CHAPTER 2

HODGE THEORY: REVIEW OF CLASSICAL RESULTS

Summary. This chapter reviews classical results of Hodge theory. It introduces the general notion of $\mathbb{C}$-Hodge structure and various extensions of this notion: polarized $\mathbb{C}$-Hodge structure and mixed $\mathbb{C}$-Hodge structure. These notions are the model (on finite dimensional vector spaces) of the corresponding notions on complex manifolds, called $\mathbb{C}$-Hodge module, polarized $\mathbb{C}$-Hodge module and mixed $\mathbb{C}$-Hodge module.

2.1. Introduction

The notion of (polarized) Hodge structure has emerged from the properties of the cohomology of smooth complex projective varieties. In this chapter, as a prelude to the theory of complex Hodge modules, we focus on the notion of (polarized) complex Hodge structure. In doing so, we forget the integral structure in the cohomology of a smooth complex projective variety, and even the rational structure and the real structure.

We are then left with a very simple structure: a complex Hodge structure is nothing but a finite-dimensional graded vector space, and a morphism between Hodge structures is a graded morphism of degree zero between these vector spaces. Hodge structures obviously form an abelian category.

A polarization is nothing but a positive definite Hermitian form on the underlying vector space, which is compatible with the grading, that is, such that the decomposition given by the grading is orthogonal with respect to the Hermitian form.

It is then clear that any Hodge substructure of a polarized Hodge structure is itself polarized by the induced Hermitian form and, as such, is a direct summand of the original polarized Hodge structure.

Why should the reader continue to read this chapter, since the main definitions and properties have been given above?

The reason is that this description does not have a good behaviour when considering holomorphic families of such object. Such families arise, for example, when considering the cohomology of the smooth varieties occurring in a flat family of smooth complex projective varieties. It is known that the grading does not deform holomorphically. Both the grading and the Hermitian form vary real-analytically, and this causes troubles when applying arguments of complex algebraic geometry.

Instead of the grading, it is then suitable to consider the two natural filtrations giving rise to this grading. One then varies holomorphically and the other one anti-holomorphically. From this richer point of view, one can introduce the notion of weight, which is fundamental in the theory, as it leads to the notion of mixed Hodge structure.

Similarly, instead of the positive definite Hermitian form, one should consider the Hermitian form which is $(-1)^p$-definite on the p-th graded term in order to have an object which varies in a locally constant way, as does the cohomology of the varieties.

This chapter moves around the notion of (polarized) complex Hodge structure by shedding light on its different aspects. We will emphasize the point of view of "triples", which will be the one chosen here for the theory of (polarized) Hodge modules.

2.2. Hodge theory on compact Riemann surfaces

Let X be a compact Riemann surface of genus $g \geqslant 0$. Let us assume for simplicity that it is connected. Then $H^0(X,\mathbb{Z})$ and $H^2(X,\mathbb{Z})$ are both isomorphic to $\mathbb{Z}$ (as X is orientable). The only interesting cohomology group is $H^1(X,\mathbb{Z})$, isomorphic to $\mathbb{Z}^{2g}$.

The Poincaré duality induces a skew-symmetric non-degenerate bilinear form

$$\langle\bullet,\bullet\rangle : H^1(X,\mathbb{Z})\otimes_{\mathbb{Z}} H^1(X,\mathbb{Z}) \xrightarrow{\ \bullet\cup\bullet\ } H^2(X,\mathbb{Z}) \xrightarrow{\ \int_{[X]}\ } \mathbb{Z}.$$

One of the main analytic results of the theory asserts that the space $H^1(X,\mathscr{O}_X)$ is finite dimensional and has dimension equal to the genus g (see e.g. [**Rey89**, Chap. IX] for a direct approach). Then, *Serre duality* $H^1(X,\mathscr{O}_X) \xrightarrow{\sim} H^0(X,\Omega^1_X)^\vee$ also gives $\dim H^0(X,\Omega^1_X) = g$. A dimension count implies then the *Hodge decomposition*

$$H^1(X,\mathbb{C}) \simeq H^{0,1}(X)\oplus H^{1,0}(X),\quad H^{0,1}(X) = H^1(X,\mathscr{O}_X),\quad H^{1,0}(X) = H^0(X,\Omega^1_X).$$

If we regard Serre duality as the pairing

$$H^{1,0}\otimes_{\mathbb{C}} H^{0,1} \xrightarrow{\ \bullet\wedge\bullet\ } H^{1,1} \xrightarrow{\ \int\ } \mathbb{C},$$

then Serre duality is equivalent to the complexified Poincaré duality pairing

$$\langle\bullet,\bullet\rangle_{\mathbb{C}} : H^1(X,\mathbb{C})\otimes_{\mathbb{C}} H^1(X,\mathbb{C}) \longrightarrow \mathbb{C},$$

as $\langle H^{1,0},H^{1,0}\rangle = 0$ and $\langle H^{0,1},H^{0,1}\rangle = 0$.

With respect to the real structure $H^1(X,\mathbb{C}) = \mathbb{C}\otimes_{\mathbb{R}} H^1(X,\mathbb{R})$, $H^{1,0}$ is conjugate to $H^{0,1}$, and using Serre duality (or Poincaré duality) we get a sesquilinear pairing

$$\mathrm{Q}^{(1)} : H^{1,0}\otimes_{\mathbb{C}} \overline{H^{1,0}} \longrightarrow \mathbb{C}.$$

Then, the Hodge-Riemann bilinear relations assert that $\mathrm{h} := i\mathfrak{c}$ is a *positive definite Hermitian form.*

2.3. Hodge theory of smooth projective varieties

Let X be a smooth complex projective variety of pure complex dimension n (i.e., each of its connected components has dimension n). It will be endowed with the usual topology, which makes it a complex analytic manifold. Classical Hodge theory asserts that each cohomology space $H^k(X,\mathbb{C})$ decomposes as the direct sum

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X), \tag{2.3.1}$$

where $H^{p,q}(X)$ stands for $H^q(X,\Omega^p_X)$ or, equivalently, for the Dolbeault cohomology space $H^{p,q}_{d''}(X)$. Although this result is classically proved by methods of analysis (Hodge theory for the Laplace operator), it can be expressed in a purely algebraic way, by means of the de Rham complex.

The holomorphic de Rham complex is the complex of sheaves $(\Omega^\bullet_X, \mathrm{d})$, where d is the differential, sending a k-form to a $(k+1)$-form. Recall (holomorphic Poincaré lemma) that $(\Omega^\bullet_X,\mathrm{d})$ is a resolution of the constant sheaf. Therefore, the cohomology $H^k(X,\mathbb{C})$ is canonically identified with the hypercohomology $\boldsymbol{H}^k\big(X,(\Omega^\bullet_X,\mathrm{d})\big)$ of the holomorphic de Rham complex.

Exercise 2.3.2 **(Algebraic de Rham complex).** Using the *Zariski topology* on X, we get an algebraic variety denoted by X^{alg}. In the algebraic category, it is also possible to define a de Rham complex, called the *algebraic de Rham complex*.

(1) Is the algebraic de Rham complex a resolution of the constant sheaf $\mathbb{C}_{X^{\mathrm{alg}}}$?
(2) Do we have $H^\bullet(X^{\mathrm{alg}},\mathbb{C}) = \boldsymbol{H}^\bullet\big(X^{\mathrm{alg}},(\Omega^\bullet_{X^{\mathrm{alg}}},\mathrm{d})\big)$?

The de Rham complex can be filtered in a natural way by sub-complexes ("filtration bête" in [**Del71b**]).

Remark 2.3.3. In general, we denote by an upper index a *decreasing filtration* and by a lower index an *increasing filtration*. Filtrations are indexed by $\mathbb{Z}$ unless otherwise specified.

We define the "stupid" (increasing) filtration on $\mathscr{O}_X$ by setting

$$F_p\mathscr{O}_X = \begin{cases} \mathscr{O}_X & \text{if } p \geqslant 0,\\ 0 & \text{if } p \leqslant -1.\end{cases}$$

Observe that, trivially, $\mathrm{d}(F_p\mathscr{O}_X \otimes_{\mathscr{O}_X} \Omega^k_X) \subset F_{p+1}\mathscr{O}_X \otimes_{\mathscr{O}_X} \Omega^{k+1}_X$. Therefore, the de Rham complex can be (decreasingly) filtered by

$$F^p(\Omega^\bullet_X,\mathrm{d}) = \{0 \longrightarrow F_{-p}\mathscr{O}_X \xrightarrow{\ \mathrm{d}\ } F_{-p+1}\mathscr{O}_X \otimes_{\mathscr{O}_X} \Omega^1_X \xrightarrow{\ \mathrm{d}\ } \cdots\}. \tag{2.3.4}$$

If $p \leqslant 0$, $F^p(\Omega^\bullet_X,\mathrm{d}) = (\Omega^\bullet_X,\mathrm{d})$, although if $p \geqslant 1$,

$$F^p(\Omega^\bullet_X,\mathrm{d}) = \{0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \underset{p}{\Omega^p_X} \longrightarrow \cdots \longrightarrow \Omega^{\dim X}_X \longrightarrow 0\}.$$

Therefore, the p-th graded complex is 0 if $p \leqslant -1$ and, if $p \geqslant 0$, it is given by

$$\operatorname{gr}_F^p(\Omega_X^\bullet, \mathrm{d}) = \{0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \underset{p}{\Omega_X^p} \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0\}.$$

In other words, the graded complex $\operatorname{gr}_F(\Omega_X^\bullet, \mathrm{d}) = \bigoplus_p \operatorname{gr}_F^p(\Omega_X^\bullet, \mathrm{d})$ is the complex $(\Omega_X^\bullet, 0)$ (i.e., the same terms as for the de Rham complex, but with differential equal to 0).

From general results on filtered complexes, the filtration of the de Rham complex induces a (decreasing) filtration on the hypercohomology spaces (that is, on the de Rham cohomology of X) and there is a spectral sequence starting from $\boldsymbol{H}^\bullet\big(X, \operatorname{gr}_F(\Omega_X^\bullet, \mathrm{d})\big)$ and abutting to $\operatorname{gr}_F H^\bullet(X, \mathbb{C})$. Let us note that $\boldsymbol{H}^\bullet\big(X, \operatorname{gr}_F(\Omega_X^\bullet, \mathrm{d})\big)$ is nothing but $\bigoplus_{p,q} H^q(X, \Omega_X^p)$.

Theorem 2.3.5. *The spectral sequence of the filtered de Rham complex on a smooth projective variety degenerates at E_1, that is,*

$$H^\bullet(X, \mathbb{C}) \simeq H^\bullet_{\mathrm{DR}}(X, \mathbb{C}) = \bigoplus_{p,q} H^q(X, \Omega_X^p).$$

Remark 2.3.6. Although the classical proof uses Hodge theory for the Laplace operator which is valid in the general case of compact Kähler manifolds, there is a purely algebraic/arithmetic proof in the projective case, due to Deligne and Illusie [**DI87**].

