CHAPTER 13

STRUCTURE AND DIRECT IMAGES OF POLARIZABLE HODGE MODULES

Summary. In this chapter, we start the proof of the two main important results concerning polarizable Hodge modules, namely, the structure theorem and the decomposition theorem. The proof will be finished in Chapter 14. Here, we will use the machinery of filtered $\mathscr{D}$-module theory and sesquilinear pairings to reduce the proofs to simpler cases, where other techniques are used. For the direct image theorem, we reduce to the case of the map from a compact Riemann surface to a point, that we have analyzed in Chapter 6, according to the results of Schmid and Zucker developed in Chapter 5. This strategy justifies the somewhat complicated and recursive definition of the category $\mathsf{pHM}(X,w)$ of polarizable Hodge modules.

13.1. Introduction

The theory of polarizable Hodge $\mathscr{D}$-modules was developed in order to give an analytic proof, relying on Hodge theory, of the decomposition theorem of the pushforward by a projective morphism of the intersection complex attached to a local system underlying a variation of polarizable Hodge structure. Two questions arise in this context:

- to relate variations of polarizable Hodge structure on a smooth analytic Zariski open subset of a complex analytic set with a polarizable Hodge module on a complex manifold containing this analytic set as a closed analytic subset (the structure theorem),
- to prove the decomposition theorem of the pushforward by a projective morphism of a polarizable Hodge module.

The proof of the decomposition theorem is obtained by reducing to the case of a constant map, by using the nearby cycle functor and its compatibility with pushforward. In the case of the constant map, one can reduce to the case where the Hodge module is a variation of polarizable Hodge structure on the complement of a normal crossing divisor in a complex manifold by using Hironaka's theorem on resolution of singularities, and the decomposition theorem already proved (by induction) for the resolution morphism. One can use a Lefschetz pencil to apply an inductive process, after having blown up the base locus of the pencil. In such a way, one is reduced to the case of the constant map on a smooth projective curve, where one can apply the Hodge-Saito theorem 6.4.18.

Another approach in the case of a constant map makes the full use of the higher dimensional analogues of the results proved in Chapter 5 for polarized variations of Hodge structure, but this needs to include in the inductive process the structure theorem for polarizable Hodge modules in the normal crossing case.

13.1.a. The structure theorem. This is the converse of Proposition 12.4.4(4). Let X be a complex manifold and let Z be an irreducible closed analytic subset of X. Let $\mathsf{VHS}_{\mathrm{gen}}(Z,w)$ the category of "generically defined variations of Hodge structure of weight w on Z".

We say that a pair (Z^o,H) consisting of a smooth Zariski-dense open subset Z^o of Z and of a variation of Hodge structure H of weight w on Z^o is equivalent to a pair (Z'^o,H') if H and H' coincide on $Z^o\cap Z'^o$. An object of $\mathsf{VHS}_{\mathrm{gen}}(Z,w)$ is such an equivalence class. Note that it has a maximal representative (by considering the union of the domains of all the representatives). A morphism between objects of $\mathsf{VHS}_{\mathrm{gen}}(Z,w)$ is defined similarly.

We also denote by $\mathsf{pVHS}_{\mathrm{gen}}(Z,w)$ the full subcategory of $\mathsf{VHS}_{\mathrm{gen}}(Z,w)$ consisting of objects which are polarizable, i.e., have a polarizable representative.

By Proposition 12.4.4(4), there is a restriction functor

$$\mathsf{pHM}_Z(X,w-\operatorname{codim}Z)^{\mathrm{left}}\longmapsto \mathsf{pVHS}_{\mathrm{gen}}(X,w).$$

Theorem 13.1.1 **(Structure theorem).** *Under these assumptions, the restriction functor* $\mathsf{pHM}_Z(X,w-\operatorname{codim}Z)^{\mathrm{left}}\mapsto \mathsf{pVHS}_{\mathrm{gen}}(X,w)$ *is an equivalence of categories.*

