal{H}^{\prime}$, that we can equip with the restriction $\nabla$ to Ker $D^{\prime \prime}$ of the connection $D^{\prime}$. Flatness of $D$ also implies that $\nabla$ is a flat holomorphic connection. Similarly, $\overline{\mathrm{Ker} D^{\prime}}$ is an holomorphic bundle, equipped with a flat holomorphic connection.

The conjugate $C^{\infty}$ bundle $\overline{\mathcal{H}}$ is equipped with the conjugate connection $\bar{D}$, which is also flat. Conjugation exchanges of course the $(1,0)$-part and the $(0,1)$-part, that is, $\bar{D}^{\prime}=\overline{D^{\prime \prime}}$ and $\overline{D^{\prime \prime}}=\overline{D^{\prime}}$. Similarly, we set $F^{\prime p} \overline{\mathcal{H}}=\overline{F^{\prime \prime p} \mathcal{H}}$, etc.

## Definition 4.1.3 (Variation of $\mathbb{C}$-Hodge structure, first definition)

A variation of $\mathbb{C}$-Hodge structure $H$ of weight $w$ consists of the data of a flat $C^{\infty}$ bundle $(\mathcal{H}, D)$, equipped with a filtration $F^{\prime \bullet} \mathcal{H}$ by holomorphic sub-bundles satisfying Griffiths transversality (4.1.1), and with a filtration $F^{\prime \prime \bullet} \mathcal{H}$ by anti-holomorphic subbundles satisfying anti-Griffiths transversality, such that the restriction of these data at each point $x \in X$ is a $\mathbb{C}$-Hodge structure of weight $w$ (Definition 2.4.2).

A morphism $\varphi: H_{1} \rightarrow H_{2}$ is a flat morphism of $C^{\infty}$-bundles compatible with both the holomorphic and the anti-holomorphic filtrations.

A polarization Q is a $D$-flat pairing $H \otimes \bar{H} \rightarrow C_{X}^{\infty}(-w)$ whose restriction to each $x \in X$ is a polarization of the Hodge structure $H_{x}$.

## Definition 4.1.4 (Variation of $\mathbb{C}$-Hodge structure, second definition)

A variation of $\mathbb{C}$-Hodge structure $H$ of weight $w$ consists of the data of a flat $C^{\infty}$ bundle $(\mathcal{H}, D)$, equipped with a Hodge decomposition by $C^{\infty}$-sub-bundles

$$
\mathcal{H}=\bigoplus_{p} \mathcal{H}^{p, w-p}
$$

satisfying Griffiths transversality:

$$
\begin{align*}
D^{\prime} \mathcal{H}^{p, q} & \subset \Omega_{X}^{1} \otimes\left(\mathcal{H}^{p, q} \oplus \mathcal{H}^{p-1, q+1}\right) \\
D^{\prime \prime} \mathcal{H}^{p, q} & \subset \overline{\Omega_{X}^{1}} \otimes\left(\mathcal{H}^{p, q} \oplus \mathcal{H}^{p+1, q-1}\right) \tag{4.1.4*}
\end{align*}
$$

A morphism $H_{1} \rightarrow H_{2}$ is a $D$-flat morphism $\left(\mathcal{H}_{1}, D\right) \rightarrow\left(\mathcal{H}_{2}, D\right)$ which is compatible with the Hodge decomposition.

A polarization is a $C^{\infty}$ Hermitian metric h on the $C^{\infty}$-bundle $\mathcal{H}$ such that

- the Hodge decomposition is orthogonal with respect to $h$,
- The polarization form Q , defined by the property that the decomposition is Q orthogonal and $\mathrm{h}_{\mid H^{p, w-p}}:=(-1)^{p} i^{w} \mathrm{Q}_{\mid \mathcal{H}^{p, w-p}}$, is a $D$-flat $\mathscr{O}_{X} \otimes_{\mathbb{C}} \mathscr{O}_{\bar{X}}$-linear pairing $\mathrm{Q}: \mathcal{H} \otimes_{\mathbb{C}} \overline{\mathcal{H}} \rightarrow \mathscr{C}_{X}^{\infty}$.

Remark 4.1.5. While it is easy, by using a partition of unity, to construct a Hermitian metric compatible with the Hodge decomposition, the condition of flatness of Q is a true constraint if $\operatorname{dim} X \geqslant 1$. For example, any flat $C^{\infty}$-bundle ( $\mathcal{H}, D$ ) can be regarded as a variation of $\mathbb{C}$-Hodge structure of type $(0,0)$, and it admits many Hermitian metrics, but the polarization condition imposes that the Hermitian metric is flat, which only occurs when the monodromy representation of the flat bundle is (conjugate to) a unitary representation.

Exercise 4.1.6. Show that Definitions 4.1.3 and 4.1 .4 are indeed equivalent. [Hint: use the standard property that, given two sub-bundles of a $C^{\infty}$ bundle $\mathcal{H}$, if the fibre of their intersection in $\mathcal{H}$ has constant dimension, then their intersection is also a sub-bundle of $\mathcal{H}$.]

Definition 4.1.7 (The categories $\operatorname{VHS}(X, \mathbb{C}, w)$ and $\operatorname{pVHS}(X, \mathbb{C}, w))$
Definitions 4.1.3 and 4.1.4 produce the category $\operatorname{VHS}(X, \mathbb{C}, w)$ of variations of $\mathbb{C}$-Hodge structures of weight $w$ on $X$. The category $\mathrm{pVHS}(X, \mathbb{C}, w)$ of polarizable variations of $\mathbb{C}$-Hodge structures of weight $w$ is the full subcategory of $\operatorname{VHS}(X, \mathbb{C}, w)$ whose objects admit a polarization.

On the other hand, the language of triples of Section 2.4.c (with a sesquilinear pairing $\mathfrak{c}$ ) enables one to keep holomorphy for both filtrations, by putting the nonholomorphic behaviour in the sesquilinear pairing $\mathfrak{c}$. This approach will be convenient in presence of singularities.

When working with a pairing $\mathfrak{c}$, we start by introducing a larger category, which can be enlarged to an abelian category by replacing the filtered flat bundles $\left(\mathcal{H}^{\prime}, \nabla, F^{\bullet} \mathcal{H}^{\prime}\right)$ and $\left(\mathcal{H}^{\prime \prime}, \nabla, F^{\bullet} \mathcal{H}^{\prime \prime}\right)$ by coherent graded $\mathscr{O}_{X}[z]$-modules with a flat $z$-connection (see Definition A.2.16).

Definition 4.1.8 (Flat filtered $\mathscr{O}$-triples (with pairing $\mathfrak{c}$ )). A filtered flat $\mathscr{O}$-triple on $X$ consists of the data of

- a pair of flat holomorphic bundles $\left(\mathcal{H}^{\prime}, \nabla\right)$ and $\left(\mathcal{H}^{\prime \prime}, \nabla\right)$ on $X$, equipped with decreasing filtrations by holomorphic sub-bundles $F^{\bullet} \mathcal{H}^{\prime}, F^{\bullet} \mathcal{H}^{\prime \prime}$ (i.e., each $\operatorname{gr}_{F}^{p} \mathcal{H}^{\prime}, \operatorname{gr}_{F}^{p} \mathcal{H}^{\prime \prime}$ is a locally free $\mathscr{O}_{X}$-module of finite rank), both filtrations satisfying Griffiths transversality (4.1.1),
- a flat $\mathscr{O}_{X} \otimes_{\mathbb{C}} \mathscr{O}_{\bar{X}}$-linear morphism $\mathfrak{c}: \mathcal{H}^{\prime} \otimes_{\mathbb{C}} \overline{\mathcal{H}^{\prime \prime}} \rightarrow \mathscr{C}_{X}^{\infty}$, i.e., for local holomorphic sections $m^{\prime}, m^{\prime \prime}$ of $\mathcal{H}^{\prime}, \mathcal{H}^{\prime \prime}$, we have

$$
\begin{aligned}
& \partial \mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)=\mathfrak{c}\left(\nabla m^{\prime}, \overline{m^{\prime \prime}}\right) \\
& \bar{\partial} \mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)=\mathfrak{c}\left(m^{\prime}, \overline{\nabla m^{\prime \prime}}\right)
\end{aligned}
$$

Remark 4.1.9 (Flatness of $\mathfrak{c}$ ). The restriction $\underline{\mathfrak{c}}$ of $\mathfrak{c}$ to the local system $\underline{\mathcal{H}^{\prime}} \otimes_{\mathbb{C}} \overline{\mathcal{H}^{\prime \prime}}$ takes values in the constant sheaf $\mathbb{C}_{X}$ since for local sections $m^{\prime}$ of $\underline{\mathcal{H}}^{\prime}$ and $m^{\prime \prime}$ of $\underline{\mathcal{H}}^{\prime \prime}$, we have, by the previous formulas, $\partial \mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)=\bar{\partial} \mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)=0$. Moreover, we can recover $\mathfrak{c}$ from its restriction $\mathfrak{c}$ by $\mathscr{O}_{X} \otimes_{\mathbb{C}} \mathscr{O}_{\bar{X}}$-linearity. As a consequence, we see that if $X$ is connected, $\mathfrak{c}$ is non-degenerate if and only if its restriction at some point $x \in X$ is a non-degenerate pairing $\mathcal{H}_{x}^{\prime} \otimes_{\mathbb{C}} \overline{\mathcal{F}_{x}^{\prime \prime}} \rightarrow \mathbb{C}$, since this obviously holds for $\mathfrak{c}$.

## Definition 4.1.10 (Variation of $\mathbb{C}$-Hodge structure, third definition)

A variation of $\mathbb{C}$-Hodge structure of weight $w$ is a filtered flat $\mathscr{O}$-triple

$$
H=\left(\left(\mathcal{H}^{\prime}, \nabla, F^{\bullet} \mathcal{H}^{\prime}\right),\left(\mathcal{H}^{\prime \prime}, \nabla, F^{\bullet} \mathcal{H}^{\prime \prime}\right), \mathfrak{c}\right)
$$

whose restriction $H_{x}=\left(\left(\mathcal{H}_{x}^{\prime}, F^{\bullet} \mathcal{H}_{X}^{\prime}\right),\left(\mathcal{H}_{x}^{\prime \prime}, F^{\bullet} \mathcal{H}_{x}^{\prime \prime}\right), \mathfrak{c}_{x}\right)$ at each $x \in X$ is a $\mathbb{C}$-Hodge triple of weight $w$. In particular, $\mathfrak{c}$ is non-degenerate.

A polarization is a flat morphism $\mathrm{Q}: H \rightarrow H^{*}(-w)$ inducing a polarization at each $x \in X$. Equivalently (see Remark 2.4.20), a polarized variation of $\mathbb{C}$-Hodge structure of weight $w$ consists of the data $\left.(H, \mathrm{Q})=\left(\mathcal{H}^{\prime}, \nabla, F^{\bullet} \mathcal{H}^{\prime}\right), \mathrm{Q}\right)$, where Q is a flat sesquilinear pairing on $\left(\mathcal{H}^{\prime}, \nabla\right)$, inducing a polarized $\mathbb{C}$-Hodge structure at eavery $x \in X$.