For every k, Poincaré duality is the non-degenerate bilinear pairing

$$\langle \bullet, \bullet \rangle_{n-k} : H^{n-k}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} H^{n+k}(X, \mathbb{Z}) \xrightarrow{\ \bullet \cup \bullet\ } H^{2n}(X, \mathbb{Z}) \xrightarrow{\ \int_{[X]}\ } \mathbb{Z}.$$

It is $(-1)^{n-k}$-symmetric. In particular (taking $k = 0$), $\langle \bullet, \bullet \rangle_n$ is a non-degenerate $(-1)^n$-symmetric bilinear form on $H^n(X, \mathbb{C})$.

Then

$$\langle H^{p',n-p'}, H^{p,n-p} \rangle_0 = 0 \quad \text{if } p + p' \neq n. \tag{2.3.7}$$

We denote by $\langle \bullet, \bullet \rangle_0$ this bilinear form made sesquilinear on the second variable:

$$\langle \bullet, \bullet \rangle_0 : H^n(X, \mathbb{C}) \otimes \overline{H^n(X, \mathbb{C})} \longrightarrow \mathbb{C}.$$

It is thus $(-1)^n$-Hermitian. From (2.3.7) and the equality $H^{n-p,p} = \overline{H^{p,n-p}}$ one deduces that the Hodge decomposition of $H^n(X, \mathbb{C})$ is $\langle \bullet, \bullet \rangle_0$-orthogonal.

Notation 2.3.8. For every $k \in \mathbb{Z}$, we set $\varepsilon(k) = (-1)^{k(k-1)/2}$. We have

$$\varepsilon(k+1) = \varepsilon(-k) = (-1)^k \varepsilon(k), \quad \varepsilon(k+\ell) = (-1)^{k\ell} \varepsilon(k)\varepsilon(\ell), \quad \varepsilon(n-k) = (-1)^k \varepsilon(n+k).$$

If we take complex coordinates $z_1, \dots, z_n$ and set $z_j = x_j + iy_j$, then

$$(\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n) \wedge (\mathrm{d}\overline{z}_1 \wedge \cdots \wedge \mathrm{d}\overline{z}_n) = (-1)^{n(n-1)/2}(2/i)^n \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n.$$

This explains the presence of $\varepsilon(n-k)$ and of powers of i in the following formulas.

Set now $H = \bigoplus_{k\in\mathbb{Z}} H^{n+k}(X,\mathbb{C})$ and let $\mathrm{Q} : H \otimes_{\mathbb{C}} \overline{H} \to \mathbb{C}$ be the sesquilinear form such that $\mathrm{Q}(H^{n-k}, \overline{H^{n+\ell}}) = 0$ if $k \neq \ell$ and, for every k, the restriction of Q to $H^{n-k}(X,\mathbb{C}) \otimes_{\mathbb{C}} \overline{H^{n+k}(X,\mathbb{C})}$ is equal to $\varepsilon(n-k)\langle\bullet, \overline{\bullet}\rangle_{n-k}$, i.e., the sesquilinear form (conjugation on the second variable) associated to the $\mathbb{C}$-bilinear form $\varepsilon(n-k)\langle\bullet,\bullet\rangle_{n-k}$.

Exercise 2.3.9. Check that $\mathrm{Q} : H \otimes_{\mathbb{C}} \overline{H} \to \mathbb{C}$ is $(-1)^n$-Hermitian.

In order to obtain positivity results, it is necessary to choose an isomorphism between the vector spaces $H^{n-k}(X,\mathbb{C})$ and $H^{n+k}(X,\mathbb{C})$ (we know that they have the same dimension, as Poincaré duality is non-degenerate). A class of good morphisms is given by the *Lefschetz operators* that we define now.

Fix an ample line bundle $\mathscr{L}$ on X (for instance, any embedding of X in a projective space defines a very ample bundle, by restricting the canonical line bundle $\mathscr{O}(1)$ of the projective space to X). The first Chern class $c_1(\mathscr{L}) \in H^2(X,\mathbb{Z})$ defines a Lefschetz operator

$$\mathrm{L}_{\mathscr{L}} := c_1(\mathscr{L}) \cup \bullet : H^j(X,\mathbb{Z}) \longrightarrow H^{j+2}(X,\mathbb{Z}). \tag{2.3.10}$$

(Note that wedging on the left or on the right amounts to the same, as c_1 has degree 2.)

The *Hard Lefschetz theorem*, usually proved together with the previous results of Hodge theory, asserts that, *for any smooth complex projective variety X, any ample line bundle $\mathscr{L}$, and any $k \geqslant 1$, the k-th power $\mathrm{L}^k_{\mathscr{L}} : H^{n-k}(X,\mathbb{Q}) \to H^{n+k}(X,\mathbb{Q})$ is an isomorphism.*[(1)] In such a case, one can choose as a Kähler form ω on X a real $(1,1)$-form whose cohomology class in $H^2(X,\mathbb{R})$ is $c_1(\mathscr{L})$, and the Lefschetz operator $\mathrm{L}_{\mathscr{L}}$ can be lifted as the operator on differential forms obtained by wedging with ω.

Since $\langle u, \overline{\mathrm{L}_{\mathscr{L}} v}\rangle = \langle \mathrm{L}_{\mathscr{L}} u, \overline{v}\rangle$ (as $\mathrm{L}_{\mathscr{L}}$ is real), and since $\varepsilon(n-k+2) = -\varepsilon(n-k)$, we conclude that $\mathrm{L}_{\mathscr{L}}$ is an infinitesimal automorphism of Q, that is,

$$\mathrm{Q}(u, \overline{\mathrm{L}_{\mathscr{L}} v}) + \mathrm{Q}(\mathrm{L}_{\mathscr{L}} u, \overline{v}) = 0. \tag{2.3.11}$$

Let us assume $k \geqslant 0$. If we fix such a Lefschetz operator, we can identify $H^{n+k}(X,\mathbb{C})$ with $H^{n-k}(X,\mathbb{C})$ by means of $\mathrm{L}^k_{\mathscr{L}}$ and get a sesquilinear form $\mathrm{Q}^{(n-k)}$ on $H^{n-k}(X,\mathbb{C})$ by setting

$$\mathrm{Q}^{(n-k)}(u, \overline{v}) := Q(u, \overline{\mathrm{L}^k_{\mathscr{L}} v}) = \varepsilon(n-k)\langle u, \overline{\mathrm{L}^k_{\mathscr{L}} v}\rangle_{n-k}. \tag{2.3.12}$$

In such a way, one obtains a $(-1)^{n-k}$-Hermitian non-degenerate bilinear form on $H^{n-k}(X,\mathbb{C})$ which satisfies, for every p,q and p',q' with $p+q = p'+q' = n-k$:

$$\mathrm{Q}^{(n-k)}(H^{p,q}, \overline{H^{p',q'}}) = 0 \quad \text{if } p+p' \neq n-k.$$

Let C be the *Weil operator*, which is the multiplication by i^{p-q} on $H^{p,q}$. Then

$$\mathrm{h}(u, \overline{v}) := \mathrm{Q}^{(n-k)}(\mathrm{C}u, \overline{v}) = \mathrm{Q}(\mathrm{C}u, \overline{\mathrm{L}^k_{\mathscr{L}} v}) \tag{2.3.13}$$

1. It is known that the same statement is not true in general if one replaces the coefficients $\mathbb{Q}$ with $\mathbb{Z}$.

is a Hermitian form on $H^{n-p}(X,\mathbb{C})$. Let us note however that this Hermitian form is possibly not positive definite in general. Let us set $\mathrm{P}^{n-k}(X,\mathbb{Q}) := \operatorname{Ker} \mathrm{L}_{\mathscr{L}}^{k+1} : H^{n-k}(X,\mathbb{Q}) \to H^{n+k+2}(X,\mathbb{Q})$: this is the *primitive part* of $H^{n-k}(X,\mathbb{Q})$ with respect to $\mathrm{L}_{\mathscr{L}}$. One notes that the Lefschetz operator has type $(1,1)$ with respect to the Hodge decomposition since it can be realized at the level of C^∞ forms by wedging with a C^∞ form of type $(1,1)$ representing $c_1(\mathscr{L})$, hence sends $H^{p,q}$ to $H^{p+1,q+1}$. Therefore, for every $k \geqslant 0$, $\mathrm{L}_{\mathscr{L}}^k$ induces and isomorphism $H^{p,q} \xrightarrow{\sim} H^{p+k,q+k}$ for every p,q with $p+q=n-k$. As a consequence, the primitive part can be decomposed as $\mathrm{P}^{n-k}(X,\mathbb{C}) = \bigoplus_{p+q=n-k} \mathrm{P}^{p,q}$, with

$$\mathrm{P}^{p,q} := \operatorname{Ker}\bigl[\mathrm{L}_{\mathscr{L}}^{k+1} : H^{p,q} \to H^{p+k+1,q+k+1}\bigr].$$

The positivity result is now stated as follows.

The restriction of the Hermitian form (2.3.13) *to* $\mathrm{P}^{n-k}(X,\mathbb{C})$ *is positive definite.* For example, when $k=0$, we get positivity on $\operatorname{Ker}[\mathrm{L}_{\mathscr{L}} : H^{p,n-p} \to H^{p+1,n-p+1}]$. In the case of curves ($n=1$), such a restriction is empty as, whatever the choice of $\mathscr{L}$ is, we have $\mathrm{L}_{\mathscr{L}} = 0$ on $H^1(X,\mathbb{C})$ (as it takes values in $H^3=0$).

2.4. Polarizable $\mathbb{C}$-Hodge structures

The previous properties of the cohomology of a projective variety can be put in an axiomatic form. This will happen to be useful as a first step to Hodge modules. We will not emphasize the rational, or even the real, aspect of this theory, and concentrate on the complex aspect only. We will make use the properties of filtered objects recalled in Appendix A.2.

2.4.a. $\mathbb{C}$-Hodge structures. This is, in some sense, a category looking like that of finite dimensional complex vector spaces. In particular, it is *abelian*, that is, the kernel and cokernel of a morphism exist in this category. This category is very useful as an intermediate category for building that of mixed Hodge structures, but the main results in Hodge theory use a supplementary property, namely the existence of a polarization (see Section 2.4.b). Let us start with the opposedness property.