For example, set $\mathcal{M}'=\mathcal{M}''=\mathscr{O}_X$, $F_0\mathscr{O}_X=\mathscr{O}_X$ and $F_{-1}\mathscr{O}_X=0$, $\mathscr{M}'=\mathscr{M}''=R_F\mathscr{O}_X=\mathscr{O}_X[z]$, $\mathfrak{c}(1,1)=1$ so that $M^*=M$, and $\mathrm{Q}:M\to M^*$ is the identity. The corresponding M is denoted by ${}_{\mathrm{H}}\mathscr{O}_X$: this is the polarized left Hodge module of weight 0, corresponding to the constant variation of Hodge structure $\mathbb{C}(0)$ on X. That it satisfies all the requirements of Definition 12.3.13 is far from obvious and is asserted by the structure theorem.

Since each polarizable Hodge module has a unique decomposition with respect to the irreducible components of its pure support, the structure theorem gives a complete description of the category $\mathsf{pHM}(X,w)$.

As we will also see later, the structure theorem enables us to prove that the pullback by a holomorphic map of complex manifolds of a polarizable Hodge module remains a polarizable Hodge module. This statement is not obvious: for the constant map $f:X\to\mathrm{pt}$, the pullback ${}_{\mathrm{H}}f^*{}_{\mathrm{H}}\mathbb{C}_{\mathrm{pt}}$ is ${}_{\mathrm{H}}\mathscr{O}_X$.

Any polarizable Hodge module is semi-simple in the category of pure Hodge modules (see Proposition 12.4.6). If X is a projective complex manifold, semi-simplicity also holds for the underlying holonomic $\mathscr{D}_X$-module, that is, the analogue of Theorem 4.3.3 holds for polarizable Hodge modules.

Theorem 13.1.2 **(Semi-simplicity).** *Assume X is projective. Let $M = (\mathscr{M}', \mathscr{M}'', \mathfrak{c})$ be a polarizable Hodge module (so that $\mathscr{M}' \simeq \mathscr{M}''$ by means of the polarization). Then the underlying holonomic $\mathscr{D}_X$-module $\mathcal{M} = \mathcal{M}' = \mathcal{M}''$ is semi-simple. Moreover, M is simple as an Hodge module if and only if the corresponding $\mathcal{M}$ is simple as an holonomic $\mathscr{D}_X$-module.*

Remark 13.1.3. On the other hand, there exist in general holonomic $\mathscr{D}_X$-modules which are simple as such, but which do not underlie a polarizable Hodge module.

13.1.b. The direct image and decomposition theorem. This theorem describes the behaviour by projective pushforward of an object of $\mathsf{pHM}(X, w)$. The case of the constant map $X \to \mathrm{pt}$ and of the left Hodge module ${}_{\mathrm{H}}\mathscr{O}_X$ corresponds to the results of Section 2.3.

Theorem 13.1.4 **(Hodge-Saito theorem).** *Let $f : X \to Y$ be a projective morphism between complex analytic manifolds and let M be a polarizable right Hodge module of weight w on X. Let $\mathscr{L}$ be an ample line bundle on X and let $\mathrm{L}_{\mathscr{L}}$ be the corresponding Lefschetz operator. Then $(\bigoplus_k {}_{\mathrm{H}}f_*^k M, \mathrm{L}_{\mathscr{L}})$ (where the k-th term is regarded in degree $-k$) is an object of $\mathsf{pHLM}(Y, w; -1)$.*

Let us make explicit this statement. As usual for a pushforward theorem, we start with a right Hodge module. We set $M = (\mathscr{M}', \mathscr{M}'', \mathfrak{c})$ and we choose a polarization Q on M, which induces an isomorphism $\mathscr{M}' \simeq \mathscr{M}''$. We set $\mathscr{M} = \mathscr{M}'$.

(a) ${}_{\mathrm{D}}f_*\mathscr{M}$, regarded as an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\widetilde{\mathscr{D}}_Y)$, is strict, that is, for every k, ${}_{\mathrm{D}}f_*^k\mathscr{M}$ is a strict graded $R_F\mathscr{D}_Y$-module. Moreover, ${}_{\mathrm{D}}f_*^k\mathscr{M}$ is strictly S-decomposable.

