## CHAPTER 3

## HODGE-LEFSCHETZ STRUCTURES


#### Abstract

Summary. We develop the notion of a Hodge-Lefschetz structure as the first example of a mixed Hodge structure. The total cohomology of a smooth complex projective variety, together with the Chern class of an ample line bundle, gives rise to the notion of graded Hodge-Lefschetz structure. On the other hand, degenerations of one-parameter families of smooth complex projective varieties are the main provider of possibly non graded Hodge-Lefschetz structures. Vanishing cycles of holomorphic functions with isolated critical points also produce such structures.


### 3.1. The Lefschetz decomposition

3.1.a. A-Lefschetz structures. We fix an abelian category A. Let $H$ be an object of A equipped with a nilpotent endomorphism N (i.e., $\mathrm{N}^{k+1}=0$ for $k$ large).

Lemma 3.1.1 (Jakobson-Morosov). There exists a unique increasing exhaustive filtration of $H$ indexed by $\mathbb{Z}$, called the monodromy filtration relative to N and denoted by M. (N)H or simply M. $H$, satisfying the following properties:
(a) For every $\ell \in \mathbb{Z}, \mathrm{N}\left(\mathrm{M}_{\ell} H\right) \subset \mathrm{M}_{\ell-2} H$,
(b) For every $\ell \geqslant 1, \mathrm{~N}^{\ell}$ induces an isomorphism $\mathrm{gr}_{\ell}^{\mathrm{M}} H \xrightarrow{\sim} \mathrm{gr}_{-\ell}^{\mathrm{M}} H$.

## Definition 3.1.2 ((Graded) Lefschetz structure).

(1) We call such a pair $(H, \mathrm{~N})$ an A -Lefschetz structure. A morphism between two such pairs is a morphism in A which commutes with the nilpotent endomorphisms.
(2) Assume moreover that $H$ is a graded object in A . We then say that $(H, \mathrm{~N})$ a graded A-Lefschetz structure if $H_{\ell}=\operatorname{gr}_{\ell}^{\mathrm{M}} H$ for every $\ell$.

For a pair $(H, \mathrm{~N})$, we will denote by $\operatorname{grN}$ the induced morphism $\mathrm{gr}_{\ell}^{\mathrm{M}} H \rightarrow \operatorname{gr}_{\ell-2}^{\mathrm{M}} H$. Therefore, an A-Lefschetz structure $(H, \mathrm{~N})$ gives rise to a graded A-Lefschetz structure, namely, the graded pair ( $\left.\mathrm{gr}^{\mathrm{M}} H, \operatorname{grN}\right)$. Any morphism $\varphi:\left(H_{1}, \mathrm{~N}_{1}\right) \rightarrow\left(H_{2}, \mathrm{~N}_{2}\right)$ is compatible with the monodromy filtrations and induces a graded morphism of degree zero $\operatorname{gr} \varphi:\left(\operatorname{gr}_{\bullet}^{\mathrm{M}} H_{1}, \operatorname{grN}_{1}\right) \rightarrow\left(\mathrm{gr}^{\mathrm{M}} H_{2}, \operatorname{grN}_{2}\right)$.

The proof of Lemma 3.1.1 is left as an exercise. In case of finite-dimensional vector spaces, one can prove the existence by using the decomposition into Jordan blocks and Example 3.1.3. In general, one proves it by induction on the index of nilpotence. The uniqueness is interesting to prove. In fact, there is an explicit formula for this filtration in terms of the kernel filtration of N and of its image filtration (see [SZ85]).

Example 3.1.3. If $H$ is a finite dimensional vector space and if N consists only of one lower Jordan block of size $k+1$, one can write the basis as $e_{k}, e_{k-2}, \ldots, e_{-k}$, with $\mathrm{N} e_{j}=e_{j-2}$. Then $\mathrm{M}_{\ell}$ is the space generated by the $e_{j}$ 's with $j \leqslant \ell$.

## Exercise 3.1.4 (Inductive construction of the monodromy filtration)

Assume $\mathrm{N}^{\ell+1}=0$ on $H$. Show the following properties:
(1) $\mathrm{M}_{\ell} H=H, \mathrm{M}_{\ell-1} H=\operatorname{Ker}^{\ell}, \mathrm{M}_{-\ell} H=\operatorname{Im~}^{\ell}, \mathrm{M}_{-\ell-1} H=0$.
(2) Set $H^{\prime}=\operatorname{Ker} \mathrm{N}^{\ell} / \operatorname{Im} \mathrm{N}^{\ell}$ and $\mathrm{N}^{\prime}: H^{\prime} \rightarrow H^{\prime}$ is induced by N . Then $\mathrm{N}^{\prime \ell}=0$ and for $j \in[-\ell+1, \ell-1], \mathrm{M}_{j} H$ is the pullback of $\mathrm{M}_{j} H^{\prime}$ by the projection $H \rightarrow H^{\prime}$.
(3) Conclude that any morphism of A-Lefschetz structures is compatible with the monodromy filtrations.

## Remark 3.1.5 (Primitive parts and Lefschetz decomposition)

For vector spaces, the choice of a splitting of the filtration (which always exists for a filtration on a finite dimensional vector space) corresponds to the choice of a Jordan decomposition of N . The decomposition (hence the splitting) is not unique, although the filtration is. In general, there exists a decomposition of the graded object, called the Lefschetz decomposition (see Figure 1). For every $\ell \geqslant 0$, we set

$$
\begin{equation*}
\mathrm{P}_{\ell}(H):=\operatorname{Ker}(\operatorname{grN})^{\ell+1}: \operatorname{gr}_{\ell}^{\mathrm{M}}(H) \longrightarrow \operatorname{gr}_{-\ell-2}^{\mathrm{M}}(H) \tag{3.1.5*}
\end{equation*}
$$

Then for every $k \geqslant 0$, we have

$$
\begin{equation*}
\operatorname{gr}_{k}^{\mathrm{M}}(H)=\underset{j \geqslant 0}{\bigoplus} \mathrm{~N}^{j} \mathrm{P}_{k+2 j}(H) \quad \text { and } \quad \operatorname{gr}_{-k}^{\mathrm{M}}(H)=\underset{j \geqslant 0}{\bigoplus} \mathrm{~N}^{k+j} \mathrm{P}_{k+2 j}(H) . \tag{3.1.5**}
\end{equation*}
$$

Exercise 3.1.6. Assume that $\ell \geqslant 0$. Show that

$$
\mathrm{P}_{\ell}(H) \oplus \mathrm{NP}_{\ell+2}(H)=\operatorname{Ker}(\mathrm{grN})^{\ell+2}: \operatorname{gr}_{\ell}^{\mathrm{M}}(H) \longrightarrow \operatorname{gr}_{-\ell-4}^{\mathrm{M}}(H)
$$

[Hint: consider the rough Lefschetz decomposition

$$
\operatorname{gr}_{\ell}(H)=\mathrm{P}_{\ell}(H) \oplus \mathrm{NP}_{\ell+2}(H) \oplus\left(\operatorname{grN}^{2} \operatorname{gr}_{\ell+4} H\right.
$$

and show that the first two terms are contained in $\operatorname{Ker}(\operatorname{grN})^{\ell+2}$, while $(\operatorname{grN})^{\ell+2}$ is injective on the third term.]

We can apply the above results to the category of Hodge structures $\mathrm{HS}(\mathbb{C}, w)$, to the category of mixed Hodge structures $\mathrm{MHS}(\mathbb{C})$, or to the category of holonomic $\mathscr{D}$-modules for instance.


Figure 1. A graphical way of representing the Lefschetz decomposition: the arrows represent the isomorphisms induced by N ; each $\mathrm{gr}_{\ell}^{\mathrm{M}}(H)$ is the direct sum of the terms of its line, where empty places are replaced with 0 .

Lemma 3.1.7. Let $H_{1}, H_{2}$ be two objects of an abelian category A, equipped with nilpotent endomorphisms $\mathrm{N}_{1}, \mathrm{~N}_{2}$. Let $\varphi:\left(H_{1}, \mathrm{~N}_{1}\right) \rightarrow\left(H_{2}, \mathrm{~N}_{2}\right)$ be a morphism which is strictly compatible with the corresponding monodromy filtrations $\mathrm{M}\left(\mathrm{N}_{1}\right), \mathrm{M}\left(\mathrm{N}_{2}\right)$. Then

$$
\operatorname{Im} \mathrm{N}_{1} \cap \operatorname{Ker} \varphi=\mathrm{N}_{1}(\operatorname{Ker} \varphi) \quad \text { and } \quad \operatorname{Im} \mathrm{N}_{2} \cap \operatorname{Im} \varphi=\mathrm{N}_{2}(\operatorname{Im} \varphi) .
$$

Proof. Let us first consider the graded morphisms $\operatorname{gr}_{\ell}^{\mathrm{M}} \varphi: \operatorname{gr}_{\ell}^{\mathrm{M}} H_{1} \rightarrow \operatorname{gr}_{\ell}^{\mathrm{M}} H_{2}$. One easily checks that it decomposes with respect to the Lefschetz decomposition. It follows that the property of the lemma is true at the graded level.

Let us now denote by $\mathrm{M}\left(\mathrm{N}_{1}\right)$. $\operatorname{Ker} \varphi$ (resp. $\mathrm{M}\left(\mathrm{N}_{2}\right)$. Coker $\varphi$ ) the induced filtration on $\operatorname{Ker} \varphi$ (resp. Coker $\varphi$ ). Since $\varphi$ is strictly compatible with $\mathrm{M}\left(\mathrm{N}_{1}\right), \mathrm{M}\left(\mathrm{N}_{2}\right)$, we have for every $\ell$ an exact sequence

$$
0 \longrightarrow \operatorname{gr}_{\ell}^{\mathrm{M}\left(\mathrm{~N}_{1}\right)} \operatorname{Ker} \varphi \longrightarrow \operatorname{gr}_{\ell}^{\mathrm{M}\left(\mathrm{~N}_{1}\right)} H_{1} \xrightarrow{\operatorname{gr}_{\ell}^{\mathrm{M}} \varphi} \operatorname{gr}_{\ell}^{\mathrm{M}\left(\mathrm{~N}_{2}\right)} H_{2} \longrightarrow \operatorname{gr}_{\ell}^{\mathrm{M}\left(\mathrm{~N}_{2}\right)} \operatorname{Coker} \varphi \longrightarrow 0
$$

from which we conclude that $\mathrm{M}\left(\mathrm{N}_{1}\right) \operatorname{Ker} \varphi\left(\right.$ resp. $\mathrm{M}\left(\mathrm{N}_{2}\right)$ Coker $\varphi$ ) satisfies the characteristic properties of the monodromy filtration of $\mathrm{N}_{1 \mid \operatorname{Ker} \varphi}$ (resp. $\mathrm{N}_{2 \mid \text { Coker } \varphi}$ ). As a consequence, $\operatorname{Ker} \varphi \cap \mathrm{M}\left(\mathrm{N}_{1}\right)_{\ell}=\mathrm{M}\left(\mathrm{N}_{1 \mid \operatorname{Ker} \varphi}\right)_{\ell}$ and $\operatorname{Im} \varphi \cap \mathrm{M}\left(\mathrm{N}_{2}\right)_{\ell}=\mathrm{M}\left(\mathrm{N}_{2 \mid \operatorname{Im} \varphi}\right)_{\ell}$ for every $\ell$.

Let us show the first equality, the second one being similar. By the result at the graded level we have

$$
\operatorname{Im} \mathrm{N}_{1} \cap \operatorname{Ker} \varphi \cap \mathrm{M}\left(\mathrm{~N}_{1}\right)_{\ell}=\mathrm{N}_{1}\left(\operatorname{Ker} \varphi \cap \mathrm{M}\left(\mathrm{~N}_{1}\right)_{\ell+2}\right)+\operatorname{Im} \mathrm{N}_{1} \cap \operatorname{Ker} \varphi \cap \mathrm{M}\left(\mathrm{~N}_{1}\right)_{\ell-1}
$$

and we can argue by induction on $\ell$ to conclude.
Lemma 3.1.8 (Strictness of $\mathrm{N}:(H, \mathrm{M} . H) \rightarrow(H, \mathrm{M}[2] . H)$ ). The morphism N , regarded as a filtered morphism $(H, \mathrm{M} . H) \rightarrow(H, \mathrm{M}[2] . H)$ is strictly compatible with the filtrations, i.e., for every $\ell, \mathrm{N}\left(\mathrm{M}_{\ell}\right)=\operatorname{Im} \mathrm{N} \cap \mathrm{M}_{\ell-2}$.

Proof. By looking at Figure 1, one shows that
(1) if $\ell \leqslant 1, \mathrm{~N}: \mathrm{M}_{\ell} H \rightarrow \mathrm{M}_{\ell-2} H$ is onto,
(2) if $\ell \geqslant-2, \mathrm{~N}: H / M_{\ell+2} H \rightarrow H / \mathrm{M}_{\ell} H$ is injective.

The lemma follows.
The following criterion is at the heart of the decomposition theorem 13.1.5, whose proof will not be reproduced here.

Theorem 3.1.9 (Deligne's criterion). Let A be an abelian category and let $\mathscr{C}^{\bullet}$ be an object of $\mathrm{D}^{\mathrm{b}}(\mathrm{A})$ endowed with an endomorphism $\mathrm{L}: \mathscr{C}^{\bullet} \rightarrow \mathscr{C}^{\bullet+2}$. Assume that $\left(\bigoplus_{k} H^{k}\left(\mathscr{C}^{\bullet}\right), \mathrm{L}\right)$ is a graded A-Lefschetz structure (see Definition 3.1.2 and set $\left.H_{-k}\left(\mathscr{C}^{\bullet}\right)=H^{k}\left(\mathscr{C}^{\bullet}\right)\right)$. Then $\mathscr{C}^{\bullet}$ is isomorphic to $\bigoplus_{k} H^{k}\left(\mathscr{C}^{\bullet}\right)[-k]$ in $\mathrm{D}^{\mathrm{b}}(\mathrm{A})$.

Remark 3.1.10 (Nilpotent $\sigma$-endomorphisms). We will have to apply the previous results in a slightly more general setting. We assume that the abelian category A is endowed with an automorphism $\sigma: \mathrm{A} \mapsto \mathrm{A}$. By a $\sigma$-endomorphism of an object $H$ of A we mean a morphism $\mathrm{N}: H \rightarrow \sigma^{-1} H$. It defines for every $k$ a morphism $\sigma^{-k} \mathrm{~N}: \sigma^{-k} H \rightarrow \sigma^{-k-1} H$. We say that N is nilpotent if there exists $k \geqslant 0$ such that $\sigma^{-k} \mathrm{~N} \circ \cdots \circ \sigma^{-1} \mathrm{~N} \circ \mathrm{~N}=0$. The previous results extend in an obvious way to the setting of $\sigma$-endomorphisms.

Let us now take up the notation of Section A.2.b on strictness. We equip the category $\operatorname{Modgr}(\widetilde{\mathscr{A}})$ of graded $\widetilde{\mathscr{A}}$-modules with the automorphism $\sigma$ shifting the grading by one, so that $\sigma(H)=H(1)$ (see Definition A.2.3). Let $H$ be an object of $\operatorname{Modgr}(\widetilde{\mathscr{A}})$ and let $\mathrm{N}: H \rightarrow H(-1)=\sigma^{-1} H$ be a nilpotent endomorphism.

Proposition 3.1.11. Let M. $(\mathrm{N}) H$ be the monodromy filtration of $(H, \mathrm{~N})$ in the abelian category $\operatorname{Modgr}(\widetilde{\mathscr{A}})$. Assume that $H$ is strict. Then the following properties are equivalent:
(1) For every $\ell \geqslant 1, \mathrm{~N}^{\ell}: H \rightarrow H(-\ell)$ is a strict morphism.
(2) For every $\ell \in \mathbb{Z}$, $\operatorname{gr}_{\ell}^{\mathrm{M}} H$ is strict.
(3) For every $\ell \geqslant 0, \mathrm{P}_{\ell} H$ is strict.

Proof. The equivalence between (2) and (3) comes from the Lefschetz decomposition in the category $\operatorname{Modgr}(\widetilde{\mathscr{A}})$.
$(2) \Rightarrow(1)$ Assume $\ell \geqslant 1$. The Lefschetz decomposition implies that each morphism $\mathrm{gr}_{-2 \ell} \mathrm{~N}^{\ell}$ on $\mathrm{gr}^{\mathrm{M}} H$ is strict. Since $\mathrm{N}^{\ell}$, regarded as a filtered morphism $(H, \mathrm{M} . H) \rightarrow$ $\left(H(-\ell), \mathrm{M}[2 \ell]_{\bullet}\right)$ is strictly compatible with the filtrations M (Lemma 3.1.8), the result follows from Lemma A.2.9(2).
$(1) \Rightarrow(2)$ We will use the inductive construction of the monodromy filtration given in Exercise 3.1.4. We argue by induction on the order of nilpotence of N. Assume that $\mathrm{N}^{\ell+1}=0$. The strictness of $H$ implies that $\mathrm{M}_{\ell} H, \mathrm{M}_{\ell-1} H, \mathrm{M}_{-\ell} H=\mathrm{gr}_{-\ell}^{\mathrm{M}} H$ and $\mathrm{P}_{\ell} H=\operatorname{gr}_{\ell}^{\mathrm{M}} H \simeq \mathrm{gr}_{-\ell}^{\mathrm{M}} H$ are strict. The strictness of and $H^{\prime}:=H / \mathrm{M}_{-\ell} H=$ Coker $\mathrm{N}^{\ell}$ follows from the strictness of $\mathrm{N}^{\ell}$. Moreover, $\left(H^{\prime}, \mathrm{N}^{\prime}\right)$ satisfies (1) with $\mathrm{N}^{\prime \ell}=0$, hence by induction each $\mathrm{gr}_{j}^{\mathrm{M}} H^{\prime}$ is strict. Now, the relation between $\mathrm{gr}_{.}^{\mathrm{M}} H^{\prime}$ and $\mathrm{gr}^{\mathrm{M}} H$ is easily seen from the Lefschetz decomposition (see Figure 1), and (2) for $\mathrm{gr}_{{ }^{\mathrm{M}}}{ }^{H}$ follows.

