CHAPTER 12

PURE HODGE MODULES

Summary. This chapter contains the definition of pure Hodge modules, without, and then with, a polarization. The actual presentation justifies the introduction of the language of triples. The main properties are:

- the category of pure Hodge modules of weight w is abelian,
- the category of pure polarized Hodge modules of weight w is semisimple.

It is convenient to also introduce Hodge-Lefschetz modules, as they appear in many intermediate steps of various proofs, due to the very definition of a Hodge module.

12.1. Introduction

Hodge $\mathscr{D}$-modules are supposed to play the role of Hodge structures with a multidimensional parameter. These objects can acquire singularities. The way each characteristic property of a Hodge structure is translated in higher dimension of the parameter space is given by the table below.

dimension 0	dimension $n \geqslant 1$
$\mathcal{H}$ a $\mathbb{C}$-vector space	$\mathcal{M}$ a holonomic $\mathscr{D}$-module
$F^\bullet\mathcal{H}$ a filtration	$F_\bullet\mathcal{M}$ a good filtration
$\mathscr{H} = R_F\mathcal{H}$	$\mathscr{M} = R_F\mathcal{M}$
$H = (\mathscr{H}', \mathscr{H}'', \mathfrak{c})$ a graded triple of $\mathbb{C}[z]$-vector spaces	$M = (\mathscr{M}', \mathscr{M}'', \mathfrak{c})$ a graded triple of $R_F\mathscr{D}$-modules
$\mathrm{Q} : H \to H^*(-w)$ a polarization	$\mathrm{Q} : M \to M^*(-w)$ a polarization

Why choosing holonomic $\mathscr{D}$-modules as analogues of $\mathbb{C}$-vector space? The reason is that the category of holonomic $\mathscr{D}$-modules is Artinian, that is, any holonomic $\mathscr{D}$-module has finite length (locally on the underlying manifold). A related reason is that its de Rham complex has constructible cohomology, generalizing the notion of local system attached to a flat bundle. Moreover, the property of holonomicity is preserved by various operations (proper pushforward, pullback by a holomorphic map), and the nearby/vanishing cycle theory (the V-filtration) is well-defined for holonomic $\mathscr{D}$-modules without any other assumption, so that the issue concerning nearby/vanishing cycles of filtered holonomic $\mathscr{D}$-modules only comes from the filtration.

In order to define the Hodge properties, we use the same method as in dimension one (see Section 6.1):

• we only consider holonomic $\mathscr{D}$-modules which are *S-decomposable*, that is, which are direct sum of $\mathscr{D}$-modules having an irreducible *pure support*, in other words supported by an irreducible closed analytic subset of the underlying manifold and having neither sub-module nor quotient module supported in a smaller subset;

• we moreover ask that the F-filtration is compatible with the decomposition by the support, in other words, the associated graded $R_F\mathscr{D}$-module is *strictly S-decomposable*;

• Fortunately, the sesquilinear pairing behaves well with respect to these constraints, so that we do not have to make more assumptions on the sesquilinear pairing. We are thus led to work in the abelian category $R_F\mathscr{D}\text{-}\mathsf{Triples}(X)$, which is a filtered analogue of the category $\mathscr{D}$-$\mathsf{Triples}$ considered in Section 10.2.c. Notice that no condition is put concerning the behaviour of the sesquilinear pairing with respect to the filtrations.

• in order to reduce the structure to a point, we use iterated nearby cycles, along a family of functions, the ideal of which define the point; therefore, one has to use the functor of nearby cycles, which is defined for objects of $R_F\mathscr{D}\text{-}\mathsf{Triples}(X)$ which are strictly $\mathbb{R}$-specializable.

The definition of a Hodge module can look frightening: in order to check that an object $M = (\mathscr{M}', \mathscr{M}'', \mathfrak{c})$ belongs to $\mathsf{HM}(X, w)$, we have to consider in an inductive way nearby cycles with respect to *all* germs of holomorphic functions.

With this respect, a variation of polarized Hodge structure is not obviously a pure Hodge module. This is true, but will have to be proved carefully.

The question should however be considered the other way round. Once we know at least one pure Hodge $\mathscr{D}$-module, we automatically know an infinity of them, by considering (monodromy-graded) nearby or vanishing cycles with respect to *any* holomorphic function. For example, once we have proved that a variation of polarized Hodge structure is a pure Hodge module, we obtain many such objects by applying the pushforward by any projective morphism, as will be shown in Chapter 13.

In the same vein, due to this inductive definition, the proof of many properties of Hodge modules can be done by induction on the dimension of the support, and this reduces to checking the property for Hodge structures.

Although the most important and useful properties of Hodge modules make use of a polarization, or more precisely of the existence of a polarization, it is interesting to start working with possibly non-polarizable Hodge modules, in order to realize the strength of the generalization of the opposedness property in higher dimensions. Nevertheless, we do not know interesting examples in dimension $\geqslant 1$ of pure Hodge modules which are not polarizable.

12.2. The ambient category

It will be convenient to work within an ambient abelian category, namely the category $R_F\mathscr{D}\text{-}\mathsf{Triples}(X)$. An object in this category consists of a pair of graded $R_F\mathscr{D}_X$-modules $\mathscr{M}', \mathscr{M}''$ and a sesquilinear pairing $\mathfrak{c}$ between the associated $\mathscr{D}_X$-modules $\mathcal{M}' := \mathscr{M}'/(z-1)\mathscr{M}'$ and $\mathcal{M}'' := \mathscr{M}''/(z-1)\mathscr{M}''$ with values in $\mathfrak{Db}_X$ (left case) or $\mathfrak{C}_X$ (right case). For M_1, M_2 objects of $R_F\mathscr{D}\text{-}\mathsf{Triples}(X)$, a morphism $\varphi : M_1 \to M_2$ is a pair (φ', φ''), with $\varphi' : \mathscr{M}'_1 \to \mathscr{M}'_2$ and $\varphi'' : \mathscr{M}''_2 \to \mathscr{M}''_1$, whose restriction to $z = 1$ is compatible with $\mathfrak{c}$. In other words, we have a natural functor "$z = 1$" from $R_F\mathscr{D}\text{-}\mathsf{Triples}(X)$ to the category $\mathscr{D}\text{-}\mathsf{Triples}(X)$ (see Definition 10.2.7). Since the various functors we have considered for graded $R_F\mathscr{D}_X$-modules restrict to the corresponding functors for $\mathscr{D}_X$-modules when we set $z = 1$, they extend in a natural way as functors defined on the category $R_F\mathscr{D}\text{-}\mathsf{Triples}$. We say that an object M of $R_F\mathscr{D}\text{-}\mathsf{Triples}(X)$ is coherent (resp. holonomic, resp. strict, resp. strictly $\mathbb{R}$-specializable, resp. strictly S-decomposable) if $\mathscr{M}'$ and $\mathscr{M}''$ are so.

We now collect various results by gathering properties of Chapters 7 and 10. We will mainly work with right $R_F\mathscr{D}_X$-modules, but we will occasionally indicate the effect of side changing.

Tate twist. For every pair of integers (k, ℓ), we set

$$(\mathscr{M}', \mathscr{M}'', \mathfrak{c})(k, \ell) = (\mathscr{M}'(k), \mathscr{M}''(-\ell), \mathfrak{c}).$$

The symmetric Tate twist (k, k) is simply denoted by (k).

Adjunction. The adjoint $(\mathscr{M}', \mathscr{M}'', \mathfrak{c})^*$ of an object of $R_F\mathscr{D}\text{-}\mathsf{Triples}(X)$ is the object $(\mathscr{M}'', \mathscr{M}', \mathfrak{c}^*)$.

12.2.a. The case of $R_F\mathscr{D}\text{-}\mathsf{Triples}(X)^{\mathrm{right}}$

Proper pushforward. Let $f : X \to Y$ be a proper morphism of complex analytic manifolds. We will consider the pushforward functor ${}_{\mathrm{H}}f_*^k$ from $R_F\mathscr{D}\text{-}\mathsf{Triples}(X)$ to $R_F\mathscr{D}\text{-}\mathsf{Triples}(Y)$. It is defined as follows, for $M = (\mathscr{M}', \mathscr{M}'', \mathfrak{c})$ (see Definition 10.3.17, with the sign of (10.3.16 $*$)).

$${}_{\mathrm{H}}f_*^k M := (\mathscr{H}^k{}_{\mathrm{D}}f_*\mathscr{M}', \mathscr{H}^{-k}{}_{\mathrm{D}}f_*\mathscr{M}'', {}_{\mathrm{T}}f_*^k\mathfrak{c}).$$

We have $({}_{\mathrm{H}}f_*^j M)(k, \ell) = {}_{\mathrm{H}}f_*^j(M(k, \ell))$ and compatibility with adjunction, after (10.3.16 $**$),

$$\tag{12.2.1} {}_{\mathrm{H}}f_*^k(M^*) = ({}_{\mathrm{H}}f_*^{-k}M)^*.$$

The Lefschetz operator. Let $\mathscr{L}$ be a line bundle on X. The Lefschetz operator is a morphism

$$\mathrm{L}_{\mathscr{L}} : {}_{\mathrm{H}}f_*^k M \longrightarrow {}_{\mathrm{H}}f_*^{k+2} M(1).$$

It is defined according to Remark A.8.17, and (10.3.20 $**$) for the sign.

Strict $\mathbb{R}$-specializability. Let $M = (\mathscr{M}', \mathscr{M}'', \mathfrak{c})$ be an object of $R_F\mathscr{D}\text{-}\mathsf{Triples}(X)_{\mathrm{coh}}$ which is strictly $\mathbb{R}$-specializable along (g) (i.e., $\mathscr{M}', \mathscr{M}''$ are so). Let $\iota_g : X \hookrightarrow X \times \mathbb{C}_t$ denote the inclusion of the graph of g. We then set, for $\lambda \in S^1$,

$$\psi_{g,\lambda} M := (\psi_{g,\lambda}\mathscr{M}', \psi_{g,\lambda}\mathscr{M}''(-1), \psi_{g,\lambda}\mathfrak{c}), \tag{12.2.2}$$

where $\psi_{g,\lambda}\mathscr{M}$ is given by Definition 7.4.1 and $\psi_{g,\lambda}\mathfrak{c}$ by Definition 10.4.18 and (10.4.7) (see also (10.4.22)). The object $\psi_{g,\lambda}M$ of $R_F\mathscr{D}\text{-}\mathsf{Triples}(X)_{\mathrm{coh}}$ is equipped with a nilpotent endomorphism $\mathrm{N} : \psi_{g,\lambda}M \to \psi_{g,\lambda}M(-1)$, following Definition 7.4.8, and (10.4.24 $*$) for the signs, that is,

$$\mathrm{N} = (\mathrm{N}', \mathrm{N}''), \quad \mathrm{N}' = -2\pi\mathsf{i}(t\eth_t - \alpha z), \quad \mathrm{N}'' = 2\pi\mathsf{i}(t\eth_t - \alpha z), \quad \lambda = \exp(2\pi\mathsf{i}\,\alpha).$$

The monodromy filtration is well-defined in the abelian category $R_F\mathscr{D}\text{-}\mathsf{Triples}(X)$, and we have (see Remark 10.4.25)

$$\begin{aligned} \mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}M &= (\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}\mathscr{M}', \mathrm{gr}_{-\ell}^{\mathrm{M}}\psi_{g,\lambda}\mathscr{M}''(-1), \mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}\mathfrak{c}),\\ \mathrm{P}_\ell\psi_{g,\lambda}M &= (\mathrm{P}_\ell\psi_{g,\lambda}\mathscr{M}', \mathrm{P}_\ell\psi_{g,\lambda}\mathscr{M}''(-1), \mathrm{P}_\ell\psi_{g,\lambda}\mathfrak{c}) \quad (\ell \geqslant 0). \end{aligned} \tag{12.2.3}$$

***Example 12.2.4* (The basic example).** We denote

$${}_{\mathrm{H}}\omega_X := (\widetilde{\omega}_X, \widetilde{\omega}_X(n), \mathfrak{c}), \tag{12.2.4 $*$}$$

where $\widetilde{\omega}_X = z^{-n}\omega_X[z]$ (see Example A.2.18) and $\mathfrak{c}$ is as in Example 10.2.14(2).