Remark 4.1.11. One can also define $\operatorname{VHS}(X, \mathbb{C}, w)$ as the full subcategory of that of filtered flat triples whose objects are variations of $\mathbb{C}$-Hodge structures of weight $w$ on $X$. The category $\mathrm{pVHS}(X, \mathbb{C}, w)$ is defined correspondingly.

The category VHS can be naturally endowed with the operations Hom, tensor product, duality, and conjugation. The full subcategory pVHS is stable by these operations, since the polarization can be constructed in a natural way in each of these operations (see Remark 2.4.21).

## Exercise 4.1.12.

(1) Show that the category $\operatorname{VHS}(X, \mathbb{C}, w)$ as defined by 4.1 .10 is equivalent to $\operatorname{VHS}(X, \mathbb{C}, w)$ as defined by 4.1.3, and hence to $\operatorname{VHS}(X, \mathbb{C}, w)$ as defined by 4.1.4. Show a similar result for $\mathrm{pVHS}(X, \mathbb{C}, w)$.
(2) Show that a polarization is an isomorphism $H \xrightarrow{\sim} H^{*}(-w)$. [Hint: use Remark 2.4.18(1).]

Corollary 4.1.13 (Abelianity). The category $\operatorname{VHS}(X, \mathbb{C}, w)$ is abelian and each morphism is strictly compatible with the Hodge filtration.

Proof. The statement is clear with Definition 4.1.4, since any morphism is bigraded with respect to the Hodge decomposition, hence so are its kernel, image and cokernel.

Exercise 4.1.14 (Abelianity and semi-simplicity). Let ( $H, \mathrm{Q}$ ) be a polarized variation of Hodge structure of weight $w$ on $X$.
(1) Show that any subobject of $H$ in $\operatorname{VHS}(X, \mathbb{C}, w)$ is a direct summand of the given variation, and that the polarization Q induces a polarization.
(2) Conclude that the full subcategory $\operatorname{pVHS}(X, \mathbb{C}, w)$ of polarizable variations of Hodge structure is abelian and semi-simple (i.e., any object decomposes as the direct sum of its irreducible components).
[Hint: Use the $C^{\infty}$ interpretation of Definition 4.1.4.]

### 4.2. The Hodge theorem

4.2.a. The Hodge theorem for unitary representations. We will extend the Hodge theorem (Theorem 2.3.5 and the results indicated after its statement concerning the polarization) to the case of the cohomology with coefficients in a unitary representation.

Let us start with a holomorphic vector bundle $\mathcal{H}^{\prime}$ of rank $d$ on a complex projective manifold $X$ equipped with a flat holomorphic connection $\nabla$. The local system $\underline{\mathcal{H}}=\mathcal{H}^{\prime \nabla}$ corresponds to a representation $\pi_{1}(X, \star) \rightarrow \mathrm{GL}_{d}(\mathbb{C})$, up to conjugation. The unitarity assumption means that we can conjugate the given representation in such a way that it takes values in the unitary group.

In other words, there exists a Hermitian metric h on the associated $C^{\infty}$-bundle $\mathcal{H}=\mathscr{C}^{\infty} \otimes_{\mathscr{O}_{X}} \mathcal{H}^{\prime}$ such that, if we denote as above by $D$ the connection on $\mathcal{H}$ defined by $D(\varphi \otimes v)=\mathrm{d} \varphi \otimes v+\varphi \otimes \nabla v$, the connection $D$ is compatible with the metric h (i.e., is the Chern connection of the metric h).

That $D$ is a connection compatible with the metric implies that its formal adjoint (with respect to the metric) is obtained with a suitably defined Hodge $\star$ operator by the formula $D^{*}=-\star D \star$. This leads to the decomposition of the space of $C^{\infty}$ $k$-forms on $X$ with coefficients in $\mathcal{H}$ (resp. $(p, q)$-forms) as the orthogonal sum of the kernel of the Laplace operator with respect to $D$ (resp. $D^{\prime}$ or $D^{\prime \prime}$ ), that is, the space of harmonic sections, and its image.

As the connection $D$ is flat, there is a $C^{\infty}$ deRham complex $\left(\mathscr{E}_{X}^{\bullet} \otimes \mathcal{H}, D\right)$, and standard arguments give

$$
H^{k}(X, \underline{\mathcal{H}})=\boldsymbol{H}^{k}\left(X, \operatorname{DR}\left(\mathcal{H}^{\prime}, \nabla\right)\right)=H^{k}\left(\Gamma\left(X,\left(\mathscr{E}_{X}^{\bullet} \otimes \mathcal{H}, D\right)\right)\right) .
$$

One can also define the Dolbeault cohomology groups by decomposing $\mathscr{E}^{\bullet}$ into $\mathscr{E}^{p, q}$ 's and by decomposing $D$ as $D^{\prime}+D^{\prime \prime}$. Then $H_{D^{\prime \prime}}^{p, q}(X, \mathcal{H})=H^{q}\left(X, \Omega_{X}^{p} \otimes \mathcal{H}^{\prime}\right)$.

As the projective manifold $X$ is Kähler, we obtain the Kähler identities for the various Laplace operators: $\Delta_{D}=2 \Delta_{D^{\prime}}=2 \Delta_{D^{\prime \prime}}$.

Then, exactly as in Theorem 2.3.5, we get:
Theorem 4.2.1. Under these conditions, one has a canonical decomposition

$$
\boldsymbol{H}^{k}\left(X, \operatorname{DR}\left(\mathcal{H}^{\prime}, \nabla\right)\right)=\bigoplus_{p+q=k} H^{p, q}(X, \mathcal{H})
$$

and $H^{q, p}(X, \mathcal{H})$ is identified with $\overline{H^{p, q}\left(X, \mathcal{H}^{\vee}\right)}$, where $\mathcal{H}^{\vee}$ is the dual bundle. ${ }^{(2)}$
The Hard Lefschetz theorem also holds in this context.
4.2.b. Variation of polarized Hodge structure on a compact Kähler manifold: the Hodge-Deligne theorem. Let us keep notation as in Section 4.2.a. We do not assume anymore that $\underline{\mathcal{H}}$ is unitary. We only assume that it underlies a variation of polarized Hodge structure of some weight $w$. In such a situation, we have a flat connection $D$ on the $C^{\infty}$-bundle $\mathcal{H}$ associated to $\mathcal{H}^{\prime}$, with $D=D^{\prime}+d^{\prime \prime}$, and we also have a Hermitian metric h on $\mathcal{H}$ associated with Q , but $D$ is possibly not compatible with the metric. The argument using the Hodge $\star$ operator is not valid anymore.

[^1]Exercise 4.2.2. Let $(H, Q)$ be a variation of polarized Hodge structure of weight $w$ on $X$ (see Definition 4.1.3). Let h be the Hermitian metric deduced from Q and let $D=D^{\prime}+D^{\prime \prime}$ be the flat $C^{\infty}$ connection. Let $\mathcal{H}=\bigoplus_{p+q=w} \mathcal{H}^{p, q}$ be the Hodge decomposition (which is h-orthogonal by construction). Show that
(1) In the Griffiths transversality relations $(4.1 .4 *)$, the composition of $D^{\prime}$ (resp. $\left.D^{\prime \prime}\right)$ with the projection on the first summand defines a ( 1,0 ) (resp. ( 0,1 ))connection $D_{\mathrm{h}}^{\prime}$ (resp. $D_{\mathrm{h}}^{\prime \prime}$ ), and that the projection to the second summand defines a $C^{\infty}$-linear morphism $\theta^{\prime}$ (resp. $\theta^{\prime \prime}$ ).
(2) Show that $D_{\mathrm{h}}:=D_{\mathrm{h}}^{\prime}+D_{\mathrm{h}}^{\prime \prime}$ is compatible with the metric h , but is possibly not flat.
(3) Show that $\theta^{\prime \prime}$ is the h-adjoint of $\theta^{\prime}$.
(4) Show that the connection $\mathcal{D}^{\prime \prime}:=D_{\mathrm{h}}^{\prime \prime}+\theta^{\prime}$ has square zero, as well as the connection $\mathcal{D}^{\prime}:=D_{\mathrm{h}}^{\prime}+\theta^{\prime \prime}$.

The decomposition $D=D^{\prime}+D^{\prime \prime}$ is replaced with the decomposition $D=\mathcal{D}^{\prime}+\mathcal{D}^{\prime \prime}$. The disadvantage is that we loose the decomposition into types $(1,0)$ and $(0,1)$, but we keep the flatness property. On the other hand, as $D_{\mathrm{h}}$ is compatible with the metric, its formal adjoint is computed with a Hodge $\star$ operator. Using the Kähler metric, one shows that $\theta$ satisfies the right relations in order to ensure the equality of Laplace operators $\Delta_{D}=2 \Delta_{\mathcal{D}^{\prime}}=2 \Delta_{\mathcal{D}^{\prime \prime}}$.

We did not really loose the decomposition into types: the operator $\mathcal{D}^{\prime \prime}$ sends a section of $\mathcal{H}^{p, q}$ to a section of $\left(\Omega_{X}^{1} \otimes \mathcal{H}^{p-1, q+1}\right)+\left(\overline{\Omega_{X}^{1}} \otimes \mathcal{H}^{p, q}\right)$. Counting the total type, we find $(p, q+1)$ for both terms. In other word, taking into account the Hodge type of a section, the operator $\mathcal{D}^{\prime \prime}$ is indeed of type $(0,1)$. A similar argument applies to $\mathcal{D}^{\prime}$.

This being understood, the arguments of Hodge theory apply to this situation as in the case considered in Section 4.2.a, to get the Hodge-Deligne theorem.

Theorem 4.2.3 (Hodge-Deligne theorem). Let (H, Q) be a polarized variation of Hodge structure of weight $w$ on a smooth complex projective variety $X$ of pure dimension $n$ and let $\mathscr{L}$ be an ample line bundle on $X$. Then $\left(H^{\bullet}(X, \underline{\mathcal{H}}), \mathrm{L}_{\mathscr{L}}\right)$ is naturally equipped with a polarizable graded ( - )Hodge-Lefschetz structure centered at $w+n$ (see Definition 3.2.4). In particular, each $H^{k}(X, \underline{\mathcal{H}})$ is endowed with a polarized $\mathbb{C}$-Hodge structure of weight $w+k$.

Sketch of proof. One realizes each cohomology class in $H^{\bullet}(X, \underline{\mathcal{H}})$ by a unique $\Delta_{\mathcal{D}^{-}}$ harmonic section, by the arguments of Hodge theory, which extend if one takes into account the total type, as above.

