CHAPTER 15

MIXED HODGE-LEFSCHETZ STRUCTURES

15.1. Relative monodromy filtration

In the same way that we have enlarged the category of Hodge structures to that of W-filtered (i.e., filtered with respect to the *weight*) Hodge structures, giving rise to mixed Hodge structures, we now consider the same procedure starting from Hodge-Lefschetz structures, by filtering with respect to the *center*. Let us therefore consider the category CHLS whose objects consist of a filtered object $C_\bullet(H, \mathrm{N})$, where the new datum (compared with that of a Hodge-Lefschetz structure) is an increasing filtration $C_\bullet H$ left invariant by N, satisfying the following condition:

$$\forall\, \ell, \quad (\mathrm{gr}_\ell^C H, \mathrm{gr}_\ell^C \mathrm{N}) \in \mathsf{HLS}(\ell). \tag{15.1.1}$$

The notion of morphism in CHLS is the obvious one (C-filtered morphisms). We say that an object C of CHLS is *graded-polarizable* if each gr_ℓ^C is a polarizable object of $\mathsf{HLS}(\ell)$.

This definition, although natural, has the following drawback: we cannot assert that H underlies a mixed Hodge structure and that $C_\bullet$ is a filtration by mixed Hodge substructures. We only know that the first nonzero C_ℓ is a Hodge-Lefschetz structure, hence a mixed Hodge structure. This leads to the following natural question.

Question 15.1.2. To find a natural condition in order to ensure that an object of CHLS is a mixed Hodge structure.

This question is important since, in the case of $\mathsf{HLS}(w)$, we were able to prove abelianity of the category and strictness of the morphisms by applying the corresponding result 2.5.5 for mixed Hodge structures (see Exercise 3.2.11(1)). An answer to (15.1.2) has been given by Deligne [**Del80**], by introducing the notion of *relative monodromy filtration* (also called relative weight filtration in [**SZ85**], which is a good reference for the main properties of such a filtration, together with [**Kas86**], [**Sai90**]).

***Definition 15.1.3* (Relative monodromy filtration).** Let H be an object of an abelian category A, equipped with a finite increasing filtration $C_\bullet H$ and a nilpotent endomorphism N, both subject to the condition that they are compatible, i.e., $\mathrm{N}C_\ell H \subset C_\ell H$ for every ℓ. A *monodromy filtration of* N *relative to the filtration* $C_\bullet H$ is an increasing filtration $\mathrm{M}_\bullet H = \mathrm{M}_\bullet(\mathrm{N}, C_\bullet)H$ of H which satisfies the following two properties:

(1) For every $k \in \mathbb{Z}$, $\mathrm{N}\mathrm{M}_k H \subset \mathrm{M}_{k-2}H$,

(2) $\mathrm{M}_\bullet(\mathrm{N}, C_\bullet)H$ induces on each $\mathrm{gr}_\ell^C H$ the monodromy filtration of $\mathrm{gr}_\ell^C \mathrm{N}$ *centered at* ℓ, i.e., the nilpotent endomorphism $\mathrm{gr}_\ell^C \mathrm{N}$ of $\mathrm{gr}_\ell^C H$ satisfies, for every $k \geqslant 0$,

$$(\mathrm{gr}_\ell^C \mathrm{N})^k : \mathrm{gr}^{\mathrm{M}}_{\ell+k} \mathrm{gr}_\ell^C H \xrightarrow{\sim} \mathrm{gr}^{\mathrm{M}}_{\ell-k} \mathrm{gr}_\ell^C H.$$

We refer to the articles mentioned above for the proof of the uniqueness of such a filtration, when it exists, and for various inductive formulas.

***Theorem 15.1.4* (Deligne, see [SZ85, Appendix]).** *Let $C_\bullet(H, \mathrm{N})$ be an object of* CHLS*. Assume that the relative monodromy filtration $\mathrm{M}_\bullet := \mathrm{M}_\bullet(\mathrm{N}, C_\bullet H)$ exists. Then $(H, \mathrm{M}_\bullet H)$ is a mixed Hodge structure, and $C_\bullet H$ (with the induced filtrations) is a filtration by mixed Hodge substructures.*

Definition 15.1.5. The category MHLS of *mixed Hodge-Lefschetz structures* is the full subcategory of CHLS consisting of objects for which the relative monodromy filtration exists.

15.2. Comments

Here come the references to the existing work which has been the source of inspiration for this chapter.