If $X = H \times \Delta_t$, we have $\psi_{t,1}\widetilde{\omega}_X = {}_{\mathrm{D}}\iota_*^0\widetilde{\omega}_H$ ($\iota : H \hookrightarrow X$), according to Remark 7.4.3, and due to Example 10.4.23, we have $\psi_{t,1}({}_{\mathrm{H}}\omega_X) = {}_{\mathrm{H}}\iota_*^0({}_{\mathrm{H}}\omega_H)$, by the choices of signs and twists in (12.2.2) and (12.2.4 $*$).

Strict Kashiwara's equivalence. For the inclusion of a closed submanifold $Y \subset X$ of codimension one, the equivalence of Proposition 7.6.2 holds for strictly $\mathbb{R}$-specializable objects of $R_F\mathscr{D}\text{-}\mathsf{Triples}(X)$ and the functor ${}_{\mathrm{H}}\iota_*^0$, according to Example 10.3.18.

Strict S-decomposability. If M is strictly S-decomposable along the divisor of a holomorphic function g (i.e., if $\mathscr{M}', \mathscr{M}''$ are so), then M decomposes as the direct sum of a minimal extension along (g) and an object supported on $g^{-1}(0)$ (this follows from Lemma 10.4.17 and Remark 7.7.16). Similarly, if M is strictly S-decomposable (i.e., if $\mathscr{M}', \mathscr{M}''$ are so), then M decomposes with respect to the pure support (this follows from Proposition 10.4.16 and Remark 7.7.16). For such an object, the functor $(\phi_{g,1}M, \mathrm{N})$ is defined in the same way as in Definition 7.4.1, Remark 10.4.21 and (10.4.29 $*$). If $\operatorname{Supp} M \subset g^{-1}(0)$, then $\phi_{g,1}M = M$ and $\mathrm{N} = 0$.

Proper pushforward and specialization. Let $f : X \to Y$ be a proper morphism, let $g' : Y \to \mathbb{C}$ be a holomorphic function and set $g = g' \circ f$. Let M be an object of $R_F\mathscr{D}\text{-}\mathsf{Triples}(X)$.

Proposition 12.2.5. *Assume that M is strictly $\mathbb{R}$-specializable along (g). Assume also that for every $k \in \mathbb{Z}$ and $\lambda \in S^1$, ${}_{\mathrm{T}}f_*^k\psi_{g,\lambda}\mathscr{M}$ and ${}_{\mathrm{T}}f_*^k\phi_{g,1}\mathscr{M}$ are strict ($\mathscr{M} = \mathscr{M}', \mathscr{M}''$). Then ${}_{\mathrm{T}}f_*^k M$ is strictly $\mathbb{R}$-specializable along (g') and, for every $\lambda \in S^1$, we have an isomorphism compatible with* N*:*

$$((-1)^{n-m}\,\mathrm{Id}, \mathrm{Id}) : {}_{\mathrm{T}}f_*^k\psi_{g,\lambda}M \xrightarrow{\sim} \psi_{g',\lambda}{}_{\mathrm{T}}f_*^k M.$$

Proof. This follows from Corollary 7.8.6 and Corollary 10.6.3. □

Pre-polarization and $(-1)^w$-Hermitian pairs. Let $w \in \mathbb{Z}$. A pre-polarization of weight w on M is a morphism

$$\mathrm{Q} = (\mathrm{Q}', \mathrm{Q}'') : M \longrightarrow M^*(-w)$$

which is $(-1)^w$-Hermitian, that is, $\mathrm{Q}^* = (-1)^w\mathrm{Q}$, i.e., $\mathscr{Q} := \mathrm{Q}'' = (-1)^w\mathrm{Q}'$. We say that is is non-degenerate if Q is an isomorphism. According to (12.2.1), it defines by pushing forward a graded morphism

$${}_{\mathrm{H}}f_*^{\bullet}\mathrm{Q} := \bigoplus_k {}_{\mathrm{H}}f_*^k\mathrm{Q} : \bigoplus_k {}_{\mathrm{H}}f_*^k M \longrightarrow \bigoplus_k {}_{\mathrm{H}}f_*^k(M^*) \simeq \bigoplus_k ({}_{\mathrm{H}}f_*^{-k}M)^*.$$

We have ${}_{\mathrm{H}}f_*^k(\mathrm{Q}^*) = ({}_{\mathrm{H}}f_*^{-k}\mathrm{Q})^*$. In particular, since Q is $(-1)^w$-Hermitian, ${}_{\mathrm{H}}f_*^{\bullet}\mathrm{Q}$ is $(-1)^w$-Hermitian in the graded sense, i.e., $({}_{\mathrm{H}}f_*^k\mathrm{Q})^* = (-1)^w\,{}_{\mathrm{H}}f_*^{-k}\mathrm{Q}$.

If M is strictly $\mathbb{R}$-specializable along (g) and $\mathrm{Q} = ((-1)^w\mathscr{Q}, \mathscr{Q})$ is a pre-polarization of weight w of M, then $((-1)^{w-1}\psi_{g,\lambda}\mathscr{Q}, \psi_{g,\lambda}\mathscr{Q})$ is a pre-polarization of weight $w-1$ of $\psi_{g,\lambda}M$.

Caveat 12.2.6. If $\mathrm{Q} = ((-1)^w\mathscr{Q}, \mathscr{Q})$, we will denote

$$\psi_{g,\lambda}\mathrm{Q} := ((-1)^{w-1}\psi_{g,\lambda}\mathscr{Q}, \psi_{g,\lambda}\mathscr{Q}).$$

This is not compatible with the definition of $\psi_{g,\lambda}$ applied to a morphism, but this will simplify the notation.

A $(-1)^w$-Hermitian pair is a pair $(\mathscr{M}, \mathfrak{c})$, where $\mathfrak{c}$ is $(-1)^w$-Hermitian on $\mathcal{M}$. It defines an object of $R_F\mathscr{D}\text{-}\mathsf{Triples}(X)^{\mathrm{right}}$ $(\mathscr{M}, \mathscr{M}(w), \mathfrak{c})$ with non-degenerate pre-polarization $((-1)^w\,\mathrm{Id}, \mathrm{Id})$. Any non-degenerate pre-polarized triple (M, Q) of weight w, with $M = (\mathscr{M}', \mathscr{M}'', \mathfrak{c})$, is isomorphic to the triple attached to a $(-1)^w$-Hermitian pair $(\mathscr{M}', \mathfrak{c}')$ for a suitable $\mathfrak{c}'$.

We will usually argue on the $(-1)^w$-Hermitian pair, which is a simpler object than the non-degenerate pre-polarized triple. For example, the pushforward of a $(-1)^w$-Hermitian pair $(\mathscr{M}, \mathfrak{c})$ is the graded $(-1)^w$-Hermitian pair $({}_{\mathrm{D}}f_*^{\bullet}\mathscr{M}, {}_{\mathrm{T}}f_*^{\bullet}\mathfrak{c})$, whose associated graded triple is $\bigoplus_k ({}_{\mathrm{D}}f_*^k \mathscr{M}, {}_{\mathrm{D}}f_*^{-k}\mathscr{M}(w), {}_{\mathrm{T}}f_*^k\mathfrak{c})$ and pre-polarization $((-1)^w \mathrm{Id}, \mathrm{Id})$ regarded, in each degree k, as an isomorphism

$$\begin{aligned}({}_{\mathrm{D}}f_*^k \mathscr{M}, {}_{\mathrm{D}}f_*^{-k}\mathscr{M}(w), {}_{\mathrm{T}}f_*^k\mathfrak{c}) \xrightarrow{\sim} ({}_{\mathrm{D}}f_*^{-k} \mathscr{M}, {}_{\mathrm{D}}f_*^{k}\mathscr{M}(w), {}_{\mathrm{T}}f_*^{-k}\mathfrak{c})^*(-w)\\ = ({}_{\mathrm{D}}f_*^k \mathscr{M}, {}_{\mathrm{D}}f_*^{-k}\mathscr{M}(w), {}_{\mathrm{T}}f_*^k\mathfrak{c}).\end{aligned}$$

Similarly, if $(\mathscr{M}, \mathfrak{c})$ is a $(-1)^w$-Hermitian pair, then $\psi_{g,\lambda}(\mathscr{M}, \mathfrak{c}) = (\psi_{g,\lambda}\mathscr{M}, \psi_{g,\lambda}\mathfrak{c})$ is a $(-1)^{w-1}$-Hermitian pair equipped with a skew-adjoint nilpotent endomorphism $\mathrm{N} : \psi_{g,\lambda}\mathscr{M} \to \psi_{g,\lambda}\mathscr{M}(-1)$. Moreover, for every $\ell \geqslant 0$, the pair $(\mathrm{P}_\ell\psi_{g,\lambda}\mathscr{M}, \mathrm{P}_\ell\psi_{g,\lambda}\mathfrak{c})$ is a $(-1)^{w-1+\ell}$-Hermitian pair.

***Example 12.2.7* (The basic example, continued).** $(\widetilde{\omega}_X, \mathfrak{c})$ is a $(-1)^n$-Hermitian pair, with associated triple ${}_{\mathrm{H}}\omega_X$ and pre-polarization $\mathrm{Q} = ((-1)^n \mathrm{Id}, \mathrm{Id})$ of weight n. Since we have ${}_{\mathrm{H}}\omega_X^*(-n) = (\widetilde{\omega}_X, \widetilde{\omega}_X(n), (-1)^n\mathfrak{c})$ (recall that $\mathfrak{c}^* = (-1)^n\mathfrak{c}$), we regard the pre-polarization of weight n as a morphism $\mathrm{Q} : {}_{\mathrm{H}}\omega_X \xrightarrow{\sim} {}_{\mathrm{H}}\omega_X^*(-n)$.