The polarization is obtained from Q and Poincaré duality as we did for Q in Section 2.3, still using the $\operatorname{sign} \varepsilon$, and from it we cook up the form Q . More precisely, the pairing

$$
\mathrm{Q}: \mathcal{H} \otimes \overline{\mathcal{H}} \longrightarrow \mathscr{C}_{X}^{\infty}
$$

induces for every $k \leqslant n$ a pairing $\mathrm{Q}^{(n-k)}$ of $\mathbb{C}$-Hodge structures

$$
H^{n-k}(X, \underline{\mathcal{H}}) \otimes H^{n+k}(X, \underline{\mathcal{H}}) \longrightarrow H^{2 n}(X, \mathbb{C})(-(w+n))=\mathbb{C}(-(w+n)) .
$$

Since the Lefschetz operator $\mathrm{L}_{\mathscr{L}}$ only acts on the forms and not on the sections of $\mathcal{H}$, it is an infinitesimal automorphism of $\mathrm{Q}^{(n-k)}$ since it is so for the Poincaré duality pairing. Then the family $\left(\mathrm{Q}^{(n-k)}\right)_{k}$ is a sesquilinear pairing on the graded (-)Hodge-Lefschetz structure $H^{\bullet}(X, \underline{\mathcal{H}})$.

Due to the Kähler identities and the commutation of $\mathrm{L}_{\mathscr{L}}$ with $\Delta_{\mathcal{D}}$, a harmonic section of $\mathscr{E}_{X}^{n-k} \otimes \mathcal{H}$ is primitive if and only if each of its components with respect to the total bigrading is so, and since $\mathrm{L}_{\mathscr{L}}$ only acts on the differential form part of such a component, this occurs if an only if its component on $\mathscr{E}_{X}^{p, q} \otimes \mathcal{H}^{a, b}$ is primitive, with $p+q=n-k$ and $a+b=w$. Fixing an h-orthonormal basis $\left(v_{i}\right)_{i}$ of $\mathcal{H}^{a, b}$, such a component can be written in a unique way as $\sum_{i} \eta_{i}^{p, q} \otimes v_{i}$ with $\eta_{i}^{p, q}$ primitive. Then the positivity property of $\mathrm{Q}^{(n-k)}$ on $\eta_{i}^{p, q} \otimes v_{i}$ amounts to the positivity of (2.3.13) on $\eta_{i}^{p, q}$.
Remarks 4.2.4. Let $H$ be a variation of polarizable $\mathbb{C}$-Hodge structure of weight $w$ on a smooth complex projective variety $X$.
(1) (The Hodge filtration) Consider the de Rham complex $\operatorname{DR}\left(\mathcal{H}^{\prime}, \nabla\right)$. According to the Griffiths transversality property, it comes equipped with a filtration, by setting (see Definition A.5.1):
$F^{p} \operatorname{DR}\left(\mathcal{H}^{\prime}, \nabla\right)=\left\{0 \rightarrow F^{p} \mathcal{H}^{\prime} \xrightarrow{\nabla} \Omega_{X}^{1} \otimes F^{p-1} \mathcal{H}^{\prime} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega_{X}^{n} \otimes F^{p-n} \mathcal{H}^{\prime} \rightarrow 0\right\}$.
The natural inclusion of complexes $F^{p} \operatorname{DR}\left(\mathcal{H}^{\prime}, \nabla\right) \hookrightarrow \mathrm{DR}\left(\mathcal{H}^{\prime}, \nabla\right)$ induces a morphism

$$
\begin{equation*}
\boldsymbol{H}^{k}\left(X, F^{p} \operatorname{DR}\left(\mathcal{H}^{\prime}, \nabla\right)\right) \longrightarrow \boldsymbol{H}^{k}\left(X, \operatorname{DR}\left(\mathcal{H}^{\prime}, \nabla\right)\right)=H^{k}(X, \underline{\mathcal{H}}) \tag{4.2.4*}
\end{equation*}
$$

whose image is the filtration $F^{\prime p} H^{k}(X, \underline{\mathcal{H}})$. Working anti-holomorphically with the filtration $F^{\prime \prime \bullet} \mathcal{H}$ by anti-holomorphic sub-bundles and the anti-holomorphic connection induced by $D^{\prime \prime}$ on $\operatorname{Ker} D^{\prime}$, one obtains the filtration $F^{\prime \prime \bullet} H^{k}(X, \underline{\mathcal{H}})$. The HodgeDeligne theorem implies that these filtrations are $w+k$-opposed.
(2) (Degeneration at $E_{1}$ of the Hodge-to-de Rham spectral sequence)

Moreover, for every $p, k$, the morphism $(4.2 .4 *)$ is injective. In other words, the filtered complex $\boldsymbol{R} \Gamma\left(X, F^{\bullet} \operatorname{DR}\left(\mathcal{H}^{\prime}, \nabla\right)\right)$ is strict (see Section A.2.b). Let us denote by $\theta$ the morphism induced by $\nabla: \theta=\operatorname{gr}^{1} \nabla: \operatorname{gr}_{F}^{p} \mathcal{H}^{\prime} \rightarrow \Omega_{X}^{1} \otimes \operatorname{gr}_{F}^{p-1} \mathcal{H}^{\prime}$. This morphism is $\mathscr{O}_{X}$-linear and is equal to the restriction of $\theta^{\prime}$ to Ker $D_{\mathrm{h}}^{\prime \prime}$ (see Exercise 4.2.2). The graded complex $\operatorname{gr}_{F}^{p} \mathrm{DR}\left(\mathcal{H}^{\prime}, \nabla\right)$ is the complex
$\operatorname{gr}_{F}^{p} \mathrm{DR}\left(\mathcal{H}^{\prime}, \nabla\right)=\left\{0 \rightarrow \operatorname{gr}_{F}^{p} \mathcal{H}^{\prime} \xrightarrow{\theta} \Omega_{X}^{1} \otimes \operatorname{gr}_{F}^{p-1} \mathcal{H}^{\prime} \xrightarrow{\theta} \cdots \xrightarrow{\theta} \Omega_{X}^{n} \otimes \operatorname{gr}_{F}^{p-n} \mathcal{H}^{\prime} \rightarrow 0\right\}$.
Since each term of this complex is $\mathscr{O}_{X}$-locally free of finite rank and since $\theta$ is $\mathscr{O}_{X}$-linear, the hypercohomology spaces $\boldsymbol{H}^{k}\left(X, \operatorname{gr}_{F}^{p} \operatorname{DR}\left(\mathcal{H}^{\prime}, \nabla\right)\right)$ are finitedimensional. Setting $\operatorname{gr}_{F}=\bigoplus_{p} \operatorname{gr}_{F}^{p}$, we introduce the Dolbeault complex

$$
\operatorname{Dol}\left(\operatorname{gr}_{F} \mathcal{H}^{\prime}, \theta\right):=\left\{0 \rightarrow \operatorname{gr}_{F} \mathcal{H}^{\prime} \xrightarrow{\theta} \Omega_{X}^{1} \otimes \operatorname{gr}_{F} \mathcal{H}^{\prime} \xrightarrow{\theta} \cdots \xrightarrow{\theta} \Omega_{X}^{n} \otimes \operatorname{gr}_{F} \mathcal{H}^{\prime} \rightarrow 0\right\} .
$$

The strictness property is then equivalent to

$$
\forall p, k, \quad \operatorname{gr}_{F}^{p} H^{k}(X, \underline{\mathcal{H}})=\boldsymbol{H}^{k}\left(X, \operatorname{gr}_{F}^{p} \operatorname{DR}\left(\mathcal{H}^{\prime}, \nabla\right)\right),
$$

where $\operatorname{gr}_{F}^{p} H^{k}(X, \underline{\mathcal{H}}) \simeq H^{p, w+k-p}(X, \underline{\mathcal{H}})$. This property is also equivalent to

$$
\forall k, \quad \operatorname{dim} H^{k}(X, \underline{\mathcal{H}})=\operatorname{dim} \boldsymbol{H}^{k}\left(X, \operatorname{Dol}\left(\operatorname{gr}_{F} \mathcal{H}, \theta\right)\right)
$$

This statement is obtained by standard arguments of Hodge theory applied to the operators $D, \mathcal{D}^{\prime}, \mathcal{D}^{\prime \prime}$ and their Laplacians.
(3) For any smooth projective variety $X$, the space $H^{0}(X, \underline{\mathcal{H}})$ is primitive (for any $\mathscr{L}$ ) and, given a polarization of $H$, the polarization of the pure $\mathbb{C}$-Hodge structure of weight $w$ is independent of the choice of $\mathscr{L}$.

If $X$ is a compact Riemann surface, then $H^{1}(X, \underline{\mathcal{H}})$ is also primitive, and there is no need to choose a polarization bundle $\mathscr{L}$ in order to obtain the polarized pure $\mathbb{C}$-Hodge structure on $H^{1}(X, \underline{\mathcal{H}})$.
(4) (The fixed-part theorem) The maximal constant subsheaf of $\underline{\mathcal{H}}$ is the constant subsheaf with stalk $H^{0}(X, \underline{\mathcal{H}})$ at each point, by means of a natural injective morphism $H^{0}(X, \underline{\mathcal{H}}) \otimes_{\mathbb{C}} \mathbb{C}_{X} \rightarrow \underline{\mathcal{H}}$. By the Hodge-Deligne theorem 4.2.3, $H^{0}(X, \underline{\mathcal{H}}) \otimes_{\mathbb{C}} \mathbb{C}_{X}$ is endowed with a constant variation of Hodge structure of weight $w$. We claim that the previous morphism is compatible with the Hodge filtrations, i.e., is a morphism in $\operatorname{VHS}(X, \mathbb{C}, w)$, that is, the morphism

$$
\varphi: H^{0}(X, \underline{\mathcal{H}}) \otimes_{\mathbb{C}} \mathscr{O}_{X} \longrightarrow \underline{\mathcal{H}} \otimes_{\mathbb{C}} \mathscr{O}_{X}=\mathcal{H}^{\prime}
$$

is compatible with the Hodge filtration $F^{\bullet}$ on both terms.
Since $X$ is compact and $F^{p} \mathcal{H}^{\prime}$ is $\mathscr{O}_{X}$-coherent (being $\mathscr{O}_{X}$-locally free of finite rank), the space $H^{0}\left(X, F^{p} \mathcal{H}^{\prime}\right)$ is finite dimensional, and we have a natural injective morphism

$$
H^{0}\left(X, F^{p} \mathcal{H}^{\prime}\right) \otimes_{\mathbb{C}} \mathscr{O}_{X} \longrightarrow F^{p} \mathcal{H}^{\prime}
$$

by sending a global section of $F^{p} \mathcal{H}^{\prime}$ to its germ at every point of $X$. On the other hand, regarding $F^{p} \mathcal{H}^{\prime}$ as a complex with only one term in degree zero, we have an obvious morphism of complexes

$$
F^{p} \operatorname{DR} \mathcal{H}^{\prime} \longrightarrow F^{p} \mathcal{H}^{\prime}
$$

which induces a morphism $\boldsymbol{H}^{0}\left(X, F^{p} \operatorname{DR} \mathcal{H}^{\prime}\right) \rightarrow H^{0}\left(X, F^{p} \mathcal{H}^{\prime}\right)$, from which we obtain a morphism

$$
\boldsymbol{H}^{0}\left(X, F^{p} \mathrm{DR} \mathcal{H}^{\prime}\right) \otimes_{\mathbb{C}} \mathscr{O}_{X} \longrightarrow F^{p} \mathcal{H}^{\prime}
$$

For $p$ small enough so that $F^{p} \mathcal{H}^{\prime}=\mathcal{H}^{\prime}$, we recover the morphism $\varphi$ above. By the degeneration property (2), $\boldsymbol{H}^{0}\left(X, F^{p} \mathrm{DR} \mathcal{H}^{\prime}\right)$ is identified with $F^{p} H^{0}(X, \underline{\mathcal{H}})$, hence the assertion.