Definition 3.1.12 (Adjoint and pre-polarization). Assume that $A$ is equipped with a contravariant involution $*: \mathrm{A} \mapsto \mathrm{A}^{\mathrm{op}}$ (i.e., such that $* *=\mathrm{Id}$ ).

- The adjoint of an A-Lefschetz structure $(H, \mathrm{~N})$ is the A-Lefschetz structure $(H, \mathrm{~N})^{*}:=\left(H^{*},-\mathrm{N}^{*}\right)$. We have $\operatorname{gr}_{\ell}^{\mathrm{M}}\left(H^{*}\right)=\left(\mathrm{gr}_{-\ell}^{\mathrm{M}} H\right)^{*}$ and, for a morphism, $\operatorname{gr}_{\ell}\left(\varphi^{*}\right)=\left(\mathrm{gr}_{-\ell} \varphi\right)^{*}$.
- The adjoint of a graded A-Lefschetz structure $(H, \mathrm{~N})$ is the graded A-Lefschetz structure $\left(H_{\bullet}, \mathrm{N}\right)^{*}:=\left(H_{\bullet}^{*},-\mathrm{N}^{*}\right)$, with $H_{\ell}^{*}:=\left(H_{-\ell}\right)^{*}$.
- A pre-polarization of an A-Lefschetz structure $(H, \mathrm{~N})$ is an isomorphism $\mathrm{k}:(H, \mathrm{~N}) \rightarrow(H, \mathrm{~N})^{*}$. It is said to be $\pm$-Hermitian if $\mathrm{k}^{*}= \pm \mathrm{k}$.

If k is a pre-polarization of $(H, \mathrm{~N})$, then for every $\ell, \mathrm{gr}_{\ell}^{\mathrm{M}} \mathrm{k}$ is an isomorphism $\operatorname{gr}_{\ell}^{\mathrm{M}} H \rightarrow\left(\mathrm{gr}_{-\ell}^{\mathrm{M}} H\right)^{*}$ which satisfies $\left(\mathrm{gr}_{\ell}^{\mathrm{M}} \mathrm{k}\right)^{*}=\mathrm{gr}_{-\ell}^{\mathrm{M}}\left(\mathrm{k}^{*}\right)$.

## 3.1.b. Nearby/vanishing Lefschetz quivers

By a nearby/vanishing quiver on an abelian category A we mean a data ( $H, G, \mathrm{c}, \mathrm{v}$ ) consisting of a pair $(H, G)$ of objects of $A$ and a pair of morphisms

such that $\mathrm{c} \circ \mathrm{v}$ is nilpotent (on $G$ ) and $\mathrm{v} \circ \mathrm{c}$ is nilpotent (on $H$ ). We denote by $\mathrm{N}_{H}, \mathrm{~N}_{G}$ the corresponding nilpotent endomorphisms.

We say that a nearby/vanishing quiver $(H, G, \mathrm{c}, \mathrm{v})$ is a middle extension if c is an epimorphism and v is a monomorphism. We say that it has a punctual support if $H=0 .{ }^{(1)}$ We say that a nearby/vanishing quiver $(H, G, \mathrm{c}, \mathrm{v})$ is Support-decomposable, or simply $S$-decomposable, if it can be decomposed as the direct sum of a middle extension quiver and a quiver with punctual support.

## Exercise 3.1.14.

(1) Show that the nearby/vanishing quivers on $A$ form an abelian category in an obvious way.
(2) Show that there is no nonzero morphism from a middle extension to an object with punctual support.
(3) Show that there is no nonzero morphism from an object with punctual support to a middle extension.
(4) Show that a nearby/vanishing quiver ( $H, G, \mathrm{c}, \mathrm{v}$ ) is S-decomposable if and only if $G=\operatorname{Imc} \oplus \operatorname{Ker} v$. Show then that the decomposition is unique.
(5) Show that the latter condition is also equivalent to the conjunction of the following two conditions:

- The natural morphism $\operatorname{Im}(\mathrm{v} \circ \mathrm{c}) \rightarrow \operatorname{Im} \mathrm{v}$ is an isomorphism.
- The natural morphism $\operatorname{Ker} \mathrm{c} \rightarrow \operatorname{Ker}(\mathrm{v} \circ \mathrm{c})$ is an isomorphism.

Let $(H, \mathrm{~N})$ be an A-Lefschetz structure. Set $G=\operatorname{Im} \mathrm{N}$ and $\mathrm{N}_{G}=\mathrm{N}_{\mid G}$. The nearby/vanishing quiver

is called the middle extension quiver attached to $\left(H, \mathrm{~N}_{H}\right)$.
Lemma 3.1.16 (The middle extension quiver). We have the following properties.
(a) $\mathrm{M} .\left(\mathrm{N}_{G}\right)=G \cap \mathrm{M}[1] .(\mathrm{N})=\mathrm{N}(\mathrm{M}[-1] .(\mathrm{N}))$.
(b) $\mathrm{c}\left(\mathrm{M}_{\bullet} H\right) \subset \mathrm{M}_{\bullet-1} G, \mathrm{v}\left(\mathrm{M}_{\bullet} G\right) \subset \mathrm{M}_{\bullet-1} H$,
(c) the morphisms

$$
\mathrm{c}:(H, \mathrm{M} \cdot(\mathrm{~N})) \longrightarrow\left(G, \mathrm{M}[1] \cdot\left(\mathrm{N}_{G}\right)\right) \quad \text { and } \quad \mathrm{v}:\left(G, \mathrm{M}_{\bullet}\left(\mathrm{N}_{G}\right)\right) \longrightarrow(H, \mathrm{M}[1] \cdot(\mathrm{N}))
$$

are strictly filtered and the associated graded morphisms are the corresponding canonical morphisms at the graded level. More precisely,
(d) $\operatorname{gr}(\mathrm{c})$ is an epimorphism $\operatorname{gr}_{\ell+1}^{\mathrm{M}}(H) \rightarrow \operatorname{gr}_{\ell}^{\mathrm{M}}(G)$ and $\operatorname{gr}(\mathrm{v})$ is a monomorphism $\operatorname{gr}_{\ell}^{\mathrm{M}}(G) \rightarrow \operatorname{gr}_{\ell-1}^{\mathrm{M}}(H)$,

[^0](e) $\operatorname{gr}_{\ell}^{\mathrm{M}}(G) \xrightarrow{\sim} \operatorname{Im}\left[\operatorname{grN}: \operatorname{gr}_{\ell+1}^{\mathrm{M}}(H) \rightarrow \operatorname{gr}_{\ell-1}^{\mathrm{M}}(H)\right] \simeq \begin{cases}\operatorname{gr}_{\ell+1}^{\mathrm{M}}(H) & \text { if } \ell \geqslant 0, \\ \operatorname{gr}_{\ell-1}^{\mathrm{M}}(H) & \text { if } \ell \leqslant 0,\end{cases}$
(f) $\mathrm{P}_{\ell}(G) \simeq \operatorname{grN}\left(\mathrm{P}_{\ell+1}(H)\right)$ for $\ell \geqslant 0$.

Proof. Assume that $\ell \geqslant 0$. We first check that the morphism $\mathrm{N}^{\ell}: \operatorname{Im} \mathrm{N} \cap \mathrm{M}_{\ell-1}(\mathrm{~N}) \rightarrow$ $\operatorname{Im} \mathrm{N} \cap \mathrm{M}_{-\ell-1}(\mathrm{~N})$ is an isomorphism. By Lemma 3.1.8, this amounts to showing that $\mathrm{N}^{\ell}: \mathrm{N}\left(\mathrm{M}_{\ell+1}\right) \rightarrow \mathrm{N}\left(\mathrm{M}_{-\ell+1}\right)$ is an isomorphism. This is a consequence of the following properties: $\mathrm{N}: \mathrm{M}_{\ell+1} \rightarrow \mathrm{~N}\left(\mathrm{M}_{\ell+1}\right)$ is an isomorphism, $\mathrm{N}: \mathrm{M}_{-\ell+1} \rightarrow \mathrm{M}_{-\ell-1}$ is onto, and $\mathrm{N}^{\ell+1}: \mathrm{M}_{\ell+1} \rightarrow \mathrm{M}_{-\ell-1}$ is an isomorphism. Now, (b) and (c) follow from the strictness of $\mathrm{N}:\left(H, \mathrm{M}_{\bullet} H\right) \rightarrow(H, \mathrm{M}[2] . H)$. The remaining part of the lemma is straightforward.

Lemma 3.1.17 (Pre-polarization of $\left(G, \mathrm{~N}_{G}\right)$ ). Let $\left((H, \mathrm{~N}),\left(G, \mathrm{~N}_{G}\right), \mathrm{c}, \mathrm{v}\right)$ be a middle extension quiver, and let k be a pre-polarization of $(H, \mathrm{~N})$. There exists a unique pre-polarization $\mathrm{k}_{G}$ of $\left(G, \mathrm{~N}_{G}\right)$ which satisfies

$$
\mathrm{k}_{G} \circ \mathrm{c}=\mathrm{v}^{*} \circ \mathrm{k} .
$$

Moreover, its adjoint $\mathrm{k}_{G}^{*}$ satisfies

$$
\mathrm{k}_{G}^{*} \circ \mathrm{c}=-\mathrm{v}^{*} \circ \mathrm{k}^{*}
$$

In particular, if k is $\pm$-Hermitian, then $\mathrm{k}_{G}$ is $\mp$-Hermitian.
Proof. We have $G^{*}=H^{*} / \operatorname{Ker} \mathrm{N}^{*}=: \operatorname{Coim}\left(\mathrm{N}^{*}\right)$, and a diagram


In order to check the existence and uniqueness of $\mathrm{k}_{G}$, it suffices to prove that $\mathrm{v}^{*} \circ \mathrm{k}_{\mid \mathrm{Ker} \mathrm{N}}=0$. Since c is an epimorphism, $\mathrm{c}^{*}$ is a monomorphism and it is enough to check that $c^{*} v^{*} \circ \mathrm{k}_{\mid \operatorname{KerN}}=0$. This follows from $c^{*} \mathrm{v}^{*} \circ \mathrm{k}=\mathrm{N}^{*} \circ \mathrm{k}=-\mathrm{k} \circ \mathrm{N}$.

It follows that $\mathrm{c}^{*} \circ \mathrm{k}_{G}^{*}=\mathrm{k}^{*} \circ \mathrm{v}$, and thus

$$
\mathrm{c}^{*} \circ \mathrm{k}_{G}^{*} \circ \mathrm{c}=\mathrm{k}^{*} \circ \mathrm{~N}=-\mathrm{N}^{*} \circ \mathrm{k}^{*}=-\mathrm{c}^{*}\left(\mathrm{v}^{*} \circ \mathrm{k}^{*}\right) .
$$

Since $\mathrm{c}^{*}$ is a monomorphism we conclude that $\mathrm{k}_{G}^{*} \circ \mathrm{c}=-\mathrm{v}^{*} \circ \mathrm{k}^{*}$.
3.1.c. Graded $\mathbb{C}$-Lefschetz structures and representations of $\mathfrak{s l}_{2}$. We now consider the category A of finite dimensional $\mathbb{C}$-vector spaces. Let $\left(\mathcal{H}_{\mathbf{0}}, \mathrm{N}\right)$ be a graded Lefschetz structure on a $\mathbb{C}$-vector space $\mathcal{H}=\bigoplus_{\ell} \mathcal{H}_{\ell}$. Hence N is a morphism $\mathcal{H}_{\ell} \rightarrow$ $\mathcal{H}_{\ell-2}$, and for every $\ell \geqslant 1, \mathrm{~N}^{\ell}: \mathcal{H}_{\ell} \rightarrow \mathcal{H}_{-\ell}$ is an isomorphism, so that $\mathcal{H}$ comes with its Lefschetz decomposition in terms of the primitive spaces $\mathrm{P} \mathcal{H}_{\ell}(\ell \geqslant 0)$.

Remark 3.1.18. The Lefschetz operator in cohomology (see (2.3.10)) arises as a graded morphism of degree two $\mathrm{L}: H^{\ell} \rightarrow H^{\ell+2}$. Given a graded vector space $\mathcal{H}=\bigoplus_{\ell} \mathcal{H}^{\ell}$ endowed with a graded endomorphism of degree two, we associate to it the graded space $\bigoplus_{\ell} \mathcal{H}_{\ell}$, with $\mathcal{H}_{\ell}:=\mathcal{H}^{-\ell}$, to recover the previous setting.

The following operations can be done on graded $\mathbb{C}$-Lefschetz structures:
(a) Morphisms. A morphism $\varphi:\left(\mathcal{H}_{1, \bullet}, \mathrm{~N}_{1}\right) \rightarrow\left(\mathcal{H}_{2, \bullet}, \mathrm{~N}_{2}\right)$ is a graded morphism $\mathcal{H}_{1, \bullet} \rightarrow \mathcal{H}_{2, \bullet}$ of degree zero which commutes with $\mathrm{N}_{1}, \mathrm{~N}_{2}$.
(b) Conjugation. We set $\overline{\left(\mathcal{H}_{\bullet}, \mathrm{N}\right)}=\left(\overline{\mathcal{H}_{\bullet}}, \overline{\mathrm{N}}\right)$.
(c) Tensor product. We define $\left(\mathcal{H}_{\bullet}, \mathrm{N}\right)=\left(\mathcal{H}_{1, \bullet}, \mathrm{~N}_{1}\right) \otimes\left(\mathcal{H}_{2, \bullet}, \mathrm{~N}_{2}\right)$ by

$$
\mathcal{H}_{\ell}=\bigoplus_{\ell_{1}+\ell_{2}=\ell} \mathcal{H}_{1, \ell_{1}} \otimes \mathcal{H}_{2, \ell_{2}} \quad \text { and } \quad \mathrm{N}=\left(\mathrm{N}_{1} \otimes \mathrm{Id}\right) \oplus\left(\operatorname{Id} \otimes \mathrm{N}_{2}\right)
$$

(d) Duality. We define the dual $\left(\mathcal{H}_{\bullet}, \mathrm{N}\right)^{\vee}$ as $\left(\mathcal{H}_{\bullet}^{\vee},-\mathrm{N}^{\vee}\right)$, with $\mathcal{H}_{\ell}^{\vee}:=\left(\mathcal{H}_{-\ell}\right)^{\vee}$.
(e) Hermitian adjunction. The Hermitian adjoint $\left(\mathcal{H}_{\bullet}, \mathrm{N}\right)^{*}$ is defined as $\left(\mathcal{H}_{\bullet}^{*},-\mathrm{N}^{*}\right)$, with $\mathcal{H}_{\ell}^{*}:=\left(\mathcal{H}_{-\ell}\right)^{*}=\left(\overline{\mathcal{H}_{-\ell}}\right)^{\vee}$ and $\mathrm{N}^{*}:=\overline{\mathrm{N}}^{\vee}$.
(f) Internal Hom. We set

$$
\operatorname{Hom}\left(\left(\mathcal{H}_{1, \bullet}, \mathrm{~N}_{1}\right),\left(\mathcal{H}_{2, \bullet}, \mathrm{~N}_{2}\right)\right)=\left(\mathcal{H}_{1, \bullet}, \mathrm{~N}_{1}\right)^{\vee} \otimes\left(\mathcal{H}_{2, \bullet}, \mathrm{~N}_{2}\right)
$$

Remark 3.1.19 (Compatibility with grading). Of course, one can define similar operations for $\mathbb{C}$-Lefschetz structures without grading, and one recovers the previous ones by grading with respect to the monodromy filtration. Note however that the kernel, image and cokernel of morphisms behave well only with the assumption of strictness with respect to M. This is the case for N itself, and for the morphisms c and v , as remarked in Lemma 3.1.8 and Lemma 3.1.16. In particular, we have

$$
\operatorname{gr}_{\bullet}^{\mathrm{M}}\left(\operatorname{Im} \mathrm{~N}, \mathrm{~N}_{\operatorname{Im} \mathrm{N}}\right)=\left((\operatorname{Im} \operatorname{grN})_{\bullet}, \operatorname{grN}_{\operatorname{Im} \operatorname{grN}}\right) .
$$

For example, the tensor product $(\mathcal{H}, \mathrm{N})=\left(\mathcal{H}_{1}, \mathrm{~N}_{1}\right) \otimes\left(\mathcal{H}_{2}, \mathrm{~N}_{2}\right)$ is defined as $\mathcal{H}=\mathcal{H}_{1} \otimes \mathcal{H}_{2}$ and $\mathrm{N}=\left(\mathrm{N}_{1} \otimes \mathrm{Id}\right) \oplus\left(\mathrm{Id} \otimes \mathrm{N}_{2}\right)$. We have $\mathrm{M}_{\ell}(\mathrm{N})=\sum_{\ell_{1}+\ell_{2}=\ell} \mathrm{M}_{\ell_{1}}\left(\mathrm{~N}_{1}\right) \otimes$ $\mathrm{M}_{\ell_{2}}\left(\mathrm{~N}_{2}\right)$ : this is proved by showing that the right-hand side satisfies the characteristic properties of the monodromy filtration. It follows that $\left(\mathrm{gr}_{\cdot}^{\mathrm{M}} \mathcal{H}, \operatorname{grN}\right)=$ $\left(\mathrm{gr}_{1, \bullet}^{\mathrm{M}} \mathcal{H}, \operatorname{grN}_{1}\right) \otimes\left(\mathrm{gr}_{2, \bullet}^{\mathrm{M}} \mathcal{H}, \operatorname{grN}_{2}\right)$ as defined by (c) above.