12.2.b. The case of $R_F\mathscr{D}$-$\mathsf{Triples}(X)^{\mathrm{left}}$

Side-changing. Given an object M of $R_F\mathscr{D}$-$\mathsf{Triples}(X)^{\mathrm{left}}$, we obtain an object of $R_F\mathscr{D}$-$\mathsf{Triples}(X)^{\mathrm{right}}$ by setting

$$M^{\mathrm{right}} := {}_{\mathrm{H}}\omega_X \otimes M,$$

where the tensor product is defined termwise in an obvious way (for the sesquilinear pairing, we use (10.2.5 $*$)), that is,

$$(\mathscr{M}', \mathscr{M}'', \mathfrak{c})^{\mathrm{right}} := (\mathscr{M}'^{\mathrm{right}}, \mathscr{M}''^{\mathrm{right}}(n), \mathfrak{c}^{\mathrm{right}}).$$

The right-to-left transformation is obtained similarly. The functors on the category $R_F\mathscr{D}$-$\mathsf{Triples}(X)^{\mathrm{left}}$ are defined in a way compatible with those on $R_F\mathscr{D}$-$\mathsf{Triples}(X)^{\mathrm{right}}$ through this side-changing operation.

***Example 12.2.8* (Variations of Hodge structures as objects of $R_F\mathscr{D}$-$\mathsf{Triples}(X)^{\mathrm{left}}$)**

Let $H = \big((\mathcal{H}', \nabla, F^\bullet\mathcal{H}'), (\mathcal{H}'', \nabla, F^\bullet\mathcal{H}''), \mathfrak{c}\big)$ be a variation of Hodge structure of weight w as in Definition 4.1.10. Set $\mathscr{H}' := R_F\mathcal{H}'$, $\mathscr{H}'' := R_F\mathscr{H}''$, and equip them with their natural left $R_F\mathscr{D}_X$-module structure induced by ∇ (due to the Griffiths transversality property). Then $(\mathscr{H}', \mathscr{H}'', \mathfrak{c})$ is an object of $R_F\mathscr{D}$-$\mathsf{Triples}(X)^{\mathrm{left}}$. By the side-changing functor defined above, we obtain the object $(\mathscr{H}', \mathscr{H}'', \mathfrak{c})^{\mathrm{right}}$ of $R_F\mathscr{D}$-$\mathsf{Triples}(X)^{\mathrm{right}}$.

Proper pushforward. The proper pushforward is defined with a shifted grading (when compared with the right case). For $f : X \to Y$ proper and M object of $R_F\mathscr{D}$-$\mathsf{Triples}(X)^{\mathrm{left}}$, we set in degree k:

$${}_{\mathrm{H}}f_*^{n-m+k}M = ({}_{\mathrm{D}}f_*^{n-m+k}\mathscr{M}', {}_{\mathrm{D}}f_*^{n-m-k}\mathscr{M}''(m-n), {}_{\mathrm{T}}f_*^{n-m+k}\mathfrak{c}),$$

by using (10.3.23 $*$) for the sesquilinear pairing. Since, by Definition A.8.22, $({}_{\mathrm{D}}f_*^{n-m+k}\mathscr{M})^{\mathrm{right}} = {}_{\mathrm{D}}f_*^k(\mathscr{M}^{\mathrm{right}})$ and, by (10.3.23 $**$), $({}_{\mathrm{T}}f_*^{n-m+k}\mathfrak{c})^{\mathrm{right}} = {}_{\mathrm{T}}f_*^k(\mathfrak{c}^{\mathrm{right}})$, we deduce that

$$({}_{\mathrm{H}}f_*^{n-m+k}M)^{\mathrm{right}} = {}_{\mathrm{H}}f_*^{k}(M^{\mathrm{right}}).$$

The compatibility with adjunction analogue of (12.2.1) is now given by the isomorphism

$$((-1)^{n-m}\,\mathrm{Id},\mathrm{Id}) : {}_{\mathrm{H}}f_*^{n-m+k}(M^*) \xrightarrow{\sim} ({}_{\mathrm{H}}f_*^{n-m-k}M)^*(n-m).$$

The Lefschetz operator $\mathrm{L}_{\mathscr{L}} : {}_{\mathrm{H}}f_*^{n-m+k}M \to {}_{\mathrm{H}}f_*^{n-m+k+2}M(1)$ is defined as in the right case, and is compatible with side-changing.

Strict $\mathbb{R}$-specializability and strict S-decomposability. Let $M = (\mathscr{M}', \mathscr{M}'', \mathfrak{c})$ be an object of $R_F\mathscr{D}\text{-}\mathsf{Triples}(X)^{\mathrm{left}}_{\mathrm{coh}}$ which is strictly $\mathbb{R}$-specializable along (g). We then set, for $\lambda \in S^1$,

$$\psi_{g,\lambda}M := (\psi_{g,\lambda}\mathscr{M}', \psi_{g,\lambda}\mathscr{M}'', \psi_{g,\lambda}\mathfrak{c}), \tag{12.2.9}$$

where $\psi_{g,\lambda}\mathscr{M}$ is given by Remark 7.4.5 and $\psi_{g,\lambda}\mathfrak{c}$ by (10.4.34 $*$). By the side-changing property for $\psi_{g,\lambda}\mathscr{M}$ (Lemma 7.4.6) and for $\psi_{g,\lambda}\mathfrak{c}$ (10.4.35 $*$), we obtain

$$\psi_{g,\lambda}(M^{\mathrm{right}}) = (\psi_{g,\lambda}M)^{\mathrm{right}}.$$

The action of N is compatible with the side-changing.

12.3. Definition and properties of pure Hodge modules

The notion of a (polarized) pure Hodge module will be defined by induction on the dimension of the support.

Let X be a complex analytic manifold and let $w \in \mathbb{Z}$. We will define by induction on $d \in \mathbb{N}$ the category $\mathsf{HM}_{\leqslant d}(X, w)$ of *Hodge modules of weight w on X, having a support of dimension $\leqslant d$.* This will be a subcategory of the category $R_F\mathscr{D}\text{-}\mathsf{Triples}(X)$ introduced in Section 12.2.

12.3.a. Right pure Hodge modules.

We first start with the definition of $\mathsf{HM}_{\leqslant d}(X, w)^{\mathrm{right}}$.

***Definition 12.3.1* (Pure Hodge modules (right case)).** The category $\mathsf{HM}_{\leqslant d}(X, w)^{\mathrm{right}}$ is the *full subcategory* of $R_F\mathscr{D}\text{-}\mathsf{Triples}(X)^{\mathrm{right}}$ for which the objects are triples $M = (\mathscr{M}', \mathscr{M}'', \mathfrak{c})$ satisfying:

(HSD) $\mathscr{M}', \mathscr{M}''$ are *holonomic*, *strictly S-decomposable*, and have support of dimension $\leqslant d$.

($\mathrm{HM}_{>0}$) For any open set $U \subset X$ and any holomorphic function $g : U \to \mathbb{C}$, for every $\lambda \in S^1$ and any integer ℓ, the triple

$$\mathrm{gr}^{\mathrm{M}}_{\ell}\psi_{g,\lambda}M = \Big(\mathrm{gr}^{\mathrm{M}}_{\ell}\psi_{g,\lambda}(\mathscr{M}'), \mathrm{gr}^{\mathrm{M}}_{-\ell}\psi_{g,\lambda}(\mathscr{M}''), \mathrm{gr}^{\mathrm{M}}_{\ell}\psi_{g,\lambda}\mathfrak{c}\Big),$$

a priori defined as an object of $R_F\mathscr{D}\text{-}\mathsf{Triples}(X)$, is an object of $\mathsf{HM}_{\leqslant d-1}(U, w-1+\ell)$.

(HM_0) For any zero-dimensional strict component $\{x_o\}$ of $\mathscr{M}'$ or $\mathscr{M}''$, we have

$$(\mathscr{M}'_{\{x_o\}}, \mathscr{M}''_{\{x_o\}}, \mathfrak{c}_{\{x_o\}}) = {}_{\mathrm{H}}\iota_{\{x_o\}*}(\mathscr{H}', \mathscr{H}'', \mathfrak{c}_o)$$

where $(\mathscr{H}', \mathscr{H}'', \mathfrak{c}_o)$ is a $\mathbb{C}$-Hodge triple of w (see Definition 2.4.28).

Let us justify all understatements made in the definition of the category $\mathsf{HM}(X,w)$. We note that we have used the $\psi_{g,\lambda}$ functor in the category $R_F\mathscr{D}$-$\mathsf{Triples}(X)$. Remark first:

Proposition 12.3.2. *If $M=(\mathscr{M}',\mathscr{M}'',\mathfrak{c})$ is an object of $\mathsf{HM}_{\leqslant d}(X,w)$, then for $\mathscr{M}=\mathscr{M}'$ or $\mathscr{M}''$, $\mathscr{M}$ is* strict*, as well as $\psi_{g,\lambda}\mathscr{M}$ and $\phi_{g,1}\mathscr{M}$ for any analytic germ g, any $\lambda\in S^1$ and any $\ell\in\mathbb{Z}$. Moreover, $\mathrm{gr}^V_\alpha({}_{\mathrm{D}}\iota_{g*}\mathscr{M})$ is strict for every $\alpha\in\mathbb{R}$.*

By the strictness property, $\mathscr{M}',\mathscr{M}''$ are the Rees modules of some coherent filtered holonomic $\mathscr{D}_X$modules $(\mathcal{M}',F^\bullet\mathcal{M}'),(\mathcal{M}'',F^\bullet\mathcal{M}'')$. In this way, we obtain higher dimensional analogues of the filtered triples of Remark 2.4.25(3).

Proof. Set $\mathscr{M}=\mathscr{M}'$ or $\mathscr{M}''$. The strictness of $\mathscr{M}$ follows from (HSD), after Corollary 7.7.14. The strictness of $\mathrm{gr}^{\mathrm{M}}_\ell\psi_{g,\lambda}\mathscr{M}$ for $\lambda\in S^1$ is by definition, since the objects of $\mathsf{HM}_{\leqslant d-1}(U,w-1+\ell)$ are strict, according to (HSD). The strictness of $\psi_{g,\lambda}\mathscr{M}$ is then clear by extension (or because $\mathscr{M}$ is strictly S-decomposable). To get the strictness of $\mathrm{gr}^V_\alpha({}_{\mathrm{D}}\iota_{g*}\mathscr{M})$ for every $\alpha\notin\mathbb{N}$, use 7.3.28(c) and (d).