(b) Each ${}_{\mathrm{H}}f_*^k M$ is a polarizable Hodge module of weight $w + k$ on Y.

(c) For every $k \geqslant 0$, the Lefschetz operator $\mathrm{L}_{\mathscr{L}}$ induces isomorphisms in $\mathsf{HM}(Y, w + k)$ (this is known as the *relative hard Lefschetz theorem*):

$$\mathrm{L}_{\mathscr{L}}^k : {}_{\mathrm{D}}f_*^k\mathscr{M} \xrightarrow{\sim} {}_{\mathrm{D}}f_*^{-k}\mathscr{M}(-k),$$

so that $(\bigoplus_k {}_{\mathrm{H}}f_*^k M, \mathrm{L}_{\mathscr{L}})$ (where the k-th term is regarded in degree $-k$) is an object of $\mathsf{HLM}(Y, w; -1)$, that is, a graded Hodge-Lefschetz module of type $\varepsilon = -1$, centered at w.

(d) By means of the identification ${}_{\mathrm{H}}f_*^k(M^*) \simeq ({}_{\mathrm{H}}f_*^{-k}M)^*$ (see (12.2.1)), the graded morphism $\bigoplus_k {}_{\mathrm{H}}f_*^k\mathrm{Q}$ induces a polarization of the graded Hodge-Lefschetz module $(\bigoplus_k {}_{\mathrm{H}}f_*^k M, \mathrm{L}_{\mathscr{L}})$.

One of the most notable consequence of the direct image theorem is the decomposition theorem.

Theorem 13.1.5 **(Decomposition Theorem).** *Let $f : X \to Y$ be a projective morphism of complex manifolds. Let $\mathscr{M}$ be a $\widetilde{\mathscr{D}}_X$-module underlying a polarizable Hodge module. Then the complex ${}_{\mathrm{D}}f_*\mathscr{M}$ in $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\widetilde{\mathscr{D}}_Y)$ decomposes (in a non-canonical way)* *as* $\bigoplus_k {}_{\mathrm{D}}f^k_*\mathscr{M}[-k]$*. Similarly, if* $\mathcal{M} = \mathscr{M}/(z-1)\mathscr{M}$ *is the underlying* $\mathscr{D}_X$*-module,* ${}_{\mathrm{D}}f_*\mathcal{M} \simeq \bigoplus_k {}_{\mathrm{D}}f^k_*\mathcal{M}[-k]$ *in* $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_Y)$*.*

When both X and Y are projective, we can combine Theorems 13.1.5, 13.1.4 and 13.1.2 to obtain:

Corollary 13.1.6. *Let* $f : X \to Y$ *be a morphism between projective complex manifolds and let* $\mathcal{M}$ *be a semi-simple holonomic* $\mathscr{D}_X$*-module underlying a polarizable Hodge module. Then* ${}_{\mathrm{D}}f_*\mathcal{M}$ *decomposes non-canonically as* $\bigoplus_k {}_{\mathrm{D}}f^k_*\mathcal{M}[-k]$*, and each* ${}_{\mathrm{D}}f^k_*\mathcal{M}$ *is itself a semi-simple holonomic* $\mathscr{D}_Y$*-module.*

13.1.c. Strategy of the proof

Sketch for Theorem 13.1.4. That holonomy is preserved by proper pushforward is recalled in Remark A.10.27. We will now focus on the other properties defining a polarizable Hodge module. The proof of Theorem 13.1.4 is by induction on the pair

$$(\dim \operatorname{Supp} M, \dim \operatorname{Supp} {}_{\mathrm{H}}f_*M)$$

ordered lexicographically in the following way.

- The case where X is a compact Riemann surface and $f : X \to \mathrm{pt}$ is the constant map has been treated in Chapter 6 for both theorems 13.1.1 and 13.1.4 (see Corollary 6.4.16 and the Hodge-Saito theorem in dimension one 6.4.18, i.e., the Hodge-Zucker theorem 5.1.1).
- $(13.1.4)_{(n,m)} \Rightarrow (13.1.4)_{(n+1,m+1)}$ is proved in Section 13.2.
- $(13.1.4)_{(\leqslant n-1,0)}$ & $(13.1.4)_{(1,0)}$ with $\operatorname{Supp} M$ smooth $\Rightarrow (13.1.4)_{(n,m)}$ for $n \geqslant 1$ is proved in Section 13.3 by using the method of Lefschetz pencils.