Similarly, $(\mathcal{H}, \mathrm{N})^{*}:=\left(\mathcal{H}^{*},-\mathrm{N}^{*}\right)$ has monodromy filtration $\mathrm{M}_{\ell}\left(-\mathrm{N}^{*}\right)=\mathrm{M}_{-\ell}(\mathrm{N})^{*}$ and we have $\operatorname{gr}_{\bullet}^{\mathrm{M}}(\mathcal{H}, \mathrm{N})^{*}=\left[\operatorname{gr}_{-\mathrm{\bullet}}^{\mathrm{M}}(\mathcal{H}, \mathrm{N})\right]^{*}$ as defined by (e) above.

Recall that $\mathfrak{s l}_{2}(\mathbb{C})$ is the Lie algebra generated by the three elements usually denoted by X, Y, H which satisfy the relations

$$
[\mathrm{X}, \mathrm{Y}]=\mathrm{H}, \quad[\mathrm{H}, \mathrm{X}]=2 \mathrm{X}, \quad[\mathrm{H}, \mathrm{Y}]=-2 \mathrm{Y}
$$

## Lemma 3.1.20.

(1) Let $\left(\mathcal{H}_{\mathbf{0}}, \mathrm{N}\right)$ be a graded $\mathbb{C}$-Lefschetz structure. There exists a unique representation of $\mathfrak{s l}_{2}$ on $\mathcal{H}$ (i.e., a Lie algebra morphism $\mathfrak{s l}_{2} \rightarrow \operatorname{End}(\mathcal{H})$ ) such that $Y$ acts by N and such that $\mathcal{H}_{\ell}$ is the eigenspace corresponding to the eigenvalue $\ell$ of H for every $\ell \in \mathbb{Z}$. Moreover, X sends $\mathcal{H}_{\ell}$ to $\mathcal{H}_{\ell+2}$ for every $\ell \in \mathbb{Z}$ and, for $\ell \geqslant 0$, we
have $\mathrm{P}_{\ell} \mathcal{H}=\operatorname{Ker}\left[\mathrm{X}: \mathcal{H}_{\ell} \rightarrow \mathcal{H}_{\ell+2}\right]$. Lastly, any endomorphism $Z \in \operatorname{End}(\mathcal{H})$ which commutes with Y and H also commutes with X .
(2) Conversely, given $\mathrm{X}, \mathrm{Y}, \mathrm{H} \in \operatorname{End}(\mathcal{H})$ satisfying the relations of $\mathfrak{s l}_{2}$ and such that H is semi-simple with integral eigenvalues, the grading by eigenspaces $\mathcal{H}_{\mathbf{0}}$, together with Y , defines a graded $\mathbb{C}$-Lefschetz structure.

We abuse notation by denoting by the same letters $\mathrm{X}, \mathrm{Y}, \mathrm{H}$ and their images in $\operatorname{End}(\mathcal{H})$.

Sketch of proof. If X exists, the relation $[\mathrm{H}, \mathrm{X}]=2 \mathrm{X}$ implies that X sends $\mathcal{H}_{\ell}$ to $\mathcal{H}_{\ell+2}$ for every $\ell \in \mathbb{Z}$. For $\ell \geqslant 0$, let $\boldsymbol{e}_{\ell}$ be a basis of $\mathrm{P} \mathcal{H}_{\ell}$. Then one can easily find constants $a_{j, \ell}$ such that for every $e_{\ell} \in \boldsymbol{e}_{\ell}$, if we define X by $\mathrm{X}\left(\mathrm{Y}^{j} e_{\ell}\right)=a_{j, \ell} \mathrm{Y}^{j-1} e_{\ell}$ (with the convention that $\mathrm{Y}^{-1} e_{\ell}=0$ ), then X satisfies the desired relations. For the uniqueness it suffices to check that if $[Z, \mathrm{Y}]=0$ and $[\mathrm{H}, Z]=2 Z$, then $Z=0$. For $\ell \geqslant 0$, the composition $\mathrm{Y}^{\ell+2} Z: \mathrm{P} \mathcal{H}_{\ell} \rightarrow \mathcal{H}_{-\ell-2}$, being equal to $Z \mathrm{Y}^{\ell+2}$, is zero, so $Z$ is zero on $\mathrm{P} \mathcal{H}_{\ell}$. It is then easy to conclude that $Z$ is zero on each $\mathrm{Y}^{j} \mathrm{P} \mathcal{H}_{\ell}(j \geqslant 0)$.

Let now $Z \in \operatorname{End}(\mathcal{H})$ be such that $Z$ commutes with Y and H . Then for every $t \in \mathbb{R},\left(e^{-t Z} \mathrm{X} e^{t Z}, \mathrm{Y}, \mathrm{H}\right)$ is also an $\mathfrak{s l}_{2}$ triple, and by uniqueness the function $t \mapsto$ $e^{-t Z} \mathrm{X} e^{t Z}-\mathrm{X}$ is identically zero, so $[\mathrm{X}, Z]=0$.

The second part of the lemma is proved similarly.

## Exercise 3.1.21.

(1) Show the following identities in $\operatorname{End}(\mathcal{H})$ :

$$
\begin{array}{ll}
e^{\mathrm{Y}} \mathrm{H} e^{-\mathrm{Y}}=\mathrm{H}+2 \mathrm{Y}, & e^{-\mathrm{X}} \mathrm{Y} e^{\mathrm{X}}=\mathrm{Y}-\mathrm{H}-\mathrm{X}, \\
e^{-\mathrm{X}} \mathrm{H} e^{\mathrm{X}}=\mathrm{H}+2 \mathrm{X}, & e^{\mathrm{Y}} \mathrm{X} e^{-\mathrm{Y}}=\mathrm{X}-\mathrm{H}-\mathrm{Y} .
\end{array}
$$

[Hint: denote by ad $\mathrm{Y}: \operatorname{End}(\mathcal{H}) \rightarrow \operatorname{End}(\mathcal{H})$ the Lie algebra morphism $A \mapsto[\mathrm{Y}, A]$; show that $e^{\mathrm{Y}} \mathrm{H} e^{-\mathrm{Y}}=e^{\operatorname{ad} \mathrm{Y}}(\mathrm{H})=\mathrm{H}+[\mathrm{Y}, \mathrm{H}]+\frac{1}{2}[\mathrm{Y},[\mathrm{Y}, \mathrm{H}]]+\cdots$ and conclude for the first equality; argue similarly for the other ones.]
(2) Set $\mathrm{w}:=e^{-\mathrm{X}} e^{\mathrm{Y}} e^{-\mathrm{X}} \in \operatorname{Aut}(\mathcal{H})$. Show that

$$
\mathrm{wH}=-\mathrm{Hw}, \quad \mathrm{wX}=-\mathrm{Y} w, \quad \mathrm{wY}=-\mathrm{X} w .
$$

Conclude that w sends $\mathcal{H}_{\ell}$ to $\mathcal{H}_{-\ell}$ for every $\ell$.
(3) For $\ell \geqslant 0$, show that $\mathrm{w}_{\mid \mathrm{P}_{\ell} \mathcal{H}}=\mathrm{Y}_{\mid \mathrm{P} \ell}^{\ell} \mathcal{H} / \ell$ !. [Hint: use (2) to avoid any computation.]
(4) Deduce that, for $\ell \geqslant 0$ and $0 \leqslant k \leqslant \ell$,

$$
\mathrm{w}_{\mid \mathrm{P}_{\ell} \mathcal{H}}^{k}=\frac{(-1)^{k}}{\ell!} \mathrm{X}^{k} \mathrm{Y}_{\mid \mathrm{P}_{\ell} \mathcal{H}}^{\ell}=\frac{(-1)^{k}}{(\ell-k)!} \mathrm{Y}_{\mid \mathrm{P}_{\ell} \mathcal{H}}^{\ell-k}
$$

## Exercise 3.1.22 (The $\mathfrak{s l}_{2}$ representation on $\operatorname{End}(\mathcal{H})$ ).

(1) Let $\left(\mathcal{H}_{\bullet}, \mathrm{N}\right)$ be a graded $\mathbb{C}$-Lefschetz structure. Consider the grading End. $(\mathcal{H})$ defined by $\operatorname{End}_{\ell}(\mathcal{H}):=\bigoplus_{k} \operatorname{Hom}\left(\mathcal{H}_{k}, \mathcal{H}_{k+\ell}\right)$, and the nilpotent endomorphism ad $\mathrm{N}=$ $[\mathrm{N}, \bullet]$. Show that this defines the $\mathfrak{s l}_{2}$ representation for which H acts by ad $\mathrm{H}, \mathrm{X}$ by $\operatorname{ad} \mathrm{X}$, and w by $\mathrm{Adw}:=\mathrm{w} \cdot \mathrm{w}^{-1}$.
(2) Show that if $d \in \operatorname{End}_{-\ell}(\mathcal{H})(\ell \geqslant 0)$ commutes with N , then $\mathrm{w}^{-1} d \mathrm{w} \in \operatorname{End}_{\ell}(\mathcal{H})$ belongs to $\mathrm{P}_{\ell} \operatorname{End}(\mathcal{H})$, i.e., commutes with X .

Exercise 3.1.23. Let $\varphi:\left(\mathcal{H}_{1, \bullet}, \mathrm{~N}_{1}\right) \rightarrow\left(\mathcal{H}_{2, \bullet}, \mathrm{~N}_{2}\right)$ be a morphism of graded $\mathbb{C}$-Lefschetz structures.
(1) Show that $\varphi$ commutes with the action of X. [Hint: equip $\operatorname{Hom}\left(\mathcal{H}_{1}, \mathcal{H}_{2}\right)$ with an $\mathfrak{s l}_{2}$-action as in 3.1.22(1) above; with respect to this action, show that $\mathrm{H}(\varphi)=0$, i.e., $\varphi \in \operatorname{Hom}_{0}\left(\mathcal{H}_{1}, \mathcal{H}_{2}\right)$, and $\mathrm{Y}(\varphi)=0$, i.e., $\mathrm{N}_{2} \circ \varphi-\varphi \circ \mathrm{N}_{1}=0$, and deduce that $\varphi \in \mathrm{P}_{0} \operatorname{Hom}\left(\mathcal{H}_{1}, \mathcal{H}_{2}\right)$; conclude that $\mathrm{X}(\varphi)=0$.]
(2) Show that the internal Hom of 3.1.c(f) is nothing but $\mathrm{P}_{0} \operatorname{Hom}\left(\mathcal{H}_{1}, \mathcal{H}_{2}\right)$.

## 3.1.d. Polarized (graded) $\mathbb{C}$-Lefschetz structures

## Definition 3.1.24 (Polarization of a (graded) $\mathbb{C}$-Lefschetz Hodge structure)

(1) A polarization of a graded $\mathbb{C}$-Lefschetz structure $\left(\mathcal{H}_{\bullet}, \mathrm{N}\right)$ is a morphism (in fact an isomorphism, see Proposition 3.1.26 below)

$$
\mathrm{k}:\left(\mathcal{H}_{\bullet}, \mathrm{N}\right) \longrightarrow\left(\mathcal{H}_{\bullet}, \mathrm{N}\right)^{*}
$$

such that, for every $\ell \geqslant 0$, the induced composed morphism

$$
\mathrm{P}_{\ell} \mathrm{k}:=\mathrm{N}^{* \ell} \circ \mathrm{k}: \mathrm{P}_{\ell} \longrightarrow\left(\mathrm{N}^{* \ell} \mathrm{P} \mathcal{H}_{\ell}^{*}\right)=\left(\mathrm{P}_{\ell} \mathcal{H}\right)^{*}
$$

is a positive definite Hermitian form.
(2) A polarization of a $\mathbb{C}$-Lefschetz structure $(\mathcal{H}, \mathrm{N})$ is a morphism (in fact an isomorphism)

$$
\mathrm{k}:(\mathcal{H}, \mathrm{N}) \longrightarrow(\mathcal{H}, \mathrm{N})^{*}
$$

such that the graded morphism $\operatorname{gr}_{\bullet}^{\mathrm{M}} \mathrm{k}: \operatorname{gr}_{\bullet}^{\mathrm{M}}(\mathcal{H}, \mathrm{N}) \rightarrow \operatorname{gr}_{\bullet}^{\mathrm{M}}\left[(\mathcal{H}, \mathrm{N})^{*}\right]=\left[\operatorname{gr}_{-}^{\mathrm{M}}(\mathcal{H}, \mathrm{N})\right]^{*}$ is a polarization of the graded $\mathbb{C}$-Lefschetz structure $\operatorname{gr}_{\bullet}^{\mathrm{M}}(\mathcal{H}, \mathrm{N})$ (see Remark 3.1.19).

Let us make explicit this definition in the graded case. Since k is a morphism, it is graded. If we regard k as a sesquilinear pairing $\mathrm{k}: \mathcal{H} \otimes \overline{\mathcal{H}} \rightarrow \mathbb{C}$, this means that $\mathrm{k}=0$ when restricted to $\mathcal{H}_{k} \otimes \overline{\mathcal{H}_{\ell}}$ with $\ell \neq-k$. For every $\ell \in \mathbb{Z}$, the restriction $\mathrm{k}_{\ell}: \mathcal{H}_{\ell} \rightarrow \mathcal{H}_{\ell}^{*}=\left(\mathcal{H}_{-\ell}\right)^{*}$ of k is a sesquilinear pairing $\mathrm{k}_{\ell}: \mathcal{H}_{\ell} \otimes \overline{\mathcal{H}_{-\ell}} \rightarrow \mathbb{C}$ which satisfies $\mathrm{k}_{\ell}(u, \overline{\mathrm{~N} v})=-\mathrm{k}_{\ell+2}(\mathrm{~N} u, \bar{v})$ for $u \in \mathcal{H}_{\ell}, v \in \mathcal{H}_{-\ell+2}$. Lastly, the positivity condition reads as follows: for $\ell \geqslant 0$, the form $\mathrm{P}_{\ell} \mathrm{k}(u, \bar{v}):=\mathrm{k}_{\ell}\left(u, \overline{\mathrm{~N}^{\ell} v}\right)$ on $\mathrm{P}_{\ell} \mathcal{H}$ is Hermitian and positive definite.
$\boldsymbol{R e m a r k}$ 3.1.25. The conditions above imply that the adjoint $\mathrm{k}_{\ell}^{*}: \mathcal{H}_{-\ell} \otimes \overline{\mathcal{H}_{\ell}} \rightarrow \mathbb{C}$ is equal to $(-1)^{\ell} \mathrm{k}_{-\ell}$. Indeed, assume $\ell \geqslant 0$ for example. For $j, k \geqslant 0$, for $u_{\ell+2 j} \in$ $\mathrm{P} \mathcal{H}_{\ell+2 j}$ and $v_{\ell+2 k} \in \mathrm{P} \mathcal{H}_{\ell+2 k}$, we have

$$
\mathrm{k}_{\ell}\left(\mathrm{N}^{j} u_{\ell+2 j}, \overline{\mathrm{~N}^{\ell+k} v_{\ell+2 k}}\right)=0 \quad \text { if } k \neq j
$$

since $\mathrm{N}^{\ell+2 i+1}$ vanishes on $\mathrm{PH}_{\ell+2 i}$, and

$$
\begin{aligned}
\mathrm{k}_{\ell}\left(\mathrm{N}^{j} u_{\ell+2 j}, \overline{\mathrm{~N}^{\ell+j} v_{\ell+2 j}}\right) & =(-1)^{j} \mathrm{k}_{\ell+2 j}\left(u_{\ell+2 j}, \overline{\mathrm{~N}^{\ell+2 j} v_{\ell+2 j}}\right) \\
& =(-1)^{j} \overline{\mathrm{k}_{\ell+2 j}\left(v_{\ell+2 j}, \overline{\mathrm{~N}^{\ell+2 j} u_{\ell+2 j}}\right.} \\
& \left.=(-1)^{\ell} \overline{\mathrm{k}_{-\ell}\left(\mathrm{N}^{\ell+j} v_{\ell+2 j}, \overline{\mathrm{~N}^{j} u_{\ell+2 j}}\right.}\right)
\end{aligned}
$$

## Proposition 3.1.26 (A positive definite Hermitian form on $\mathcal{H}$ )

Let $\left(\mathcal{H}_{\bullet}, \mathrm{N}\right)$ be a graded $\mathbb{C}$-Lefschetz structure. Let Y act by N and let H correspond to the grading. Let $\mathrm{k}:\left(\mathcal{H}_{\bullet}, \mathrm{N}\right) \rightarrow\left(\mathcal{H}_{\bullet}, \mathrm{N}\right)^{*}$ be a morphism. Then k is a polarization of $\left(\mathcal{H}_{\bullet}, \mathrm{N}\right)$ if and only if the sesquilinear form $\mathrm{h}(u, \bar{v}):=\mathrm{k}(u, \overline{\mathrm{w} v})$ is Hermitian positive definite on $\mathcal{H}$. If k is a polarization, then k is an isomorphism and the Lefschetz decomposition of $\mathcal{H}$ is orthogonal with respect to h .