Let us show the strictness of $\phi_{g,1}\mathscr{M}$ (hence that of $\mathrm{gr}^V_\alpha({}_{\mathrm{D}}\iota_{g*}\mathscr{M})$ for every $\alpha\in\mathbb{N}$, according to 7.3.28(d)). We can assume that $\mathscr{M}$ has pure support, according to (HSD). If $g\equiv0$ on the support of $\mathscr{M}$, then the strictness of $\phi_{g,1}\mathscr{M}$ is a consequence of the strictness of $\mathscr{M}$, by Kashiwara's equivalence 7.6.2. Otherwise, we know by strict the pure support condition along g (see Proposition 7.7.2(1)) that $\mathrm{var}:\phi_{g,1}\mathscr{M}\to\psi_{g,1}\mathscr{M}(-1)$ is injective, hence the strictness of $\phi_{g,1}\mathscr{M}$ is a consequence of that of $\psi_{g,1}\mathscr{M}$. □

We note also that we have locally finite strict S-decompositions $\mathscr{M}'=\oplus_{Z'}\mathscr{M}'_{Z'}$ and $\mathscr{M}''=\oplus_{Z''}\mathscr{M}''_{Z''}$ where Z' belongs to the set of strict irreducible components of $\mathscr{M}'$ and Z'' to that of $\mathscr{M}''$. For any open set $U\subset X$, the irreducible components of all $Z'\cap U$ form the set of strict components of $\mathscr{M}'_{|U}$, and similarly for $\mathscr{M}''$. For $g:U\to\mathbb{C}$, we have $\psi_{g,\lambda}\mathscr{M}'_{Z'_U}=0$ for every $\lambda\in S^1$ if g vanishes identically on the strict component Z'_U of $\mathscr{M}'_{|U}$, and has support of codimension one in Z'_U otherwise. The support of $\psi_{g,\lambda}\mathscr{M}'_{|U}$ has therefore dimension $\leqslant d-1$.

According to Remark 7.7.16, the strict components of $\mathcal{M}'=\mathscr{M}'/(z-1)\mathscr{M}'$ are those of $\mathscr{M}'$, and similarly for $\mathscr{M}''$. Moreover, by Proposition 10.4.16, the component $\mathfrak{c}_{Z',Z''}$ of $\mathfrak{c}$ on $\overline{\mathcal{M}''_{Z''}}\otimes_{\mathbb{C}}\mathcal{M}'_{Z'}$ vanishes unless $Z'=Z''$. We denote therefore by $\mathfrak{c}_Z$ the component of $\mathfrak{c}$ when $Z=Z'=Z''$ is a common strict component of $\mathscr{M}'$ and $\mathscr{M}''$. We thus have an S-decomposition

$$(\mathscr{M}',\mathscr{M}'',\mathfrak{c})=\oplus_Z(\mathscr{M}'_Z,\mathscr{M}''_Z,\mathfrak{c}_Z)\tag{12.3.3}$$

indexed by the set of strict components of $\mathscr{M}'$ or $\mathscr{M}''$. We will see below (Corollary 12.3.5) that the set of strict components is the same for $\mathscr{M}'$ and $\mathscr{M}''$, and that each $(\mathscr{M}'_Z,\mathscr{M}''_Z,\mathfrak{c}_Z)$ is a pure Hodge module of weight w.

With such a notation, (HM_0) is concerned with the zero-dimensional strict components, which are not seen by $(\mathrm{HM}_{>0})$. Let us choose local coordinates $x_1,\dots,x_n$

at x_o. Then (HM_0) says that

$$\mathscr{M}'_{\{x_o\}} = \mathscr{H}' \otimes_{\mathbb{C}} \mathbb{C}[\eth_{x_1}, \dots, \eth_{x_n}], \quad \mathscr{M}''_{\{x_o\}} = \mathscr{H}'' \otimes_{\mathbb{C}} \mathbb{C}[\eth_{x_1}, \dots, \eth_{x_n}],$$

and $\mathfrak{c}_{\{x_o\}}$ is obtained by $\mathscr{D}_{(\overline{X},X)}$-linearity from its restriction to $\mathcal{H}' \otimes_{\mathbb{C}} \overline{\mathcal{H}''}$. There, it is equal to $\mathfrak{c}_o \cdot \delta_{x_o}$, where δ_{x_o} denotes the Dirac current at x_o and $\mathfrak{c}_o : \mathcal{H}' \otimes_{\mathbb{C}} \overline{\mathcal{H}''} \to \mathbb{C}$ satisfies the w-opposedness property of Definition 2.4.27 with respect to the filtrations on $\mathcal{H}', \mathcal{H}''$ determined by the strict objects $\mathscr{H}', \mathscr{H}''$.

It is easy to see now that the set of zero-dimensional strict components is the same for $\mathscr{M}'$ and $\mathscr{M}''$: if $\{x_o\}$ is not a strict component of $\mathscr{M}''$ for example, then $\mathscr{M}''_{\{x_o\}} = 0$ and thus $\mathscr{H}'' = 0$. As $\mathfrak{c}_o$ is non-degenerate, this implies that $\mathscr{H}' = 0$, therefore $\mathscr{M}'_{\{x_o\}} = 0$ and $\{x_o\}$ is not a strict component of $\mathscr{M}'$.

We will now give the basic properties of right pure Hodge modules.

12.3.b. Locality. For any open set $U \subset X$, there exists a natural restriction functor

$$\mathsf{HM}_{\leqslant d}(X,w) \xrightarrow{\rho_U} \mathsf{HM}_{\leqslant d}(U,w).$$

Moreover, if M is any object of $R_F\mathscr{D}$-$\mathsf{Triples}(X)$ such that, for every open set U of a covering of X, $M_{|U}$ is an object of $\mathsf{HM}_{\leqslant d}(U,w)$, then M is an object of the category $\mathsf{HM}_{\leqslant d}(X,w)$.

12.3.c. Stability by direct summand

Proposition 12.3.4. *If $M = M_1 \oplus M_2$ is an object of $\mathsf{HM}_{\leqslant d}(X,w)$, then each M_i ($i = 1,2$) also.*

Proof. The property of holonomicity restricts to direct summands, as well as the property of strict specializability (Exercise 7.3.31(1)) and the property of strict S-decomposability (Lemma 7.7.8(2)). Then we argue by induction on d for ($\mathrm{HM}_{>0}$). For (HM_0), we use Lemma 2.4.31. ☐

Corollary 12.3.5. *If $M = (\mathscr{M}', \mathscr{M}'', \mathfrak{c})$ is an object of the category $\mathsf{HM}_{\leqslant d}(X,w)$, then the strict components of $\mathscr{M}'$ and $\mathscr{M}''$ are the same and the S-decomposition* (12.3.3) *holds in $\mathsf{HM}_{\leqslant d}(X,w)$. Moreover, $\mathsf{HM}_{\leqslant d}(X,w)$ is the direct sum of the full subcategories $\mathsf{HM}_Z(X,w)$ consisting of objects having pure support on the irreducible closed analytic subset $Z \subset X$ of dimension $\leqslant d$.*

Proof. We assume that there is a strict component Z' of $\mathscr{M}'$ which is not a strict component of $\mathscr{M}''$. Then we have an object $(\mathscr{M}'_{Z'}, 0, 0)$ in $\mathsf{HM}_{\leqslant d}(X,w)$, according to the previous proposition. We wish to show that $\mathscr{M}'_{Z'} = 0$, and it is enough, by the condition of the pure support, to show the vanishing on the smooth locus of Z'. We can thus reduce to the case where $Z' = X$, according to Proposition 7.7.10.

We now argue by induction on $\dim X$, the case $\dim X = 0$ having been treated above. Let t be a local coordinate on X. Arguing as in Corollary 7.7.14, one checks that $\mathscr{M}'_X/t\mathscr{M}'_X = \psi_{t,1}\mathscr{M}'_X$, and that $\psi_{t,\lambda}\mathscr{M}'_X = 0$ for $\lambda \in S^1 \smallsetminus \{1\}$, as well as $\phi_{t,1}\mathscr{M}'_X = 0$. It follows that $\mathrm{N} = 0$, so $\psi_{t,1}\mathscr{M}'_X$ is strictly S-decomposable, according to (HSD). By induction, the object $\psi_{t,1}(\mathscr{M}'_X, 0, 0)$ is zero. Hence $\mathscr{M}'_X/t\mathscr{M}'_X = 0$, and by applying Nakayama's lemma as in Corollary 7.7.14, we obtain $\mathscr{M}_X = 0$.

The remaining statement is easy. □

12.3.d. Kashiwara's equivalence. Let ι_Y denote the inclusion of Y as a closed analytic submanifold of the analytic manifold X and let ${}_{\mathrm{H}}\iota^0_{Y*}$ be the pushforward functor for $R_F\mathscr{D}$-triples.

Proposition 12.3.6. *The functor ${}_{\mathrm{H}}\iota^0_{Y*}$ induces an equivalence between $\mathsf{HM}(Y, w)$ and $\mathsf{HM}_Y(X, w)$ (objects supported on Y).*

Proof. By locality (Section 12.3.b), the result is local and it is enough to check the case of a smooth hypersurface $Y \times \{0\} \subset Y \times \mathbb{C} = X$. Moreover, the point is to prove essential surjectivity. We prove it for $\mathsf{HM}_{\leqslant d}(Y, w)$ and $\mathsf{HM}_{Y,\leqslant d}(X, w)$ by induction on d. The case $d = 0$ is by definition. For $d \geqslant 1$, we note that the functor ${}_{\mathrm{H}}\iota^0_{Y*}$ and its inverse functor preserve (HSD), by Proposition 7.6.5 and Proposition 7.7.10. Then $(\mathrm{HM})_{>0}$ holds by induction on d and Proposition 7.6.5. □

12.3.e. Generic structure of Hodge modules

Proposition 12.3.7. *Let M be an object of $\mathsf{HM}(X, w)^{\mathrm{left}}$ having pure support on the irreducible closed analytic set $Z \subset X$. Then there exists an open dense set $Z^o \subset Z$ and a variation of Hodge structure H of weight $w - \operatorname{codim} Z$ on Z^o, such that $M_{|Z^o} = {}_{\mathrm{H}}\iota^{-\operatorname{codim} Z}_{Z^o*} H$. In particular, if $Z = X$, then $M_{|X^o}$ is a variation of Hodge structure of weight w.*

Note that we use Definition 4.1.10 for a variation of Hodge structure, in order to have an object similar to the object M.

Proof. Set $M = (\mathscr{M}', \mathscr{M}'', \mathfrak{c})$. Restrict first to a smooth open set of Z and apply Kashiwara's equivalence to reduce to the case when $Z = X$. On some dense open set X^o of X, the characteristic variety of $\mathscr{M}'$ and $\mathscr{M}''$ is contained in the zero section. By Exercise A.10.16 and Proposition 7.7.10 (that we can apply because of (HSD)), $\mathscr{M}'_{|X^o}$ and $\mathscr{M}''_{|X^o}$ are $\widetilde{\mathscr{O}}_{X^o}$-locally free of finite rank. Then, $\mathfrak{c}_{|X^o}$ takes values in $\mathscr{C}^\infty_{|X^o}$ (see Lemma 10.2.6). We now restrict to X^o.