Sketch for the structure theorem 13.1.1. We notice first that the restriction functor $\mathsf{HM}_Z(X,w)^{\mathrm{left}} \to \mathsf{pVHS}_{\mathrm{gen}}(Z, w - \operatorname{codim} Z)$ is faithful. Indeed, let M_1, M_2 be objects of $\mathsf{HM}_Z(X,w)$ and let $\varphi, \varphi' : M_1 \to M_2$ be morphisms between them, which coincide on some Z^o. Then the image of $\varphi - \varphi'$ is an object of $\mathsf{HM}(X,w)$, according to Proposition 12.3.9, and is supported on $Z \smallsetminus Z^o$, hence is zero according to the definition of the pure support.

Due to the faithfulness, we note that the question is local: for fullness, if a morphism between the restriction to some Z^o of two polarized Hodge modules locally extends on Z, then it globally extends by uniqueness of the extension; then, two local extensions as polarized Hodge modules of a variation of polarized Hodge structure coincide (through the extension of the identity morphism on some Z^o) and we can glue local extensions.

For the essential surjectivity we start from a variation of polarized Hodge structure on some smooth Zariski-dense open subset $Z^o \subset Z$. We choose a projective morphism $f : Z' \to X$ with Z' smooth and connected, such that f is an isomorphism $Z'^o := f^{-1}(Z^o) \to Z^o$, and such that $Z' \smallsetminus Z'^o$ is a divisor with normal crossing. Assume we have extended the variation on Z'^o as a polarized Hodge module on Z' with pure support Z', we apply to the latter the direct image theorem 13.1.4 for f, and get the desired polarized Hodge module as the component of this direct image ${}_{\mathrm{H}}f_*^0$ having pure support Z. We argue similarly for the fullness: if any morphism defined on some Z^o can be extended as a morphism between the extended objects on Z', we push it forward by f and restrict it as a morphism between the corresponding components.

We are thus reduced to the case where $Z = X$ and the variation exists on $X^o := X \smallsetminus D$, where D is a divisor with normal crossings. Moreover, the question is local. By using the asymptotic theory of variations of Hodge structure we construct coherent $\mathscr{D}_X$-modules $\mathscr{M}', \mathscr{M}''$ and we prove that the sesquilinear pairing $\mathfrak{c}^o$ takes values in the sheaf of moderate distributions along D.

13.2. Behaviour of the properties (HSD), ($\mathrm{HM}_{>0}$) and ($\mathrm{PHM}_{>0}$) by projective pushforward

Let $f : X \to Y$ be a projective morphism between complex manifolds, let t be a holomorphic function on Y and set $g = t \circ f : X \to \mathbb{C}$. The question we wish to consider being local on Y, it is not restrictive to assume that t is part of a local coordinate system on Y. Let $\mathscr{L}$ be a relatively ample line bundle on X. In other words, we choose a relative embedding

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & Y \times \mathbb{P}^N \arrow[d] \\ & Y \end{tikzcd}$$

so that $\mathscr{L}$ comes by pullback from an ample line bundle on $\mathbb{P}^N$.

Let $M = (\mathscr{M}', \mathscr{M}'', \mathfrak{c})$ be a an object of $R_F\mathscr{D}\text{-}\mathsf{Triples}(X)_{\mathrm{coh}}$. We assume that M is strictly $\mathbb{R}$-specializable along (g) and is a minimal extension along (g), that is, can is onto and var is injective (see Definition 7.7.3).

Let $\mathrm{Q} : M \to M^*(-w)$ be a $(-1)^w$-Hermitian morphism.