Definition 2.4.1 **(Opposite filtrations).** Let us fix $w \in \mathbb{Z}$. Given two decreasing filtrations $F'^{\bullet}\mathcal{H}, F''^{\bullet}\mathcal{H}$ of a vector space $\mathcal{H}$ by vector subspaces, we say that the filtrations $F'^{\bullet}\mathcal{H}$ and $F''^{\bullet}\mathcal{H}$ are *w-opposite* if

$$\begin{cases} F'^{p}\mathcal{H} \cap F''^{w-p+1}\mathcal{H} = 0 \\ F'^{p}\mathcal{H} + F''^{w-p+1}\mathcal{H} = \mathcal{H} \end{cases} \qquad \text{for every } p \in \mathbb{Z},$$

i.e., $F'^{p}\mathcal{H} \oplus F''^{w-p+1}\mathcal{H} \xrightarrow{\sim} \mathcal{H}$ for every $p \in \mathbb{Z}$.

Definition 2.4.2 **($\mathbb{C}$-Hodge structure).** A $\mathbb{C}$*-Hodge structure of weight* $w \in \mathbb{Z}$

$$H = (\mathcal{H}, F'^{\bullet}\mathcal{H}, F''^{\bullet}\mathcal{H})$$

consists of a complex vector space $\mathcal{H}$ equipped with two decreasing filtrations $F'^{\bullet}\mathcal{H}$ and $F''^{\bullet}\mathcal{H}$ which are w-opposite. A morphism between $\mathbb{C}$-Hodge structures is a linear morphism between the underlying vector spaces compatible with both filtrations. We denote by $\mathsf{HS}(\mathbb{C})$ the category of $\mathbb{C}$-Hodge structures of some weight w and by $\mathsf{HS}(\mathbb{C}, w)$ the full category whose objects have weight w.

***Exercise 2.4.3* (The category $\mathsf{HS}(\mathbb{C}, w)$ is abelian).**

(1) Given two decreasing filtrations $F'^{\bullet}\mathcal{H}, F''^{\bullet}\mathcal{H}$ of a vector space $\mathcal{H}$ by vector subspaces, show that the following properties are equivalent:

(a) the filtrations $F'^{\bullet}\mathcal{H}$ and $F''^{\bullet}\mathcal{H}$ are w-*opposite*;

(b) setting $\mathcal{H}^{p,w-p} = F'^{p}\mathcal{H} \cap F''^{w-p}\mathcal{H}$, then $\mathcal{H} = \bigoplus_p \mathcal{H}^{p,w-p}$.

(2) (*Strictness of morphisms*) Show that a morphism $\varphi : H_1 \to H_2$ between objects of $\mathsf{HS}(\mathbb{C}, w)$ preserves the decomposition (1b) as well. Conclude that it is *strictly* compatible with both filtrations, that is, $\varphi(F^{\bullet}\mathcal{H}_1) = \varphi(\mathcal{H}_1) \cap F^{\bullet}\mathcal{H}_2$ (with $F = F'$ or $F = F''$). Deduce that, if H' is a sub-object of H in $\mathsf{HS}(\mathbb{C}, w)$, i.e., there is a morphism $H' \to H$ in $\mathsf{HS}(\mathbb{C}, w)$ whose induced morphism $\mathcal{H}' \to \mathcal{H}$ is injective, then $F^{\bullet}\mathcal{H}' = \mathcal{H}' \cap F^{\bullet}\mathcal{H}$ for $F = F'$ and $F = F''$, and $\mathcal{H}'^{p,q} = \mathcal{H}' \cap \mathcal{H}^{p,q}$.

(3) (*Abelianity*) Conclude that the category $\mathsf{HS}(\mathbb{C}, w)$ is *abelian*.

Let us emphasize this statement.

Proposition 2.4.4. *The category $\mathsf{HS}(\mathbb{C}, w)$ of complex Hodge structures of weight w is abelian, and any morphism is strictly compatible with both filtrations and with the decomposition.* □

***Proposition 2.4.5* (Morphisms in $\mathsf{HS}(\mathbb{C})$).**

(1) *Let $\varphi : H' \to H$ be a morphism between objects of $\mathsf{HS}(\mathbb{C}, w)$ such that the induced morphism $\mathcal{H}' \to \mathcal{H}$ is injective. Then $F^{\bullet}\mathcal{H}' = \varphi^{-1}F^{\bullet}\mathcal{H}$ and φ is a monomorphism in $\mathsf{HS}(\mathbb{C}, w)$. If moreover the induced morphism $\mathcal{H}' \to \mathcal{H}$ is an isomorphism, then φ is an isomorphism in $\mathsf{HS}(\mathbb{C}, w)$.*

(2) *There is no non-zero morphism $\varphi : H_1 \to H_2$ in $\mathsf{HS}(\mathbb{C})$ if $w_1 > w_2$.*

Proof.

(1) The first point is immediate since φ is graded of degree zero with respect to the Hodge grading.

(2) The image of $\mathcal{H}_1^{p,w_1-p}$ is contained in $F'^{p}\mathcal{H}_2 \cap F''^{w_1-p}\mathcal{H}_2$, hence in $F'^{p}\mathcal{H}_2 \cap F''^{w_2+1-p}\mathcal{H}_2$ since $w_1 > w_2$, and the latter space is zero by Definition 2.4.1. □

***Exercise 2.4.6* (Non-abelianity).** Is the category $\mathsf{HS}(\mathbb{C})$ abelian? [*Hint*: consider a linear morphism $\mathcal{H}_1^{1,0} \oplus \mathcal{H}_1^{0,1} \to \mathcal{H}_2^{2,0} \oplus \mathcal{H}_2^{1,1} \oplus \mathcal{H}_2^{0,2}$ sending $\mathcal{H}_1^{1,0}$ into $\mathcal{H}_2^{2,0} \oplus \mathcal{H}_2^{1,1}$ and $\mathcal{H}_1^{0,1}$ into $\mathcal{H}_2^{1,1} \oplus \mathcal{H}_2^{0,2}$, and check when it is strict.]

***Remark 2.4.7* (A geometric interpretation of a bi-filtered vector space)**

Introduce a new variable z and consider, in the free $\mathbb{C}[z, z^{-1}]$-module $\mathscr{H} :=$

$\mathbb{C}[z,z^{-1}]\otimes_{\mathbb{C}}\mathcal{H}$, the object $\mathscr{F}':=\bigoplus_p F'^p\mathcal{H}z^{-p}$; show that $\mathscr{F}'$ is a free $\mathbb{C}[z]$-submodule of $\mathscr{H}$ which generates $\mathscr{H}$, that is, $\mathscr{H}=\mathbb{C}[z,z^{-1}]\otimes_{\mathbb{C}[z]}\mathscr{F}'$. Similarly, denote by $\mathscr{F}''$ the object $\bigoplus_q F''^q\mathcal{H}z^q$; show that $\mathscr{F}''$ is a free $\mathbb{C}[z^{-1}]$-submodule of $\mathscr{H}$ which generates $\mathscr{H}$, that is, $\mathscr{H}=\mathbb{C}[z,z^{-1}]\otimes_{\mathbb{C}[z^{-1}]}\mathscr{F}''$. Using the gluing

$$\begin{tikzcd}
\mathbb{C}[z,z^{-1}]\otimes_{\mathbb{C}[z]}\mathscr{F}' \arrow[r, "\sim"] \arrow[rr, bend left, "\sim"] & \mathscr{H} & \mathbb{C}[z,z^{-1}]\otimes_{\mathbb{C}[z^{-1}]}\mathscr{F}'' \arrow[l, "\sim"']
\end{tikzcd}$$

the pair $(\mathscr{F}',\mathscr{F}'')$ defines an algebraic vector bundle $\mathscr{F}$ on $\mathbb{P}^1$ of rank $\dim\mathcal{H}$. Show that the properties 2.4.3(1a) and (1b) are also equivalent to

(c) The vector bundle $\mathscr{F}$ is isomorphic to $\mathscr{O}_{\mathbb{P}^1}(w)^{\dim\mathcal{H}}$.

Exercise 2.4.8.

(1) (Another proof of 2.4.5(2)) Show that a morphism in $\mathsf{HS}(\mathbb{C})$ induces a morphism between the associated vector bundles on $\mathbb{P}^1$ (see Remark 2.4.7). Conclude that there is no non-zero morphism if $w_1>w_2$. [*Hint*: use standard properties of vector bundles on $\mathbb{P}^1$.]

(2) Let H_1 and H_2 be objects of $\mathsf{HS}(\mathbb{C},w)$, let $\mathscr{F}_1,\mathscr{F}_2$ be the associated $\mathscr{O}_{\mathbb{P}^1}$-modules (see Remark (2.4.7)) and let H be a bi-filtered vector space whose associated $\mathscr{O}_{\mathbb{P}^1}$-module $\mathscr{F}$ is an extension of $\mathscr{F}_1,\mathscr{F}_2$ in the category of $\mathscr{O}_{\mathbb{P}^1}$-modules. Show that H is an object of $\mathsf{HS}(\mathbb{C},w)$. [*Hint*: use standard properties of vector bundles on $\mathbb{P}^1$.]

***Exercise 2.4.9* (Operations on filtrations and opposedness).**

(1) (*Exchange of filtrations*) Set $\mathrm{Exch}(F'^\bullet\mathcal{H},F''^\bullet\mathcal{H}):=(F''^\bullet\mathcal{H},F'^\bullet\mathcal{H})$. Show that, if $(F'^\bullet\mathcal{H},F''^\bullet\mathcal{H})$ are w-opposite, then so are $\mathrm{Exch}(F'^\bullet\mathcal{H},F''^\bullet\mathcal{H})$.

(2) (*Tensor product*) If one defines the filtration on the tensor product as

$$F^p(\mathcal{H}_1\otimes\mathcal{H}_2)=\sum_{p_1+p_2=p}F^{p_1}\mathcal{H}_1\otimes F^{p_2}\mathcal{H}_2,$$

then, if $(F'^\bullet\mathcal{H}_1,F''^\bullet\mathcal{H}_1)$ are w_1-opposite and $(F'^\bullet\mathcal{H}_2,F''^\bullet\mathcal{H}_2)$ are w_2-opposite, show that $(F'^\bullet(\mathcal{H}_1\otimes\mathcal{H}_2),F''^\bullet(\mathcal{H}_1\otimes\mathcal{H}_2))$ are (w_1+w_2)-opposite.

(3) (*Hom*) If one defines the filtration on the space of linear morphisms as

$$F^p\operatorname{Hom}(\mathcal{H}_1,\mathcal{H}_2)=\{f\in\operatorname{Hom}(\mathcal{H}_1,\mathcal{H}_2)\mid\forall\,k\in\mathbb{Z},\ f(F^k\mathcal{H}_1)\subset F^{p+k}\mathcal{H}_2\},$$

show that, with the assumption of (1), $(F'^\bullet\operatorname{Hom}(\mathcal{H}_1,\mathcal{H}_2),F''^\bullet\operatorname{Hom}(\mathcal{H}_1,\mathcal{H}_2))$ are (w_2-w_1)-opposite.