Proof. For the first statement and the "only if" part, it is enough to check that, for $u, v \in \mathcal{H}_{\ell}$, we have $\mathrm{k}(v, \overline{\mathrm{w} u})=\overline{\mathrm{k}(u, \overline{\mathrm{w} v})}$ and $u \neq 0, \mathrm{k}(u, \overline{\mathrm{w} u})>0$. Let us set $u=\sum_{j \geqslant 0} \mathrm{~N}^{j} u_{\ell+2 j}$ with $u_{\ell+2 j} \in \mathrm{P} \mathcal{H}_{\ell+2 j}$ and let us decompose $v$ similarly. Then we have

$$
\begin{aligned}
\mathrm{k}(u, \overline{\mathrm{w} v}) & =\sum_{j, k \geqslant 0} \mathrm{k}\left(\mathrm{~N}^{j} u_{\ell+2 j}, \overline{\mathrm{wN}^{k} v_{\ell+2 k}}\right)=\sum_{j, k \geqslant 0} \frac{(-1)^{k}}{(\ell+k)!} \mathrm{k}\left(\mathrm{~N}^{j} u_{\ell+2 j}, \overline{\mathrm{~N}^{\ell+k} v_{\ell+2 k}}\right) \\
& =\sum_{j, k \geqslant 0} \frac{(-1)^{j+k}}{(\ell+k)!} \mathrm{k}\left(u_{\ell+2 j}, \overline{\mathrm{~N}^{\ell+j+k} v_{\ell+2 k}}\right) .
\end{aligned}
$$

Notice that $\mathrm{k}\left(u_{\ell+2 j}, \overline{\mathrm{~N}^{\ell+j+k} v_{\ell+2 k}}\right)=0$ for $k \neq j$. As a consequence,

$$
\mathrm{k}(u, \overline{\mathrm{w} v})=\sum_{j \geqslant 0} \frac{1}{(\ell+j)!} \mathrm{k}\left(u_{\ell+2 j}, \overline{\mathrm{~N}^{\ell+2 j} v_{\ell+2 j}}\right),
$$

and the assertion follows. The "if" part is obtained similarly, and the h-orthogonality of the Lefschetz decomposition is proved similarly. Since $h$ is positive definite, it is non-degenerate, hence so is k , that is, $\mathrm{k}: \mathcal{H} \rightarrow \mathcal{H}^{*}$ is an isomorphism.

Proposition 3.1.27 (Polarization of $\operatorname{Im} \mathrm{N})$. Let $(\mathcal{H}, \mathrm{N}, \mathrm{k})$ be a polarized $\mathbb{C}$-Lefschetz structure and set $\left(\mathcal{G}, \mathrm{N}_{\mathcal{G}}\right)=\operatorname{Im} \mathrm{N}$. Let us define $\mathrm{k}_{\mathcal{G}}:\left(\mathcal{G}, \mathrm{N}_{\mathcal{G}}\right) \rightarrow\left(\mathcal{G}, \mathrm{N}_{\mathcal{G}}\right)^{*}$ by

$$
\mathrm{k}_{\mathcal{G}}(u, \bar{v}):=\mathrm{k}(x, \overline{\mathrm{~N} y})=-\mathrm{k}(\mathrm{~N} x, \bar{y}), \quad x, y \in \mathcal{H}, u=\mathrm{N} x, v=\mathrm{N} y
$$

i.e., $\mathrm{k}_{\mathcal{G}}$ is the pre-polarization induced by k on $\mathcal{G}$ (see Lemma 3.1.17). Then $\mathrm{k}_{\mathcal{G}}$ is a polarization of $\left(\mathcal{G}, \mathrm{N}_{\mathcal{G}}\right)$.

Proof. That N is skew-Hermitian with respect to k clearly implies that $\mathrm{k}_{\mathcal{G}}(u, \bar{v})$ does not depend on the choice of $x, y$ and $\mathrm{N}_{\mathcal{G}}$ is skew-Hermitian with respect to $\mathrm{k}_{\mathcal{G}}$. We have $\left(\mathrm{gr}^{\mathrm{M}} \mathrm{k}_{\mathrm{grM}_{\mathcal{G}}}=\operatorname{gr}^{\mathrm{M}}\left(\mathrm{k}_{\mathcal{G}}\right)\right.$, and we still denote it by $\mathrm{k}_{\mathcal{G}}$. For $u, v \in \mathrm{P}_{\ell} \mathcal{G}(\ell \geqslant 0)$, we have $u=\mathrm{N} x, v=\mathrm{N} y$, with $x, y \in \mathrm{P}_{\ell+1} \mathcal{H}$, according to Lemma 3.1.16(f). Hence

$$
\mathrm{P}_{\ell} \mathrm{k}_{\mathcal{G}}(u, \bar{v})=\mathrm{k}_{\mathcal{G}}\left(u, \overline{\mathrm{~N}^{\ell} v}\right)=\mathrm{k}\left(x, \overline{\mathrm{~N}^{\ell+1} y}\right)=\mathrm{P}_{\ell+1} \mathrm{k}(x, \bar{y}) .
$$

Let $(\mathcal{H}, \mathrm{N}, \mathrm{k})$ and $\left(\mathcal{H}^{\prime}, \mathrm{N}^{\prime}, \mathrm{k}^{\prime}\right)$ be polarized graded $\mathbb{C}$-Lefschetz structures. Let $\mathrm{c}, \mathrm{v}$ be graded morphisms of degree -1 :

$$
\mathrm{c}: \mathcal{H}_{\ell} \longrightarrow \mathcal{H}_{\ell-1}^{\prime}, \quad \mathrm{v}: \mathcal{H}_{\ell}^{\prime} \longrightarrow \mathcal{H}_{\ell-1}
$$

such that $\mathrm{N}=\mathrm{v} \circ \mathrm{c}$ and $\mathrm{N}^{\prime}=\mathrm{c} \circ \mathrm{v}$.
Proposition 3.1.28. Assume that, for every $\ell$, the following diagram commutes:

that is, the two sesquilinear forms

$$
\begin{array}{r}
\mathrm{k}_{\ell}(\cdot, \overline{\mathrm{v} \bullet}): \mathcal{H}_{\ell} \otimes \overline{\mathcal{H}_{-\ell+1}^{\prime}} \longrightarrow \mathbb{C} \\
\mathrm{k}_{\ell-1}^{\prime}(\mathrm{c} \cdot, \bar{\bullet}): \mathcal{H}_{\ell} \otimes \overline{\mathcal{H}_{-\ell+1}^{\prime}} \longrightarrow \mathbb{C}
\end{array}
$$

are equal. Then we have a decomposition

$$
\mathcal{H}^{\prime}=\operatorname{Im} \mathrm{c} \oplus \operatorname{Ker} \mathrm{v}
$$

as a graded $\mathbb{C}$-Lefschetz structure.
Proof. Recall that $\mathrm{N}: \mathcal{H}_{\ell} \rightarrow \mathcal{H}_{\ell-2}$ is injective for $\ell \geqslant 1$, and similarly for $\mathrm{N}^{\prime}$. Therefore, $\mathrm{c}: \mathcal{H}_{\ell} \rightarrow \mathcal{H}_{\ell-1}^{\prime}$ and $\mathrm{v}: \mathcal{H}_{\ell}^{\prime} \rightarrow \mathcal{H}_{\ell-1}$ are also injective for $\ell \geqslant 1$. Notice also that, for $\ell \geqslant 1, \mathrm{c}\left(\mathrm{P} \mathcal{H}_{\ell}\right) \subset \operatorname{Ker}\left[\mathrm{N}^{\prime \ell}: \mathcal{H}_{\ell-1}^{\prime} \rightarrow \mathcal{H}_{-2 \ell-1}^{\prime}\right]$, and similarly for $\mathrm{v}\left(\mathrm{P} \mathcal{H}_{\ell-1}^{\prime}\right)$. From Exercise 3.1.6 we deduce that, for $\ell \geqslant 0$,

$$
\begin{align*}
& \mathrm{c}\left(\mathrm{P} \mathcal{H}_{\ell}\right) \subset \mathrm{P}_{\ell-1}^{\prime} \oplus \mathrm{N}^{\prime}\left(\mathrm{PH}_{\ell+1}^{\prime}\right)  \tag{3.1.29}\\
& \mathrm{v}\left(\mathrm{P} \mathcal{H}_{\ell}^{\prime}\right) \subset \mathrm{P} \mathcal{H}_{\ell-1} \oplus \mathrm{~N}\left(\mathrm{P} \mathcal{H}_{\ell+1}\right) \tag{3.1.30}
\end{align*}
$$

if we set $\mathrm{PH}_{-1}=0$ and $\mathrm{PH}_{-1}^{\prime}=0$. Let us check this for $\mathrm{v}\left(\mathrm{P} \mathcal{H}_{0}^{\prime}\right)$ for example: we have $\mathrm{v}\left(\mathrm{P}_{0}^{\prime}\right) \subset \mathcal{H}_{-1} \cap \operatorname{Ker} \mathrm{~N}$ since $\mathrm{NvP}_{\mathcal{H}_{0}^{\prime}}=\mathrm{vN}^{\prime} \mathrm{P} \mathcal{H}_{0}^{\prime}=0$; on the other hand, $\mathcal{H}_{-1}=\mathrm{NPH}_{1} \oplus \mathrm{~N}^{2} \mathrm{P} \mathcal{H}_{3} \oplus \cdots$ and $\mathcal{H}_{-1} \cap \operatorname{Ker} \mathrm{~N}=\mathrm{NPH}_{1}$, hence the assertion.

We will prove by induction the following two properties.
(a) for all $\ell \geqslant 0, \mathrm{c}\left(\mathrm{P} \mathcal{H}_{\ell}\right) \subset \mathrm{PH}_{\ell-1}^{\prime}$,
(b) for all $\ell \geqslant 2, \mathrm{c}\left(\mathrm{P} \mathcal{H}_{\ell}\right)=\mathrm{PH}_{\ell-1}^{\prime}$.

Step one: Proof of $(\mathrm{b})_{\ell+2} \Rightarrow(\mathrm{a})_{\ell}(\ell \geqslant 0)$. By $(\mathrm{b})_{\ell+2}$ we have $\mathrm{P} \mathcal{H}_{\ell+1}^{\prime}=\mathrm{c}\left(\mathrm{P} \mathcal{H}_{\ell+2}\right)$, so

$$
\mathrm{c}\left(\mathrm{P} \mathcal{H}_{\ell}\right) \subset \mathrm{P} \mathcal{H}_{\ell-1}^{\prime} \oplus \mathrm{cN}\left(\mathrm{P} \mathcal{H}_{\ell+2}\right) .
$$

Since $\mathrm{c}\left(\mathrm{P} \mathcal{H}_{\ell}\right) \subset \operatorname{Ker} \mathrm{N}^{\ell} \mathrm{v}$ and, by (3.1.30), $\mathrm{P}_{\mathcal{H}_{\ell-1}^{\prime}} \subset \operatorname{Ker} \mathrm{N}^{\ell} \mathrm{v}$, it is enough to prove Ker $\mathrm{N}^{\ell} \mathrm{v} \cap \operatorname{cN}\left(\mathrm{P} \mathcal{H}_{\ell+2}\right)=0$, that is, Ker $\mathrm{N}^{\ell+2} \cap \mathrm{PH}_{\ell+2}=0$, which is by definition.

Step two: Proof that $\mathrm{v}\left(\mathrm{P} \mathcal{H}_{\ell}^{\prime}\right) \cap \mathrm{P} \mathcal{H}_{\ell-1}=0$. The assumption implies that, with obvious notation,

$$
\mathrm{k}\left(u_{\ell-1}, \overline{\mathrm{v} w_{-\ell+2}^{\prime}}\right)=\mathrm{k}^{\prime}\left(\mathrm{c}_{\ell-1}, \overline{w_{-\ell+2}^{\prime}}\right) .
$$

We have to prove

$$
u_{\ell}^{\prime} \in \mathrm{P} \mathcal{H}_{\ell}^{\prime} \text { and } \mathrm{v} u_{\ell}^{\prime} \in \mathrm{P} \mathcal{H}_{\ell-1} \Longrightarrow u_{\ell}^{\prime}=0
$$

Assume $u_{\ell}^{\prime} \neq 0$. By the positivity property for $\mathrm{k}^{\prime}$, we have $\mathrm{k}^{\prime}\left(u_{\ell}^{\prime}, \overline{\mathrm{N}^{\prime \ell} u_{\ell}^{\prime}}\right)>0$. On the other hand, $\mathrm{v} u_{\ell}^{\prime} \in \mathrm{P} \mathcal{H}_{\ell-1}$, so $\mathrm{k}\left(\mathrm{v} u_{\ell}^{\prime}, \overline{\mathrm{N}^{\ell-1} \mathrm{v} u_{\ell}^{\prime}}\right) \geqslant 0$. We have

$$
\begin{aligned}
\mathrm{k}\left(\mathrm{v} u_{\ell}^{\prime}, \overline{\mathrm{N}^{\ell-1} \mathrm{v} u_{\ell}^{\prime}}\right) & =\mathrm{k}^{\prime}\left(\mathrm{N}^{\prime} u_{\ell}^{\prime}, \overline{\mathrm{N}^{\ell-1} u_{\ell}^{\prime}}\right) \quad \text { (by the assumption) } \\
& =-\mathrm{k}^{\prime}\left(u_{\ell}^{\prime}, \overline{\mathrm{N}^{\ell} u_{\ell}^{\prime}}\right) \quad(\text { by } 3.2 .19(1)) \\
& <0,
\end{aligned}
$$

a contradiction.
Step three: Proof that $\mathrm{PH}_{\ell}=0 \Rightarrow \mathrm{P} \mathcal{H}_{\ell-1}^{\prime}=0$ for $\ell \geqslant 2$. By (3.1.30), $\mathrm{P} \mathcal{H}_{\ell}=0$ implies $\mathrm{v}\left(\mathrm{PH}_{\ell-1}^{\prime}\right) \subset \mathrm{P}_{\ell-2}$, hence $\mathrm{v}\left(\mathrm{PH}_{\ell-1}^{\prime}\right)=0$ by the previous step. Since $\ell \geqslant 2$, this implies that $\mathrm{P} \mathcal{H}_{\ell-1}^{\prime}=0$ because v is injective on $\mathcal{H}_{\ell-1}^{\prime}$.

As a consequence, $(\mathrm{b})_{\ell}$ holds for $\ell \gg 0$. Indeed, for $\ell \gg 0$ we have $\mathrm{P} \mathcal{H}_{\ell}=0$, so (b) $)_{\ell}$ amounts to $\mathrm{PH}_{\ell-1}^{\prime}=0$.
Step four: Proof that $(\mathrm{a})_{\ell} \Rightarrow(\mathrm{b})_{\ell}$ for $\ell \geqslant 2$. Let $u_{\ell-1}^{\prime} \in \mathrm{P} \mathcal{H}_{\ell-1}^{\prime}$. We have $\mathrm{v} u_{\ell-1}^{\prime} \in$ $\mathrm{P} \mathcal{H}_{\ell-2} \oplus \mathrm{NPH}_{\ell}$ by (3.1.30), that is, $\mathrm{v} u_{\ell-1}^{\prime}=u_{\ell-2}+\mathrm{vc} u_{\ell}$. By $(\mathrm{a})_{\ell}, \mathrm{c} u_{\ell} \in \mathrm{P} \mathcal{H}_{\ell-1}^{\prime}$. Therefore, since $\mathrm{v}\left(u_{\ell-1}^{\prime}-\mathrm{c} u_{\ell}\right)=u_{\ell-2} \in \mathrm{PH}_{\ell-2}$ and since $\ell \geqslant 2$, Step two implies $u_{\ell-2}=0$, and by the injectivity of v on $\mathcal{H}_{\ell-1}^{\prime}$, this implies $u_{\ell-1}^{\prime}=\mathrm{c} u_{\ell}$. This ends the proof of (a) and (b).

We can now conclude the proof of the proposition. We notice that (a) implies that c decomposes with respect to the Lefschetz decomposition. Similarly, Step two together with (3.1.30) implies that $\mathrm{v}\left(\mathrm{PH}_{\ell}^{\prime}\right) \subset \mathrm{NP}_{\ell+1}$, so v is also compatible with the Lefschetz decomposition. Proving the decomposition $\mathcal{H}^{\prime}=\operatorname{Imc} \oplus \operatorname{Ker} v$ amounts thus to proving it on each primitive subspace $\mathrm{P}_{\ell}^{\prime}(\ell \geqslant 0)$. For $\ell \geqslant 1$, we have $\mathrm{PH}_{\ell}^{\prime}=\mathrm{c}\left(\mathrm{PH}_{\ell+1}\right)$ by $(\mathrm{b})_{\ell+1}$, and $\operatorname{Kerv}=0$ so the decomposition is trivial. We are left to prove

$$
\mathrm{P} \mathcal{H}_{0}^{\prime}=\mathrm{c}\left(\mathrm{P} \mathcal{H}_{1}\right) \oplus \operatorname{Ker}_{\mid \mathrm{P} \mathcal{H}_{0}^{\prime}}
$$

This follows from the exercise below.
Exercise 3.1.31. Let $P_{1}, P_{2}, P_{3}$ be objects of an abelian category A. Let $c: P_{1} \rightarrow P_{2}$ and $v: P_{2} \rightarrow P_{3}$ be two morphisms such that $v \circ c: P_{1} \rightarrow P_{3}$ is an isomorphism. Show that $P_{2}=\operatorname{Im} c \oplus \operatorname{Ker} v$. [Hint: check that it amounts to proving that the composed morphism $\varphi: \operatorname{Im} c \rightarrow P_{2} / \operatorname{Ker} v$ is an isomorphism; use the commutative diagram

show that $\operatorname{Ker} \varphi=\operatorname{Ker} v \circ \varphi=c(\operatorname{Ker} v \circ c)=0$, and similarly, $\operatorname{Im} v \circ \varphi=\operatorname{Im} v \circ \varphi \circ c=$ $\operatorname{Im} v \circ c=P_{3}$, hence conclude that $v \circ \varphi$ is an epimorphism, then that $v$ is both an epimorphism and a monomorphism, thus an isomorphism, and $\varphi$ is an isomorphism.]