As a consequence, if a global horizontal section of $\left(\mathcal{H}^{\prime}, \nabla\right)$, i.e., a global section of $\mathcal{\mathcal { H }}$, regarded as a global section of $\mathcal{H}^{\prime}$, is in $F^{p} \mathcal{H}^{\prime}$ at one point, it is a global section of $F^{p} \mathcal{H}^{\prime}$.

Arguing similarly with the anti-holomorphic Hodge filtration, and then with the Hodge decomposition of the $C^{\infty}$-bundle $\mathcal{H}$, we find that the natural injective morphism

$$
H^{0}(X, \underline{\mathcal{H}}) \otimes_{\mathbb{C}} \mathscr{C}_{X}^{\infty} \longrightarrow \underline{\mathcal{H}} \otimes_{\mathbb{C}} \mathscr{C}_{X}^{\infty}=\mathcal{H}
$$

is compatible with the Hodge decomposition of each term. As a consequence, for any global horizontal section of $(\mathcal{H}, D)$, i.e., any global section of $\underline{\mathcal{H}}$, regarded as a global section of $\mathcal{H}$, the Hodge $(p, q)$-components are also $D$-horizontal. In particular, if the global section is of type $(p, q)$ at one point, it is of type $(p, q)$ at every point of $X$.

## 4.2.c. Unitary representation on a complex manifold with a complete metric

The compactness assumption in Hodge theory is not mandatory. One can relax it, provided that the metric on the manifold remains complete (see e.g. [Dem96, §12]). Let us indicate the new phenomena that occur in the setting of Section 4.2.a.

One works with $C^{\infty}$ sections $v$ of $\mathscr{E}_{X}^{\bullet} \otimes \mathcal{H}$ which are globally $L^{2}$ with respect to the metric h and to the complete metric on $X$, and whose differential $D v$ is $L^{2}$. The analysis of the Laplace operator is now similar to that of the compact case. One uses a $L^{2}$ de Rham complex and a $L^{2}$ Dolbeault complex (i.e., one puts a $L^{2}$ condition on sections and their derivatives).

One missing point, however, is the finite dimensionality of the $L^{2}$-cohomologies involved. In the compact case, it is ensured, for instance, by the finiteness of the Betti cohomology $H^{k}(X, \underline{\mathcal{H}})$. So the theorem is stated as

Theorem 4.2.5. Let $(X, \omega)$ be a complete Kähler manifold and $\left(\mathcal{H}^{\prime}, \nabla\right)$ be a holomorphic bundle with a flat connection $\nabla$ corresponding to a unitary representation $\mathcal{H}^{\prime \nabla}$ of $\pi_{1}(X, \star)$. Let $(\mathcal{H}, D)$ be the associated flat $C^{\infty}$ bundle. Then, with the assumption that all the terms involved are finite dimensional, one has a canonical isomorphisms

$$
\left.H_{L^{2}}^{k}(X, \mathcal{H}, D) \simeq \bigoplus_{p+q=k} H_{L^{2}}^{p, q}\left(X, \mathcal{H}, D^{\prime \prime}\right), \quad H_{L^{2}}^{q, p}(X, \mathcal{H}, D) \simeq \overline{H_{L^{2}}^{p, q}(X, \mathcal{H} \vee}, D^{\vee}\right)
$$

It remains to relate the $L^{2}$ de Rham cohomology with topology. If we are lucky, then this will not only provide a relation with Betti cohomology, but the Betti cohomology will be finite dimensional and this will also provide the finiteness assumption needed for the $L^{2}$ de Rham cohomology.

There will also be a need for the finiteness of the $L^{2}$ Dolbeault cohomology. In the case that will occupy us later, where $X$ is a punctured compact Riemann surface, this will be done by relating $L^{2}$ Dolbeault cohomology with the cohomology of a coherent sheaf on the compact Riemann surface.

We will indicate in Sections 5.3 and 5.4 the way to solve these two problems in dimension one, by means of the $L^{2}$ Poincaré lemma and the $L^{2}$ Dolbeault lemma.

### 4.3. Semi-simplicity

4.3.a. A review on completely reducible representations. We review here some classical results concerning the theory of finite-dimensional linear representations. Let $\Pi$ be a group and let $\rho$ be a linear representation of $\Pi$ on a finite-dimensional $\boldsymbol{k}$-vector space $V$. In other words, $\rho$ is a group homomorphism $\Pi \rightarrow \mathrm{GL}(V)$. We will say that $V$ is a $\Pi$-module (it would be more correct to introduce the associative algebra $\boldsymbol{k}[\Pi]$ of the group $\Pi$, consisting of $\boldsymbol{k}$-linear combinations of the elements of $\Pi$, and to speak of a left $\boldsymbol{k}[\Pi]$-module). The subspaces of $V$ stable by $\rho(\Pi)$ correspond thus to the sub- $\Pi$-modules of $V$.

We say that a $\Pi$-module $V$ is irreducible if it does not admit any nontrivial sub- $\Pi$ module. Then, any homomorphism between two irreducible $\Pi$-modules is either zero, or an isomorphism (Schur's lemma). If $k$ is algebraically closed, any automorphism of an irreducible $\Pi$-module is a nonzero multiple of the identity (consider an eigenspace of the automorphism).

Proposition 4.3.1. Given a $\Pi$-module $V$, the following properties are equivalent:
(1) The $\Pi$-module $V$ is semi-simple, i.e., every sub-П-module has a supplementary sub-П-module.
(2) The $\Pi$-module $V$ est completely reducible, i.e., $V$ has a decomposition (in general non unique) into the direct sum of irreducible sub-П-modules.
(3) The $\Pi$-module $V$ is generated by its irreducible sub-П-modules.

Proof. The only nonobvious point is $(3) \Rightarrow(1)$. Let then $W$ be a sub- $\Pi$-module of $V$. We will show the result by induction on $\operatorname{codim} W$, this being clear for $\operatorname{codim} W=0$. If codim $W \geqslant 1$, there exists by assumption a nontrivial irreducible sub- $\Pi$-module $V_{1} \subset V$ not contained in $W$. Since $V_{1}$ is irreducible, we have $W \cap V_{1}=\{0\}$, so $W_{1}:=$ $W \oplus V_{1}$ is a sub- $\Pi$-module of $V$ to which one can apply the inductive assumption. If $W_{1}^{\prime}$ is a supplementary $\Pi$-module of $W_{1}$, then $W^{\prime}=W_{1}^{\prime} \oplus V_{1}$ is a supplementary $\Pi$-module of $W$.

It follows then from Schur's lemma that a completely reducible $\Pi$-module has a unique decomposition as the direct sum

$$
V=\bigoplus_{i} V_{i}=\bigoplus_{i}\left(V_{i}^{o} \otimes E_{i}\right)
$$

in which the isotypic components $V_{i}$ are sub- $\Pi$-modules of the form $V_{i}^{o} \otimes E_{i}$, where $V_{i}^{o}$ is an irreducible $\Pi$-module, $V_{i}^{o}$ is not isomorphic to $V_{j}^{o}$ for $i \neq j$, and $E_{i}$ is a trivial $\Pi$-module, i.e., on which $\Pi$ acts by the identity.

One also notes that if $W$ is a sub- $\Pi$-module of a completely reducible $\Pi$-module $V$, then $W$ is completely reducible and its isotypical decomposition is

$$
W=\bigoplus_{i}\left(W \cap V_{i}\right),
$$

in which $W \cap V_{i}=V_{i}^{o} \otimes F_{i}$ for some subspace $F_{i}$ of $E_{i}$. A $\Pi$-module supplementary to $W$ can be obtained by choosing for every $i$ a $\boldsymbol{k}$-vector space supplementary to $F_{i}$ in $E_{i}$.

Remarks 4.3.2. The previous properties have easy consequences.
(1) A $\boldsymbol{k}$-vector space $V$ is a semi-simple $\Pi$-module if and only if the associated complex space $V_{\mathbb{C}}=\mathbb{C} \otimes V$ is a semi-simple $\Pi$-module (for the complexified representation).

Indded, let us first recall that the group $\operatorname{Aut}(\mathbb{C})$ acts on $V_{\mathbb{C}}$ : if $\varepsilon=\left(\varepsilon_{1}, \ldots, \varepsilon_{n}\right)$ is some $\boldsymbol{k}$-basis of $V$ then, for $a_{1}, \ldots, a_{n} \in \mathbb{C}$ and $\sigma \in \operatorname{Aut}(\mathbb{C})$, one sets $\sigma\left(\sum_{i} a_{i} \varepsilon_{i}\right)=$ $\sum_{i} \sigma\left(a_{i}\right) \varepsilon_{i}$. A subspace $W_{\mathbb{C}}$ of $V_{\mathbb{C}}$ is "defined over $\boldsymbol{k}$ ", i.e., of the form $\mathbb{C} \otimes W$ for some sub-espac $W$ of $V$, if and only if it is stable by any automorphism $\sigma \in \operatorname{Aut}(\mathbb{C})$ : indeed, if $d=\operatorname{dim}_{\mathbb{C}} W_{\mathbb{C}}$, one can find, up to renumbering the basis $\varepsilon$, a basis $e_{1}, \ldots, e_{d}$ of $W_{\mathbb{C}}$ such that

$$
\begin{aligned}
e_{1} & =\varepsilon_{1}+a_{1,2} \varepsilon_{2}+\cdots+a_{1, d} \varepsilon_{d}+\cdots+a_{1, n} \varepsilon_{n} \\
e_{2} & = \\
\vdots & \varepsilon_{2}+\cdots+a_{1, d} \varepsilon_{d}+\cdots+a_{2, n} \varepsilon_{n} \\
& \\
e_{d} & =
\end{aligned}
$$

with $a_{i, j} \in \mathbb{C}$; one then shows by descending induction on $i \in\{d, \ldots, 1\}$ that, if $W_{\mathbb{C}}$ is stable by $\operatorname{Aut}(\mathbb{C})$, then $a_{i, j}$ are invariant by any automorphism of $\mathbb{C}$ over $\boldsymbol{k}$, i.e., belong to $\boldsymbol{k}$ since $\mathbb{C}$ is separable over $\boldsymbol{k}$.