(4) (*Dual*) Equip the vector space $\mathbb{C}$ with the trivial filtrations $F'^0=F''^0=\mathbb{C}$ and $F'^1=F''^1=0$. They are 0-opposite. Conclude that the dual space $\mathcal{H}^\vee$ is naturally equipped with a pair of $(-w)$-opposite filtrations, defined by

$$F'^p\mathcal{H}^\vee=(F'^{-p+1}\mathcal{H})^\perp,\qquad F''^p\mathcal{H}^\vee=(F''^{-p+1}\mathcal{H})^\perp,$$

and we have

$$\operatorname{gr}^p_{F'}\mathcal{H}^\vee\simeq\bigl(\operatorname{gr}^{-p}_{F'}\mathcal{H}\bigr)^\vee,\qquad\operatorname{gr}^p_{F''}\mathcal{H}^\vee\simeq\bigl(\operatorname{gr}^{-p}_{F''}\mathcal{H}\bigr)^\vee.$$

(5) (*Conjugation*) Let $\overline{\mathcal{H}}$ be the complex conjugate of $\mathcal{H}$. Consider the bi-filtered vector space $\overline{H}$:

$$\overline{H} := (\overline{\mathcal{H}}, \overline{F''^{\bullet}\mathcal{H}}, \overline{F'^{\bullet}\mathcal{H}}). \tag{2.4.9 $*$}$$

Show that $\overline{H} \in \mathsf{HS}(\mathbb{C}, w)$ and $\overline{\mathcal{H}^{p,w-p}} = \overline{\mathcal{H}}^{w-p,p}$.

(6) (*Adjunction*) Define the adjoint Hodge structure H^* as the conjugate dual Hodge structure $\overline{H}^{\vee}$. Deduce that it is an object of $\mathsf{HS}(\mathbb{C}, -w)$.

Remark 2.4.10 **(Tate twist).** Given a $\mathbb{C}$-Hodge structure H of weight w and integers k, ℓ, we set $H(k,\ell) := (\mathcal{H}, F[k]'^{\bullet}\mathcal{H}, F[\ell]''^{\bullet}\mathcal{H})$. Then $H(k,\ell)$ is a $\mathbb{C}$-Hodge structure of weight $w-k-\ell$. This leads to an equivalence between the category $\mathsf{HS}(\mathbb{C}, w)$ with $\mathsf{HS}(\mathbb{C}, w-k-\ell)$ (morphisms are unchanged). Let us note in particular that $H^*(k,\ell) = H(-k,-\ell)^*$.

Notation 2.4.11 **(for the symmetric Tate twist).** In various formulas, a symmetric Tate twist (k,k) occurs, corresponding to the Tate twist (k) in $\mathbb{Q}$-Hodge theory. In order to keep this analogy clear, we will keep the notation (k) instead of (k,k). In particular, $H^*(k) = H(-k)^*$.

Remark 2.4.12 **(An ambient abelian category).** In order to regard all categories $\mathsf{HS}(\mathbb{C}, w)$ $(w \in \mathbb{Z})$ as full subcategories of a single *abelian* category, one has to modify a little the presentation of $\mathsf{HS}(\mathbb{C}, w)$. The starting point is that the category of filtered vector spaces and filtered morphisms is not abelian, and one can use the Rees trick already used in Remark (2.4.7) (see Definition A.2.3) to replace it with an abelian category.

A finite dimensional $\mathbb{C}$-vector space $\mathcal{H}$ with an exhaustive filtration $F^{\bullet}\mathcal{H}$ defines a free graded $\mathbb{C}[z]$-module $\mathscr{H}$ of finite rank by the formula $\mathscr{H} = \bigoplus_p F^p\mathcal{H}z^{-p}$ (the term $F^p\mathcal{H}z^{-p}$ is in degree p). On the other hand, the category $\mathsf{Mod}_{\mathrm{gr\,ft}}(\mathbb{C}[z])$ of graded $\mathbb{C}[z]$-modules of finite type (whose morphisms are graded of degree zero) is abelian, but not all its objects are free. The free modules in this category are also called *strict objects*. Strict objects are in one-two-one correspondence with filtered vector spaces: from a strict object $\mathscr{H}$ one recovers the vector space $\mathcal{H} := \mathscr{H}/(z-1)\mathscr{H}$, and the grading $\mathscr{H} = \bigoplus \mathscr{H}^p$ induces a filtration $F^p\mathcal{H} := \mathscr{H}^p/\mathscr{H}^p \cap (z-1)\mathscr{H}$.

Similarly, we say that a morphism in this category is *strict* if its kernel and cokernel are strict. A morphism between strict objects corresponds to a filtered morphism between the corresponding filtered vector spaces. A morphism between strict objects is strict if and only if its cokernel is strict.

To a bi-filtered vector space $(\mathcal{H}, F'^{\bullet}\mathcal{H}, F''^{\bullet}\mathcal{H})$ we associate the following pair of filtered vector spaces:

- $(\mathcal{H}', F^{\bullet}\mathcal{H}') := (\mathcal{H}, F'^{\bullet}\mathcal{H})$,
- $(\mathcal{H}'', F^{\bullet}\mathcal{H}'') := (\overline{\mathcal{H}}, \overline{F''^{\bullet}\mathcal{H}})$.

We thus have an isomorphism $\gamma : \mathcal{H}' \xrightarrow{\sim} \overline{\mathcal{H}''}$ (the identity). We associate to $(\mathcal{H}, F'^{\bullet}\mathcal{H}, F''^{\bullet}\mathcal{H})$ the object $(\mathscr{H}', \mathscr{H}'', \gamma)$. In such a way, we embed the (nonabelian) category of bi-filtered vector spaces (and morphisms compatible with both filtrations) as a full subcategory of the category of triples $(\mathscr{H}', \mathscr{H}'', \gamma)$ consisting of two graded $\mathbb{C}[z]$-modules $\mathscr{H}', \mathscr{H}''$ and an isomorphism $\gamma : \mathcal{H}' \xrightarrow{\sim} \overline{\mathcal{H}''}$. Morphisms are pairs of graded morphisms (φ', φ'') of degree zero whose restriction to $z = 1$ are compatible with γ. One recovers a bi-filtered vector space if $\mathscr{H}', \mathscr{H}''$ are *strict* (i.e., $\mathbb{C}[z]$-flat, see Exercise A.2.5) by setting $\mathcal{H} = \mathcal{H}'$, by getting the filtrations $F^\bullet\mathcal{H}', \overline{F^\bullet\mathcal{H}''}$ from $\mathscr{H}', \mathscr{H}''$, and by transporting them to $\mathcal{H}$ by the isomorphisms Id and γ^{-1}.

***Remark 2.4.13* (Complex Hodge structures and representations of S^1)**
A $\mathbb{C}$-Hodge structure of weight 0 on a complex vector space $\mathcal{H}$ is nothing but a grading of this space indexed by $\mathbb{Z}$, and a morphism between such Hodge structures is nothing but a graded morphism of degree zero. Indeed, in weight 0, the summand $\mathcal{H}^{p,-p}$ can simply be written $\mathcal{H}^p$. This grading defines a continuous representation $\rho : S^1 \to \mathrm{Aut}(\mathcal{H})$ by setting $\rho(\lambda)_{|\mathcal{H}^p} = \lambda^p \,\mathrm{Id}_{\mathcal{H}^p}$.

Conversely, any continuous representation $\rho : S^1 \to \mathrm{Aut}(\mathcal{H})$ is of this form. This can be seen as follows. Since S^1 is compact, one can construct a Hermitian metric on $\mathcal{H}$ which is invariant by any $\rho(\lambda)$. It follows that each $\rho(\lambda)$ is semi-simple and there is a common eigen-decomposition of $\mathcal{H}$. The eigenvalues are continuous characters on S^1. Any such character χ takes the form $\chi(\lambda) = \lambda^p$ (note first that $|\chi| = 1$ since $|\chi(S^1)|$ is compact in $\mathbb{R}_+^*$ and, if $|\chi(\lambda_o)| \neq 1$, then $|\chi(\lambda_o^k)| = |\chi(\lambda_o)|^k$ tends to 0 or ∞ if $k \to \infty$; therefore, χ is a continuous group homomorphism $S^1 \to S^1$, and the assertion is standard).

Recall (Schur's lemma) that the center of $\mathrm{Aut}(\mathcal{H})$ is $\mathbb{C}^* \,\mathrm{Id}$. We claim that a continuous representation $\widetilde{\rho} : S^1 \to \mathrm{Aut}(\mathcal{H})/\mathbb{C}^* \,\mathrm{Id}$ determines a $\mathbb{C}$-Hodge structure of weight 0, up to a shift by an integer of the indices. In other words, one can lift $\widetilde{\rho}$ to a representation ρ. We first note that the morphism

$$\begin{aligned} \mathbb{R}_+^* \times \mathrm{Ker}\,|\mathrm{det}| &\longrightarrow \mathrm{Aut}(\mathcal{H}) \\ (c, T) &\longmapsto c^{1/d}\, T \quad (d = \dim \mathcal{H}) \end{aligned}$$

is an isomorphism. It follows that $\mathrm{Ker}\,|\mathrm{det}| \to \mathrm{Aut}(\mathcal{H})/\mathbb{R}_+^* \,\mathrm{Id}$ is an isomorphism. Similarly, $\mathrm{Ker}\,|\mathrm{det}|/S^1 \,\mathrm{Id} \simeq \mathrm{Aut}(\mathcal{H})/\mathbb{C}^* \,\mathrm{Id}$. It follows that any continuous representation $\widetilde{\rho}$ lifts as a continuous representation $\widehat{\rho} : S^1 \to \mathrm{Aut}(\mathcal{H})/S^1 \,\mathrm{Id}$. Given a Hermitian metric h and $[T] \in \mathrm{Aut}(\mathcal{H})/S^1 \,\mathrm{Id}$, then $h(Tu, \overline{Tv})$ does not depend on the lift T of $[T]$ in $\mathrm{Aut}(\mathcal{H})$, and one can thus construct a $\widehat{\rho}$-invariant metric on $\mathcal{H}$. The eigenspace decomposition is well-defined, although the eigenvalues of $\widehat{\rho}(\lambda)$ are defined up to a multiplicative constant. One can fix the constant to one of some eigenspace, and argue as above for the other eigenspaces. The lift is not unique, and the indeterminacy produces a shift in the filtration.

Example 2.4.14.

(1) Let X be a smooth complex projective variety. Then $H^k(X, \mathbb{C})$ defines a $\mathbb{C}$-Hodge structure of weight k by setting $F'^p H^k(X, \mathbb{C}) = F^p H^k(X, \mathbb{C})$ and

$F''^q H^k(X,\mathbb{C}) = \overline{F^q H^k(X,\mathbb{C})}$ and by using the isomorphism $\overline{H^k(X,\mathbb{C})} \simeq H^k(X,\mathbb{C})$ coming from the real structure $H^k(X,\mathbb{C}) \simeq \mathbb{C} \otimes_{\mathbb{R}} H^k(X,\mathbb{R})$.