## Definition 3.1.32 (Differential polarized graded $\mathbb{C}$-Lefschetz structure)

Let $\left(\mathcal{H}_{0}, \mathrm{~N}, \mathrm{k}\right)$ be a polarized graded $\mathbb{C}$-Lefschetz structure. A differential on $\left(\mathcal{H}_{\bullet}, \mathrm{N}, \mathrm{k}\right)$ is a graded morphism $d: \mathcal{H}_{\bullet} \rightarrow \mathcal{H}_{\bullet-1}$, which is self-adjoint with respect to k and such that $d \circ d=0$ and $[\mathrm{N}, d]=0$.

Let $\left(\mathcal{H}_{\bullet}, \mathrm{N}, \mathrm{k}, d\right)$ be a differential polarized graded $\mathbb{C}$-Lefschetz structure. The cohomology $\operatorname{Ker} d / \operatorname{Im} d$ is naturally graded, and N induces a nilpotent endomorphism on it, which is a graded morphism of degree -2 , since N commutes with $d$. Moreover, since $d$ is k -self-adjoint, k induces a sesquilinear pairing on $\operatorname{Ker} d / \operatorname{Im} d$.

Proposition 3.1.33. If $\left(\mathcal{H}_{.}, \mathrm{N}, \mathrm{k}, d\right)$ is a differential polarized graded $\mathbb{C}$-Lefschetz structure, then its cohomology $\operatorname{Ker} d / \operatorname{Im} d$, endowed with the previous grading, nilpotent endomorphism and sesquilinear pairing, is a polarized graded $\mathbb{C}$-Lefschetz structure.

Proof. The point is to prove that, for $\ell \geqslant 1, \mathrm{~N}^{\ell}:(\operatorname{Ker} d / \operatorname{Im} d)_{\ell} \rightarrow(\operatorname{Ker} d / \operatorname{Im} d)_{-\ell}$ is an isomorphism. Let h denote the positive definite Hermitian form $\mathrm{k}(u, \overline{\mathrm{w} v})$. Then the h-adjoint $d^{\star}$ of $d$ is equal to $\mathrm{w}^{-1} d \mathrm{w}$ and has degree 1 . Consider the "Laplacian" $\Delta:=d d^{\star}+d^{\star} d$. It is graded of degree zero. Due to the positivity of h , we have, in a way compatible with the grading,

$$
\operatorname{Ker} d / \operatorname{Im} d=\operatorname{Ker} d \cap \operatorname{Ker} d^{\star}=\operatorname{Ker} \Delta, \quad \mathcal{H}=\operatorname{Ker} \Delta \stackrel{\perp}{\oplus} \operatorname{Im} \Delta
$$

where the sum is orthogonal with respect to $h$. It enough to prove that $\Delta$ commutes with N , since this implies that N preserves this decomposition, and thus $\mathrm{N}^{\ell}:(\operatorname{Ker} \Delta)_{\ell} \rightarrow(\operatorname{Ker} \Delta)_{-\ell}$ is an isomorphism. Similarly, $\mathrm{P}(\operatorname{Ker} \Delta)_{\ell}=\mathrm{P} \mathcal{H}_{\ell} \cap \operatorname{Ker} \Delta$, and the Lefschetz decomposition of $\mathcal{H}$ also decomposes with respect to the above decomposition of $\mathcal{H}$. Since all direct sums are h-orthogonal, we deduce that $\mathrm{h}_{\mid \mathrm{P}(\operatorname{Ker} \Delta)_{\ell}}$ is positive definite, and it is equal to the pairing induced by $\mathrm{k}\left(u, \overline{\mathrm{~N}^{\ell} v}\right)$ on $\mathrm{P}(\operatorname{Ker} \Delta)_{\ell}$, proving that k induces a polarization on $((\operatorname{Ker} d / \operatorname{Im} d) ., \mathrm{N})$.

Let us consider the graded subspace $\mathcal{D}=\mathbb{C} d^{\star} \oplus \mathbb{C} d$ of End $\mathcal{H}$. It is stable by the action of $\mathfrak{s l}_{2}$ and, with respect to it, we have $\mathcal{D}=\mathcal{D}_{1} \oplus \mathcal{D}_{-1}, \mathrm{X}\left(d^{\star}\right)=0, d=$ $\mathrm{w}\left(d^{\star}\right)=\mathrm{Y}\left(d^{\star}\right)$ (see Exercise 3.1.22). Let us consider the composition morphism Comp : $\mathcal{D} \otimes \mathcal{D} \rightarrow$ End $\mathcal{H}$. Note that $\mathcal{D} \otimes \mathcal{D}$ is an $\mathfrak{s l}_{2}$-representation, defined by 3.1.c(c) and Lemma 3.1.20(1). The composition clearly commutes with H. It also commutes with Y, since, for $\varphi, \varphi^{\prime} \in \operatorname{End}(\mathcal{H})$, we have $\left[\mathrm{N}, \varphi^{\prime} \varphi\right]=\left[\mathrm{N}, \varphi^{\prime}\right] \varphi+\varphi^{\prime}[\mathrm{N}, \varphi]$. It commutes then with X (by the same argument or by Exercise 3.1.23(1)). The image of $d^{\star} \otimes d+d \otimes d^{\star}$ is equal to $\Delta$. We wish to prove that $\Delta \in \mathrm{P}_{0}$ End $\mathcal{H}$ (see Exercise 3.1.23(2)). Since Comp sends $\mathrm{P}_{0}(\mathcal{D} \otimes \mathcal{D})$ into $\mathrm{P}_{0}$ End $\mathcal{H}$, the assertion will follow from the property

$$
\begin{equation*}
d^{\star} \otimes d+d \otimes d^{\star} \in \mathrm{P}_{0}(\mathcal{D} \otimes \mathcal{D})+\text { Ker Comp. } \tag{3.1.34}
\end{equation*}
$$

The Lefschetz decomposition of the four-dimensional vector space $\mathcal{D} \otimes \mathcal{D}$ is easy to describe (a particular case of the Clebsch-Gordan formula):

- $(\mathcal{D} \otimes \mathcal{D})_{2}=\mathbb{C}\left(d^{\star} \otimes d^{\star}\right)$,
- $(\mathcal{D} \otimes \mathcal{D})_{-2}=\mathbb{C}(d \otimes d)$,
- $(\mathcal{D} \otimes \mathcal{D})_{0}=\mathrm{Y} \mathbb{C}\left(d^{\star} \otimes d^{\star}\right) \oplus \mathrm{P}_{0}(\mathcal{D} \otimes \mathcal{D})$.

The assumption $d \circ d=0$ implies that $\operatorname{Comp}(\mathcal{D} \otimes \mathcal{D})_{-2}=0$, hence $\operatorname{Comp}(\mathcal{D} \otimes \mathcal{D})_{2}=0$, $\operatorname{CompY}(\mathcal{D} \otimes \mathcal{D})_{2}=0$. In other words, $\mathcal{D} \otimes \mathcal{D}=\mathrm{P}_{0}(\mathcal{D} \otimes \mathcal{D})+$ Ker Comp, so (3.1.34) is clear.
3.1.e. Bi-graded $\mathbb{C}$-Lefschetz structures. We will encounter the following bigraded situation when dealing with spectral sequences. A bi-graded $\mathbb{C}$-Lefschetz structure on a vector space $\mathcal{H}$ consists of the data of a bi-grading $\mathcal{H}_{\bullet, \bullet}$ and of two commuting nilpotent endomorphisms $\mathrm{N}_{1}, \mathrm{~N}_{2}$ such that, for every $\ell_{1}, \ell_{2},\left(\mathcal{H}_{\bullet}, \ell_{2}, \mathrm{~N}_{1}\right)$ and $\left(\mathcal{H}_{\ell_{1}, \boldsymbol{\bullet}}, \mathrm{~N}_{2}\right)$ are graded $\mathbb{C}$-Lefschetz structures. We thus have two $\mathfrak{s l}_{2}$ actions that we denote by $\left(\mathrm{X}_{1}, \mathrm{Y}_{1}, \mathrm{H}_{1}\right)$ and $\left(\mathrm{X}_{2}, \mathrm{Y}_{2}, \mathrm{H}_{2}\right)$. The assumption means that $\mathrm{Y}_{1}\left(\right.$ resp. $\left.\mathrm{H}_{1}\right)$ commutes with $\mathrm{Y}_{2}$ and $\mathrm{H}_{2}$. Lemma 3.1.20 implies that $\mathrm{Y}_{1}$ (resp. $\mathrm{H}_{1}$ ) also commutes with $\mathrm{X}_{2}$, and then that $\mathrm{X}_{2}$ commutes with $\mathrm{X}_{1}$. In particular, $\mathrm{w}_{1}$ and $\mathrm{w}_{2}$ commute.

We note that $\mathrm{X}:=\mathrm{X}_{1}+\mathrm{X}_{2}, \mathrm{Y}:=\mathrm{Y}_{1}+\mathrm{Y}_{2}$ and $\mathrm{H}:=\mathrm{H}_{1}+\mathrm{H}_{2}$ form an $\mathfrak{s l}_{2}$-triple, hence (Lemma 3.1.20) defines a graded $\mathbb{C}$-Lefschetz structure, with $\mathcal{H}_{\ell}=\bigoplus_{\ell_{1}+\ell_{2}=\ell} \mathcal{H}_{\ell_{1}, \ell_{2}}$. The corresponding w is $\mathrm{w}_{1} \mathrm{w}_{2}$, due to the commutation properties. For $\ell_{1}, \ell_{2} \geqslant 0$, the bi-primitive subspace $\mathrm{PH}_{\ell_{1}, \ell_{2}}$ of $\mathcal{H}_{\ell_{1}, \ell_{2}}$ is $\operatorname{Ker} \mathrm{N}_{1}^{\ell_{1}+1} \cap \operatorname{Ker} \mathrm{~N}_{2}^{\ell_{2}+1}$ and we have a corresponding Lefschetz decomposition. We define the adjoint bi-graded $\mathbb{C}$-Lefschetz structure $\left(\mathcal{H}_{\bullet, \bullet}, \mathrm{N}_{1}, \mathrm{~N}_{2}\right)^{*}$ as $\left(\mathcal{H}_{\bullet, \bullet}^{*},-\mathrm{N}_{1}^{*},-\mathrm{N}_{2}^{*}\right)$ with $\left(\mathcal{H}_{\ell_{1}, \ell_{2}}^{*}=\left(\mathcal{H}_{-\ell_{1},-\ell_{2}}\right)^{*}\right.$. The notion of polarization is similarly adapted from Definition 3.1.24.

Proposition 3.1.35. Let $\left(\mathcal{H}_{\bullet, \bullet}, \mathrm{N}_{1}, \mathrm{~N}_{2}, \mathrm{k}\right)$ be a polarized bi-graded $\mathbb{C}$-Lefschetz structure. Then the associated graded $\mathbb{C}$-Lefschetz structure $\left(\mathcal{H}_{\bullet}, \mathrm{N}\right)$, equipped with the same sesquilinear pairing k , is a polarized graded $\mathbb{C}$-Lefschetz structure.

Proof. By the same argument as in Proposition 3.1.26, one checks that the sesquilinear form $\mathrm{k}\left(u, \overline{\mathrm{w}_{1} \mathrm{w}_{2} v}\right)$ is Hermitian positive definite on $\mathcal{H}$. Since $\mathrm{w}=\mathrm{w}_{1} \mathrm{w}_{2}$ Proposition 3.1.26 in the reverse direction gives the assertion.

Remark 3.1.36. The use of w enables us not to compute $\mathrm{P} \mathcal{H}_{\ell}$ explicitly (with respect to N).

We now have the bi-graded analogue of Proposition 3.1.33. Let ( $\left.\mathcal{H}_{\bullet}, \bullet, \mathrm{N}_{1}, \mathrm{~N}_{2}, \mathrm{k}\right)$ be a polarized bi-graded $\mathbb{C}$-Lefschetz structure. A differential $d$ on it is a bi-graded morphism of bi-degree $(-1,-1)$, which is self-adjoint with respect to k , and which commutes with $\mathrm{N}_{1}, \mathrm{~N}_{2}$.

Proposition 3.1.37. If ( $\left.\mathcal{H}_{\bullet, \bullet}, \mathrm{N}_{1}, \mathrm{~N}_{2}, \mathrm{k}, d\right)$ be a differential polarized bi-graded $\mathbb{C}$ Lefschetz structure, then its cohomology $\operatorname{Ker} d / \operatorname{Im} d$, endowed with the natural
bi-grading, nilpotent endomorphisms and sesquilinear pairing, is a polarized bi-graded $\mathbb{C}$-Lefschetz structure.

Proof. As above, we consider the positive definite Hermitian form $\mathrm{k}(u, \overline{\mathrm{w} v})$ with $\mathrm{w}:=$ $\mathrm{w}_{1} \mathrm{w}_{2}$ and the corresponding Laplacian $\Delta=d d^{\star}+d^{\star} d$, with $d=\mathrm{w}^{-1} d \mathrm{w}$. Then $\Delta$ is bi-graded of bi-degree zero. As in Proposition 3.1.33, we consider the bi-graded space $\mathcal{D}=\mathbb{C} d^{\star} \oplus \mathbb{C} \mathrm{Y}_{1}\left(d^{\star}\right) \oplus \mathrm{Y}_{2}\left(d^{\star}\right) \oplus \mathbb{C} d$, with $d=\mathrm{Y}_{1} \mathrm{Y}_{2}\left(d^{\star}\right)$. Arguing similarly, we only need to prove that

$$
\begin{equation*}
\left(d \otimes d^{\star}+d^{\star} \otimes d\right) \in \mathrm{P}_{0,0}(\mathcal{D} \otimes \mathcal{D})+\text { Ker Comp } \tag{3.1.38}
\end{equation*}
$$

where $\mathrm{P}_{0,0}(\mathcal{D} \otimes \mathcal{D})=\operatorname{Ker} \mathrm{Y}_{1} \cap \operatorname{Ker} \mathrm{Y}_{2} \cap(\mathcal{D} \otimes \mathcal{D})_{(0,0)}$. We have
Ker Comp $\ni \mathrm{Y}_{1} \mathrm{Y}_{2}\left(d^{\star} \otimes d^{\star}\right)=\left(d \otimes d^{\star}+d^{\star} \otimes d\right)+\left[\left(\mathrm{Y}_{1} d^{\star} \otimes \mathrm{Y}_{2} d^{\star}\right)+\left(\mathrm{Y}_{2} d^{\star} \otimes \mathrm{Y}_{1} d^{\star}\right)\right]$.
On the other hand,

$$
\begin{aligned}
\mathrm{Y}_{1}\left[\left(\mathrm{Y}_{1} d^{\star} \otimes \mathrm{Y}_{2} d^{\star}\right)+\left(\mathrm{Y}_{2} d^{\star} \otimes \mathrm{Y}_{1} d^{\star}\right)\right] & =\left(\mathrm{Y}_{1} d^{\star} \otimes \mathrm{Y}_{1} \mathrm{Y}_{2} d^{\star}\right)+\left(\mathrm{Y}_{1} \mathrm{Y}_{2} d^{\star} \otimes \mathrm{Y}_{1} d^{\star}\right) \\
& =\mathrm{Y}_{1}\left[\left(d^{\star} \otimes \mathrm{Y}_{1} \mathrm{Y}_{2} d^{\star}\right)+\left(\mathrm{Y}_{1} \mathrm{Y}_{2} d^{\star} \otimes d^{\star}\right)\right] \\
& =\mathrm{Y}_{1}\left(d \otimes d^{\star}+d^{\star} \otimes d\right),
\end{aligned}
$$

and similarly with $\mathrm{Y}_{2}$, so we obtain

$$
\left(d \otimes d^{\star}+d^{\star} \otimes d\right)-\left[\left(\mathrm{Y}_{1} d^{\star} \otimes \mathrm{Y}_{2} d^{\star}\right)+\left(\mathrm{Y}_{2} d^{\star} \otimes \mathrm{Y}_{1} d^{\star}\right)\right] \in \mathrm{P}_{0,0}(\mathcal{D} \otimes \mathcal{D})
$$

and therefore (3.1.38) holds.

### 3.2. Polarizable Hodge-Lefschetz structures

3.2.a. Hodge-Lefschetz structures. We adapt the general framework of Section 3.1 on the Lefschetz decomposition to the case of Hodge structures. The ambient abelian category A is the category of triples considered in Remark 2.4.12 or that of triples of Definition 2.4.24, and we choose for $\sigma$ the symmetric Tate twist (1) (see Notation 2.4.26).

In the case of Hodge structures, as we expect that the nilpotent operator $\mathrm{N}: \mathcal{H} \rightarrow \mathcal{H}$ will send $F^{k}$ into $F^{k-1}$ (an infinitesimal version of Griffiths transversality property, see Section 4.1), we will regard N as a morphism $H \rightarrow H(-1)$, using the symmetric Tate twist notation. As a consequence, N commutes with the Weil operator C.