Proposition 13.2.1. *Together with these assumptions, let us moreover assume that*

(a) $\dim(\operatorname{Supp} M \cap g^{-1}(0)) \leqslant d$,

(b) *M satisfies* ($\mathrm{HM}_{>0}$) *relatively to g (Definition* 12.3.1*), that is, for any $\lambda \in S^1$ and any integer ℓ, $\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}M$ is an object of $\mathsf{HM}_{\leqslant d}(X, w-1+\ell)$,*

(c) *(M, Q) satisfies* ($\mathrm{PHM}_{>0}$) *relatively to g (Definition* 12.4.1*), that is, for any $\lambda \in S^1$ and any integer $\ell \geqslant 0$, the morphism $\mathrm{P}_\ell\psi_{g,\lambda}\mathrm{Q}$ induces a polarization of the object $\mathrm{P}_\ell\psi_{g,\lambda}M$ of $\mathsf{HM}_{\leqslant d}(X, w-1+\ell)$.*

In other words, we assume that $(\mathrm{gr}_\bullet^{\mathrm{M}}\psi_{g,\lambda}M, \mathrm{N}, \mathrm{gr}_\bullet\psi_{g,\lambda}\mathrm{Q})$ is a polarized graded Hodge-Lefschetz triple of type $\varepsilon = 1$, centered at $w-1$.

Then, if Theorem 13.1.4 *holds in dimension $\leqslant d$, the following holds for every $k \in \mathbb{Z}$.*

(1) *${}_{\mathrm{H}}f_*^k M$ is strictly $\mathbb{R}$-specializable and strictly S-decomposable along (t),*

(2) $\big(\bigoplus_{k,\ell}\mathrm{gr}^{\mathrm{M}}_{-\ell}\psi_{t,\lambda}({}_{\mathrm{H}}f^k_*M),(\mathrm{L}_{\mathscr{L}},\mathrm{N}),\mathrm{gr}^{\mathrm{M}}_{-\ell}\psi_{t,\lambda}({}_{\mathrm{H}}f^k_*\mathrm{Q})\big)$ *is a polarized bi-graded Hodge-Lefschetz triple of type* $\varepsilon=(-1,1)$*, centered at* $w-1$*.*

Proof. One of the points to understand is the way to pass from properties of ${}_{\mathrm{H}}f^k_*\mathrm{gr}^{\mathrm{M}}_{-\ell}\psi_{g,\lambda}M$ to properties of $\mathrm{gr}^{\mathrm{M}}_{-\ell}\psi_{t,\lambda}({}_{\mathrm{H}}f^k_*M)$. Although we know that $\psi_{t,\lambda}({}_{\mathrm{H}}f^k_*M)\xrightarrow{\sim}{}_{\mathrm{H}}f^k_*\psi_{g,\lambda}M$ if the latter is strict, according to Proposition 12.2.5, we have to check the strictness property. Moreover, we are left with the question of passing from ${}_{\mathrm{H}}f^k_*\mathrm{gr}^{\mathrm{M}}_{-\ell}$ to $\mathrm{gr}^{\mathrm{M}}_{-\ell}{}_{\mathrm{H}}f^k_*$. Here, we do not have a commutation property, but we will use Corollary 12.4.12 to analyze the corresponding spectral sequence. At this point, the existence of a polarization is essential. The strict S-decomposability is not obvious either, and the polarization also plays an essential role for proving it.

Since we assume that Theorem 13.1.4 holds for objects in $\mathsf{pHM}_{\leqslant d}(X)$ and since $\dim(\operatorname{Supp}M\cap g^{-1}(0))\leqslant d$, we deduce that, for every $\lambda\in S^1$,

$$\Big(\bigoplus_{k,\ell}{}_{\mathrm{H}}f^k_*\mathrm{gr}^{\mathrm{M}}_{-\ell}\psi_{g,\lambda}M,(\mathrm{L}_{\mathscr{L}},{}_{\mathrm{H}}f^k_*\mathrm{gr}\mathrm{N}),{}_{\mathrm{H}}f^k_*\mathrm{gr}^{\mathrm{M}}_{-\ell}\psi_{g,\lambda}\mathrm{Q}\Big)$$

is a polarized object of $\mathsf{HLM}(Y,w-1;-1,1)$ if we keep here the grading convention used in Corollary 12.4.12. This corollary implies that