(2) Let $f : X \to Y$ be a morphism between smooth projective varieties. Then the induced morphism $f^* : H^k(Y,\mathbb{C}) \to H^k(X,\mathbb{C})$ is a morphism of Hodge structures of weight k.

Exercise 2.4.15 **(The Hodge polynomial).** Let H be a Hodge structure of weight w with Hodge decomposition $\mathcal{H} = \bigoplus_{p+q=w} \mathcal{H}^{p,q}$. The Hodge polynomial $P_h(H) \in \mathbb{Z}[u,v,u^{-1},v^{-1}]$ is the two-variable Laurent polynomial defined as $\sum_{p,q\in\mathbb{Z}} h^{p,q} u^p v^q$ with $h^{p,q} = \dim \mathcal{H}^{p,q}$. This is a homogeneous Laurent polynomial of degree w. Show the following formulas:

$$\begin{aligned} P_h(H_1 \otimes H_2)(u,v) &= P_h(H_1)(u,v) \cdot P_h(H_2)(u,v), \\ P_h(\mathrm{Hom}(H_1,H_2))(u,v) &= P_h(H_1)(u^{-1},v^{-1}) \cdot P_h(H_2)(u,v), \\ P_h(H^\vee)(u,v) &= P_h(H)(u^{-1},v^{-1}), \\ P_h(H(k))(u,v) &= P_h(H)(u,v) \cdot (uv)^{-k}. \end{aligned}$$

2.4.b. Polarized/polarizable $\mathbb{C}$-Hodge structures. In the same way Hodge structures look like complex vector spaces, *polarized $\mathbb{C}$-Hodge structures* look like vector spaces equipped with a positive definite Hermitian form. Any such object can be decomposed into an orthogonal direct sum of irreducible objects, which have dimension one (this follows from the classification of positive definite Hermitian forms). We will see that this remains true for polarized $\mathbb{C}$-Hodge structures (but this does not remain true, fortunately, in higher dimensions). From a categorical point of view, i.e., when considering morphisms between objects, it will be convenient not to restrict to morphisms compatible with polarizations (see Remark 2.4.21).

Definition 2.4.16 **(Polarization of a $\mathbb{C}$-Hodge structure, first definition)**
Given a Hodge structure H of weight w, regarded as a grading $\mathcal{H} = \bigoplus_p \mathcal{H}^{p,w-p}$ of the finite-dimensional $\mathbb{C}$-vector space $\mathcal{H}$, a *polarization* is a positive definite Hermitian form h on $\mathcal{H}$ such that the grading is h-orthogonal (so h induces a positive definite Hermitian form on each $\mathcal{H}^{p,w-p}$).

Although this definition is natural and quite simple, it does not extend "flatly" in higher dimension, and this leads to emphasize the polarization Q below, which is also the right object to consider when working with $\mathbb{Q}$-Hodge structures.

Definition 2.4.17 **(Polarization of a $\mathbb{C}$-Hodge structure, second definition)**
Let $H = (\mathcal{H}, F'^{\bullet}\mathcal{H}, F''^{\bullet}\mathcal{H})$ be a $\mathbb{C}$-Hodge structure of weight w. A *polarization* of H is a $\mathbb{C}$-bilinear pairing $\mathrm{Q} : \mathcal{H} \otimes_{\mathbb{C}} \overline{\mathcal{H}} \to \mathbb{C}$ (i.e., a sesquilinear pairing on $\mathcal{H}$) satisfying

(1) Q is $(-1)^w$-Hermitian, i.e., $\mathrm{Q}^*(y,\overline{x}) := \overline{\mathrm{Q}(x,\overline{y})} = (-1)^w \mathrm{Q}(y,\overline{x})$ for all $x,y \in \mathcal{H}$,
(2) $\mathrm{Q}(F'^p\mathcal{H}, \overline{F''^q\mathcal{H}}) = 0$ and $\mathrm{Q}(F''^p\mathcal{H}, \overline{F'^q\mathcal{H}}) = 0$ for $p+q > w$,

(3) the pairing $\mathrm{h}(x,\overline{y}) = \mathrm{Q}(\mathrm{C}x,\overline{y})$ on $\mathcal{H}$ is (Hermitian) positive definite. (Recall that $\mathrm{C} : \mathcal{H} \to \mathcal{H}$ is equal to i^{p-q} on $\mathcal{H}^{p,q}$.)

Remarks 2.4.18 **(Polarized $\mathbb{C}$-Hodge structures).** Let H be a $\mathbb{C}$-Hodge structure of weight w with polarization Q.

(1) The conditions imply that the pairing $\mathrm{Q} : \mathcal{H} \otimes_{\mathbb{C}} \overline{\mathcal{H}} \to \mathbb{C}$ is *non-degenerate.*

(2) It follows from 2.4.17(2) and (3) that the decomposition $\mathcal{H} = \bigoplus_p \mathcal{H}^{p,w-p}$ is Q-orthogonal and h-orthogonal, so a polarization in the sense of the second definition gives rise to a polarization in the sense of the first one. Conversely, from h as in the first definition one defines Q by $\mathrm{Q}(x,\overline{y}) = \mathrm{h}(\mathrm{C}^{-1}x,\overline{y})$ and, the decomposition being Q-orthogonal, one recovers a polarization in the sense of the second definition.

Note also that the second part of Condition 2.4.17(2) follows from the first one and 2.4.17(1). On the other hand, Condition 2.4.17(1) can be deduced from (2) and (3).

(3) Let us equip $\mathbb{C}$ with the trivial Hodge structure of weight 0 by setting $F'^p\mathbb{C} = F''^p\mathbb{C} = \mathbb{C}$ if $p \leqslant 0$ and $F'^p\mathbb{C} = F''^p\mathbb{C} = 0$ for $p \geqslant 1$, that we simply denote by $\mathbb{C}$. We can then considered its twist $\mathbb{C}(-w)$ as in Notation 2.4.11, which is a $\mathbb{C}$-Hodge structure of weight $2w$. Condition 2.4.17(2) can simply be expressed by saying that

$$\mathrm{Q} : H \otimes \overline{H} \longrightarrow \mathbb{C}(-w)$$

is a morphism in $\mathsf{HS}(\mathbb{C})$. Note here that $\overline{H}$ is defined by (2.4.9 $*$), so that the left-hand term has weight $2w$, according to Exercise 2.4.9(2), and Q is in fact a morphism in $\mathsf{HS}(\mathbb{C}, 2w)$.

(4) Let H^* denote the adjoint complex Hodge structure (Exercise 2.4.9(6)). Condition 2.4.17(2) simply says that Q is a morphism $H \to H^*(-w)$. Its adjoint morphism Q^* is a morphism $H(w) \to H^*$, that we can also regard as a morphism $H \to H^*(-w)$. Condition 2.4.17(1) can then be expressed by saying that Q is $(-1)^w$-Hermitian as such, that is, $\mathrm{Q}^* = (-1)^w\mathrm{Q}$.

(5) It is usual, when considering $\mathbb{Q}$- or $\mathbb{R}$-Hodge structures, for which the Tate twist $(-w)$ also replaces the standard $\mathbb{Q}$-structure on $\mathbb{C}$ by $(2\pi\mathsf{i})^{-w}\mathbb{Q}$, to replace Q with $\mathrm{S} := (2\pi\mathsf{i})^{-w}\mathrm{Q}$. In our complex setting, replacing Q with S does not bring new information.

(6) If Q is a polarization of H, then $(-1)^w\mathrm{Q}$ is a polarization of $\operatorname{Exch} H$ (see 2.4.9(1)) since when we exchange filtrations, $\mathcal{H}^{p,q}$ is transformed into $\mathcal{H}^{q,p}$.

(7) Similarly, defining the form $\overline{\mathrm{Q}} : \overline{\mathcal{H}} \otimes_{\mathbb{C}} \mathcal{H} \to \mathbb{C}$ by $\overline{\mathrm{Q}}(\overline{x}, y) = \overline{\mathrm{Q}(y,\overline{x})}$, one checks that $(-1)^w\overline{\mathrm{Q}}$ a polarization of $\overline{H}$, as defined by (2.4.9 $*$). As a consequence, $\overline{\mathrm{Q}}$ is a polarization of $\operatorname{Exch}\overline{H}$.

Exercise 2.4.19 **(Polarization and Tate twist).** Show that, if (H,Q) is a polarized Hodge structure of weight w, then $(H(k,\ell), i^{k-\ell}\mathrm{Q})$ is a polarized Hodge structure of weight $w-k-\ell$.

Remark 2.4.20 **(Simplified data for a polarized Hodge structure)**

The definition of a polarized Hodge structure as a pair (H,Q) contains some redundancy. However, it has the advantage of exhibiting the underlying Hodge structure. We give a simplified presentation, which only needs one filtration, together with the sesquilinear form Q.

A polarized Hodge structure of weight w can be described as the data of

(i) a filtered vector space $(\mathcal{H}, F^\bullet\mathcal{H})$,

(ii) a sesquilinear pairing $\mathrm{Q} : \mathcal{H} \otimes_{\mathbb{C}} \overline{\mathcal{H}} \to \mathbb{C}$, i.e., a morphism $\mathrm{Q} : \mathcal{H} \to \mathcal{H}^*$,

subject to the following conditions:

(1) Q is $(-1)^w$-Hermitian non-degenerate, i.e., induces an isomorphism $\mathcal{H} \xrightarrow{\sim} \mathcal{H}^*$ satisfying $\mathrm{Q}^* = (-1)^w\mathrm{Q}$,

(2) if $F^\bullet\mathcal{H}^*$ is the filtration on the adjoint space $\mathcal{H}^*$ naturally defined by $F^\bullet\mathcal{H}$, then $F^\bullet\mathcal{H}$ is 0-opposite to the filtration $\mathrm{Q}^{-1}(F^\bullet\mathcal{H}^*)$ (which corresponds thus to $F''[w]^\bullet\mathcal{H}$),

(3) the positivity condition 2.4.17(3) holds.

Note that (2) means that the filtration $F''^\bullet\mathcal{H}$ defined by

$$F''^{w-p+1}\mathcal{H} = F^p\mathcal{H}^{\perp_{\mathrm{Q}}}$$

is w-opposite to $F^p\mathcal{H}$, and the corresponding decomposition is Q-orthogonal.