Let t be a local coordinate. Then $\mathrm{gr}^0_V\mathscr{M} = \mathscr{M}/t\mathscr{M}$ for $\mathscr{M} = \mathscr{M}', \mathscr{M}''$. After Examples 10.4.36 and 10.4.37, $\mathrm{gr}^0_V\mathfrak{c}$ is the restriction of $\mathfrak{c}$ to $t = 0$ as a C^∞ function. We conclude that $\psi_{t,1}M$ is the pushforward ${}_{\mathrm{T}}\iota^{-1}_{*} M_{|t=0}$. It is also pure of weight $w - 1$ since N is easily seen to be zero. Therefore, $M_{|t=0}$ is pure of weight w and, by induction on $\dim X$, is a variation of Hodge structure of weight w. The assertion follows. □

12.3.f. Morphisms

Proposition 12.3.8. *There is no nonzero morphism (in $R_F\mathscr{D}$-$\mathsf{Triples}(X)$) from an object in the category $\mathsf{HM}(X, w)$ to an object in $\mathsf{HM}(X, w')$ if $w > w'$.*

Proof. Let $\varphi : M_1 \to M_2$ be such a morphism. According to Corollary 12.3.5, we can assume that both have the irreducible closed analytic set Z as their pure support. As the result is clear for smooth Hodge structures (see Proposition 2.4.5(2)), it follows from Proposition 12.3.7 that the support of $\operatorname{Im}\varphi$ is strictly smaller than Z. By definition of the pure support (see Definition 7.7.9), this implies that $\operatorname{Im}\varphi = 0$. $\square$

Proposition 12.3.9. *The category* $\mathsf{HM}(X, w)$ *is abelian, all morphisms are strict and strictly specializable.*

Proof. Let us introduce the subcategory $\mathsf{WHM}(X)$ of $R_F\mathscr{D}$-$\mathsf{Triples}(X)$, the objects of which are triples with a finite filtration $W_\bullet$ indexed by $\mathbb{Z}$ such that, for every ℓ, gr_ℓ^W is in $\mathsf{HM}(X, \ell)$. The morphisms in $\mathsf{WHM}(X)$ are the morphisms of $R_F\mathscr{D}$-$\mathsf{Triples}(X)$ which respect the filtration W. Let us consider both properties:

(a_d) $\mathsf{HM}_{\leqslant d}(X, w)$ abelian, all morphisms are strict and strictly $\mathbb{R}$-specializable;

(b_d) $\mathsf{WHM}_{\leqslant d}(X)$ abelian and morphisms are strict and strictly compatible with the filtration W.

Let us start with (a_0). Abelianity and strictness follow from Kashiwara's equivalence 12.3.6 since, by Exercise 2.4.3(2), the same properties hold in dimension zero. By using Proposition 7.6.5, one checks that strict $\mathbb{R}$-specializability of morphisms holds if and only if it holds in dimension zero, where the property is trivially true.

(a_d) $\Rightarrow$ (b_d). We note first that, by Proposition 12.3.2 and Lemma A.2.9(1), the $\widetilde{\mathscr{D}}_X$-modules which are components of an object in $\mathsf{WHM}_{\leqslant d}(X)$ are strict. According to Propositions 12.3.8 and A.2.10, (a_d) implies that the category $\mathsf{WHM}_{\leqslant d}(X)$ is abelian and that morphisms are strictly compatible with W. Using Lemma A.2.9(2), we conclude that all morphisms are strict.

(b_{d-1}) $\Rightarrow$ (a_d) for $d \geqslant 1$. The question is local. Let $\varphi = (\varphi', \varphi'') : M_1 \to M_2$ be a morphism of pure Hodge modules of weight w. According to Proposition 12.3.8, we can assume that all the $\widetilde{\mathscr{D}}_X$-modules involved have pure support Z (closed irreducible analytic subset of X) of dimension d. We will first show that $\operatorname{Ker}\varphi$ and $\operatorname{Coker}\varphi$ are also strictly $\mathbb{R}$-specializable, S-decomposable and have pure support Z.

Let g be the germ of an analytic function not vanishing identically on Z. Then, setting $W_\bullet := \mathrm{M}[w-1]_\bullet$, $\big(\psi_{g,\lambda}M_1, W_\bullet\big), \big(\psi_{g,\lambda}M_2, W_\bullet\big)$ are objects of $\mathsf{WHM}_{\leqslant d-1}(X)$ for $\lambda \in S^1$ by definition, so (b_{d-1}) implies that $\psi_{g,\lambda}\varphi', \psi_{g,\lambda}\varphi''$ are strict. We will show below that

- $\phi_{g,1}\varphi'$ and $\phi_{g,1}\varphi''$ are strict (hence so are $\operatorname{gr}_\alpha^V {}_{\mathrm{D}}\iota_{g*}\varphi', \operatorname{gr}_\alpha^V {}_{\mathrm{D}}\iota_{g*}\varphi''$ for every $\alpha \in \mathbb{R}$, according to 7.3.24(3) and (4)),
- can is onto for $\operatorname{Ker}\varphi'$ and $\operatorname{Ker}\varphi''$, and
- var is injective for $\operatorname{Coker}\varphi'$ and $\operatorname{Coker}\varphi''$.

The first assertion will be enough to show that φ' and φ'' are strictly $\mathbb{R}$-specializable, hence $\operatorname{Ker}\varphi', \dots, \operatorname{Coker}\varphi''$ are also strictly $\mathbb{R}$-specializable (Proposition 7.3.33). The two other assertions will insure that these modules satisfy Properties 7.7.2(1) and (2), hence are strictly S-decomposable along $\{t=0\}$ and have neither sub nor quotient module supported on $Z\cap\{g=0\}$. Applying this for any such g implies that $\operatorname{Ker}\varphi',\dots,\operatorname{Coker}\varphi''$ are strictly S-decomposable and have pure support Z. Now, $\operatorname{Ker}\varphi',\dots,\operatorname{Coker}\varphi''$ are clearly holonomic, hence they are also strict (see Corollary 7.7.14). We now have obtained that φ is strict and strictly specializable.

Let us come back to the proof of the previous three assertions. As var is injective for $\mathscr{M}'$ and $\mathscr{M}''$, we identify $\phi_{g,1}\varphi'$ to the restriction of $\psi_{g,1}\varphi'$ on $\operatorname{Im}\mathrm{N}\subset\psi_{g,1}\mathscr{M}'_1(-1)$, and similarly for φ''. By the inductive assumption, the morphism

$$\mathrm{N}:\big(\psi_{g,1}M_k,\mathrm{M}_\bullet(\mathrm{N})\big)\longrightarrow\big(\psi_{g,1}M_k(-1),\mathrm{M}[2]_\bullet(\mathrm{N})\big)$$

is strict, for $k=1,2$ (we know that it is strictly compatible with the filtration M, according to Lemma 3.1.15. Moreover, $\operatorname{Im}\mathrm{N}$ is an object of $\mathsf{WHM}_{\leqslant d-1}(X)$. Note also that, according to Lemma 3.1.15, we have $\mathrm{N}(\mathrm{M}_\bullet\psi_{g,1}M_k)=\mathrm{M}[1]_\bullet(\mathrm{N}_{|\operatorname{Im}\mathrm{N}})$. Using once more this inductive assumption, the restriction of $\psi_{g,1}\varphi$ on $\operatorname{Im}\mathrm{N}$ is strict, hence the first point.

In order to show the other assertions, consider the following diagram of exact sequences (and the similar diagram for φ''):

$$\begin{tikzcd}
0 \arrow[r] & \psi_{g,1}\operatorname{Ker}\varphi' \arrow[r] \arrow[d, "\mathrm{can}"] & \psi_{g,1}\mathscr{M}'_1 \arrow[r, "\psi_{g,1}\varphi'"] \arrow[d, two heads, "\mathrm{can}"] \arrow[dd, bend right, "\mathrm{N}_1"'] & \psi_{g,1}\mathscr{M}'_2 \arrow[r] \arrow[d, two heads, "\mathrm{can}"] \arrow[dd, bend right, "\mathrm{N}_2"'] & \psi_{g,1}\operatorname{Coker}\varphi' \arrow[r] \arrow[d, two heads, "\mathrm{can}"] & 0\\
0 \arrow[r] & \phi_{g,1}\operatorname{Ker}\varphi' \arrow[r] \arrow[d, hook, "\mathrm{var}"] & \phi_{g,1}\mathscr{M}'_1 \arrow[r, "\phi_{g,1}\varphi'"] \arrow[d, hook, "\mathrm{var}"] & \phi_{g,1}\mathscr{M}'_2 \arrow[r] \arrow[d, hook, "\mathrm{var}"] & \phi_{g,1}\operatorname{Coker}\varphi' \arrow[r] \arrow[d, "\mathrm{var}"] & 0\\
0 \arrow[r] & \psi_{g,1}\operatorname{Ker}\varphi' \arrow[r] & \psi_{g,1}\mathscr{M}'_1 \arrow[r, "\psi_{g,1}\varphi'"] & \psi_{g,1}\mathscr{M}'_2 \arrow[r] & \psi_{g,1}\operatorname{Coker}\varphi' \arrow[r] & 0
\end{tikzcd}$$

where the shift (-1) of the lower line is omitted in the notation, for simplicity. We have to prove that the left up can is onto and that the right down var is injective. This amounts to showing that $\operatorname{Im}\mathrm{N}_1\cap\operatorname{Ker}\psi_{g,1}\varphi'=\mathrm{N}_1(\operatorname{Ker}\psi_{g,1}\varphi')$ (because this is equivalent to $\operatorname{Im}\mathrm{can}\cap\operatorname{Ker}\phi_{g,1}\varphi'=\mathrm{can}(\operatorname{Ker}\phi_{g,1}\varphi')$) and $\operatorname{Im}\mathrm{N}_2\cap\operatorname{Im}\psi_{g,1}\varphi'=\mathrm{N}_2(\operatorname{Im}\psi_{g,1}\varphi')$. This follows from Lemma 3.1.7 applied to the germs of the various sheaves.

To end the proof of $(\mathrm{b}_{d-1})\Rightarrow(\mathrm{a}_d)$, it remains to be proved that $\operatorname{Ker}\varphi$ and $\operatorname{Coker}\varphi$ satisfy $(\mathrm{HM}_{>0})$. It follows from the abelianity of $\mathsf{WHM}_{\leqslant d-1}(X)$ and from the strict $\mathbb{R}$-specializability of φ that $\psi_{g,\lambda}\operatorname{Ker}\varphi$ and $\psi_{g,\lambda}\operatorname{Coker}\varphi$ (with $\lambda\in S^1$) are in $\mathsf{WHM}_{\leqslant d-1}(X)$ and, as we have seen in Lemma 3.1.7, the weight filtration is the monodromy filtration. This gives $(\mathrm{HM}_{>0})$, concluding the proof of Proposition 12.3.9. $\square$

Corollary 12.3.10. *Given any morphism $\varphi:M_1\to M_2$ between objects of $\mathsf{HM}(X,w)$ and any germ g of holomorphic function on X, then, for every $\lambda\in S^1$, the specialized morphism $\psi_{g,\lambda}\varphi$ is strictly compatible with the monodromy filtration $\mathrm{M}_\bullet$ and, for every*

$\ell \in \mathbb{Z}$, $\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}\varphi$ *decomposes with respect to the Lefschetz decomposition, i.e.,*

$$\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}\varphi = \begin{cases} \bigoplus_{k\geqslant 0} \mathrm{N}^k \mathrm{P}_{\ell+2k}\psi_{g,\lambda}\varphi & (\ell \geqslant 0), \\ \bigoplus_{k\geqslant 0} \mathrm{N}^{k-\ell} \mathrm{P}_{-\ell+2k}\psi_{g,\lambda}\varphi & (\ell \leqslant 0). \end{cases}$$

In particular we have

$$\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}\operatorname{Ker}\varphi = \operatorname{Ker}\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}\varphi$$

and similarly for Coker*, where, on the left side, the filtration* $\mathrm{M}_\bullet$ *is that induced naturally by* $\mathrm{M}_\bullet\psi_{g,\lambda}M_1$ *or, equivalently, the monodromy filtration of* N *acting on* $\psi_{g,\lambda}\operatorname{Ker}\varphi = \operatorname{Ker}\psi_{g,\lambda}\varphi$. □

Corollary 12.3.11. *If* M *is in* $\mathsf{HM}_{\leqslant d}(X,w)$*, then the Lefschetz decomposition for* $\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}M$ *(with* $\lambda\in S^1$*) holds in* $\mathsf{HM}_{\leqslant d-1}(X,w-1+\ell)$*.*

Proof. Indeed, $\mathrm{N}:\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}M \to \mathrm{gr}_{\ell-2}^{\mathrm{M}}\psi_{g,\lambda}M(-1)$ is a morphism in the category $\mathsf{HM}_{\leqslant d-1}(X,w-1+\ell)$, which is abelian, so the primitive part is an object of this category, and therefore each term of the Lefschetz decomposition is also an object of this category. □

12.3.g. Vanishing cycles. Let M be an object of $\mathsf{HM}_{\leqslant d}(X,w)$. By definition, for any locally defined analytic function g, the object $(\psi_{g,1}M, \mathrm{M}[w-1]_\bullet(\mathrm{N}))$ is an object of $\mathsf{WHM}_{\leqslant d}(X)$.