$$\Big(\bigoplus_{k,\ell}\mathrm{gr}^{\mathrm{M}}_{-\ell}\,{}_{\mathrm{H}}f^k_*\psi_{g,\lambda}M,(\mathrm{L}_{\mathscr{L}},\mathrm{gr}\,{}_{\mathrm{H}}f^k_*\mathrm{N}),\mathrm{gr}^{\mathrm{M}}_{-\ell}\,{}_{\mathrm{H}}f^k_*\psi_{g,\lambda}\mathrm{Q}\Big)$$

is a polarized object of $\mathsf{HLM}(Y,w-1;-1,1)$. In particular, each $\mathrm{gr}^{\mathrm{M}}_{-\ell}\,{}_{\mathrm{H}}f^k_*\psi_{g,\lambda}M$ is strict, and therefore so is ${}_{\mathrm{H}}f^k_*\psi_{g,\lambda}M$.

A similar argument applies to the vanishing cycles, according to Proposition 12.4.13 (up to changing $w-1$ to w), and therefore each ${}_{\mathrm{H}}f^k_*\phi_{g,1}M$ is strict. We can now apply Corollary 7.8.6 to conclude that ${}_{\mathrm{H}}f^k_*M$ is strictly $\mathbb{R}$-specializable along (t) for every k. We also conclude from Proposition 12.2.5 that

$$(\psi_{g,\lambda}\,{}_{\mathrm{H}}f^k_*M,\mathrm{N})={}_{\mathrm{H}}f^k_*(\psi_{g,\lambda}M,\mathrm{N}).$$

At this point, we have proved that

$$\Big(\bigoplus_{k,\ell}\mathrm{gr}^{\mathrm{M}}_{-\ell}\psi_{g,\lambda}\,{}_{\mathrm{H}}f^k_*M,(\mathrm{L}_{\mathscr{L}},\mathrm{gr}\mathrm{N}),\mathrm{gr}^{\mathrm{M}}_{-\ell}\psi_{g,\lambda}\,{}_{\mathrm{H}}f^k_*\mathrm{Q}\Big)$$

is a polarized object of $\mathsf{HLM}(Y,w-1;-1,1)$.

What can we say about vanishing cycles? Note that, since we have defined vanishing cycles $\phi_{g,1}$ of $\mathscr{D}$-triples only when the $\mathscr{D}$-triple is S-decomposable along (g) (see Remark 10.4.21), we cannot mention $\phi_{g,1}{}_{\mathrm{H}}f^k_*M$ before having proven that M is strictly S-decomposable. In any case, we have a quiver

$$\begin{tikzcd}[column sep=huge] \psi_{g,1}\,{}_{\mathrm{H}}f^k_*M={}_{\mathrm{H}}f^k_*\psi_{g,1}M \arrow[r, bend left, "{}_{\mathrm{H}}f^k_*\,\mathrm{can}"] & {}_{\mathrm{H}}f^k_*\phi_{g,1}M \arrow[l, bend left, "{}_{\mathrm{H}}f^k_*\,\mathrm{var}", "(-1)"'] \end{tikzcd}\tag{13.2.2}$$

that we ultimately want to be the nearby/vanishing quiver of ${}_{\mathrm{H}}f^k_*M$.

Nevertheless, the same argument as for nearby cycles gives that

$$(\phi_{g,1}\,{}_{\mathrm{H}}f_*^k\mathscr{M},\mathrm{N}) = {}_{\mathrm{H}}f_*^k(\phi_{g,1}\mathscr{M},\mathrm{N}),\quad \mathscr{M}=\mathscr{M}',\mathscr{M}'',$$

and

$$\Big(\bigoplus_{k,\ell}\mathrm{gr}^{\mathrm{M}}_{-\ell}\,{}_{\mathrm{H}}f_*^k\phi_{g,1}M,(\mathrm{L}_{\mathscr{L}},\mathrm{gr}\,{}_{\mathrm{H}}f_*^k\mathrm{N}),\mathrm{gr}^{\mathrm{M}}_{-\ell}\,{}_{\mathrm{H}}f_*^k\phi_{g,1}\mathrm{Q}\Big)$$