Let us now prove the assertion. Let us first assume that $V$ is irreducible and let us consider the subspace $W_{\mathbb{C}}$ of $V_{\mathbb{C}}$ generated by the sub- $\Pi$-modules of minimal dimension (hence irreducible). Since the representation of $\Pi$ is defined over $\boldsymbol{k}$, if $E_{\mathbb{C}}$ is a $\Pi$-module, so is $\sigma\left(E_{\mathbb{C}}\right)$ for every $\sigma \in \operatorname{Aut}(\mathbb{C})$; therefore the space $W_{\mathbb{C}}$ is invariant by $\operatorname{Aut}(\mathbb{C})$, in other words takes the form $\mathbb{C} \otimes W$ for some subspace $W$ of $V$. It is clear that $W$ is a sub- $\Pi$-module of $V$, hence $W=V$. According to $4.3 .1(3), V_{\mathbb{C}}$ is semi-simple.

Conversely, let us assume that $V_{\mathbb{C}}$ is semi-simple. Let us choose a $\boldsymbol{k}$-linear form $\ell: \mathbb{C} \rightarrow \boldsymbol{k}$ such that $\ell(1)=1$. It defines a $\boldsymbol{k}$-linear map $L: V_{\mathbb{C}} \rightarrow V$ which is $\Pi$-invariant and which induces the identity on $V$. Let $W$ be a sub- $\Pi$-module of $V$. We have a $\Pi$-invariant projection $V_{\mathbb{C}} \rightarrow W_{\mathbb{C}}$, hence a composed projection $p$ which is $\Pi$-invariant:

from which one obtains a $\Pi$-module supplementary to $W$ in $V$.
(2) If $\Pi^{\prime \prime} \rightarrow \Pi$ is a surjective group-homomorphism and $\rho^{\prime \prime}$ is the composed representation, alors $V$ is a semi-simple $\Pi$-module if and only if it is a semi-simple
$\Pi^{\prime \prime}$-module. Indeed, the $\Pi$-module structure only depends on the image $\rho(\Pi) \subset$ GL( $V$ ).
(3) Let $\Pi^{\prime} \triangleleft \Pi$ be a normal subgroup, and let $V$ be a $\Pi$-module. Then, if $V$ is semi-simple as a $\Pi$-module, it so as a $\Pi^{\prime}$-module. Indeed, if $V^{\prime}$ is an irreducible sub- $\Pi^{\prime}$-module of $V$, then $\rho(\pi) V^{\prime}$ remains so for every $\pi \in \Pi$. If $V$ is $\Pi$-irreducible and if $V^{\prime}$ is a nonzero irreducible sub- $\Pi^{\prime}$-module, the sub- $\Pi^{\prime}$-module generated by the $\rho(\pi) V^{\prime}$ is a $\Pi$-module, hence coincides with $V$. As a consequence, $V$ is generated by its irreducible sub- $\Pi^{\prime}$-modules, hence is $\Pi^{\prime}$-semi-simple, according to 4.3.1(3).
(4) A real representation $\Pi \rightarrow \operatorname{Aut}\left(V_{\mathbb{R}}\right)$ is simple if and only if the associated complexified representation $\Pi \rightarrow \operatorname{Aut}\left(V_{\mathbb{C}}\right)$ has at most two simple components. [Hint: any simple component of the complexified representation can be summed with its conjugate to produce a sub-representation of the real representation.]
4.3.b. The semi-simplicity theorem. Let $X$ be a smooth projective variety. Let $H=\left(\mathcal{H}, F^{\prime \bullet} \mathcal{H}, F^{\prime \prime \bullet} \mathcal{H}, D, \mathrm{Q}\right)$ be a variation of polarized $\mathbb{C}$-Hodge structure of weight $w$ on $X$ (see Definition 4.1.3), and let $\underline{\mathcal{H}}=\operatorname{Ker} D$ be the associated complex local system.

Theorem 4.3.3. Under these assumptions, the complex local system $\underline{\mathcal{H}}$ is semi-simple.
Let us already note that the result is easy for unitary local systems (underlying thus polarized variations of type $(0,0)$, as in Section 4.2.a). The general case will use the objects introduced in Section 4.2.b, and will not be specific to variations of polarized Hodge structures. The proof of the semi-simplicity theorem will apply to more general objects called harmonic bundle, and will be given in Sections 4.3.d4.3.f. Moreover, we can relax the property that the smooth variety is projective, and only assume that it is a compact Kähler manifold, since we will only use the Kähler identities.

## Remarks 4.3.4.

(1) If $\underline{\mathcal{H}}$ is obtained from a local system $\underline{\mathcal{H}}_{\mathbb{Q}}$ defined over $\mathbb{Q}$, then $\underline{\mathcal{H}}_{\mathbb{Q}}$ is also semi-simple as such, according to Remark 4.3.2(1).
(2) We claim that, under these assumptions, the polarization is unique up to a positive multiplicative constant. Indeed, the polarization induces a morphism $\underline{\mathcal{H}} \rightarrow \underline{\mathcal{H}^{*}}$ between two irreducible local systems, hence is uniquely determined as such up to a nonzero multiplicative constant. The positivity condition on the polarization (i.e., on the associated Hermitian form h) implies that the constant must be positive.

## 4.3.c. Structure of variations of polarized $\mathbb{C}$-Hodge structure

Let $X$ be a complex manifold. We will say that two variations of polarized $\mathbb{C}$-Hodge structures are equivalent if one is obtained from the other one by a Tate twist $(k, \ell)$ (see Exercise 2.4.19) and by multiplying the polarization form by a positive constant.

Lemma 4.3.5. There exists at most one equivalence class of variations of polarized $\mathbb{C}$-Hodge structure on a simple (i.e., irreducible) $\mathbb{C}$-local system $\underline{\mathcal{H}}$ on a compact complex manifold $X$.

Remark 4.3.6. A criterion for the existence of a variation of polarized $\mathbb{C}$-Hodge structure on a simple $\mathbb{C}$-local system $\underline{\mathcal{H}}$ is given in $[\operatorname{Sim} 92, \S 4]$ in terms of rigidity.

Proof. If we are given two polarizable variations of $\mathbb{C}$-Hodge structure on an irreducible local system $\underline{\mathcal{H}}$, we deduce such a polarizable variation on $\mathscr{E} n d(\underline{\mathcal{H}})$ (Remark 4.1.11), and the dimension-one vector space $\operatorname{End}(\underline{\mathcal{H}}):=H^{0}(X, \mathscr{E} n d(\underline{\mathcal{H}}))$ is endowed with a $\mathbb{C}$-Hodge structure of some type $(k, \ell)$ by the Hodge-Deligne theorem 4.2.3. The identity morphism $\operatorname{Id}_{\mathcal{H}} \in \operatorname{End}(\underline{\mathcal{H}})$ defines thus a morphism of type $(k, \ell)$ between the two variations. Therefore, the first one is obtained from the second one by a Tate twist $(k, \ell)$. It remains to check that, on a given polarizable variation of $\mathbb{C}$-Hodge structure on an irreducible local system $\underline{\mathcal{H}}$, there exists exactly one polarization up to a positive multiplicative constant. Note that such a polarization is an isomorphism $\underline{\mathcal{H}} \xrightarrow{\sim} \underline{\mathcal{H}}^{*}$, so one polarization is obtained from another one by multiplying by a nonzero constant. This constant must be positive, by the positivity property of the associated Hermitian form.

Let $X$ be a compact complex manifold and let $H=\left(\mathcal{H}, F^{\bullet} \mathcal{H}, F^{\prime \prime \bullet} \mathcal{H}, D, \mathrm{Q}\right)$ be a variation of polarized $\mathbb{C}$-Hodge structure of weight $w$ on $X$. If the associated local system $\underline{\mathcal{H}}$ is semi-simple, which is the case when $X$ is Kähler, according to Theorem 4.3.3, it decomposes as $\underline{\mathcal{H}}=\bigoplus_{\alpha \in A} \underline{\mathcal{H}}_{\alpha}^{n_{\alpha}}$, where $\underline{\mathcal{H}}_{\alpha}$ are irreducible and pairwise non isomorphic, and $\underline{\mathcal{H}}_{\alpha}^{n_{\alpha}}$ means the direct sum of $n_{\alpha}$ copies of $\underline{\mathcal{H}}_{\alpha}$. Similarly, $(\mathcal{H}, D)=\bigoplus_{\alpha \in A}\left(\mathcal{H}_{\alpha}, D\right)^{n_{\alpha}}$, and the polarization Q , being $D$-horizontal, decomposes with respect to $\alpha \in A$ as $\mathrm{Q}=\bigoplus \mathrm{Q}_{\alpha, n_{\alpha}}$. Let us set $\mathcal{H}_{\alpha}^{o}:=\mathbb{C}^{n_{\alpha}}$ and let us write $\underline{\mathcal{H}}=\bigoplus_{\alpha \in A} \mathcal{H}_{\alpha}^{o} \otimes \underline{\mathcal{H}}_{\alpha}$. If we are given a basis $\mathrm{Q}_{\alpha}$ of the dimension-one vector space $\operatorname{Hom}\left(\mathcal{H}_{\alpha}, \mathcal{H}_{\alpha}^{*}\right)$, there exists a unique morphism $\mathrm{Q}_{\alpha}^{o} \in \operatorname{Hom}\left(\mathcal{H}_{\alpha}^{o}, \mathcal{H}_{\alpha}^{o *}\right)$ such that $\mathrm{Q}_{\alpha, n_{\alpha}}=\mathrm{Q}_{\alpha}^{o} \otimes \mathrm{Q}_{\alpha}$.