Remark 2.4.21 **(Category of polarizable $\mathbb{C}$-Hodge structures)**

A $\mathbb{C}$-Hodge structure may be polarized by many polarizations. At many places, we do not want to make a choice of a polarization, and it is enough to know that there exists one. Nevertheless, any $\mathbb{C}$-Hodge structure admits at least one polarization, as is obvious from Definition 2.4.16. Notice that this property will not remain true in higher dimension and this will lead us to distinguish the full subcategory of polarizable (instead of polarized) objects (see Definition 4.1.7). This not needed here.

The notion of polarizable Hodge structure is interesting in the case of $\mathbb{Q}$-Hodge structures, when one insists that the polarization Q is defined over $\mathbb{Q}$, as in the geometric setting of §2.3. Such a polarization does not necessarily exists.

Recall that the category $\mathsf{HS}(\mathbb{C})$ is endowed with tensor product, Hom, duality and conjugation. If we are moreover given a polarization of the source terms of these operations, we naturally obtain a polarization on the resulting $\mathbb{C}$-Hodge structure. For example, if $H = H_1 \otimes H_2$, then $\mathcal{H}^{p,w-p} = \bigoplus_{p_1+p_2=p} \mathcal{H}_1^{p_1,w_1-p_1} \otimes \mathcal{H}_2^{p_2,w_2-p_2}$ and the positive definite Hermitian forms $\mathrm{h}_1, \mathrm{h}_2$ induce such a form h on each $\mathcal{H}_1^{p_1,w-p_1} \otimes \mathcal{H}_2^{p_2,w-p_2}$, and thus on $\mathcal{H}^{p,w-p}$ by imposing that the above decomposition is h-orthogonal.

As a consequence, the category of polarizable $\mathbb{Q}$-Hodge structures (that we did not need to define precisely) is also endowed with such operations.

Exercise 2.4.22 **(Polarization on $\mathbb{C}$-Hodge sub-structures).** Let Q be a polarization (Definition 2.4.17) of a $\mathbb{C}$-Hodge structure H of weight w. Let H_1 be a $\mathbb{C}$-Hodge substructure of weight w of H (see Proposition 2.4.5(1)).

(1) Show that the restriction Q_1 of Q to $\mathcal{H}_1$ is a polarization of H_1. [*Hint*: use that the restriction of a positive definite Hermitian form to a subspace remains positive definite.]

(2) Deduce that (H_1, Q_1) is a direct summand of (H, Q) in the category of polarized $\mathbb{C}$-Hodge structures of weight w. [*Hint*: define $\mathcal{H}_2$ to be $\mathcal{H}_1^\perp$, where the orthogonal is taken with respect to Q; use (1) to show that $(\mathcal{H}, \mathrm{Q}) = (\mathcal{H}_1, \mathrm{Q}_1) \oplus (\mathcal{H}_2, \mathrm{Q}_2)$; show similarly that $\mathcal{H}_2^{p,w-p} := \mathcal{H}_2 \cap \mathcal{H}^{p,w-p} = \mathcal{H}_1^{p,w-p,\perp}$ for every p and conclude that H_2 is a $\mathbb{C}$-Hodge structure of weight w, which is polarized by Q_2.]

Exercise 2.4.23 **(Simple and semi-simple $\mathbb{C}$-Hodge structures)**
Show the following:

(1) A $\mathbb{C}$-Hodge structure H of weight w is simple (i.e., does not admit any nontrivial $\mathbb{C}$-Hodge sub-structure) if and only if $\dim_{\mathbb{C}} \mathcal{H} = 1$.

(2) Any $\mathbb{C}$-Hodge structure is semi-simple as such.

2.4.c. The category of triples. We will introduce another language for dealing with polarized complex Hodge structures. This is similar to the presentation given in Remark 2.4.12, but compared with it, we replace $\mathcal{H}''$ with its dual $\mathcal{H}''^\vee$. This approach will be useful in higher dimensions.

Definition 2.4.24 **(Triples).** The category $\mathsf{Triples}$ is the category whose objects $\mathscr{T} = (\mathscr{H}', \mathscr{H}'', \mathfrak{c})$ consist of a pair of $\mathbb{C}[z]$-modules of finite type $\mathscr{H}', \mathscr{H}''$ and a sesquilinear pairing $\mathfrak{c} : \mathcal{H}' \otimes \overline{\mathcal{H}''} \to \mathbb{C}$ between the associated vector spaces, and whose morphisms $\varphi : \mathscr{T}_1 \to \mathscr{T}_2$ are pairs (φ', φ'') of morphisms (graded of degree zero)

$$\varphi' : \mathscr{H}_1' \longrightarrow \mathscr{H}_2', \qquad \varphi'' : \mathscr{H}_2'' \longrightarrow \mathscr{H}_1'' \tag{2.4.24 $*$}$$

such that, for every $v_1' \in \mathcal{H}_1'$ and $v_2'' \in \mathcal{H}_2''$, denoting by $[\varphi'], [\varphi'']$ the morphisms induced by φ', φ'' on $\mathcal{H}', \mathcal{H}''$, we have

$$\mathfrak{c}_1(v_1', \overline{[\varphi''](v_2'')}) = \mathfrak{c}_2([\varphi'](v_1'), \overline{v_2''}). \tag{2.4.24 $**$}$$

Remarks 2.4.25 **(Operations on the category** $\mathsf{Triples}$**).**

(1) The category of triples is abelian, the "prime" part is covariant, while the "double-prime" part is contravariant. For example, the triple $\operatorname{Ker}\varphi$ is the triple $(\operatorname{Ker}\varphi', \operatorname{Coker}\varphi'', \mathfrak{c}_1^\varphi)$, where $\mathfrak{c}_1^\varphi$ is the pairing between $\operatorname{Ker}\varphi'$ and $\operatorname{Coker}\varphi''$ induced by $\mathfrak{c}_1$, which is well-defined because of (2.4.24 $**$). Similarly, we have $\operatorname{Coker}\varphi = (\operatorname{Coker}\varphi', \operatorname{Ker}\varphi'', \mathfrak{c}_2^\varphi)$, $\operatorname{Im}\varphi = (\operatorname{Im}\varphi', \mathscr{H}_2''/\operatorname{Ker}\varphi'', \mathfrak{c}_2^\varphi)$.

(2) An increasing filtration $W_\bullet\mathscr{T}$ of a triple $\mathscr{T}$ consists of increasing filtrations $W_\bullet\mathscr{H}', W_\bullet\mathscr{H}''$ such that $\mathfrak{c}(W_\ell\mathcal{H}', \overline{W_{-\ell-1}\mathcal{H}''}) = 0$ for every ℓ. Then $\mathfrak{c}$ induces a pairing $\mathfrak{c}_\ell : W_\ell\mathcal{H}' \otimes \overline{\mathcal{H}''/W_{-\ell-1}\mathcal{H}''} \to \mathbb{C}$. We set $W_\ell\mathscr{T} = (W_\ell\mathscr{H}', \mathscr{H}''/W_{-\ell-1}\mathscr{H}'', \mathfrak{c}_\ell)$. We have $\mathrm{gr}_\ell^W\mathscr{T} = (\mathrm{gr}_\ell^W\mathscr{H}', \mathrm{gr}_{-\ell}^W\mathscr{H}'', \mathfrak{c}_\ell)$.

(3) We say that a triple is *strict* if $\mathscr{H}', \mathscr{H}''$ are strict. Strict triples are in one-two-one correspondence with *filtered triples* $T = (F^\bullet\mathcal{H}', F^\bullet\mathcal{H}'', \mathfrak{c})$. We say that a morphism $\varphi : \mathscr{T}_1 \to \mathscr{T}_2$ is strict if its components φ', φ'' are strict. Strict morphisms between strict triples are in one-to-one correspondence with strict morphisms between filtered triples.

(4) The difference between the construction made in Remark 2.4.12 is that the "double prime" part is now contravariant, and the isomorphism γ is replaced with a pairing. This gives more flexibility since the pairing is not assumed non-degenerate a priori. We say that a triple $\mathscr{T}$ is *non-degenerate* if $\mathfrak{c}$ is so. If $\mathscr{T} = (\mathscr{H}', \mathscr{H}'', \mathfrak{c})$ is strict and non-degenerate, one can associate a triple like in Remark 2.4.12 by replacing $(\mathcal{H}'', F^\bullet\mathcal{H}'')$ defined from $\mathscr{H}''$ with $(\mathcal{H}''^*, F^\bullet\mathcal{H}''^*)$ and by defining γ as the isomorphism $\mathcal{H}' \to \mathcal{H}''^*$ obtained from $\mathfrak{c}$.

(5) Let $(\mathscr{T}, W_\bullet\mathscr{T})$ be a W-filtered triple as in (2). Assume that $\mathscr{T}$ and all $\mathrm{gr}_\ell^W\mathscr{T}$ are strict, i.e., all inclusions $W_\ell\mathscr{T} \hookrightarrow W_{\ell+1}\mathscr{T}$ are strict morphisms. Then $\mathrm{gr}_\ell^W\mathscr{H}'$ is the Rees object attached with the filtered vector space

$$F^p\mathrm{gr}_\ell^W\mathcal{H}' := \frac{F^p\mathcal{H}' \cap W_\ell\mathcal{H}'}{F^p\mathcal{H}'' \cap W_{\ell-1}\mathcal{H}'},$$

and a similar equality for $\mathrm{gr}_{-\ell}^W\mathscr{H}'$.

(6) The *adjoint* of a triple $\mathscr{T} = (\mathscr{H}', \mathscr{H}'', \mathfrak{c})$ is the triple $\mathscr{T}^* := (\mathscr{H}'', \mathscr{H}', \mathfrak{c}^*)$, where $\mathfrak{c}^*$ is defined by

$$\mathfrak{c}^*(v'', \overline{v'}) := \overline{\mathfrak{c}(v', \overline{v''})}.$$

We have $\mathscr{T}^{**} = \mathscr{T}$. If $\varphi = (\varphi', \varphi'') : \mathscr{T}_1 \to \mathscr{T}_2$ is a morphism, its adjoint $\varphi^* : \mathscr{T}_2^* \to \mathscr{T}_1^*$ is the morphism (φ'', φ').

The adjoint of a strict triple $\mathscr{T}$ is also strict, and $\mathscr{T}^*$ corresponds to the filtered triple $T^* := (F^\bullet\mathcal{H}'', F^\bullet\mathcal{H}', \mathfrak{c}^*)$.

(7) Given a pair of integers (k, ℓ), the twist $\mathscr{T}(k, \ell)$ is defined by

$$\mathscr{T}(k, \ell) := (z^k\mathscr{H}', z^{-\ell}\mathscr{H}'', \mathfrak{c}).$$

We have $(\mathscr{T}(k, \ell))^* = \mathscr{T}^*(-\ell, -k)$. If $\varphi : \mathscr{T}_1 \to \mathscr{T}_2$ is a morphism, then it is also a morphism $\mathscr{T}_1(k, \ell) \to \mathscr{T}_2(k, \ell)$.