Definition 3.2.1 (Hodge-Lefschetz structure). Set $\varepsilon= \pm 1$. Let $H=\left(\mathcal{H}, F^{\prime \bullet} \mathcal{H}, F^{\prime \prime \bullet} \mathcal{H}\right)$ be a bi-filtered vector space and let $\mathrm{N}: H \rightarrow H(-\varepsilon)$ be a nilpotent endomorphism. Let M. $H$ denote the monodromy filtration in the abelian category of filtered triples (see Remark 2.4.12). We say that $(H, \mathrm{~N})$ is an $\varepsilon$ Hodge-Lefschetz structure centered at $w$ if for every $\ell$, the object $\mathrm{gr}_{\ell}^{\mathrm{M}} H$ belongs to $\mathrm{HS}(\mathbb{C}, w+\varepsilon \ell)$.

By a Hodge-Lefschetz structure, we simply mean an $\varepsilon$ Hodge-Lefschetz structure with $\varepsilon=+1$.

For an $\varepsilon$ Hodge-Lefschetz structure $(H, \mathrm{~N})$, the restriction to $z=1$ of $\mathrm{M} . H$ is the monodromy filtration M. $\mathcal{H}$ associated to the restriction of N to $\mathcal{H}$. Moreover (see Remark 2.4.25(5)), for $F=F^{\prime}$ or $F^{\prime \prime}$, the filtration $F^{\bullet} \operatorname{gr}_{\ell}^{\mathrm{M}} \mathcal{H}$ is the filtration naturally induced by $F^{\bullet} H$ on $\operatorname{gr}_{\ell}^{\mathrm{M}} \mathcal{H}$, that is,

$$
F^{p} \operatorname{gr}_{\ell}^{\mathrm{M}} \mathcal{H}:=\frac{F^{p} \mathcal{H} \cap \mathrm{M}_{\ell} \mathcal{H}}{F^{p} \mathcal{H} \cap \mathrm{M}_{\ell-1} \mathcal{H}}
$$

Let us consider the graded object $\mathrm{gr}^{\mathrm{M}} H=\bigoplus_{\ell} \operatorname{gr}_{\ell}^{\mathrm{M}} H$. It is an object of $\mathrm{HS}(\mathbb{C})$, that is, the Hodge structure is pure on each graded piece, with a weight depending on the grading index. This graded space is also equipped with a nilpotent $\varepsilon$-endomorphism of degree -2 , that we denote by $\operatorname{grN}$, which sends $\mathrm{gr}_{\ell}^{\mathrm{M}} H$ to $\operatorname{gr}_{\ell-2}^{\mathrm{M}} H(-\varepsilon)$ for every $\ell$, and which is naturally induced by N . From the definition above, it has the following property: for every $\ell \in \mathbb{Z}$,

$$
\begin{equation*}
\operatorname{grN}: \operatorname{gr}_{\ell}^{\mathrm{M}} H \longrightarrow \mathrm{gr}_{\ell-2}^{\mathrm{M}} H(-\varepsilon) \tag{3.2.2}
\end{equation*}
$$

is a morphism of Hodge structures. In particular (according to Proposition 2.4.5(1)), for every $\ell \geqslant 1$,

$$
\begin{equation*}
(\operatorname{grN})^{\ell}: \operatorname{gr}_{\ell}^{\mathrm{M}} H \longrightarrow \mathrm{gr}_{-\ell}^{\mathrm{M}} H(-\varepsilon \ell) \tag{3.2.3}
\end{equation*}
$$

is an isomorphism and, for every $\ell \geqslant 0$, the primitive subspace $\operatorname{Ker}(\mathrm{grN})^{\ell+1}$ is a Hodge substructure of weight $\ell$ in $\operatorname{gr}_{\ell}^{\mathrm{M}} H$. It will be denoted by $\mathrm{P}_{\ell} H$.

Definition 3.2.4. We say that the $\varepsilon$ Hodge-Lefschetz structure is graded if it is endowed with a grading, i.e., an isomorphism with its graded structure with respect to the monodromy filtration.

Exercise 3.2.5. Show that a graded $\varepsilon$ Hodge-Lefschetz structure is completely determined by the Hodge structures $\mathrm{P}_{\ell} H(\ell \geqslant 0)$.

Exercise 3.2.6. The goal of this exercise is to show that any Hodge-Lefschetz structure is isomorphic (non-canonically) to its associated graded Hodge-Lefschetz structure with respect to the monodromy filtration. In (1)-(4) below, the filtration $F$ is either $F^{\prime}$ or $F^{\prime \prime}$.
(1) For every $\ell \geqslant 0$ and $p$, choose a section $s_{j, p}: \operatorname{gr}_{F}^{p} \mathrm{P}_{\ell} \mathcal{H} \rightarrow F^{p} \mathrm{M}_{\ell} \mathcal{H}$ of the projection $F^{p} \mathrm{M}_{\ell} \mathcal{H} \rightarrow \operatorname{gr}_{F}^{p} \operatorname{gr}_{\ell}^{\mathrm{M}} \mathcal{H}$ and show that $\operatorname{Im} \mathrm{N}^{\ell+1} s_{\ell, p} \subset F^{p-\ell-1} \mathrm{M}_{-\ell-3} \mathcal{H}$. The next questions aim at modifying this section in such a way that its image is contained in $\operatorname{Ker} \mathrm{N}^{\ell+1}$.
(2) Show that, for every $j \geqslant 0$, and any $p, \ell \geq 0$

$$
F^{p-\ell-1} \mathrm{M}_{-\ell-3-j} \mathcal{H} \subset \mathrm{~N}^{\ell+j+3} F^{p+j+2} \mathrm{M}_{\ell+j+3} \mathcal{H}+F^{p-\ell-1} \mathrm{M}_{-\ell-3-(j+1)} \mathcal{H}
$$

(3) Conclude that, for every $j \geqslant 0$,

$$
F^{p-\ell-1} \mathrm{M}_{-\ell-3-j} \mathcal{H} \subset \mathrm{~N}^{\ell+1} F^{p} \mathrm{M}_{\ell-1} \mathcal{H}+F^{p-\ell-1} \mathrm{M}_{-\ell-3-(j+1)} \mathcal{H}
$$

(4) Show that if for some $j \geqslant 0$ we have constructed a section $s_{\ell, p}^{(j)}$ such that $\operatorname{Im} \mathrm{N}^{\ell+1} s_{\ell, p}^{(j)} \subset F^{p-\ell-1} \mathrm{M}_{-\ell-3-j} \mathcal{H}$, then one can find a section $s_{\ell, p}^{(j+1)}$ such that $\operatorname{Im} \mathrm{N}^{\ell+1} s_{\ell, p}^{(j+1)} \subset F^{p-\ell-1} \mathrm{M}_{-\ell-3-(j+1)} \mathcal{H}$. Use then $s_{\ell, p}=s_{\ell, p}^{(0)}$ to obtain a section $s_{\ell, p}^{(\infty)}$ such that $\mathrm{N}^{\ell+1} s_{\ell, p}^{(\infty)}=0$.
(5) Use the Lefschetz decomposition to obtained the desired isomorphism.

## Remark 3.2.7 (Hodge-Lefschetz structures are mixed Hodge structures)

The symmetry between the cases $\varepsilon=+1$ and $\varepsilon=-1$ is only apparent. The Hodge-Lefschetz structures are examples of mixed Hodge structures, with (increasing) weight filtration $W$. defined by

$$
W_{w+\ell} H=\mathrm{M}_{\ell} H
$$

The symmetry (3.2.3) reads

$$
\left(\operatorname{grN}^{\ell}: \operatorname{gr}_{w+\ell}^{W} H \xrightarrow{\sim} \operatorname{gr}_{w-\ell}^{W} H(-\ell)\right.
$$

justifying the expression "centered at $w$ ". On the other hand, the $\varepsilon$ Hodge-Lefschetz structures with $\varepsilon=-1$ are not necessarily mixed Hodge structures. They are so in the graded case. In fact, we will only encounter graded $\varepsilon$ Hodge-Lefschetz structures when $\varepsilon=-1$. On the other hand, we will encounter (non graded) Hodge-Lefschetz structures in the theory of nearby/vanishing cycles, see Section 7.4.

Example 3.2.8. The cohomology $H_{\bullet}=H^{n-\bullet}(X, \mathbb{C})$ of a smooth complex projective variety, equipped with the nilpotent endomorphism $\mathrm{N}_{\mathscr{L}}=\mathrm{L}_{\mathscr{L}}$, is naturally graded. We define the filtration $F^{\bullet} H^{\bullet}(X, \mathbb{C})$ as being the direct sum of the Hodge filtration on each term. Then $\operatorname{gr}_{\ell}^{\mathrm{M}} H=H^{n-\ell}(X, \mathbb{C})$ equipped with its filtration is a Hodge structure of weight $n-\ell$. The cohomology $H^{n-\bullet}(X, \mathbb{C})$ is thus a graded (-)HodgeLefschetz structure centered at $n=\operatorname{dim} X$.

Exercise 3.2.9 (Tate twist of Hodge-Lefschetz structures). Define the Tate twist ( $k, \ell$ ) of an $\varepsilon$ Hodge-Lefschetz structure $(H, \mathrm{~N})$ centered at $w$ as $(H(k, \ell), \mathrm{N})$ and leaving N unchanged. Show that $(H, \mathrm{~N})(k, \ell)$ is an $\varepsilon$ Hodge-Lefschetz structure centered at $w-\varepsilon(k+\ell)$.

Definition 3.2.10 (Category of Hodge-Lefschetz structures). The category HLS of Hodge-Lefschetz structures is the category whose objects consist of Hodge-Lefschetz structures $(\varepsilon=+1)$ centered at some $w \in \mathbb{Z}$, and whose morphisms are morphisms of mixed Hodge structures compatible with N . The category $\operatorname{HLS}(w)$ is the full sub-category consisting of objects centered at $w$.

Similarly, the category of graded $\varepsilon$ Hodge-Lefschetz structures is the category whose objects consist of graded $\varepsilon$ Hodge-Lefschetz structures and whose morphisms are graded morphisms of Hodge structures, of degree zero with respect to the grading.

Exercise 3.2.11 (The category $\mathrm{HLS}(w)$ is abelian). Show the folowing properties.
(1) In the category HLS, any morphism is strict with respect to the filtrations $F^{\bullet}$ and the filtration $W_{\bullet}$. [Hint: Use Proposition 2.5.5.]
(2) $\mathrm{N}:(H, \mathrm{~N}) \rightarrow(H, \mathrm{~N})(-1)$ is a morphism in this category. In particular, $\mathrm{N}\left(F^{p} \mathcal{H}\right)=F^{p-1} \mathcal{H} \cap \operatorname{Im} \mathrm{~N}$ for $F=F^{\prime}$ or $F^{\prime \prime}$.
(3) The filtration M. (N) $H$ is a filtration in the category of mixed Hodge structures.
(4) Consider the category $\mathrm{MHS}^{\oplus}$ whose objects are $H^{\oplus}:=\bigoplus_{k, \ell \in \mathbb{Z}} H(k, \ell)$, where $H$ is a mixed Hodge structure, and morphisms $\varphi^{\oplus}: H_{1}^{\oplus} \rightarrow H_{2}^{\oplus}$ are the direct sums of the same morphism of mixed Hodge structures $\varphi: H_{1} \rightarrow H_{2}\left(k_{o}, \ell_{o}\right)$ for some $\left(k_{o}, \ell_{o}\right)$, twisted by any $(k, \ell) \in \mathbb{Z}$. Show that
(a) the category $\mathrm{MHS}^{\oplus}$ is abelian,
(b) for $(H, \mathrm{~N})$ in $\mathrm{HLS}(w), \mathrm{N}$ defines a nilpotent endomorphism $\mathrm{N}^{\oplus}$ in the category $\mathrm{MHS}^{\oplus}$ on $H^{\oplus}$,
(c) $\bigoplus_{k, \ell} \mathrm{M} .(\mathrm{N}) H(k, \ell)$ is the monodromy filtration of $\mathrm{N}^{\oplus}$ in the abelian category $\mathrm{MHS}^{\oplus}$.
(5) Let $\varphi:\left(H_{1}, \mathrm{~N}_{1}\right) \rightarrow\left(H_{2}, \mathrm{~N}_{2}\right)$ be a morphism in HLS. Then $\varphi=0$ if $w_{1}>w_{2}$. [Hint: Use that $\varphi$ is compatible with both M. and $W_{.}$.]
(6) Let $\varphi$ be a morphism in $\operatorname{HLS}(w)$. Show that $\varphi$ is strictly compatible with M.. Conclude that HLS $(w)$ is abelian.
(7) Show that, for such a $\varphi$, the conclusion of Lemma 3.1.7 holds in the category of mixed Hodge structures. [Hint: Use the auxiliary category $\mathrm{MHS}^{\oplus}$ and the nilpotent endomorphisms $\mathrm{N}_{1}^{\oplus}, \mathrm{N}_{2}^{\oplus}$.]
(8) Similar results hold for graded $\varepsilon$ Hodge-Lefschetz structures.

Exercise 3.2.12 (Sub-Hodge-Lefschetz structure). Let $\left(H_{1}, \mathrm{~N}_{1}\right) \rightarrow\left(H_{2}, \mathrm{~N}_{2}\right)$ be a morphism in $\operatorname{HLS}(w)$ which is injective on $\mathcal{H}_{1}$. Show that it is a monomorphism (i.e., the filtrations $F^{\bullet} \mathcal{H}_{1}$ and M. $\mathcal{H}_{1}$ are those induced from $H_{2}$ ). [Hint: use Proposition 2.4.5(1).]

Show a similar result for graded $\varepsilon$ Hodge-Lefschetz structures.
Proposition 3.2.13. Let $(H, \mathrm{~N})$ be an object in $\mathrm{HLS}(w)$. Then $\left(G:=\operatorname{Im} \mathrm{N}, \mathrm{N}_{G}\right)$ is an object of $\operatorname{HLS}(w+1)$.

Proof. The image of N is considered in the abelian category A considered at the beginning of this section, but we can as well work with the image of $\mathrm{N}^{\oplus}$ in the abelian category $\mathrm{MHS}^{\oplus}$, that is, considering the image of $H$ in MHS. The proposition amounts to identifying the weight filtration $W_{\bullet} G:=\mathrm{N}\left(W_{\bullet} H\right)$ with $\mathrm{M}_{w+1+\ell} G$. This follows from Lemma 3.1.16 applied to the category $\mathrm{MHS}^{\oplus}$.

The definition of a nearby/vanishing Hodge-Lefschetz quiver will be a little different from the general definition (3.1.13) of a nearby/vanishing quiver, since we will impose that the nilpotent morphisms $\mathrm{N}_{H}, \mathrm{~N}_{G}$ are those of the corresponding Hodge-Lefschetz structures, hence are (1)-morphisms (we use the terminology of Remark 3.1.10).

## Definition 3.2.14 (Nearby/vanishing Hodge-Lefschetz quiver)

A nearby/vanishing Hodge-Lefschetz quiver centered at $w$ consists of data

$$
(H, \mathrm{~N}),(G, \mathrm{~N}), \mathrm{c}, \mathrm{v},
$$

such that

- $(H, \mathrm{~N})$ is a Hodge-Lefschetz structure centered at $w$,
- $(G, \mathrm{~N})$ is a Hodge-Lefschetz structure centered at $w+1$,
- we have morphisms of Hodge-Lefschetz structures:

$$
\mathrm{c}:(H, \mathrm{~N}) \longrightarrow(G, \mathrm{~N}), \quad \mathrm{v}:(G, \mathrm{~N}) \longrightarrow(H, \mathrm{~N})(-1) \quad \text { (right case) }
$$

- $\mathrm{c} \circ \mathrm{v}=\mathrm{N}_{G}$ and $\mathrm{v} \circ \mathrm{c}=\mathrm{N}_{H}$.

We will use the notation reminiscent to that of (3.1.13):


Example 3.2.16. According to Proposition 3.2.13, the data

$$
\left((H, \mathrm{~N}),\left(\operatorname{Im} \mathrm{N}, \mathrm{~N}_{\mid \mathrm{Im} \mathrm{~N}}\right), \mathrm{N}, \text { incl. }\right)
$$

form a middle extension quiver. Similarly, we have the notion of S-decomposable quiver.

Exercise 3.2.17. Show that the properties of Exercise 3.1.14 extend to nearby/vanishing Hodge-Lefschetz quivers and that, forgetting the filtration one recovers a nearby/vanishing quiver of $\mathbb{C}$-vector spaces.
3.2.b. Polarization. Let $(H, \mathrm{~N})$ be an $\varepsilon$ Hodge-Lefschetz structure centered at $w$, which is graded in the $(-)$ case. The adjoint object $(H, \mathrm{~N})^{*}$ is the object $\left(H^{*},-\mathrm{N}^{*}\right)$, where $\mathrm{N}^{*}$ is the nilpotent endomorphism adjoint to N . One checks, by using the characteristic property of the monodromy filtration, that $\mathrm{gr}_{\ell}^{\mathrm{M}}\left(H^{*}\right)=\left(\mathrm{gr}_{-\ell}^{\mathrm{M}} H\right)^{*}$. Since $\operatorname{gr}_{-\ell}^{\mathrm{M}} H$ is pure of weight $w-\varepsilon \ell, \operatorname{gr}_{\ell}^{\mathrm{M}}\left(H^{*}\right)$ is pure of weight $-w+\varepsilon \ell$, hence $H^{*}$ is a mixed Hodge structure with weight filtration $\mathrm{M}\left(\mathrm{N}^{*}\right)[w]$, and $(H, \mathrm{~N})^{*}$ is a Hodge-Lefschetz structure centered at $-w$.