is a polarized object of $\mathsf{HLM}(Y,w;-1,1)$ (take care of the position of ${}_{\mathrm{H}}f_*^k$). For $k\leqslant 0$, let us denote by $\mathrm{P}_{\mathrm{L}}\,{}_{\mathrm{H}}f_*^k$ the primitive part with respect to $\mathrm{L}_{\mathscr{L}}$. We deduce from (13.2.2) a graded Lefschetz quiver with vertices $\bigoplus_\ell\mathrm{gr}^{\mathrm{M}}_\ell\mathrm{P}_{\mathrm{L}}\,{}_{\mathrm{H}}f_*^k\psi_{g,1}M$ and $\bigoplus_\ell\mathrm{gr}^{\mathrm{M}}_\ell\mathrm{P}_{\mathrm{L}}\,{}_{\mathrm{H}}f_*^k\phi_{g,1}M$, to which we can apply Proposition 12.4.11, that is, the analogue of Proposition 3.2.28. We conclude that

$$\bigoplus_\ell\mathrm{gr}^{\mathrm{M}}_\ell\mathrm{P}_{\mathrm{L}}\,{}_{\mathrm{H}}f_*^k\phi_{g,1}M=\mathrm{Im}\,\mathrm{gr}\mathrm{P}_{\mathrm{L}}\,{}_{\mathrm{H}}f_*^k\,\mathrm{can}\oplus\mathrm{Ker}\,\mathrm{gr}\mathrm{P}_{\mathrm{L}}\,{}_{\mathrm{H}}f_*^k\,\mathrm{var}\,.$$

We also see that $\mathrm{P}_{\mathrm{L}}\,{}_{\mathrm{H}}f_*^k\psi_{g,\lambda}M$ and $\mathrm{P}_{\mathrm{L}}\,{}_{\mathrm{H}}f_*^k\phi_{g,1}M$ are objects $\mathsf{WHM}(Y)$, with weight filtration equal to the monodromy filtration shifted by $w-1$ and w respectively. It follows that the morphisms $\mathrm{P}_{\mathrm{L}}\,{}_{\mathrm{H}}f_*^k\,\mathrm{can}$ and $\mathrm{P}_{\mathrm{L}}\,{}_{\mathrm{H}}f_*^k\,\mathrm{var}$ are strictly compatible with the monodromy filtration (after a suitable shift), and therefore the decomposition above reads

$$\mathrm{gr}^{\mathrm{M}}\mathrm{P}_{\mathrm{L}}\,{}_{\mathrm{H}}f_*^k\phi_{g,1}M=\mathrm{gr}^{\mathrm{M}}\,\mathrm{Im}\,\mathrm{P}_{\mathrm{L}}\,{}_{\mathrm{H}}f_*^k\,\mathrm{can}\oplus\mathrm{gr}^{\mathrm{M}}\,\mathrm{Ker}\,\mathrm{P}_{\mathrm{L}}\,{}_{\mathrm{H}}f_*^k\,\mathrm{var},$$

from which we deduce

$$\mathrm{P}_{\mathrm{L}}\,{}_{\mathrm{H}}f_*^k\phi_{g,1}M=\mathrm{Im}\,\mathrm{P}_{\mathrm{L}}\,{}_{\mathrm{H}}f_*^k\,\mathrm{can}\oplus\mathrm{Ker}\,\mathrm{P}_{\mathrm{L}}\,{}_{\mathrm{H}}f_*^k\,\mathrm{var},$$

and finally, by applying the Lefschetz decomposition for L,

$${}_{\mathrm{H}}f_*^k\phi_{g,1}M=\mathrm{Im}\,{}_{\mathrm{H}}f_*^k\,\mathrm{can}\oplus\mathrm{Ker}\,{}_{\mathrm{H}}f_*^k\,\mathrm{var},\quad\forall\,k\in\mathbb{Z}.$$