If $\mathscr{T}$ is strict with associated filtered triple T, the twisted object $\mathscr{T}(k, \ell)$ is also strict and its associated filtered triple is

$$T(k, \ell) := (F[k]^\bullet\mathcal{H}', F[-\ell]^\bullet\mathcal{H}'', \mathfrak{c}).$$

This is compatible with the Tate twist as defined in Remark 2.4.10, by means of the equivalence of Lemma 2.4.30 below. (Recall that $F[k]^p := F^{p+k}$.)

Notation 2.4.26. As in Notation 2.4.11, we simply use the notation (w) for the symmetric Tate twist (w, w).

Definition 2.4.27 **(w-opposedness condition).** Let T be a filtered triple and let $w \in \mathbb{Z}$. The filtration $F^\bullet\mathcal{H}''$ naturally induces a filtration $F^\bullet\mathcal{H}''^*$ on the adjoint space $\mathcal{H}''^* = \overline{\mathcal{H}''^\vee}$. We say that T satisfies the w-opposedness condition if $\mathfrak{c}$ is non-degenerate and if the filtration $F^\bullet\mathcal{H}'$ is w-opposite to the filtration obtained from $F^\bullet\mathcal{H}''^*$ by means of the isomorphism $\mathcal{H}' \xrightarrow{\sim} \mathcal{H}''^*$ induced by $\mathfrak{c}$.

***Definition 2.4.28* (ℂ-Hodge triples).** The category of $\mathbb{C}$-Hodge triples of weight $w \in \mathbb{Z}$ is the full subcategory of triples (i.e., morphisms are described by (2.4.24 $*$) and satisfying (2.4.24 $**$)) whose objects are strict and satisfy the w-opposedness condition.

***Remark 2.4.29* (Adjunction and Tate twist).** The category of $\mathbb{C}$-Hodge triples of weight w is left invariant by the adjunction functor 2.4.25(6) and, for a $\mathbb{C}$-Hodge triple H of weight w, the Tate twisted triple $H(k,\ell)$ is a $\mathbb{C}$-Hodge triple H of weight $w-(k+\ell)$.

Any $\mathbb{C}$-Hodge triple of weight w is isomorphic to a $\mathbb{C}$-Hodge triple of weight w obtained from a $\mathbb{C}$-Hodge structure of weight w as in the preliminary remark. Both categories are in fact equivalent, and so this new category is also abelian. It is also stable by direct summand in the category of triples.

Lemma 2.4.30. *The correspondence*

$$H = (\mathcal{H}, F'^{\bullet}\mathcal{H}, F''^{\bullet}\mathcal{H}) \longmapsto T = \big((\mathcal{H}', F^{\bullet}\mathcal{H}'), (\mathcal{H}'', F^{\bullet}\mathcal{H}''), \mathfrak{c}\big),$$

obtained by setting

$$(\mathcal{H}', F^{\bullet}\mathcal{H}') := (\mathcal{H}, F'^{\bullet}\mathcal{H}), \quad (\mathcal{H}'', F^{\bullet}\mathcal{H}'') := (\mathcal{H}^*, F''^{\bullet}\mathcal{H}^*), \quad \mathfrak{c} := \langle\bullet,\bullet\rangle : \mathcal{H}\otimes\mathcal{H}^{\vee} \to \mathbb{C}$$

(recall that $F''^{\bullet}\mathcal{H}^*$ is obtained by duality from $\overline{F''^{\bullet}\mathcal{H}}$) is an equivalence between $\mathsf{HS}(\mathbb{C}, w)$ *and the category of* $\mathbb{C}$*-Hodge triples of weight* w. ☐

From now on, we will not distinguish between $\mathbb{C}$-Hodge structures of weight w and $\mathbb{C}$-Hodge triples of weight w, and we will often write H instead of T.

Lemma 2.4.31. *Assume we have a decomposition $\mathscr{T} = \mathscr{T}_1 \oplus \mathscr{T}_2$ of triples. If $\mathscr{T}$ is* $\mathbb{C}$*-Hodge of weight w, so are $\mathscr{T}_1$ and $\mathscr{T}_2$.*

Proof. Firstly, $\mathscr{T}_1$ and $\mathscr{T}_2$ must be strict, hence correspond to filtered triples T_1, T_2. The non-degeneracy of $\mathfrak{c}_1$ and $\mathfrak{c}_2$ is also clear. Lastly, we use the interpretation 2.4.7(c) of w-opposedness and the standard property that, if a vector bundle on $\mathbb{P}^1$ is isomorphic to $\mathscr{O}_{\mathbb{P}^1}(w)^d$, then any direct summand is isomorphic to a power of $\mathscr{O}_{\mathbb{P}^1}(w)$. ☐

***Exercise 2.4.32* (Stability by extension).** Let $0 \to \mathscr{T}_1 \to \mathscr{T} \to \mathscr{T}_2 \to 0$ be a short exact sequence of triples. Show that, if $\mathscr{T}_1, \mathscr{T}_2$ are $\mathbb{C}$-Hodge triples of weight w, then so is $\mathscr{T}$. [*Hint*: by using the interpretation 2.4.7(c) of w-opposedness, reduce the question to showing that, if a locally free $\mathscr{O}_{\mathbb{P}^1}$-module is an extension of two trivial bundles $\mathscr{O}_{\mathbb{P}^1}^{d_1}$ and $\mathscr{O}_{\mathbb{P}^1}^{d_2}$, then it is itself a trivial bundle.]

***Definition 2.4.33* (Pre-polarization of weight w of a triple).** A *pre-polarization* of weight w of a triple $\mathscr{T}$ is an isomorphism

$$\mathrm{Q} = (\mathcal{Q}', \mathcal{Q}'') : \mathscr{T} \xrightarrow{\sim} \mathscr{T}^*(-w)$$

which is $(-1)^w$-Hermitian, in the sense that its adjoint

$$\mathrm{Q}^* = (\mathcal{Q}'', \mathcal{Q}') : (\mathscr{T}^*(-w))^* = \mathscr{T}(w) \longrightarrow \mathscr{T}^*,$$

which defines a morphism denoted in the same way $\mathrm{Q}^* : \mathscr{T} \to \mathscr{T}^*(-w)$, satisfies

$$\mathrm{Q}^* = (-1)^w \mathrm{Q}, \quad i.e., \ \mathcal{Q}'' = (-1)^w \mathcal{Q}' =: \mathcal{Q}.$$

Let $\mathscr{T}$ be a $\mathbb{C}$-Hodge triple of weight w and let $\mathrm{Q} = ((-1)^w \mathcal{Q}, \mathcal{Q})$ be a pre-polarization of $\mathscr{T}$ of weight w. It defines an isomorphism of filtered vector spaces

$$\mathcal{Q} : (\mathcal{H}', F^\bullet \mathcal{H}') \longrightarrow (\mathcal{H}'', F^\bullet \mathcal{H}'')(-w) \tag{2.4.34}$$

which is compatible with $\mathfrak{c}$ and $\mathfrak{c}^*$: this means that, for every pair $v_1', v_2' \in \mathcal{H}'$, we have

$$\mathfrak{c}(v_1', \overline{\mathcal{Q} v_2'}) = (-1)^w \mathfrak{c}^*(\mathcal{Q} v_1', \overline{v_2'}) =: (-1)^w \overline{\mathfrak{c}(v_2'), \overline{\mathcal{Q} v_1'})}, \tag{2.4.35}$$

which is equivalent to the property that the pairing

$$\mathrm{Q}(\bullet, \overline{\bullet}) := \mathfrak{c}(\bullet, \overline{\mathcal{Q}\bullet}) : \mathcal{H}' \otimes \overline{\mathcal{H}'} \longrightarrow \mathbb{C} \tag{2.4.36}$$

is $(-1)^w$-Hermitian in the usual sense.

***Definition 2.4.37* (Polarization of a $\mathbb{C}$-Hodge triple).** Let $\mathscr{T}$ be a $\mathbb{C}$-Hodge triple of weight w. A polarization of $\mathscr{T}$ is a pre-polarization Q of weight w of the underlying triple such that $((\mathcal{H}', F^\bullet \mathcal{H}'), \mathrm{Q})$, with Q defined by (2.4.36), is a polarized $\mathbb{C}$-Hodge structure of weight w in the sense of Remark 2.4.20.

The relation with polarized $\mathbb{C}$-Hodge structures can now be expressed in a simpler way.

Proposition 2.4.38. *Let $\mathscr{T} = (\mathscr{H}', \mathscr{H}'', \mathfrak{c})$ be an object of* Triples*. It is a polarizable $\mathbb{C}$-Hodge triple of weight w if and only if it is isomorphic (in* Triples*) to the object $(\mathscr{H}', \mathscr{H}'(w), \mathfrak{c}')$ for some suitable $\mathfrak{c}'$, such that $\mathscr{H}'$ is strict and the corresponding $((\mathcal{H}', F^\bullet \mathcal{H}'), \mathfrak{c}')$ is a polarized $\mathbb{C}$-Hodge structure of weight w as in Remark* 2.4.20.

Proof. In one direction, let Q be a polarization of $\mathscr{T}$. Then $(\mathrm{Id}, \mathcal{Q})$ is an isomorphism $\mathscr{T} \xrightarrow{\sim} (\mathscr{H}', \mathscr{H}'(w), \mathfrak{c}')$ with $\mathfrak{c}'(\bullet, \overline{\bullet}) = \mathfrak{c}(\bullet, \overline{\mathcal{Q}\bullet})$, according to (2.4.24 $**$), and by definition, $\mathscr{H}'$ is strict and the corresponding filtered vector space $(\mathcal{H}', F^\bullet \mathcal{H}')$ is such that $((\mathcal{H}', F^\bullet \mathcal{H}'), \mathfrak{c}')$ is a polarized $\mathbb{C}$-Hodge structure of weight w. Conversely, given a polarized $\mathbb{C}$-Hodge structure $((\mathcal{H}', F^\bullet \mathcal{H}'), \mathrm{Q})$ of weight w, one checks that $((\mathcal{H}', F^\bullet \mathcal{H}'), (\mathcal{H}', F[w]^\bullet \mathcal{H}'), \mathrm{Q})$ is a $\mathbb{C}$-Hodge triple of weight w and that $((-1)^w \mathrm{Id}, \mathrm{Id})$ is a polarization of it. If

$$\varphi = (\varphi', \varphi'') : \mathscr{T} \xrightarrow{\sim} ((\mathcal{H}', F^\bullet \mathcal{H}'), (\mathcal{H}', F[w]^\bullet \mathcal{H}'), \mathrm{Q})$$

is an isomorphism in Triples, then $\mathscr{T}$ is a $\mathbb{C}$-Hodge triple and, setting $\mathcal{Q} := \varphi''^{-1} \varphi'$, $\mathrm{Q} := ((-1)^w \mathcal{Q}, \mathcal{Q})$ is a polarization of $\mathscr{T}$. $\square$

2.5. Mixed Hodge structures

Our aim is to apply the construction of Section A.2.c with the categories $\mathsf{A}_j = \mathsf{HS}(\mathbb{C}, j)$. The first question is to find a suitable abelian category A. The category $\mathsf{HS}(\mathbb{C})$ of Hodge structures of arbitrary weight is not suitable, since it is not abelian (see Exercise 2.4.6). Instead, we will use the category $\mathsf{Triples}$ of Definition 2.4.24, and we will regard an object of $\mathsf{HS}(\mathbb{C}, j)$ as a $\mathbb{C}$-Hodge triple of weight j.