Proposition 4.3.7. Under these conditions, the following holds:
(1) For every $\alpha \in A$, there exists a unique equivalence class of variation of polarized $\mathbb{C}$-Hodge structure of weight $w$ on $\mathcal{\mathcal { H }}_{\alpha}$.
(2) For every $\alpha \in A$, let us fix a representative $H_{\alpha}=\left(\mathcal{H}_{\alpha}, F^{\prime \bullet} \mathcal{H}_{\alpha}, F^{\prime \prime \bullet} \mathcal{H}_{\alpha}, D, \mathrm{Q}_{\alpha}\right)$ of such an equivalence class. There exists then a polarized $\mathbb{C}$-Hodge structure

$$
H_{\alpha}^{o}=\left(\mathcal{H}_{\alpha}^{o}, F^{\prime \bullet} \mathcal{H}_{\alpha}^{o}, F^{\prime \prime \bullet} \mathcal{H}_{\alpha}^{o}, \mathrm{Q}_{\alpha}^{o}\right)
$$

of weight 0 with $\operatorname{dim} \mathcal{H}_{\alpha}^{o}=n_{\alpha}$ such that

$$
\begin{equation*}
H=\bigoplus_{\alpha \in A}\left(H_{\alpha}^{o} \otimes_{\mathbb{C}} H_{\alpha}\right) \tag{4.3.7*}
\end{equation*}
$$

Proof of Proposition 4.3.7.
(1) The uniqueness statement in 4.3.7(1) is given by Lemma 4.3.5. In order to prove the existence in 4.3.7(1), it is enough to exhibit for every $\alpha \in A$ a sub-variation of Hodge structure of $H$ of weight $w$ with underlying local system $\mathcal{\mathcal { H }}_{\alpha}$. The polarization Q will then induce a polarization $\mathrm{Q}_{\alpha}$, according to Exercise 4.1.14(1). For that purpose, it is enough to exhibit $\underline{\mathcal{H}}_{\alpha}$ as the image of an endomorphism $\underline{\mathcal{H}} \rightarrow \underline{\mathcal{H}}$ which is compatible with the Hodge structures: by abelianity (Corollary 4.1.13), this image is an object of $\operatorname{VHS}(X, \mathbb{C}, w)$. Let us therefore analyze $\operatorname{End}(\underline{\mathcal{H}})=H^{0}(X, \mathscr{E} n d(\underline{\mathcal{H}}))$.

If we set $\mathcal{H}_{\alpha}^{o}=\mathbb{C}^{n_{\alpha}}$, so that $\underline{\mathcal{H}}=\bigoplus_{\alpha}\left(\mathcal{H}_{\alpha}^{o} \otimes_{\mathbb{C}} \underline{\mathcal{H}}_{\alpha}\right)$, we have an algebra isomorphism $\operatorname{End}(\underline{\mathcal{H}}) \simeq \prod_{\alpha} \operatorname{End}\left(\mathcal{H}_{\alpha}^{o}\right)\left(\right.$ where $u_{\alpha} u_{\beta}=0$ if $u_{\alpha} \in \mathcal{H}_{\alpha}^{o}, u_{\beta} \in \mathcal{H}_{\beta}^{o}$ and $\left.\alpha \neq \beta\right)$. We know that the local system $\mathscr{E} n d(\underline{\mathcal{H}})$ underlies a variation of polarized $\mathbb{C}$-Hodge structure of weight 0 . Therefore, $\operatorname{End}(\underline{\mathcal{H}})$ underlies a $\mathbb{C}$-Hodge structure of weight 0 by the Hodge-Deligne theorem 4.2.3. It is then enough to show that each $\mathcal{H}_{\alpha}^{o}$ underlies $a \mathbb{C}$-Hodge structure $H_{\alpha}^{o}$ of weight 0 such that the equality $\operatorname{End}(\underline{\mathcal{H}})=\prod_{\alpha} \operatorname{End}\left(\mathcal{H}_{\alpha}^{o}\right)$ is compatible with the Hodge structures on both terms. Indeed, choose then any rank-one endomorphism $p_{\alpha}$ of some nonzero vector space $\mathcal{H}_{\alpha}^{o,(k,-k)}$. Extend it as a rank-one endomorphism of $\mathcal{H}_{\alpha}^{o}$ of type $(0,0)$ by mapping every other summand $\mathcal{H}_{\alpha}^{o,(\ell,-\ell)}$ to zero, and extend it similarly as a rank-one endomorphism of $\bigoplus_{\beta} \mathcal{H}_{\beta}^{o}$ of type $(0,0)$. One obtains thus a rank-one endomorphism in $\operatorname{End}(\underline{\mathcal{H}})^{0,0}$. With respect to this identification, its image is $\left(\operatorname{Im} p_{\alpha}\right) \otimes_{\mathbb{C}} \underline{\mathcal{H}}_{\alpha} \simeq \underline{\mathcal{H}}_{\alpha}$, as wanted.

Let us prove the assertion, which reduces to proving the existence of a grading of each $\mathcal{H}_{\alpha}^{o}$ giving rise to the Hodge grading of $\operatorname{End}(\underline{\mathcal{H}})$. By the product formula above, the $\mathbb{C}$-algebra $\operatorname{End}(\underline{\mathcal{H}})$ is semi-simple, with center $Z=\prod_{\alpha} \mathbb{C} \cdot \mathrm{Id}_{\mathcal{H}_{\alpha}^{o}}$. An algebra automorphism $\varphi$ of $\operatorname{End}(\underline{\mathcal{H}})$ induces an automorphism of the ring $Z$, whose matrix in the basis above only consists of zeros and ones. By the Skolem-Noether theorem (see e.g. [Bou12, §14, No 5, Th. 4]), algebra automorphisms for which the corresponding matrix is the identity are interior automorphisms, that is, products of interior automorphisms of each $\operatorname{End}\left(\mathcal{H}_{\alpha}^{o}\right)$. Any algebra automorphism can be composed with an automorphism with matrix having block entries Id or 0 in order that the matrix on $Z$ is the identity. As a consequence, the identity component of the group of algebra automorphisms $\operatorname{Aut}{ }^{\text {alg }}(\operatorname{End}(\underline{\mathcal{H}}))$ is identified with $\prod_{\alpha \in A}\left(\operatorname{Aut}\left(\mathcal{H}_{\alpha}^{o}\right) / \mathbb{C}^{*} \operatorname{Id}_{\alpha}\right)$.

As in Remark 2.4.13, the $\mathbb{C}$-Hodge structure of weight 0 on $\operatorname{End}(\underline{\mathcal{H}})$ defines a continuous representation $\rho: S^{1} \rightarrow \operatorname{Aut}(\operatorname{End}(\underline{\mathcal{H}}))$, such that $\rho(\lambda)=\lambda^{p}$ on $\operatorname{End}(\underline{\mathcal{H}})^{p,-p}$. Since the grading is compatible with the algebra structure, the continuous representation $\rho$ takes values in the group of algebra automorphisms Aut ${ }^{\text {alg }}(\operatorname{End}(\underline{\mathcal{H}}))$. Since $\rho(1)=I d$, it takes values in the identity component of $\operatorname{Aut}^{\text {alg }}(\operatorname{End}(\underline{\mathcal{H}}))$, i.e., in $\prod_{\alpha \in A}\left(\operatorname{Aut}\left(\mathcal{H}_{\alpha}^{o}\right) / \mathbb{C}^{*} \operatorname{Id}_{\alpha}\right)$. By the argument given in Remark 2.4.13, it defines a grading, up to a shift, on each $\mathcal{H}_{\alpha}^{o}$, as wanted.
(2) Let us now endow $\mathcal{H}_{\alpha}^{o}$ with a polarized $\mathbb{C}$-Hodge structure of weight 0 so that $(4.3 .7 *)$ holds. We already have obtained a grading, i.e., a $\mathbb{C}$-Hodge structure of weight 0 . In order to obtain a polarization of this $\mathbb{C}$-Hodge structure satisfying $(4.3 .7 *)$, we note that $\mathcal{H}_{\alpha}^{o}=\operatorname{Hom}\left(\underline{\mathcal{H}}_{\alpha}, \underline{\mathcal{H}}\right)$, and since $\mathscr{H} \operatorname{om}\left(\underline{\mathcal{H}}_{\alpha}, \underline{\mathcal{H}}\right)$ underlies a variation of polarized Hodge structure of weight 0 according to 4.3.7(1), $\mathcal{H}_{\alpha}^{o}$ comes equipped
with a polarized Hodge structure of weight 0 . By definition, the natural morphism $\mathcal{H}_{\alpha}^{o} \otimes \underline{\mathcal{H}}_{\alpha} \rightarrow \underline{\mathcal{H}}$ underlies a morphism of variations of polarized Hodge structure.
4.3.d. Harmonic bundles. Let $X$ be a complex manifold, let $(\mathcal{H}, D)$ be a flat $C^{\infty}$ bundle on $X$, and let h be a Hermitian metric on $\mathcal{H}$. We decompose $D$ into its $(1,0)$ and $(0,1)$ parts: $D=D^{\prime}+D^{\prime \prime}$.

Lemma 4.3.8. Given $(\mathcal{H}, D, \mathrm{~h})$, there exists a unique connection $D_{\mathrm{h}}=D_{\mathrm{h}}^{\prime}+D_{\mathrm{h}}^{\prime \prime}$ on $\mathcal{H}$ and a unique $C^{\infty}$-linear morphism $\theta=\theta^{\prime}+\theta^{\prime \prime}: \mathcal{H} \rightarrow \mathscr{E}_{X}^{1} \otimes \mathcal{H}$ satisfying the following properties:
(1) $D_{\mathrm{h}}$ is a Chern connection for h , i.e., $\mathrm{dh}(u, \bar{v})=\mathrm{h}\left(D_{\mathrm{h}} u, \bar{v}\right)+\mathrm{h}\left(u, \overline{D_{\mathrm{h}} v}\right)$, or equivalently, decomposing into types,

$$
\mathrm{d}^{\prime} \mathrm{h}(u, \bar{v})=\mathrm{h}\left(D_{\mathrm{h}}^{\prime} u, \bar{v}\right)+\mathrm{h}\left(u, \overline{D_{\mathrm{h}}^{\prime \prime} v}\right), \quad \mathrm{d}^{\prime \prime} \mathrm{h}(u, \bar{v})=\mathrm{h}\left(D_{\mathrm{h}}^{\prime \prime} u, \bar{v}\right)+\mathrm{h}\left(u, \overline{D_{\mathrm{h}}^{\prime} v}\right)
$$

(2) $\theta$ is self-adjoint with respect to h , i.e., $\mathrm{h}(\theta u, \bar{v})=\mathrm{h}(u, \overline{\theta v})$, or equivalently, decomposing into types,

$$
\mathrm{h}\left(\theta^{\prime} u, \bar{v}\right)=\mathrm{h}\left(u, \overline{\theta^{\prime \prime} v}\right), \quad \mathrm{h}\left(\theta^{\prime \prime} u, \bar{v}\right)=\mathrm{h}\left(u, \overline{\theta^{\prime} v}\right)
$$

(3) $D=D_{\mathrm{h}}+\theta$, or equivalently, decomposing into types,

$$
D^{\prime}=D_{\mathrm{h}}^{\prime}+\theta^{\prime}, \quad D^{\prime \prime}=D_{\mathrm{h}}^{\prime \prime}+\theta^{\prime \prime}
$$

Remark 4.3.9. In 4.3.8(1), we have extended the metric h in a natural way as a sesquilinear operator $\left(\mathscr{E}_{X}^{1} \otimes \mathcal{H}\right) \otimes \overline{\mathcal{H}} \rightarrow \mathscr{E}_{X}^{1}$ resp. $\mathcal{H} \otimes\left(\overline{\mathscr{E}_{X}^{1} \otimes \mathcal{H}}\right) \rightarrow \mathscr{E}_{X}^{1}$.