Corollary 12.3.12 **(Vanishing cycles).** *For such an* M*, the object* $(\phi_{g,1}M, \mathrm{M}[w]_\bullet(\mathrm{N}))$ *is in* $\mathsf{WHM}_{\leqslant d}(X)$*. Moreover, the morphisms* can, var *are filtered morphisms*

$$(\psi_{g,1}M, \mathrm{M}[w-1]_\bullet(\mathrm{N})) \xrightarrow{\text{can}} (\phi_{g,1}M, \mathrm{M}[w]_\bullet(\mathrm{N}))$$
$$(\phi_{g,1}M, \mathrm{M}[w]_\bullet(\mathrm{N})) \xrightarrow{\text{var}} (\psi_{g,1}M(-1), \mathrm{M}[w+1]_\bullet(\mathrm{N})),$$

hence are morphisms in $\mathsf{WHM}(X)$*, and similarly for* $\mathrm{gr}_{-1}^{\mathrm{M}}\mathrm{can}$ *and* $\mathrm{gr}_{-1}^{\mathrm{M}}\mathrm{var}$*.*

Proof. We can assume that M has pure support an irreducible closed analytic subset Z of X. If $g\equiv 0$ on Z, then the result follows from Kashiwara's equivalence.

Assume now that $g\not\equiv 0$ on Z. The object $\phi_{g,1}M$ is equipped with the filtration $W_\bullet\phi_{g,1}M$ naturally induced by $\mathrm{M}[w-1]_\bullet(\mathrm{N})\psi_{g,1}M$. As such, it is identified with the image of $\mathrm{N}:\big(\psi_{g,1}M, \mathrm{M}[w-1]_\bullet(\mathrm{N})\big)\to\big(\psi_{g,1}M(-1), \mathrm{M}[w+1]_\bullet(\mathrm{N})\big)$, hence is an object of $\mathsf{WHM}(X)$, because this category is abelian.

The result now follows from Lemma 3.1.15, which gives in particular that $W_\bullet\phi_{g,1}M = \mathrm{M}[w]_\bullet(\mathrm{N})\phi_{g,1}M$. □

12.3.h. Left pure Hodge modules. The definition of left pure Hodge modules differs from that of right pure Hodge modules by a shift in dimension zero, due to the fact that the pushforward by an embedding $Y\hookrightarrow X$ shifts the weight by $-\operatorname{codim}Y$ in the left case, while it has no effect in the right case.

***Definition 12.3.13* (Pure Hodge modules (left case)).** The category $\mathsf{HM}_{\leqslant d}(X,w)^{\mathrm{left}}$ is the *full subcategory* of $R_F\mathscr{D}$-$\mathsf{Triples}(X)^{\mathrm{left}}$ for which the objects are triples $M = (\mathscr{M}',\mathscr{M}'',\mathfrak{c})$ satisfying:

(HSD) $\mathscr{M}',\mathscr{M}''$ are *holonomic*, *strictly S-decomposable*, and have support of dimension $\leqslant d$.

($\mathrm{HM}_{>0}$) For any open set $U\subset X$ and any holomorphic function $g:U\to\mathbb{C}$, for every $\lambda\in S^1$ and any integer ℓ, the triple

$$\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}M := \Bigl(\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}(\mathscr{M}'),\mathrm{gr}_{-\ell}^{\mathrm{M}}\psi_{g,\lambda}(\mathscr{M}''),\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}\mathfrak{c}\Bigr),$$

a priori defined as an object of $R_F\mathscr{D}$-$\mathsf{Triples}(X)$, is an object of $\mathsf{HM}_{\leqslant d-1}(U,w-1+\ell)$.

(HM_0) For any zero-dimensional strict component $\{x_o\}$ of $\mathscr{M}'$ or $\mathscr{M}''$, we have

$$(\mathscr{M}'_{\{x_o\}},\mathscr{M}''_{\{x_o\}},\mathfrak{c}_{\{x_o\}}) = {}_{\mathrm{D}}\iota^{-n}_{\{x_o\}*}(\mathscr{H}',\mathscr{H}'',\mathfrak{c}_o),$$

where $(\mathscr{H}',\mathscr{H}'',\mathfrak{c}_o)$ is a $\mathbb{C}$-Hodge triple of weight $w+n$ (see Definition 2.4.28) and the pushforward is taken in the left setting, that we can also write $[{}_{\mathrm{D}}\iota^0_{\{x_o\}*}(\mathscr{H}',\mathscr{H}'',\mathfrak{c}_o)]^{\mathrm{left}}$.

***Proposition 12.3.14* (Side-changing for HM).** *The side-changing functor*

$$R_F\mathscr{D}\text{-}\mathsf{Triples}(X)^{\mathrm{left}} \longmapsto R_F\mathscr{D}\text{-}\mathsf{Triples}(X)^{\mathrm{right}}$$

induces an equivalence $\mathsf{HM}_{\leqslant d}(X,w)^{\mathrm{left}}\simeq\mathsf{HM}_{\leqslant d}(X,w+\dim X)^{\mathrm{right}}$. □

As a consequence, all properties of right pure Hodge modules are transferred to left pure Hodge modules. Let us simply mention Kashiwara's equivalence for the inclusion $\iota_Y:Y\hookrightarrow X$.

***Proposition 12.3.15* (Kashiwara's equivalence for left pure Hodge modules)**
The functor ${}_{\mathrm{H}}\iota_{Y*}^{-\operatorname{codim}Y}$ *induces an equivalence*

$$\mathsf{HM}(Y,w)\simeq\mathsf{HM}_Y(X,w-\operatorname{codim}Y). \qquad \square$$

12.3.i. Graded Hodge-Lefschetz modules. We now go back to the right setting. Given $\varepsilon=\pm1$, we can define the category $\mathsf{HLM}_{\leqslant d}(X,w;\varepsilon)$ of graded Hodge-Lefschetz modules as in Section 3.2.a: the objects are pairs (M,L), with $M=\oplus M_j$, and M_j are objects of $\mathsf{HM}_{\leqslant d}(X,w+\varepsilon j)$; L is a graded morphism $M\to M[2](-\varepsilon)$ of degree -2, that is, for every j, L induces a morphism $M_j\to M_{j-2}(-\varepsilon)$, such that, for $j\geqslant0$, $\mathrm{L}^j:M_j\to M_{-j}(-\varepsilon j)$ is an isomorphism. We note that $\mathrm{P}M_j$ is an object of $\mathsf{HM}_{\leqslant d}(X,w+\varepsilon j)$, by Proposition 12.3.9, and the Lefschetz decomposition of M_j holds in $\mathsf{HM}_{\leqslant d}(X,w+\varepsilon j)$; moreover, the category $\mathsf{HLM}_{\leqslant d}(X,w;\varepsilon)$ is abelian, any morphism is graded with respect to the Lefschetz decomposition, and moreover is strict and strictly specializable, as follows from Proposition 12.3.9.

Remark 12.3.16. Let M be an object of $\mathsf{HM}_{\leqslant d}(X,w)$ and let $g:X\to\mathbb{C}$ be a holomorphic function. Then for every $\lambda\in S^1$, $(\bigoplus_\ell\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}M,\mathrm{N})$ is an object of the category $\mathsf{HLM}_{\leqslant d-1}(X,w-1;1)$ and $(\mathrm{gr}_\bullet^{\mathrm{M}}\phi_{g,1}M,\mathrm{gr}_{-2}^{\mathrm{M}}\mathrm{N})$ is an object of $\mathsf{HLM}_{\leqslant d}(X,w;1)$.

More generally, for every $k \geqslant 0$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_k) = (\pm 1, \dots, \pm 1)$, we can define the category $\mathsf{HLM}_{\leqslant d}(X, w; \boldsymbol{\varepsilon})$ of k-graded Hodge-Lefschetz modules: the objects are tuples (M, L), with $\mathrm{L} = (\mathrm{L}_1, \dots, \mathrm{L}_k)$, $M = \oplus_{\boldsymbol{j} \in \mathbb{Z}^k} M_{\boldsymbol{j}}$, each $M_{\boldsymbol{j}}$ is an object in $\mathsf{HM}_{\leqslant d}\big(X, w + \boldsymbol{\varepsilon} \cdot \boldsymbol{j}\big)$ with $\boldsymbol{\varepsilon} \cdot \boldsymbol{j} := \sum_i \varepsilon_i j_i$, the morphisms L_i should pairwise commute, be of k-degree $(0, \dots, -2, \dots, 0)$ and for every $\boldsymbol{j}$ with $j_i \geqslant 0$, $\mathrm{L}_i^{j_i}$ should induce an isomorphism from $M_{\boldsymbol{j}}$ to the component where j_i is replaced with $-j_i$; the primitive part $\mathrm{P}M_{\boldsymbol{j}}$, for $j_1, \dots, j_k \geqslant 0$, is the intersection of the $\operatorname{Ker} \mathrm{L}_i^{j_i+1}$ and we have a Lefschetz multi-decomposition, with respect to which any morphism is multi-graded. The category is abelian, and any morphism is strict and strictly $\mathbb{R}$-specializable.