Let

$$
\mathrm{Q}: H \otimes \bar{H} \longrightarrow \mathbb{C}(-w)
$$

be a bi-filtered morphism. Assume that N is an infinitesimal automorphism of Q , that is, $\mathrm{Q}(\cdot, \overline{\mathrm{N} \bullet})+\mathrm{Q}(\mathrm{N} \bullet, \bar{\bullet})=0$. Equivalently, we can regard Q as a morphism

$$
\mathrm{Q}:(H, \mathrm{~N}) \longrightarrow(H, \mathrm{~N})^{*}(-w)
$$

Then, for every $\ell, \mathrm{Q}$ induces a morphism of $\mathbb{C}$-Hodge structures

$$
\operatorname{gr}_{\ell}^{\mathrm{M}} \mathrm{Q}: \operatorname{gr}_{\ell}^{\mathrm{M}} H \longrightarrow \operatorname{gr}_{\ell}^{\mathrm{M}}\left(H^{*}\right)(-w)=\left(\operatorname{gr}_{-\ell}^{\mathrm{M}} H\right)^{*}(-w)
$$

that is, a sesquilinear pairing of $\mathbb{C}$-Hodge structures

$$
\operatorname{gr}_{\ell}^{\mathrm{M}} \mathrm{Q}: \overline{\operatorname{gr}_{-\ell}^{\mathrm{M}} H} \otimes \operatorname{gr}_{\ell}^{\mathrm{M}} H \longrightarrow \mathbb{C}(-w)
$$

Let us assume $\ell \geqslant 0$. Composing $\operatorname{gr}_{\ell}^{\mathrm{M}} \mathrm{Q}$ with $\mathrm{N}^{* \ell}$ gives a morphism

$$
\mathrm{N}^{* \ell} \circ \operatorname{gr}_{\ell}^{\mathrm{M}} \mathrm{Q}: \operatorname{gr}_{\ell}^{\mathrm{M}} H \longrightarrow \operatorname{gr}_{-\ell}^{\mathrm{M}}\left(H^{*}\right)(-w-\varepsilon \ell)=\left(\operatorname{gr}_{\ell}^{\mathrm{M}} H\right)^{*}(-w-\varepsilon \ell)
$$

In other words, if we set

$$
\mathrm{Q}_{\ell, \ell}(x, \bar{y}):=\operatorname{gr}_{\ell}^{\mathrm{M}} \mathrm{Q}\left(x, \overline{(\operatorname{grN})^{\ell} y}\right)
$$

we define a pairing

$$
\mathrm{Q}_{\ell, \ell}: \operatorname{gr}_{\ell}^{\mathrm{M}} H \otimes \overline{\operatorname{gr}_{\ell}^{\mathrm{M}} H} \longrightarrow \mathbb{C}(-w-\varepsilon \ell)
$$

and we can restrict this pairing to the primitive part to get a pairing

$$
\mathrm{P}_{\ell} \mathrm{Q}: \mathrm{P}_{\ell} H \otimes \overline{\mathrm{P}_{\ell} H} \longrightarrow \mathbb{C}(-w-\varepsilon \ell)
$$

Lemma 3.2.18. Q is non-degenerate if and only if $\mathrm{P}_{\ell} \mathrm{Q}$ are non-degenerate for $\ell \geqslant 0$.
Proof. Since $(H, \mathrm{~N})^{*}$ is a Hodge-Lefschetz structure centered at $-w,(H, \mathrm{~N})^{*}(-w)$ is such a structure centered at $w$, and $\mathrm{Q}: H \rightarrow H^{*}(-w)$ is a morphism of mixed Hodge structures. Therefore, it is strictly compatible with the weight filtrations, and Q is an isomorphism if and only if grQ is an isomorphism between the corresponding graded Hodge-Lefschetz structures. By the Lefschetz decomposition, this is equivalent to $\mathrm{P}_{\ell} \mathrm{Q}$ being an isomorphism.

## Definition 3.2.19 (Polarization of a Hodge-Lefschetz structure)

We say that the pairing $\mathrm{Q}: H \otimes \bar{H} \rightarrow \mathbb{C}(-w)$ is a polarization of the $\varepsilon$ HodgeLefschetz structure $(H, \mathrm{~N})$ centered at $w$ (graded in the ( - ) case) if
(1) N is an infinitesimal automorphism of Q ,
(2) $\mathrm{P}_{\ell} \mathrm{Q}$ is a polarization of the Hodge structure $\mathrm{P}_{\ell} H$ of weight $w+\varepsilon \ell$ for every $\ell \geqslant 0$.

Exercise 3.2.20. Show that if Q is a polarization of $(H, \mathrm{~N})$, then
(1) Q is a polarization of $(H, \mathrm{~N})(k)$ for every $k \in \mathbb{Z}$,
(2) Q is nondegenerate, [Hint: use Lemma 3.2.18]
(3) Q is $(-1)^{w}$-Hermitian. [Hint: Argue as in Remark 3.1.25.]

Exercise 3.2.21. Let $((H, N), \mathrm{Q})$ be a polarized Hodge-Lefschetz structure centered at $w$, and let $\left(H_{1}, \mathrm{~N}\right)$ be a sub-object in $\operatorname{HLS}(w)$ (see Exercise 3.2.12). Show that Q induces a polarization $\mathrm{Q}_{1}$ on $\left(H_{1}, \mathrm{~N}\right)$ and that $\left(\left(H_{1}, \mathrm{~N}\right), \mathrm{Q}_{1}\right)$ is a direct summand of $((H, \mathrm{~N}), \mathrm{Q})$. Conclude that the category $\mathrm{pHLS}(w)$ of polarizable Hodge-Lefschetz structures of weight $w$ is semi-simple. [Hint: Use Exercise 3.2.12 to show that $\mathrm{P}_{\ell} H^{\prime}$ is a sub-object of $\mathrm{P}_{\ell} H$ in $\mathrm{HS}(w+\ell)$, and conclude with Exercise 2.4.22.]

## Remark 3.2.22 (Simplified data for a polarized Hodge-Lefschetz structure)

We can simplify the data of a polarized Hodge-Lefschetz structure centered at $w$ by giving $\left(\left(\mathcal{H}, F^{\bullet} \mathcal{H}\right), \mathrm{N}\right)\left(\right.$ with $\mathrm{N}:\left(\mathcal{H}, F^{\bullet} \mathcal{H}\right) \rightarrow\left(\mathcal{H}, F^{\bullet} \mathcal{H}\right)(-1)$ ) and an isomorphism $\mathrm{Q}:\left(\left(\mathcal{H}, F^{\bullet} \mathcal{H}\right), \mathrm{N}\right) \rightarrow\left(\left(\mathcal{H}, F^{\bullet} \mathcal{H}\right), \mathrm{N}\right)^{*}(-w)$ in such a way that, defining $F^{\prime \prime \bullet} \mathcal{H}$ as in Remark 2.4.20, we obtain data ( $H, \mathrm{~N}, \mathrm{Q}$ ) as in Definition 3.2.19.

Remark 3.2.23 (Polarized mixed Hodge structure). The terminology polarized mixed Hodge structure is also used in the literature for a polarized Hodge-Lefschetz structure.

Remark 3.2.24. The definition of polarized graded (-)Hodge-Lefschetz structure is the analogue of the geometric Hodge structure of Section 2.3: the cohomology $H_{\bullet}:=$ $H^{n-\bullet}(X, \mathbb{C})$, equipped with $\mathrm{N}_{\mathscr{L}}=\mathrm{L}_{\mathscr{L}}$, is a polarizable graded $(-)$ Hodge-Lefschetz structure centered at $n$.

Proposition 3.2.25 (Polarization on $\operatorname{Im} \mathrm{N})$. Let (H,N) be a Hodge-Lefschetz structure centered at $w$ with polarization Q . Let $\left(G, \mathrm{~N}_{G}\right)=\left(\operatorname{Im} \mathrm{N}, \mathrm{N}_{\mid \mathrm{Im} \mathrm{N}}\right)$ be the image of N regarded as an object of $\mathrm{HS}(w+1)$ (Proposition 3.2.13), and define $\mathrm{Q}_{G}$ on $\mathcal{G}$ by

$$
\mathrm{Q}_{G}(\mathrm{~N} x, \overline{\mathrm{~N} y}):=\mathrm{Q}(x, \overline{\mathrm{~N} y})=-\mathrm{Q}(\mathrm{~N} x, \bar{y})
$$

(see Lemma 3.1.17). Then $\mathrm{Q}_{G}$ is a polarization of $\left(G, \mathrm{~N}_{G}\right)$.
Proof. Recall that, for $\ell \geqslant 0, g r N$ induces an isomorphism $\mathrm{P}_{\ell+1} H \xrightarrow{\sim} \mathrm{P}_{\ell} G$ (Lemma 3.1.16(f)). Since $\mathrm{Q}, \mathrm{v}$, c and $\mathrm{c}^{*}$ are morphisms of mixed Hodge structures, so is $\mathrm{Q}_{G}$, and we argue as in Lemma 3.2.18 to obtain that $\operatorname{grQ}_{G}$ is equal to $(\mathrm{grQ})_{G}$. We can thus assume that $(H, \mathrm{~N}, \mathrm{Q})$, and thus $\left(G, \mathrm{~N}_{G}, \mathrm{Q}_{G}\right)$, are graded. For $u, v \in \mathrm{P}_{\ell} \mathcal{G}$ and $x, y \in \mathrm{P}_{\ell+1} \mathcal{H}$ with $u=\mathrm{N} x, v=\mathrm{N} y$, we have

$$
\begin{aligned}
\mathrm{P}_{\ell} \mathrm{Q}_{G}(\mathrm{C} u, \bar{v}):=\mathrm{Q}_{G}\left(\mathrm{CN} x, \overline{\mathrm{~N}^{\ell}(\mathrm{N} y)}\right) & =\mathrm{Q}_{G}\left(\mathrm{NC} x, \overline{\mathrm{~N}^{\ell}\left(\mathrm{N}^{\ell} y\right)}\right. \\
& =\mathrm{Q}\left(\mathrm{C} x, \overline{\mathrm{~N}^{\ell+1} y}\right)=\mathrm{P}_{\ell+1} \mathrm{Q}(\mathrm{C} x, \bar{y})
\end{aligned}
$$

Remark 3.2.26 (Hodge-Lefschetz triples). Let us make explicit the notion of (graded) $\varepsilon$ Hodge-Lefschetz structure in the language of filtered triples (see Section 2.4.c). Let $H=\left(\left(\mathcal{H}^{\prime}, F^{\bullet} \mathcal{H}^{\prime}\right),\left(\mathcal{H}^{\prime \prime}, F^{\bullet} \mathcal{H}^{\prime \prime}\right), \mathfrak{c}\right)$ be a filtered triple, equipped with a nilpotent endomorphism $\mathrm{N}=\left(\mathrm{N}^{\prime}, \mathrm{N}^{\prime \prime}\right): H \rightarrow H(-\varepsilon)$, that is,

$$
\mathrm{N}^{\prime}:\left(\mathcal{H}^{\prime}, F^{\bullet} \mathcal{H}^{\prime}\right) \longrightarrow\left(\mathcal{H}^{\prime}, F[-\varepsilon]^{\bullet} \mathcal{H}^{\prime}\right) \quad \text { and } \quad \mathrm{N}^{\prime \prime}:\left(\mathcal{H}^{\prime \prime}, F^{\bullet} \mathcal{H}^{\prime \prime}\right) \longrightarrow\left(\mathcal{H}^{\prime \prime}, F[-\varepsilon]^{\bullet} \mathcal{H}^{\prime \prime}\right)
$$

are two filtered nilpotent morphisms which satisfy, when forgetting the filtration,

$$
\mathfrak{c}\left(v^{\prime}, \overline{\mathrm{N}^{\prime \prime} v^{\prime \prime}}\right)=\mathfrak{c}\left(\mathrm{N}^{\prime} v^{\prime}, \overline{v^{\prime \prime}}\right)
$$

The adjoint $(H, \mathrm{~N})^{*}$ of $(H, \mathrm{~N})$ is $\left(H^{*},-\mathrm{N}^{*}\right)$, where $H^{*}$ is the adjoint of $H$ and $\mathrm{N}^{*}$ is the adjoint of the morphism N , regarded as a morphism $H^{*} \rightarrow H^{*}(-\varepsilon)$. The monodromy filtration is defined in the abelian category Triples, and we say that $(H, \mathrm{~N})$ is an
 every $\ell$. In such a case, for every $j, k \in \mathbb{Z}$,

$$
(H, \mathrm{~N})(j, k):=\left(\left(\mathcal{H}^{\prime}, F[j]^{\bullet} \mathcal{H}^{\prime}, \mathrm{N}^{\prime}\right),\left(\mathcal{H}^{\prime \prime}, F[-k]^{\bullet} \mathcal{H}^{\prime \prime}, \mathrm{N}^{\prime \prime}\right), \mathfrak{c}\right)
$$

is an $\varepsilon$ Hodge-Lefschetz triple centered at $w-\varepsilon(k+\ell)$ and $(H, \mathrm{~N})^{*}$ is an $\varepsilon$ HodgeLefschetz triple centered at $-w$, with monodromy filtration satisfying $\operatorname{gr}_{\ell}^{\mathrm{M}}\left(H^{*}\right)=$ $\left(\mathrm{gr}_{-\ell}^{\mathrm{M}} H\right)^{*}$.

We note that $\mathfrak{c}$ induces zero on $\mathrm{M}_{\ell} \mathcal{H}^{\prime} \otimes \overline{\mathrm{M}_{-\ell-1} \mathcal{F}^{\prime \prime}}$ for every $\ell$, hence induces a pairing

$$
\mathfrak{c}_{\ell,-\ell}: \operatorname{gr}_{\ell}^{\mathrm{M}} \mathcal{H}^{\prime} \otimes_{\mathbb{C}} \overline{\operatorname{gr}_{-\ell}^{\mathrm{M}} \mathcal{H}^{\prime \prime}} \longrightarrow \mathbb{C}
$$

We have $\left(\operatorname{gr}_{\ell}^{\mathrm{M}} H\right)^{\prime}=\operatorname{gr}_{\ell}^{\mathrm{M}} \mathcal{H}^{\prime}$, while $\left(\operatorname{gr}_{\ell}^{\mathrm{M}} H\right)^{\prime \prime}=\operatorname{gr}_{-\ell}^{\mathrm{M}} \mathcal{H}^{\prime \prime}$, both with their induced filtrations. On the other hand, $\operatorname{grN}:=\left(\mathrm{grN}^{\prime}, \operatorname{grN}^{\prime \prime}\right)$ is a morphism

$$
\left(\mathrm{gr}_{\ell}^{\mathrm{M}} \mathcal{H}^{\prime}, \mathrm{gr}_{-\ell}^{\mathrm{M}} \mathcal{H}^{\prime \prime}\right) \longrightarrow\left(\operatorname{gr}_{\ell-2}^{\mathrm{M}} \mathcal{H}^{\prime}, \mathrm{gr}_{-\ell+2}^{\mathrm{M}} \mathcal{H}^{\prime \prime}\right)
$$

which is compatible with $\mathfrak{c}_{\ell,-\ell}$ and $\mathfrak{c}_{\ell-2,-\ell+2}$. Therefore, the data $\left(\mathrm{gr}^{\mathrm{M}} H, \operatorname{grN}\right)$ defined as

$$
\begin{gathered}
\bigoplus_{\ell}\left(\operatorname{gr}_{\ell}^{\mathrm{M}} \mathcal{H}^{\prime}, F^{\bullet} \operatorname{gr}_{\ell}^{\mathrm{M}} \mathcal{H}^{\prime}\right), \quad \bigoplus_{\ell}\left(\operatorname{gr}_{-\ell}^{\mathrm{M}} \mathcal{H}^{\prime \prime}, F^{\bullet} \operatorname{gr}_{-\ell}^{\mathrm{M}} \mathcal{H}^{\prime \prime}\right) \\
\operatorname{grc}:=\bigoplus_{\ell} \mathfrak{c}_{\ell,-\ell}, \quad \operatorname{grN}:=\left(\operatorname{grN}^{\prime}, \operatorname{grN}^{\prime \prime}\right)
\end{gathered}
$$

form a graded $\varepsilon$ Hodge-Lefschetz triple. In particular it is non-degenerate, which implies that $\mathfrak{c}$ itself is non-degenerate. Its adjoint $\left(\mathrm{gr}_{\bullet}^{\mathrm{M}} H, \mathrm{grN}\right) *$ is also a graded $\varepsilon$ Hodge-Lefschetz triple.

A pre-polarization $\mathrm{Q}=\left(\mathrm{Q},(-1)^{w} Q\right)$ of an $\varepsilon \operatorname{Hodge}$-Lefschetz triple $(H, \mathrm{~N})$ centered at $w$ is a $(-1)^{w}$-Hermitian morphism $(H, \mathrm{~N}) \rightarrow(H, \mathrm{~N})^{*}(-w)$, i.e., a morphism $H \rightarrow$ $H^{*}(-w)$ such that

$$
\mathrm{Q} \circ \mathrm{~N}+\mathrm{N}^{*} \circ \mathrm{Q}: H \longrightarrow H^{*}(-w-\varepsilon)
$$

is zero, that is, $\mathcal{Q} \circ \mathrm{N}^{\prime}+\mathrm{N}^{\prime \prime} \circ \mathcal{Q}: \mathcal{H}^{\prime} \rightarrow \mathcal{H}^{\prime \prime}$ is zero. Since Q is a morphism, it is compatible with the monodromy filtrations and Q induces a pre-polarization $\mathrm{gr}^{\mathrm{M}} \mathrm{Q}$ : $\operatorname{gr}_{{ }^{\mathrm{M}}} H \rightarrow \mathrm{gr}_{\bullet}^{\mathrm{M}}\left(H^{*}\right)(-w)=\left(\mathrm{gr}_{-}^{\mathrm{M}} H\right)^{*}(-w)$ of the $\varepsilon$ Hodge-Lefschetz triple $\left(\mathrm{gr}^{\mathrm{M}} H, \mathrm{grN}\right)$ (check the behaviour of the filtrations $F^{\bullet}$ ). We can then continue the definition of a polarization as in Section 3.2.b.