Lemma 2.5.1. *For every $j \in \mathbb{Z}$, the category $\mathsf{HS}(\mathbb{C}, j)$ is a full subcategory of $\mathsf{Triples}$ which satisfies the following properties.*

(1) *$\mathsf{HS}(\mathbb{C}, j)$ is stable by* Ker *and* Coker *in* $\mathsf{Triples}$.
(2) *For every $j > k$,* $\mathrm{Hom}_{\mathsf{Triples}}(\mathsf{HS}(\mathbb{C}, j), \mathsf{HS}(\mathbb{C}, k)) = 0$.

Proof. The first point follows from the abelianity of the full subcategory $\mathsf{HS}(\mathbb{C}, j)$ of triples, and the second one is Proposition 2.4.5(2). □

We will denote by $\mathsf{HS}_\bullet(\mathbb{C})$ the data $\big(\mathsf{Triples}, \mathsf{HS}(\mathbb{C}, j)_{j\in\mathbb{Z}}\big)$.

Definition 2.5.2 **(Mixed Hodge structures).** The category $\mathsf{MHS}(\mathbb{C})$ is the category $\mathsf{WHS}_\bullet(\mathbb{C})$.

Corollary 2.5.3 **(of Proposition A.2.10).** *The category $\mathsf{MHS}(\mathbb{C})$ is abelian, and morphisms are strictly compatible with $W_\bullet$.* □

Remark 2.5.4. Let us make explicit the notion of mixed Hodge structure.

(1) A *mixed $\mathbb{C}$-Hodge structure* consists of

(a) a finite dimensional $\mathbb{C}$-vector space $\mathcal{H}$ equipped with an exhaustive increasing filtration $W_\bullet\mathcal{H}$ indexed by $\mathbb{Z}$,

(b) decreasing filtrations $F^\bullet\mathcal{H}$ ($F = F'$ or F''),

such that each quotient space $\mathrm{gr}^W_\ell\mathcal{H} := W_\ell\mathcal{H}/W_{\ell-1}\mathcal{H}$, when equipped with the *induced filtrations*

$$F^p\mathrm{gr}^W_\ell\mathcal{H} := \frac{F^p\mathcal{H}_\mathbb{C} \cap W_\ell\mathcal{H}}{F^p\mathcal{H}_\mathbb{C} \cap W_{\ell-1}\mathcal{H}}$$

is a $\mathbb{C}$-Hodge structure of weight ℓ. From the point of view of $\mathbb{C}$-Hodge triples, a mixed $\mathbb{C}$-Hodge triple consists of a W-filtered triple $(H, W_\bullet H)$ such that H is strict and each $\mathrm{gr}_\ell H$ is a $\mathbb{C}$-Hodge triple of weight ℓ. In particular it is strict, hence Remark 2.4.25(5) applies.

(2) A morphism of mixed $\mathbb{C}$-Hodge structures

$$(H_1, W_\bullet H_1) \longrightarrow (H_2, W_\bullet H_2)$$

is a morphism $\mathcal{H}_1 \to \mathcal{H}_2$ which is compatible with the filtrations $W_\bullet$ and with the filtrations $F'^\bullet, F''^\bullet$. Equivalently, it consists of a pair of bi-filtered morphisms

$$\begin{cases} (\mathcal{H}'_1, F^\bullet\mathcal{H}'_1, W_\bullet\mathcal{H}'_1) \to (\mathcal{H}'_2, F^\bullet\mathcal{H}'_2, W_\bullet\mathcal{H}'_2), \\ (\mathcal{H}''_2, F^\bullet\mathcal{H}''_2, W_\bullet\mathcal{H}''_2) \to (\mathcal{H}''_1, F^\bullet\mathcal{H}''_1, W_\bullet\mathcal{H}''_1) \end{cases}$$

compatible with $\mathfrak{c}_1, \mathfrak{c}_2$.

(3) The category $\mathsf{MHS}(\mathbb{C})$ of mixed Hodge structures defined by 2.5.2, i.e., as in (1) and (2), is equipped with endofunctors, the Tate twists (k,ℓ) $(k,\ell\in\mathbb{Z})$ defined by

$$(H,W_\bullet H)(k,\ell):=\big((H(k,\ell),W[-(k+\ell)]_\bullet H(k,\ell))\big).$$

(4) We say that a mixed Hodge structure H is

- *pure* (of weight w) if $\operatorname{gr}^W_\ell H=0$ for $\ell\neq w$,
- *graded-polarizable* if $\operatorname{gr}^W_\ell H$ is polarizable for every $\ell\in\mathbb{Z}$.

Proposition 2.5.5. *Any morphism in the abelian category* $\mathsf{MHS}(\mathbb{C})$ *is strictly compatible with both filtrations* $F^\bullet$ *and* $W_\bullet$.

Proof. Note that for every morphism φ, the graded morphism $\operatorname{gr}^W_\ell\varphi$ is F-strict, according to Exercise 2.4.3(2). The proof is then by induction on the length of $W_\bullet$, by considering the diagram (A.2.11). Since the sequence of cokernels is exact, the cokernel of φ_j is strict, and we can apply the criterion of Exercise A.2.8(3). $\square$

Remark 2.5.6 **(Graded polarizable mixed Hodge structures)**
Since any $\mathbb{C}$-Hodge structure is polarizable, any mixed Hodge structure is *graded-polarizable.* However, in the case of mixed $\mathbb{Q}$-Hodge structures, it is useful to distinguish the full subcategory of graded-polarizable mixed $\mathbb{Q}$-Hodge structures.

The main result in the theory of mixed Hodge structures is due to Deligne [**Del71b, Del74**].

Theorem 2.5.7 **(Hodge-Deligne Theorem, mixed case).** *Let X be a complex quasi-projective variety. Then the cohomology $H^\bullet(X,\mathbb{C})$ and the cohomology with compact supports $H^\bullet_{\mathrm{c}}(X,\mathbb{C})$ admit a canonical (graded-polarizable) mixed Hodge structure.*

2.6. Comments

Sections 2.2 and 2.3 give a very brief abstract of classical Hodge theory, for which various references exist: Hodge's book [**Hod41**] is of course the first one; more recently, Griffiths and Harris' book [**GH78**], Demailly's introductory article [**Dem96**] and Voisin's book [**Voi02**] are modern references. The point of view of an abstract Hodge structure, as emphasized by Deligne in [**Del71a, Del71b**], is taken up in Peters and Steenbrink's book [**PS08**], which we have tried to follow with respect to notation at least.

In Hodge theory, the $\mathbb{Q}$-structure (or, better, the $\mathbb{Z}$-structure) is usually emphasized, as both Hodge and $\mathbb{Q}$-structures give information on the transcendental aspects of algebraic varieties, by means of the periods for example. It may then look strange to focus only, as we did in this chapter, and as is also done in [**Kas86, KK87**] and [**SV11**], on one aspect of the theory, namely that of complex Hodge structures, where the $\mathbb{Q}$-structure is absent, and so is any real structure. The main reason is that this is a preparation to the theory in higher dimensions, where the analytic and the rational structures diverge with respect to the tools needed for expressing them. On the one hand, the analytic part of the theory needs the introduction of holonomic $\mathscr{D}$-modules (replacing $\mathbb{C}$-vector spaces), while on the other hand the rational structure makes use of the theory of $\mathbb{Q}$-perverse sheaves (replacing $\mathbb{Q}$-vector spaces). The relation between both theories is provided by the Riemann-Hilbert correspondence, in the general framework developed by Kashiwara [**Kas84**] and Mebkhout [**Meb84a, Meb84b**] (see also [**Meb89**] and [**Meb04**]). The theory of Hodge modules developed by Saito [**Sai88, Sai90**] combines both structures, as desirable, but this leads to developing fine comparison results between the analytic and the rational theory by means of the Riemann-Hilbert correspondence. This is done in [**Sai88**] and also in [**Sai89a**]. In order to simplify the text and focus on the very Hodge aspects of the theory, we have chosen not to introduce the Riemann-Hilbert correspondence in this text, which explains that the $\mathbb{Q}$-structure of Hodge modules will not be considered.

Developing the theory from the complex point of view only also has the advantage of emphasizing the relation with the theory of twistor $\mathscr{D}$-modules, as developed in [**Sab05, Moc02, Moc07, Moc15**]. In fact, the idea of introducing a sesquilinear pairing $\mathfrak{c}$ is inspired by the latter theory, where one does not expect any $\mathbb{Q}$ or $\mathbb{R}$-structure in general, and where one is obliged to develop the theory with a complex approach only. The category of triples introduced in Section 2.4.c mimics the notion of twistor structure, introduced by Simpson in [**Sim97**], and adapted for a higher dimensional use in [**Sab05**]. The somewhat strange idea to replace an isomorphism by a sesquilinear pairing is motivated by the higher dimensional case, already for a variation of Hodge structure, where among the two filtrations considered in Definition 2.4.1, one varies in a holomorphic way and the other one in an anti-holomorphic way. Also, the idea of emphasizing the Rees module of a filtration, as in Remark 2.4.12, is much inspired by the theory of twistor $\mathscr{D}$-modules. Note however that the sesquilinear pairing $\mathfrak{c}$ (Definition 2.4.24) does not play exactly the same role in both theories, and it is easier to manipulate in Hodge theory.

Also, in complex Hodge theory, the Tate twist is more flexible since we can reduce to weight zero any complex Hodge structure of weight $w \in \mathbb{Z}$. However, we will not use this possibility in order to keep the relation with standard Hodge theory as close as possible.

Mixed Hodge structures are quickly introduced in Section 2.5. This fundamental notion, envisioned by Grothendieck as part of the realization properties of a theory of motives, and realized by Deligne in [**Del71a, Del71b, Del74**], is explained carefully in [**PS08**, Chap. 3]. In the theory of pure Hodge modules, it only appears through the disguise of a Hodge-Lefschetz structure considered in Chapter 3. It will be taken up in Part IV on mixed Hodge modules.