Lemma 12.3.17. *Let (M, L) be an object of the category $\mathsf{HLM}_{\leqslant d}(X, w; \boldsymbol{\varepsilon})$. Then, for every $\lambda \in S^1$, the specialized object $\big(\mathrm{gr}^{\mathrm{M}}_{\bullet} \psi_{t,\lambda} M, (\mathrm{gr}^{\mathrm{M}}_{\bullet} \psi_{t,\lambda} \mathrm{L}, \mathrm{N})\big)$ is an object of $\mathsf{HLM}_{\leqslant d-1}\big(X, w-1; (\boldsymbol{\varepsilon}, 1)\big)$ and $\mathrm{P}_{\mathrm{L}} \mathrm{gr}^{\mathrm{M}}_{\ell} \psi_{t,\lambda} M_{\boldsymbol{j}} = \mathrm{gr}^{\mathrm{M}}_{\ell} \psi_{t,\lambda} \mathrm{P}_{\mathrm{L}} M_{\boldsymbol{j}}$, where P_{L} denotes the multi-primitive part with respect to L.*

Proof. The lemma is a direct consequence of the strict compatibility of the $\psi_{t,\lambda} \mathrm{L}_i$ with the monodromy filtration $\mathrm{M}(\mathrm{N})$, as follows from Proposition 12.3.9. □

Lemma 12.3.18. *The category $\mathsf{HLM}_{\leqslant d}(X, w; \boldsymbol{\varepsilon})$ has an inductive definition analogous to that of $\mathsf{HM}_{\leqslant d}(X, w)$, where one replaces the condition $(\mathrm{HM}_{>0})$ with the condition $(\mathsf{HLM}_{>0})$, asking that $\big(\mathrm{gr}^{\mathrm{M}}_{\bullet} \psi_{t,\lambda}(M, \mathrm{L}), \mathrm{N}\big)$ is an object of $\mathsf{HLM}_{\leqslant d-1}\big(X, w-1; (\boldsymbol{\varepsilon}, 1)\big)$, and the condition (HM_0) with the analogous property (HLM_0).*

Proof. According to the previous lemma, it is enough to show that, if (M, L) satisfies the inductive conditions, then it is an object of $\mathsf{HLM}_{\leqslant d}(X, w; \boldsymbol{\varepsilon})$. This is done by induction on d, the case $d = 0$ being easy. One shows first that each $M_{\boldsymbol{j}}$ is in $\mathsf{HM}_{\leqslant d}(X, w + \boldsymbol{\varepsilon} \cdot \boldsymbol{j})$ for every $\boldsymbol{j}$ and that $\psi_{t,\lambda} \mathrm{L}_i^{j_i}$ is an isomorphism from $\psi_{t,\lambda} M_{\boldsymbol{j}}$ to $\psi_{t,\lambda} M_{j_1, \dots, -j_i, \dots, j_k}(-\varepsilon_i j_i)$ for every $i = 1, \dots, k$, any $\boldsymbol{j}$ with $j_i \geqslant 0$, any local coordinate t and any $\lambda \in S^1$. Considering the decomposition with respect to the support, one deduces that $\mathrm{L}_i^{j_i}$ is an isomorphism from $M_{\boldsymbol{j}}$ to $M_{j_1, \dots, -j_i, \dots, j_k}(-\varepsilon_i j_i)$. □

***Exercise 12.3.19* (Left Hodge-Lefschetz modules).** Define the category $\mathsf{HLM}_{\leqslant d}(X, w; \boldsymbol{\varepsilon})^{\mathrm{left}}$ and show the equivalence

$$\mathsf{HLM}_{\leqslant d}(X, w; \boldsymbol{\varepsilon})^{\mathrm{left}} \xrightarrow[\sim]{\text{left-to-right}} \mathsf{HLM}_{\leqslant d}(X, w + \dim X; \boldsymbol{\varepsilon})^{\mathrm{right}}.$$

12.4. Polarization

12.4.a. Polarized/polarizable Hodge modules. We also define the notion of polarization by induction on the dimension of the support.

***Definition 12.4.1* (Polarization).** A *polarization* of an object M of $\mathsf{HM}_{\leqslant d}(X, w)$ is a morphism $\mathrm{Q} : M \to M^*(-w)$ which is $(-1)^w$-Hermitian (i.e., a pre-polarization of weight w in the sense of Section 12.2.a) such that:

($\mathrm{PHM}_{>0}$) for any open set $U \subset X$ and any holomorphic function $g : U \to \mathbb{C}$, for any $\lambda \in S^1$ and any integer $\ell \geqslant 0$, the morphism [1] $\mathrm{P}_\ell\psi_{g,\lambda}\mathrm{Q}$ induces a polarization of the object $\mathrm{P}_\ell\psi_{g,\lambda}M$ of $\mathsf{HM}_{\leqslant d-1}(U, w-1+\ell)$,

(PHM_0) for any zero-dimensional strict component $\{x_o\}$ of M, we have $\mathrm{Q} = {}_{\mathrm{D}}\iota^0_{\{x_o\}*}\mathrm{Q}_o$, where Q_o is a polarization of the zero-dimensional Hodge structure $(\mathscr{H}', \mathscr{H}'', \mathfrak{c}_o)$ of weight w (see Condition (HM_0)) as in Definition 2.4.36.

Remarks 12.4.2.

(1) We note that Condition ($\mathrm{PHM}_{>0}$) is meaningful because of Exercise 10.4.10.

(2) Conditions ($\mathrm{PHM}_{>0}$) and (PHM_0) imply that Q is an isomorphism $M \xrightarrow{\sim} M^*(-w)$. Indeed, one can assume that the pure support of M has only one component Z. If it is of dimension zero, we apply Remark 2.4.18(1). In dimension $\geqslant 1$, it is enough to prove that Q is an isomorphism on a dense open set of Z since, by definition of the pure support, this implies that Q is an isomorphism. By Proposition 12.3.7, we are reduced to the case where M underlies a variation of Hodge structure of weight w (up to side-changing), for which we refer to Exercise 4.1.12(2).

(3) We say that an object M of $\mathsf{HM}_{\leqslant d}(X, w)$ is *polarizable* if it admits a polarization Q. For a given polarization Q, the pair (M, Q) is called a *polarized Hodge module*. Often we do not need to make precise the polarization, but it is important to understand the behaviour of a given polarization by means of various functors applied to a polarizable Hodge module in order to be able to conclude, by exhibiting in this way a polarization, that the image through these functors remains polarizable.

We will denote by $\mathsf{pHM}_{\leqslant d}(X, w)$ the full subcategory of $\mathsf{HM}_{\leqslant d}(X, w)$ of polarizable objects. Notice that, with this definition, morphisms are not supposed to be compatible with the polarizations if one is given such polarizations on the source and the target. According to Corollary 7.7.13, for an object M in $\mathsf{pHM}_{\leqslant d}(X, w)$ and a polarization Q on it, we have an S-decomposition

$$(M, \mathrm{Q}) = \bigoplus_Z (M_Z, \mathrm{Q}_Z). \tag{12.4.3}$$

The following proposition is easy:

Proposition 12.4.4.

(1) *In the situation of Proposition* 12.3.4*, if a polarization* Q *is the direct sum of two morphisms* Q_1 *and* Q_2*, then each* Q_i *is a polarization of* M_i*.*

(2) *Corollary* 12.3.5 *holds for* $\mathsf{pHM}_{\leqslant d}(X, w)$*.*

(3) *Kashiwara's equivalence of Section* 12.3.d *holds for* $\mathsf{pHM}(X, w)$*.*

(4) *Proposition* 12.3.7 *holds for* $\mathsf{pHM}_Z(X, w)$*.* □

Remark 12.4.5 **($(-1)^w$-Hermitian Hodge pair).** Let (M, Q) be a polarized Hodge triple of weight w. Since Q is non-degenerate (Remark 12.4.2(2)), (M, Q) is isomorphic to a

1. See Caveat 12.2.6.

polarized Hodge triple $(\mathscr{M}', \mathscr{M}'(w), \mathfrak{c}')$ of weight w, for some suitable $\mathfrak{c}'$, with polarization $((-1)^w \operatorname{Id}, \operatorname{Id})$. We call $(-1)^w$*-Hermitian Hodge pair* a pair $(\mathscr{M}, \mathfrak{c})$ such that $((\mathscr{M}, \mathscr{M}(w), \mathfrak{c}), ((-1)^w \operatorname{Id}, \operatorname{Id}))$ is a polarized Hodge triple of weight w. In particular, the sesquilinear pairing $\mathfrak{c}$ is $(-1)^w$-Hermitian on $\mathcal{M}$.

If $(\mathscr{M}, \mathfrak{c})$ is a $(-1)^w$-Hermitian Hodge pair, then for every $\ell \geqslant 0$, $\mathrm{P}_\ell\psi_{g,\lambda}(\mathscr{M}, \mathfrak{c}) := (\mathrm{P}_\ell\psi_{g,\lambda}\mathscr{M}, \mathrm{P}_\ell\psi_{g,\lambda}\mathfrak{c})$ is a $(-1)^{w-1+\ell}$-Hermitian Hodge pair.

12.4.b. Semi-simplicity

Proposition 12.4.6. *If M_1 is a subobject (in the category $\mathsf{HM}(X, w)$) of a polarized object (M, Q), then Q induces a polarization Q_1 of M_1 and (M_1, Q_1) is a direct summand of (M, Q) in $\mathsf{pHM}(X, w)$. In particular, the category $\mathsf{pHM}(X, w)$ is semisimple (all objects are semisimple and morphisms between simple objects are zero or isomorphisms).*

Proof. We can assume that the pure support of M has only one component Z. If its dimension is zero, we apply Exercise 2.4.22. If $\dim Z \geqslant 1$, we consider the exact sequences

$$\begin{array}{ccccccccc}
0 & \longleftarrow & M_1^*(-w) & \xleftarrow{\;i^*\;} & M^*(-w) & \longleftarrow & M_2^*(-w) & \longleftarrow & 0\\
 & & \Big\uparrow{\scriptstyle \mathrm{Q}_1} & & \Big\uparrow{\scriptstyle \mathrm{Q}\,\wr} & & & & \\
0 & \longrightarrow & M_1 & \xrightarrow{\;i\;} & M & \longrightarrow & M_2 & \longrightarrow & 0
\end{array}$$

where M_2 is the cokernel, in the abelian category $\mathsf{HM}(X, w)$, of $M_1 \hookrightarrow M$. We want to show first that Q_1 is an isomorphism. For this, we argue as in Remark 12.4.2(2), by reducing to checking the property on a smooth open dense set of Z, where we can apply Exercise 4.1.14(1).

We conclude that we have a projection $p = \mathrm{Q}_1^{-1} \circ i^* \circ \mathrm{Q} : M \to M_1$ such that $p \circ i = \operatorname{Id}$, and a decomposition $M = M_1 \oplus \mathrm{Q}^{-1} M_2^*(-w)$. We apply Proposition 12.4.4(1) to show that M_2 is polarizable. $\square$

12.4.c. Polarized/polarizable Hodge-Lefschetz modules. Let (M, L) be an object of $\mathsf{HLM}(X, w; \boldsymbol{\varepsilon})$ and $(M, \mathrm{L})^* := (M^*, -\mathrm{L}^*)$ denote its adjoint object. A *polarization* Q is a (multi) graded morphism $\mathrm{Q} : M \to M^*(-w)$ (i.e., Q sends $M_{\boldsymbol{\ell}}$ to $M_{\boldsymbol{\ell}}^*(-w) = (M_{-\boldsymbol{\ell}})^*(-w)$), such that each L_i is skew-adjoint with respect to Q (i.e., Q is a morphism $(M, \mathrm{L}) \to (M, \mathrm{L})^*(-w)$) and that, for every $\boldsymbol{\ell} = (\ell_1, \dots, \ell_k)$ with nonnegative components, the induced morphism (see Section 3.2.b)

$$\mathrm{L}_1^{*\ell_1} \cdots \mathrm{L}_k^{*\ell_k} \circ \mathrm{Q} : M_{\boldsymbol{\ell}} \longrightarrow (M_{\boldsymbol{\ell}})^*(-w - \boldsymbol{\varepsilon} \cdot \boldsymbol{\ell})$$

induces a polarization of the object $\mathrm{P}_{\mathrm{L}} M_{\boldsymbol{\ell}}$ of $\mathsf{HM}(X, w + \boldsymbol{\varepsilon} \cdot \boldsymbol{\ell})$.