Lastly, we remark as in Proposition 2.4.38 that any polarizable $\varepsilon$ Hodge-Lefschetz triple centered at $w$ is (graded) isomorphic to $\left(\left(\mathscr{H}^{\prime}, \mathscr{H}^{\prime}(-w), \mathfrak{c}^{\prime}\right), \mathrm{N}\right)$ with polarization $\left((-1)^{w} \mathrm{Id}, \mathrm{Id}\right)$, such that $\left(\left(\mathcal{H}^{\prime}, F^{\bullet} \mathcal{H}^{\prime}\right), \mathrm{N}, \mathfrak{c}^{\prime}\right)$ is a polarized (graded) $\varepsilon$ Hodge-Lefschetz structure centered at $w$.

Remark 3.2.27 (Stability by extension). As in Remark 3.2.26, we consider the abelian category of graded triples $H=\bigoplus_{\ell} H_{\ell}$ endowed with a nilpotent endomorphism $\mathrm{N}: H_{\ell} \rightarrow H_{\ell-2}(-\varepsilon)$. Let

$$
0 \longrightarrow\left(H^{\prime}, \mathrm{N}\right) \longrightarrow(H, \mathrm{~N}) \longrightarrow\left(H^{\prime \prime}, \mathrm{N}\right) \longrightarrow 0
$$

be an exact sequence in this category. Assume that $\left(H^{\prime}, \mathrm{N}\right),\left(H^{\prime \prime}, \mathrm{N}\right)$ are graded $\varepsilon$ Hodge-Lefschetz triples of the same weight $w$. Then $(H, \mathrm{~N})$ is of the same kind. Indeed, by Exercise 2.4.32, each summand $H_{\ell}$ is a $\mathbb{C}$-Hodge triple of weight $w+\varepsilon \ell$. It is then clear that $\mathrm{N}^{\ell}$ is an isomorphism $H_{\ell} \xrightarrow{\sim} H_{-\ell}(-\varepsilon \ell)$ if this holds on $H^{\prime}, H^{\prime \prime}$.

## 3.2.c. Polarized graded Hodge-Lefschetz structures

We now revisit Sections 3.1.d and 3.1.e for (bi-)graded Hodge-Lefschetz structures. Some parts of the proofs will be very similar. Let $(H, \mathrm{~N})$ be a graded Hodge-Lefschetz structure of type $\varepsilon$, centered at $w$. The automorphism $\mathrm{w}: \mathcal{H} \rightarrow \mathcal{H}$ sends each term of the Lefschetz decomposition into another term of this decomposition, and Exercise 3.1.21(4) gives the exact formula for each term. It follows from this formula that each component of w is a morphism of Hodge structures. Hence, for every $\ell$, w : $H_{\ell} \rightarrow H_{-\ell}(-\varepsilon \ell)$ is a morphism of Hodge structures.

Proposition 3.2.28. Let (H, N, Q) be a polarized graded Hodge-Lefschetz structure. Then each $H_{\ell}$ is a polarizable Hodge structure of weight $w+\ell \varepsilon$.

Proof. We apply Proposition 3.1.26 to $\mathrm{k}(u, \bar{v}):=\mathrm{Q}(\mathrm{C} u, \bar{v})$.
Let $(H, \mathrm{~N}, \mathrm{Q})$ and $\left(H^{\prime}, \mathrm{N}^{\prime}, \mathrm{Q}^{\prime}\right)$ be polarized graded Hodge-Lefschetz structures of type $\varepsilon=1$, centered at $w$ and $w+1$ respectively. Let $\mathrm{c}, \mathrm{v}$ be graded morphisms of degree -1 :

$$
\mathrm{c}: H_{\ell} \longrightarrow H_{\ell-1}^{\prime}, \quad \mathrm{v}: H_{\ell}^{\prime} \longrightarrow H_{\ell-1}(-1)
$$

such that $\mathrm{N}=\mathrm{v} \circ \mathrm{c}$ and $\mathrm{N}^{\prime}=\mathrm{cov}$. Note that c is a morphism of Hodge structures of weight $w+\ell$ and v a morphism of Hodge structures of weight $w+1+\ell$. For example, starting from a middle extension nearby/vanishing Hodge-Lefschetz quiver centered at $w$ as in Example 3.2.16, we replace $H, G$ by the associated graded Hodge-Lefschetz structures, and c, v by the associated graded morphism of degree one (see Lemma 3.1.16).

Proposition 3.2.29. Assume that, for every $\ell$, the following diagram commutes:

that is, the two sesquilinear forms

$$
\begin{array}{r}
\mathrm{Q}_{\ell}(\bullet, \overline{\mathrm{v} \bullet}): H_{\ell} \otimes \overline{H_{-\ell+1}^{\prime}} \longrightarrow \mathbb{C}(-w-1) \\
\mathrm{Q}_{\ell-1}^{\prime}(\mathrm{c} \bullet, \bar{\bullet}): H_{\ell} \otimes \overline{H_{-\ell+1}^{\prime}} \longrightarrow \mathbb{C}(-w-1)
\end{array}
$$

are equal. Then we have a decomposition

$$
H^{\prime}=\operatorname{Im} \mathrm{c} \oplus \operatorname{Ker} \mathrm{v}
$$

as a graded Hodge-Lefschetz structure.
Proof. We will argue in a way similar to Proposition 3.1.28. We will take care of Tate twists and we will replace the pairing $\mathrm{k}(u, \overline{\mathrm{w} v})$ with $\mathrm{Q}(\mathrm{C} u, \overline{\mathrm{w} v})$. Recall that $\mathrm{N}: H_{\ell} \rightarrow H_{\ell-2}(-1)$ commutes with the Weil operator C , as well as c: $H_{\ell} \rightarrow H_{\ell-1}^{\prime}$
and v : $H_{\ell}^{\prime} \rightarrow H_{\ell-1}(-1)$. The inclusions (3.1.29) and (3.1.30) are respectively replaced with

$$
\begin{align*}
& \mathrm{c}\left(\mathrm{P} H_{\ell}\right) \subset \mathrm{P} H_{\ell-1}^{\prime} \oplus \mathrm{N}^{\prime}\left(\mathrm{P} H_{\ell+1}^{\prime}(1)\right)  \tag{3.2.30}\\
& \mathrm{v}\left(\mathrm{P} H_{\ell}^{\prime}\right) \subset \mathrm{P} H_{\ell-1}(-1) \oplus \mathrm{N}\left(\mathrm{P} H_{\ell+1}\right) \tag{3.2.31}
\end{align*}
$$

if we set $\mathrm{PH}_{-1}=0$ and $\mathrm{PH}_{-1}^{\prime}=0$. Let us indicate the modification to be made in the proof the following two properties.
(a) for all $\ell \geqslant 0, \mathrm{c}\left(\mathrm{P} H_{\ell}\right) \subset \mathrm{P} H_{\ell-1}^{\prime}$,
(b) for all $\ell \geqslant 2, \mathrm{c}\left(\mathrm{P} H_{\ell}\right)=\mathrm{P} H_{\ell-1}^{\prime}$.

Among the four steps, only Step two has to be modified.
Step two: Proof that $\mathrm{v}\left(\mathrm{P} H_{\ell}^{\prime}\right) \cap \mathrm{P} H_{\ell-1}(-1)=0$. The assumption implies that, with obvious notation,

$$
\mathrm{Q}\left(u_{\ell-1}, \overline{\mathrm{v} w_{-\ell+2}^{\prime}}\right)=-\mathrm{Q}^{\prime}\left(\mathrm{c} u_{\ell-1}, \overline{w_{-\ell+2}^{\prime}}\right)
$$

We have to prove

$$
u_{\ell}^{\prime} \in \mathrm{PH}_{\ell}^{\prime} \text { and } \mathrm{v} u_{\ell}^{\prime} \in \mathrm{P} \mathcal{H}_{\ell-1} \Longrightarrow u_{\ell}^{\prime}=0
$$

Assume $u_{\ell}^{\prime} \neq 0$. We have $\mathrm{Q}^{\prime}\left(\mathrm{C} u_{\ell}^{\prime}, \overline{\mathrm{N}^{\prime \ell} u_{\ell}^{\prime}}\right)>0$ and $\mathrm{Q}\left(\mathrm{Cv} u_{\ell}^{\prime}, \overline{\mathrm{N}^{\ell-1} \mathrm{v} u_{\ell}^{\prime}}\right) \geqslant 0$. Then

$$
\begin{aligned}
0 \leqslant \mathrm{Q}\left(\mathrm{Cv} u_{\ell}^{\prime}, \overline{\mathrm{N}^{\ell-1} \mathrm{v} u_{\ell}^{\prime}}\right) & =\mathrm{Q}^{\prime}\left(\mathrm{CN}^{\prime} u_{\ell}^{\prime}, \overline{\mathrm{N}^{\ell-1} u_{\ell}^{\prime}}\right) \quad \text { (by the assumption) } \\
& =-\mathrm{Q}^{\prime}\left(\mathrm{C} u_{\ell}^{\prime}, \overline{\mathrm{N}^{\ell} u_{\ell}^{\prime}}\right) \quad \text { (by 3.2.19(1)) } \\
& <0,
\end{aligned}
$$

a contradiction.
End of the proof of the proposition. We can now conclude the proof of the proposition. Arguing as in the end of the proof of Proposition 3.1.28, we are left to prove

$$
\mathrm{P} H_{0}^{\prime}=\mathrm{c}\left(\mathrm{P} H_{1}\right) \oplus \operatorname{Ker}_{\mid \mathrm{P} H_{0}^{\prime}}
$$

This follows from Exercise 3.1.31 below, applied to the abelian category of Hodge structures of weight $w+1$.

Fix $\varepsilon_{1}, \varepsilon_{2}= \pm 1$. The notion of a (polarized) bigraded Hodge-Lefschetz structure $\left(H, \mathrm{~N}_{1}, \mathrm{~N}_{2}\right)$ of weight $w$ and bi-type $\left(\varepsilon_{1}, \varepsilon_{2}\right)$ is defined in a natural way, similarly to the single graded case: $\mathrm{N}_{1}$ and $\mathrm{N}_{2}$ should commute and the primitive part in $H_{\ell_{1}, \ell_{2}}$ is by definition the intersection of $\operatorname{Ker} \mathrm{N}_{1}^{\ell_{1}+1}$ and $\operatorname{Ker} \mathrm{N}_{2}^{\ell_{2}+1}$. We have the following analogue of Proposition 3.1.35. We note that w: $H_{\ell_{1}, \ell_{2}} \rightarrow H_{-\ell_{1},-\ell_{2}}\left(-\varepsilon_{1} \ell_{1}-\varepsilon_{2} \ell_{2}\right)$ is a morphism of Hodge structures for every $\ell_{1}, \ell_{2}$.

Proposition 3.2.32. Let $\left(H, \mathrm{~N}_{1}, \mathrm{~N}_{2}, \mathrm{Q}\right)$ be a polarized bi-graded Hodge-Lefschetz structure centered at $w$ and of bi-type $(\varepsilon, \varepsilon)$. Put on $H$ the grading $H_{\ell}=\oplus_{\ell+k=\ell} H_{\ell, k}$ and set $\mathrm{N}=\mathrm{N}_{1}+\mathrm{N}_{2}$. Then $\left(H, \mathrm{~N}_{1}+\mathrm{N}_{2}, \mathrm{Q}\right)$ is a polarized graded Hodge-Lefschetz structure centered at $w$ and of type $\varepsilon$.

Proof. Similar to that of Proposition 3.1.35.

Set $\varepsilon=\left(\varepsilon_{1}+\varepsilon_{2}\right) / 2 \in\{-1,0,1\}$. A differential $d$ on $\left(H, \mathrm{~N}_{1}, \mathrm{~N}_{2}, \mathrm{Q}\right)$ is a morphism of Hodge structures of bidegree $(-1,-1)$

$$
d: H_{\ell_{1}, \ell_{2}} \longrightarrow H_{\ell_{1}-1, \ell_{2}-1}(-\varepsilon)
$$

such that $d \circ d=0$, which commutes with $\mathrm{N}_{1}, \mathrm{~N}_{2}$ and is self-adjoint with respect to Q .
Proposition 3.2.33. In such a situation, the cohomology $\operatorname{Ker} d / \operatorname{Im} d$, with the induced $\mathrm{N}_{1}, \mathrm{~N}_{2}, \mathrm{Q}$, is a polarized bigraded Hodge-Lefschetz structure ( $H, \mathrm{~N}_{1}, \mathrm{~N}_{2}, \mathrm{Q}$ ) of weight $w$ and type $\varepsilon_{1}, \varepsilon_{2}$.

Proof. The proof is similar to that of Proposition 3.1.37, by considering the positive definite Hermitian form $\mathrm{Q}(\mathrm{C} u, \overline{\mathrm{w} v})$.

### 3.3. Comments

The Hard Lefschetz theorem for complex projective varieties equipped with an ample line bundle, named so after the fundamental memoir of Lefschetz [Lef24], and for which there does not exist up to now a purely topological proof (see [Lam81] for an overview of the topology of complex algebraic varieties), is intrinsically present in classical Hodge theory (see e.g. [GH78, Dem96, Voi02]). That a relative version of this theorem is instrumental in proving the decomposition theorem (one of the main goals of the theory of pure Hodge modules) had been emphasized and proved by Deligne in [Del68], by observing the criterion 3.1.9. On the other hand, the theory of degeneration of variations of polarized Hodge structures [Sch73, GS75] also gives rise to such Hodge-Lefschetz structures, not necessarily graded however. Note also that such structures have been discovered by Steenbrink [Ste77] and Varchenko [Var82] on the space of vanishing cycles attached to an isolated critical point of a holomorphic function. This property was at the source of the definition of pure Hodge modules by Saito in [Sai88].

Since the very definition of a pure Hodge module by Saito [Sai88] is modeled on the theory of degenerations, we devote a complete chapter to the notion of a Hodge-Lefschetz structure. Together with the criterion 3.1.9, three results are used in an essential way in the decomposition theorem for pure polarized Hodge modules as proved by Saito [Sai88], namely Propositions 3.2.29, 3.2.32 and 3.2.33. They are originally proved in $[\mathbf{S a i 8 8}, \S 4]$. We follow here the proof given by Guillén and Navarro Aznar in [GNA90], according to the idea, due to Deligne, of using harmonic theory in finite dimensions and the full strength of the action of $\mathrm{SL}_{2}$ by means of the Weil element denoted by w. The polarization property is often reduced to saying that the primitive part of the Hodge-Lefschetz structure is a polarized Hodge structure, and is is rarely emphasized that each graded part of a polarized graded Hodge-Lefschetz structure (like any cohomology space of a smooth complex projective variety) is also a polarized pure Hodge structure. The latter approach makes it explicit.

We have developed the notion of polarized Lefschetz structure in order not to mix two mechanisms to produce polarization: the one coming from Hodge structure (by means of the Weil operator C ) and the one coming from the nilpotent operator and of the $\mathfrak{s l}_{2}$-action. That these two mechanisms can be treated separately is already implicitly present in [GNA90]. We have made it explicit in Section 3.1, in a way that, in Section 3.2, the adaptation to the Hodge-theoretical setting becomes straightforward. Basic results on the monodromy filtration, which gives rise to the Hodge-theoretic weight filtration, are explained in [Sch73, CK82, SZ85].

The notion of a polarized Hodge-Lefschetz structure is also known under the name of polarized mixed Hodge structure [CK82], and it is also said that the nilpotent operator polarizes the mixed Hodge structure. This is justified by the fact that the choice of an ample line bundle on a smooth complex projective variety is regarded as a polarization, and it determines a polarization form on the cohomology. Such data also give rise to a nilpotent orbit (see [Sch73, CK82] and also [Kas85, Def. 2.3.1]). We do not use this terminology here, since we also want to use a Hodge-Lefschetz structure without any polarization, as we did for Hodge structures.

For the purpose of pure Hodge modules, the notion of middle extension Lefschetz quiver is a basic tool, corresponding to the notion of middle extension for perverse sheaves or holonomic $\mathscr{D}$-modules. It consists of two objects, called respectively nearby cycles and vanishing cycles related by two morphisms usually called can and var. The middle extension property is that can is an epimorphism and var is a monomorphism, so that the vanishing cycles are identified with the image of $\mathrm{N}:=$ varocan in the nearby cycles. Hodge theory for vanishing cycles can then be deduced from Hodge theory for nearby cycles, as already remarked by Kashiwara and Kawai [KK87]. In particular, Lemma 3.1.16 is much inspired from [KK87, Prop. 2.1.1], and also of [Sai88, Lem. 5.1.12].

The basic decomposition result of Exercise 3.1.31 is at the heart of the notion of Support-decomposability, which is a fundamental property of Saito's pure Hodge modules [Sai88]. Exercise 3.2.6 is taken from [Sai89b, Prop. 3.7].


[^0]:    1. The present terminology is justified by Exercise 6.4.1.