Proof. Let $D_{\mathrm{h}}$ be a Chern connection on $\mathcal{H}$. Let $A$ be a $\mathscr{C}_{X}^{\infty}$-linear morphism $A: \mathcal{H} \rightarrow \mathscr{E}_{X}^{\infty} \otimes \mathcal{H}$ which is skew-adjoint with respect to h , that is, such that $\mathrm{h}(A u, \bar{v})=-\mathrm{h}(u, \overline{A v})=0$ for every local sections $u, v$ of $\mathcal{H}$. Then the connection $D_{\mathrm{h}}+A$ is also compatible with the metric. So let us choose any Chern connection $\widetilde{D}_{\mathrm{h}}$ and let us set $A=D-\widetilde{D}_{\mathrm{h}}$. Let us decompose $A$ as $A^{+}+A^{-}$, with $A^{+}$self-adjoint and $A^{-}$skew-adjoint. We can thus set $D_{\mathrm{h}}=\widetilde{D}_{\mathrm{h}}+A^{-}$and $\theta=A^{+}$. Uniqueness is seen similarly.

Remark 4.3.10. Iterating 4.3.8(2), we find that $\theta^{\prime \prime} \wedge \theta^{\prime \prime}$ is h-adjoint to $-\theta^{\prime} \wedge \theta^{\prime}$ and $\theta^{\prime} \wedge \theta^{\prime \prime}+\theta^{\prime \prime} \wedge \theta^{\prime}$ is skew-adjoint. By applying $\mathrm{d}^{\prime}$ or $\mathrm{d}^{\prime \prime}$ to 4.3.8(1) and (2), we see that $D_{\mathrm{h}}^{\prime \prime 2}$ is adjoint to $-D_{\mathrm{h}}^{\prime 2}, D_{\mathrm{h}}^{\prime \prime}\left(\theta^{\prime}\right)$ is adjoint to $-D_{\mathrm{h}}^{\prime}\left(\theta^{\prime \prime}\right), D_{\mathrm{h}}^{\prime \prime}\left(\theta^{\prime \prime}\right)$ is adjoint to $-D_{\mathrm{h}}^{\prime}\left(\theta^{\prime}\right)$, and $D_{\mathrm{h}}^{\prime} D_{\mathrm{h}}^{\prime \prime}+D_{\mathrm{h}}^{\prime \prime} D_{\mathrm{h}}^{\prime}$ is skew-adjoint with respect to h.

Exercise 4.3.11. Let $(\mathcal{H}, D)$ be a flat bundle and let h be a Hermitian metric on $\mathcal{H}$.
(1) Show that there exist a unique ( 1,0 )-connection $\widehat{D}^{\prime}$ and a unique ( 0,1 )connection $\widehat{D}^{\prime \prime}$ such that $D^{\prime}+\widehat{D}^{\prime \prime}$ and $\widehat{D}^{\prime}+D^{\prime \prime}$ preserve the metric h.
(2) Show that $\widehat{D}^{\prime}=D_{\mathrm{h}}^{\prime}-\theta^{\prime}$ and $\widehat{D}^{\prime \prime}=D_{\mathrm{h}}^{\prime \prime}-\theta^{\prime \prime}$.
(3) We set $D^{\mathrm{c}}:=\widehat{D}^{\prime \prime}-\widehat{D}^{\prime}$. Show that $\frac{1}{2}\left(D+D^{\mathrm{c}}\right)=D_{\mathrm{h}}^{\prime \prime}+\theta^{\prime}$ and that

$$
D D^{\mathrm{c}}+D^{\mathrm{c}} D=2 D\left(\theta^{\prime}-\theta^{\prime \prime}\right)
$$

Definition 4.3.12 (Harmonic flat bundle). Let $(\mathcal{H}, D, \mathrm{~h})$ be a flat $C^{\infty}$-bundle endowed with a Hermitian metric h. We say that $(\mathcal{H}, D, \mathrm{~h})$ is a harmonic flat bundle if the operator $D_{\mathrm{h}}^{\prime \prime}+\theta^{\prime}=\frac{1}{2}\left(D+D^{\mathrm{c}}\right)$ has square 0 .

By looking at types, this is equivalent to

$$
D_{\mathrm{h}}^{\prime \prime 2}=0, \quad D_{\mathrm{h}}^{\prime \prime}\left(\theta^{\prime}\right)=0, \quad \theta^{\prime} \wedge \theta^{\prime}=0
$$

By adjunction, this implies

$$
D_{\mathrm{h}}^{\prime 2}=0, \quad D_{\mathrm{h}}^{\prime}\left(\theta^{\prime \prime}\right)=0, \quad \theta^{\prime \prime} \wedge \theta^{\prime \prime}=0
$$

Moreover, the flatness of $D$ implies then

$$
D_{\mathrm{h}}^{\prime}\left(\theta^{\prime}\right)=0, \quad D_{\mathrm{h}}^{\prime \prime}\left(\theta^{\prime \prime}\right)=0, \quad D_{\mathrm{h}}^{\prime} D_{\mathrm{h}}^{\prime \prime}+D_{\mathrm{h}}^{\prime \prime} D_{\mathrm{h}}^{\prime}=-\left(\theta^{\prime} \wedge \theta^{\prime \prime}+\theta^{\prime \prime} \wedge \theta^{\prime}\right)
$$

Lemma 4.3.13. Let $(\mathcal{H}, D)$ be a flat bundle and let h be a Hermitian metric on $\mathcal{H}$. Then

$$
\left(D_{\mathrm{h}}^{\prime \prime}+\theta^{\prime}\right)^{2}=0 \Longrightarrow D D^{\mathrm{c}}+D^{\mathrm{c}} D=0
$$

Proof. Since $D^{2}=0$, it is a matter of proving $\left(D^{\mathrm{c}}\right)^{2}=0$. From the vanishing above, we find $\left(\widehat{D}^{\prime}\right)^{2}=0,\left(\widehat{D}^{\prime \prime}\right)^{2}=0$. We also get $\widehat{D}^{\prime \prime} \widehat{D}^{\prime}+\widehat{D}^{\prime} \widehat{D}^{\prime \prime}=0$.

Let $E=\operatorname{Ker} D_{\mathrm{h}}^{\prime \prime}: H \rightarrow H$. If $(\mathcal{H}, D, \mathrm{~h})$ is harmonic, $E$ is a holomorphic vector bundle equipped with a holomorphic $\operatorname{End}(E)$-valued 1-form $\theta^{\prime}$ satisfying $\theta^{\prime} \wedge \theta^{\prime}=0$. It is called a Higgs bundle and $\theta^{\prime}$ is its associated Higgs field.

Proposition 4.3.14. Let $(\mathcal{H}, D, \mathrm{~h})$ be a flat bundle with metric underlying a polarized variation of $\mathbb{C}$-Hodge structure on $X$. Then $(\mathcal{H}, D, \mathrm{~h})$ is a harmonic flat bundle.

Proof. This is the content of Exercise 4.2.2.
4.3.e. The energy functional. We now fix the metric $h$ on the $C^{\infty}$-bundle $\mathcal{H}$. The group of $C^{\infty}$ automorphisms $g$ of $\mathcal{H}$ acts on a given connection $D$ by the formula ${ }_{g} D:=g \circ D \circ g^{-1}=D-D(g) \circ g^{-1}$, where we have extended the action of $D$ in a natural way on the bundle $\mathscr{E} n d(\mathcal{H})$. If $D$ is flat, then so is ${ }_{g} D$. We then set ${ }_{g} D={ }_{g} D_{\mathrm{h}}+{ }_{g} \theta$. Let us also set $\widehat{D}=D_{\mathrm{h}}-\theta$.
Lemma 4.3.15. We have ${ }_{g} \theta=\theta-\frac{1}{2}\left(D(g) g^{-1}+g^{*-1} \widehat{D}\left(g^{*}\right)\right)$, where $g^{*}$ is the h -adjoint of $g$.

Proof. We have

$$
{ }_{g} D=D_{\mathrm{h}}+\theta-\left(D_{\mathrm{h}}(g) g^{-1}+[\theta, g] g^{-1}\right)=D_{\mathrm{h}}-D_{\mathrm{h}}(g) g^{-1}+g^{-1} \theta g .
$$

It follows that ${ }_{g} \theta$ is the self-adjoint part of $-D_{\mathrm{h}}(g) g^{-1}+g^{-1} \theta g$, that is, taking into account that the adjoint of $D_{\mathrm{h}}(g)$ is $D_{\mathrm{h}}\left(g^{*}\right)$ (by working in a local h-orthonormal basis),

$$
\begin{equation*}
g_{g} \theta=\frac{1}{2}\left(-\left(D_{\mathrm{h}}(g) g^{-1}+g^{*-1} D_{\mathrm{h}}\left(g^{*}\right)\right)+g^{-1} \theta g+g^{*} \theta g^{*-1}\right) . \tag{4.3.16}
\end{equation*}
$$

The lemma follows from a straightforward computation.

If we fix a metric on $X$, we deduce with h a metric on $\mathscr{E}_{X}^{1} \otimes \mathcal{H}$ and then a metric $\|\cdot\|$ on $\mathscr{H} \operatorname{om}\left(\mathcal{H}, \mathscr{E}_{X}^{1} \otimes \mathcal{H}\right)$ with associated scalar product $(\cdot, \cdot)$. We then denote by $\langle\cdot, \cdot\rangle$ the integrated product using the volume form on $X$ :

$$
\langle\cdot, \cdot\rangle=\int_{X}(\cdot, \cdot) \mathrm{d} \operatorname{vol} .
$$

Definition 4.3.17. The energy of $g \in \operatorname{Aut}(\mathcal{H})$ with respect to $(\mathcal{H}, D, \mathrm{~h})$ is defined as

$$
\mathrm{E}_{(\mathcal{H}, D, \mathrm{~h})}(g):=\left\|_{g} \theta\right\|^{2}=\left\langle{ }_{g} \theta,{ }_{g} \theta\right\rangle .
$$

Let $\xi \in \operatorname{End}(\mathcal{H})$ and let $\xi=\xi^{+}+\xi^{-}$be its decomposition into its self-adjoint part $\xi^{+}=\frac{1}{2}\left(\xi+\xi^{*}\right)$ and its skew-adjoint part $\xi^{-}=\frac{1}{2}\left(\xi-\xi^{*}\right)$.