From this property we conclude in particular that $\mathscr{H}^k{}_{\mathrm{D}}f_*\mathscr{M}'$ and $\mathscr{H}^k{}_{\mathrm{D}}f_*\mathscr{M}''$ are strictly S-decomposable along (g'). In particular, $\mathscr{H}^k{}_{\mathrm{D}}f_*\mathcal{M}'$ and $\mathscr{H}^k{}_{\mathrm{D}}f_*\mathcal{M}''$ are S-decomposable along (g') for every $k\in\mathbb{Z}$. Lemma 10.4.17 shows that the sesquilinear pairing ${}_{\mathrm{T}}f_*^k$ splits correspondingly. In other words, ${}_{\mathrm{H}}f_*^kM$ is strictly S-decomposable along (g'), as wanted. Moreover, $\phi_{g,1\mathrm{H}}f_*^kM$ is then defined, and it is now clear that $\phi_{g,1}\,{}_{\mathrm{H}}f_*^kM={}_{\mathrm{H}}f_*^k\phi_{g,1}M$. ∎

Proof of $(13.1.4)_{(n,m)}\Rightarrow(13.1.4)_{(n+1,m+1)}$. Let $f:X\to Y$ be a projective morphism and let (M,Q) be a polarized object of $\mathsf{pHM}_Z(X,w)$, where Z is an irreducible analytic subset of X of dimension $n+1$. We can assume that (M,Q) is a $(-1)^w$-Hodge Hermitian pair $(\mathscr{M},\mathfrak{c})$, and we will omit Q in the notation. Assume that $f(Z)$ has dimension $m+1$ and that $(13.1.4)_{(n,m)}$ holds. Since Theorem 13.1.4 is a local statement on Y, we can work in an open neighbourhood of a point $y_o\in f(Z)$, that we can take as small as needed. By the strict S-decomposability of $(\mathscr{M},\mathfrak{c})$ on X, we can therefore assume that Z and $f(Z)$ are irreducible when restricted to a fundamental basis of neighbourhoods of y_o.

Let g' be a holomorphic function on some $\mathrm{nb}(y_o)$ and set $g=g'\circ f$. We distinguish two cases.

(1) g' vanish identically on he closed irreducible subset $f(Z)\cap\mathrm{nb}(y_o)$ of $\mathrm{nb}(y_o)$. We now omit referring to $\mathrm{nb}(y_o)$. We denote by $\iota_g : X \hookrightarrow X\times\mathbb{C}$ and by $\iota : X\times\{0\}\hookrightarrow X\times\mathbb{C}$ the inclusions, and similarly on Y. The only property to be checked relative to g' is that ${}_{\mathrm{D}}f_*^k\mathscr{M}$ is S-decomposable along (g'), which is equivalent to ${}_{\mathrm{D}}\iota_{g'*}{}_{\mathrm{D}}f_*^k\mathscr{M} = {}_{\mathrm{D}}\iota_{*}{}_{\mathrm{D}}f_*^k\mathscr{M}$ for every k (use the same argument as in Corollary 7.7.1). The left-hand term is equal to ${}_{\mathrm{D}}f_*^k{}_{\mathrm{D}}\iota_{g*}\mathscr{M}$, if we still denote by f the map $f\times\mathrm{Id}_{\mathbb{C}}$. Similarly the right-hand term is equal to ${}_{\mathrm{D}}f_*^k{}_{\mathrm{D}}\iota_{*}\mathscr{M}$, with obvious abuse of notation. Since $g\equiv 0$ on Z and $\mathscr{M}$ is assumed to be S-decomposable along (g), we have ${}_{\mathrm{D}}\iota_{g*}\mathscr{M} = {}_{\mathrm{D}}\iota_{*}\mathscr{M}$, hence the desired assertion.

(2) $g'^{-1}(0)\cap f(Z)$ has codimension one in $f(Z)$. Then $g^{-1}(0)\cap Z$ has codimension one in Z. We can thus apply Proposition 13.2.1 with $d=n$. It follows that each ${}_{\mathrm{H}}f_*^kM$ satisfies (HSD), $(\mathrm{HM})_{>0}$ and $(\mathrm{PHM})_{>0}$ with respect to g'. □

13.3. End of the proof of Theorem 13.1.4

[3]

[3] From here to the end of the chapter, work in progress, do not take it into account.

13.4. Comments

Here come the references to the existing work which has been the source of inspiration for this chapter.