Proposition 4.3.18. For $t$ varying in $\mathbb{R}$, we have

$$
\left.\frac{\mathrm{d}}{\mathrm{~d} t} \mathrm{E}_{(\mathcal{H}, D, \mathrm{~h})}\left(e^{t \xi}\right)\right|_{t=0}=2\left\langle D_{\mathrm{h}} \xi^{+}, \theta\right\rangle
$$

Proof. We have $D\left(e^{t \xi}\right) e^{-t \xi}=t D \xi \bmod t^{2}$ and $e^{-t \xi^{*}} \widehat{D}\left(e^{t \xi^{*}}\right)=t \widehat{D} \xi^{*} \bmod t^{2}$. From Lemma 4.3.15 we deduce

$$
\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{~d} t} \mathrm{E}_{(\mathcal{H}, D, \mathrm{~h})}\left(e^{t \xi}\right)\right|_{t=0} & =-\left\langle D \xi+\widehat{D} \xi^{*}, \theta\right\rangle-\left\langle\theta, D \xi+\widehat{D} \xi^{*}\right\rangle \\
& =-\left\langle D_{\mathrm{h}} \xi^{+}+\left[\theta, \xi^{-}\right], \theta\right\rangle-\left\langle\theta, D_{\mathrm{h}} \xi^{+}+\left[\theta, \xi^{-}\right]\right\rangle \\
& =-2 \operatorname{Re}\left\langle D_{\mathrm{h}} \xi^{+}, \theta\right\rangle=-2\left\langle D_{\mathrm{h}} \xi^{+}, \theta\right\rangle
\end{aligned}
$$

since $\left\langle\theta \xi^{-}, \theta\right\rangle=-\left\langle\theta, \theta \xi^{-}\right\rangle,\left\langle\xi^{-} \theta, \theta\right\rangle=-\left\langle\theta, \xi^{-} \theta\right\rangle$, and both $\theta$ and $D_{\mathrm{h}} \xi^{+}$are selfadjoint.

The property of being semi-simple or not for $(\mathcal{H}, D)$ is seen on the energy functional.
Proposition 4.3.19. Let $(\mathcal{H}, D)$ be a flat bundle. Assume that there exists a metric h such that the energy function $g \mapsto \mathrm{E}_{(\mathcal{H}, D, \mathrm{~h})}(g)$ has a critical point at $g=\mathrm{Id}$. Then $(\mathcal{H}, D)$ is semi-simple.

Proof. Let us argue by contraposition and let us assume that $(\mathcal{H}, D)$ is not semisimple. Let h be any metric on $\mathcal{H}$. We will prove that Id is not a critical point for $g \mapsto \mathrm{E}_{(\mathcal{H}, D, \mathrm{~h})}(g)$. It is enough to prove that there exists $\xi \in \operatorname{End}(\mathcal{H})$ such that the function

$$
f: \mathbb{R} \longrightarrow \mathbb{R}, \quad t \longmapsto \mathrm{E}_{(\mathcal{H}, D, \mathrm{~h})}\left(e^{t \xi}\right)
$$

has no critical point at $t=0$. By assumption, there exists a sub-bundle $\mathcal{H}_{1}$ of $\mathcal{H}$ stable by $D$ such that its orthogonal $\mathcal{H}_{2}$ is not stable by $D$. Set $n_{i}=\operatorname{rk} \mathcal{H}_{i}(i=1,2)$. With respect to this decomposition we have

$$
D=\left(\begin{array}{cc}
D_{1} & 2 \eta \\
0 & D_{2}
\end{array}\right)
$$

with $\eta: \mathcal{H}_{2} \rightarrow \mathscr{E}_{X}^{1} \otimes \mathcal{H}_{1}$ nonzero. Set $\xi=n_{2} \operatorname{Id}_{\mathcal{H}_{1}}-n_{1} \operatorname{Id}_{\mathcal{H}_{2}}$ and $g=e^{t \xi}(t \in \mathbb{R})$. We have
so

$$
\begin{aligned}
D\left(e^{t \xi}\right) e^{-t \xi} & =\left[\left(\begin{array}{cc}
0 & 2 \eta \\
0 & 0
\end{array}\right),\left(\begin{array}{cc}
e^{n_{2} t} & 0 \\
0 & e^{-n_{1} t}
\end{array}\right)\right] \cdot\left(\begin{array}{cc}
e^{-n_{2} t} & 0 \\
0 & e^{n_{1} t}
\end{array}\right) \\
& =\left(\begin{array}{ll}
0 & 2 \eta \\
0 & 0
\end{array}\right)-\left(\begin{array}{cc}
e^{-n_{2} t} & 0 \\
0 & e^{n_{1} t}
\end{array}\right)\left(\begin{array}{cc}
0 & 2 \eta \\
0 & 0
\end{array}\right)\left(\begin{array}{cc}
e^{n_{2} t} & 0 \\
0 & e^{-n_{1} t}
\end{array}\right) \\
& =\left(\begin{array}{ccc}
0 & 2\left(1-e^{-\left(n_{1}+n_{2}\right) t}\right) \eta \\
0 & 0
\end{array}\right),
\end{aligned}
$$

$$
{ }_{g} D=D-D\left(e^{t \xi}\right) e^{-t \xi}=\left(\begin{array}{cc}
D_{1} & 2 e^{-\left(n_{1}+n_{2}\right) t} \eta \\
0 & D_{2}
\end{array}\right)
$$

and

$$
{ }_{g} \theta=\left(\begin{array}{cc}
\theta_{1} & e^{-\left(n_{1}+n_{2}\right) t} \eta \\
e^{-\left(n_{1}+n_{2}\right) t} \eta^{*} & \theta_{2}
\end{array}\right) .
$$

It follows that

$$
f(t)=c_{0}+c_{1} e^{-\left(n_{1}+n_{2}\right) t}, \quad c_{0} \geqslant 0, c_{1}>0
$$

and it is clear that $f^{\prime}(0) \neq 0$.
4.3.f. Proof of the semi-simplicity theorem. In view of Proposition 4.3.19, the semi-simplicity theorem is a consequence of the following.
Proposition 4.3.20. Assume that $X$ is compact Kähler and that $(\mathcal{H}, D, \mathrm{~h})$ is a harmonic flat bundle. Then $g \mapsto \mathrm{E}_{(\mathcal{H}, D, \mathrm{~h})}(g)$ has a critical point at $g=\mathrm{Id}$.
Proof. According to Proposition 4.3.18, it is enough to show that $D_{\mathrm{h}}^{*} \theta=0$, where $D_{\mathrm{h}}^{*}$ denotes the formal adjoint of $D_{\mathrm{h}}$. Setting $D_{\mathrm{h}}^{\mathrm{c}}:=D_{\mathrm{h}}^{\prime \prime}-D_{\mathrm{h}}^{\prime}$, the Kähler identities for a Hermitian vector bundle imply that $D_{\mathrm{h}}^{*}$ is a multiple of $\left[\Lambda, D_{\mathrm{h}}^{\mathrm{c}}\right]$. Since $\theta$ is a matrix of 1 -forms and $\Lambda$ is an operator of type $(-1,-1)$, we have $\Lambda \theta=0$. On the other hand, by the properties after Definition 4.3.12, we have $D_{\mathrm{h}}^{\mathrm{c}}(\theta)=0$.

### 4.4. Comments

Although one can trace back the notion of variation of Hodge structure to the study of the Legendre family of elliptic curves in the nineteenth century, the modern approach using the Gauss-Manin connection goes back to the fundamental work of Griffiths [Gri68, Gri70a, Gri70b] motivated by the properties of the period domain (see also [Del71c], [CMSP03]), a subject that is not considered in the present text. In the work of Griffiths, the transversality property (4.1.1) has been emphasized. From the point of view of $\mathscr{D}$-modules, this property is now encoded in the notion of a coherent filtration, and is at the heart of the notion of filtered $\mathscr{D}$-module, which is part of a Hodge module as defined by Saito.

The notion of a variation of polarized Hodge structure can be regarded as equivalent to the notion of a smooth polarized Hodge module. However, this equivalence is
not obvious since the definition of a polarized Hodge module imposes properties on nearby cycles along any germ of holomorphic function, while the notion of variation only requires to consider coordinate functions.

The $C^{\infty}$ approach as in Definition 4.1.4 proves useful for extending the Hodge theorem on smooth complex projective varieties and constant coefficients to the case when the coefficient system is a unitary local system (see [Dem96]) and the more general case when it underlies a variation of polarized Hodge structure (Hodge-Deligne theorem 4.2.3 explained in the introduction of [Zuc79]). It is also well-adapted to the extension of this theorem to harmonic flat bundles, as explained by Simpson in [Sim92]. In this smooth context, the flat sesquilinear pairing $\mathfrak{c}$ gives rise in a natural way to the (non-flat in general) Hermitian Hodge metric. The fixed-part theorem, proved in Remark 4.2.4(4), is originally due to Griffiths [Gri70a] in a geometric setting, and has been proved in a more general context by Deligne [Del71b, Cor. 4.1.2], and also by Schmid [Sch73, Th. 7.22].

We have also mentioned the case of complete Kähler manifolds, going back to Andreotti and Vesentini [AV65] and Hörmander [Hör65, Hör66]. Theorem 4.2.5 is taken from [Dem96, §12B]. They are useful for understanding the $L^{2}$ approach as in Zucker's theorem 5.1.1 of [Zuc79].

It is remarkable that the local system underlying a variation of polarized Hodge structure on a smooth complex projective variety (or a compact Kähler manifold) is semi-simple. This property can be regarded as a special case of a result of Corlette [Cor88] and [Sim92], since the Hodge metric is a pluri-harmonic metric on the corresponding flat holomorphic bundle. These articles are at the source of Sections 4.3.b and 4.3.d-4.3.f.

Lastly, the structure theorem for variations of polarized Hodge structure (Proposition 4.3.7) is nothing but [Del87, Prop. 1.13].


[^0]:    1. The precise definition is as follows. Let $\overline{\mathscr{O}}_{X}$ denote the sheaf of anti-holomorphic functions on $X$ and regard $\mathscr{O}_{X}$ as an $\overline{\mathscr{O}}_{X}$-module: the action of an anti-holomorphic function $\bar{g}$ on an holomorphic function $f$ is by definition $\bar{g} \cdot f:=g f$. Then any $\overline{\mathscr{O}}_{X}$-module $E^{\prime \prime}$ determines an $\mathscr{O}_{X}$-module $\overline{E^{\prime \prime}}$ by setting $\overline{E^{\prime \prime}}:=\mathscr{O}_{X} \otimes_{\overline{\mathscr{O}}_{X}} E^{\prime \prime}$.
[^1]:    2. When we work with a variation of polarized Hodge structure, the polarization Q identifies $(\mathcal{H}, D)$ and $\left(\mathcal{H}^{\vee}, D^{\vee}\right)$ and we recover the usual conjugation relation between $H^{q, p}$ and $H^{p, q}$.