Lemma 12.4.7. *The full subcategory $\mathsf{pHLM}(X, w; \boldsymbol{\varepsilon})$ of polarizable objects of the category $\mathsf{HLM}(X, w; \boldsymbol{\varepsilon})$ has an inductive definition as in Definition 12.4.1.*

Proof. This directly follows from the commutativity of $\mathrm{P_L}$ and $\mathrm{gr}^{\mathrm{M}}_{\ell}\psi_{t,\lambda}$ shown in Lemma 12.3.17. □

Exercise 12.4.8. Show that the conclusion of Proposition 12.4.6 holds for $\mathsf{pHLM}(X,w;\boldsymbol{\varepsilon})$.

Corollary 12.4.9. *Let (M,L) be an object of $\mathsf{pHLM}(X,w;\boldsymbol{\varepsilon})$ with pure support Z and polarization Q. Let $g:U\to\mathbb{C}$ be a holomorphic function $\not\equiv 0$ on Z. Then $\big(\mathrm{gr}^{\mathrm{M}}_{\bullet}\phi_{g,1}(M,\mathrm{L}),\mathrm{N}\big)$ is an object of $\mathsf{pHLM}(X,w;\boldsymbol{\varepsilon},1)$ with polarization induced by Q.*

Proof. Apply the Lefschetz analogue of Corollary 12.3.12 and Exercise 12.4.8. □

Proposition 12.4.10. *Let (M,L) be an object of $\mathsf{pHLM}(X,w;\boldsymbol{\varepsilon})$. Then each summand $M_{\boldsymbol{\ell}}$ is an object of $\mathsf{pHM}(X,w+\boldsymbol{\varepsilon}\cdot\boldsymbol{\ell})$.*

Proof. Let us fix a polarization Q of (M,L). Let us define $\mathrm{w}=\mathrm{w}_1\cdots\mathrm{w}_k$, where w_i is relative to L_i. As in Proposition 3.2.27, one first checks that $\mathrm{w}^*\circ\mathrm{Q}$ induces a polarization of $M_{\boldsymbol{\ell}}$ for $\boldsymbol{\ell}\geqslant 0$, since w commutes with taking $\mathrm{P}_j\psi_{g,\lambda}$ for every $j\geqslant 0$ for any locally defined holomorphic function g (here, P_j is taken with respect to N). One then conclude that $M_{\boldsymbol{\ell}}$ is polarizable for arbitrary $\boldsymbol{\ell}$ by using isomorphisms induced by powers of L_i $(i=1,\dots,k)$. □

Proposition 12.4.11. *The conclusions of Propositions* 3.2.28 *and* 3.2.32 *remain valid for graded Hodge-Lefschetz modules.*

Proof.

(1) Let us begin with Proposition 3.2.28. We will denote by L,L' the nilpotent operators on M,M'. Firstly, we remark that $\mathrm{c}(M_{\ell+1})$ and $\operatorname{Ker}\mathrm{v}_{|M'_\ell}$ are objects of $\mathsf{HM}(X,w+\varepsilon\ell)$, according to Proposition 12.3.9.

Let us show that $\operatorname{Im}\mathrm{c}$ and $\operatorname{Ker}\mathrm{v}$ are subobjects of M' in $\mathsf{HLM}(X,w;\varepsilon)$, that is, for every $\ell\geqslant 0$, L'^{ℓ} induces an isomorphism $\mathrm{c}(M_{\ell+1})\xrightarrow{\sim}\mathrm{c}(M_{-\ell+1})$ and $\operatorname{Ker}\mathrm{v}_{|M'_\ell}\xrightarrow{\sim}\operatorname{Ker}\mathrm{v}_{|M'_{-\ell}}$. Clearly, $\operatorname{Ker}\mathrm{L}'^{\ell}=0$, since this holds on M'_ℓ. We need to check that $\operatorname{Coker}\mathrm{L}'^{\ell}=0$. We know in any case that $\operatorname{Coker}\mathrm{L}'^{\ell}$ is an object of $\mathsf{HM}(X,w+\varepsilon\ell)$.

We prove the assertion by induction on the dimension of the support of M'_ℓ. The case where this dimension is zero is precisely furnished by Proposition 3.2.28. We will treat the case of $\operatorname{Im}\mathrm{c}$, and the case of $\operatorname{Ker}\mathrm{v}$ is similar. We can assume that the pure support of M'_ℓ is an irreducible closed analytic subset $Z\in X$. We claim that there exists a Zariski-dense open subset $Z^o\subset Z$ where $\operatorname{Coker}\mathrm{L}'^{\ell}$ correspond to a variation of Hodge structure: if $\operatorname{Coker}\mathrm{L}'^{\ell}\neq 0$, its pure support is equal to Z, by the definition of the pure support of M'_ℓ, and the assertion follows from Proposition 12.4.4(4); otherwise, $\operatorname{Coker}\mathrm{L}'^{\ell}=0$ and the assertion is clear.

Let us choose a local coordinate t on Z^o, hence on X, and let us consider the nearby cycle functor $\psi_{t,1}$. It is enough to prove that $\psi_{t,1}\operatorname{Coker}\mathrm{L}'^{\ell}=0$, since this will prove that the restriction to $t=0$ of the corresponding variation is zero. Since $t=0$ is strictly non-characteristic for $\operatorname{Coker}\mathrm{L}'^{\ell}$, the corresponding N is zero and the monodromy filtration is trivial.

We know that c and $\psi_{t,\lambda}$ commute since c is strictly $\mathbb{R}$-specializable (see Proposition 12.3.9). It follows that, by induction, $\psi_{t,\lambda}\mathrm{L}'^{\ell} : \psi_{t,\lambda}\mathrm{c}(M_{\ell+1}) \to \psi_{t,\lambda}\mathrm{c}(M_{-\ell+1})$ is an isomorphism for every λ, ℓ. Since $\mathrm{L}'^{\ell} : \mathrm{c}(M_{\ell+1}) \to \mathrm{c}(M_{-\ell+1})$ is a morphism of Hodge structures, it is strictly $\mathbb{R}$-specializable, and thus $\psi_{t,\lambda}\operatorname{Coker}\mathrm{L}'^{\ell} = \operatorname{Coker}\psi_{t,\lambda}\mathrm{L}'^{\ell} = 0$, as wanted.

By Lemma 12.4.8, $\operatorname{Im}\mathrm{c}$ and $\operatorname{Ker}\mathrm{v}$ decompose as direct sums of simple objects in $\mathsf{HLM}(X, w; \varepsilon)$, so their intersection is an object in the same category. By the same argument as above, using induction on the dimension, the intersection $\operatorname{Im}\mathrm{c} \cap \operatorname{Ker}\mathrm{v}$ vanishes. Similarly, the direct summand of $\operatorname{Im}\mathrm{c}\oplus\operatorname{Ker}\mathrm{v}$ in M' is an object of $\mathsf{HLM}(X, w; \varepsilon)$ and also vanishes by induction. We therefore have a decomposition $M' = \operatorname{Im}\mathrm{c}\oplus\operatorname{Ker}\mathrm{v}$ in $\mathsf{HLM}(X, w; \varepsilon)$.

(2) Let us now consider Proposition 3.2.32. So, let $\big((M_{j_1,j_2})_{\boldsymbol{j}\in\mathbb{Z}^2}, \mathrm{L}_1, \mathrm{L}_2\big)$ be an object of $\mathsf{pHLM}(X, w; \varepsilon_1, \varepsilon_2)$ with a polarization Q. We assume that it comes equipped with a bi-graded differential, which is a morphism $d : M_{j_1,j_2} \to M_{j_1-1,j_2-1}(-\varepsilon)$ ($\varepsilon := (\varepsilon_1 + \varepsilon_2)/2$) in $\mathsf{HM}(X, w + \varepsilon_1 j_1 + \varepsilon_2 j_2)$, of bi-degree $(-1, -1)$, which commutes with L_1 and L_2 and is self-adjoint with respect to Q. In particular, d is strict and strictly specializable (Proposition 12.3.9) and we have, for any germ g of holomorphic function, any $\lambda \in S^1$ and any $\ell \geqslant 0$,

$$\mathrm{P}_\ell\psi_{g,\lambda}(\operatorname{Ker} d/\operatorname{Im} d) = \operatorname{Ker}(\mathrm{P}_\ell\psi_{g,\lambda}d)/\operatorname{Im}(\mathrm{P}_\ell\psi_{g,\lambda}d)$$

(see Corollary 12.3.10). By induction on the dimension of the support, we can assert that $\big(\mathrm{P}_\ell\psi_{g,\lambda}(\operatorname{Ker} d/\operatorname{Im} d), \mathrm{P}_\ell\psi_{g,\lambda}\mathrm{L}\big)$ is an object of $\mathsf{HLM}(X, w - 1 + \ell; \boldsymbol{\varepsilon})$ with polarization $\mathrm{P}_\ell\psi_{g,\lambda}\mathrm{Q}$, and we conclude with Lemma 12.4.7. □

***Corollary 12.4.12* (Degeneration of a spectral sequence).** *Let $(M^\bullet, d)$ be a bounded complex in $R_F\mathscr{D}$-$\mathsf{Triples}(X)$, with $d : M^j \to M^{j+1}$ and $d \circ d = 0$. Let us assume that it is equipped with the following data:*

(a) *a morphism of complexes $\mathrm{Q} : (M^\bullet, d) \to (M^\bullet, d)^*(-w)$ which is $(-1)^w$-Hermitian, that is, for every k, a morphism $\mathrm{Q} : M^k \to (M^{-k})^*(-w)$ which is compatible with d and d^*, and such that $\mathrm{Q}^* = (-1)^w\mathrm{Q}$,*

(b) *a morphism $\mathrm{L} : (M^\bullet, d) \to (M^{\bullet+2}(1), d)$ which is skew-adjoint with respect to Q,*

(c) *a morphism $\mathrm{N} : (M^\bullet, d) \to (M^\bullet(-1), d)$ which is nilpotent, commutes with L, and skew-adjoint with respect to Q, with monodromy filtration of $\mathrm{M}_\bullet(\mathrm{N})$.*

Let us consider the spectral sequence associated to the filtered complex $(\mathrm{M}_{-\ell}M^\bullet, d)$ with $E_1^{\ell,j-\ell} = \mathscr{H}^j\mathrm{gr}^{\mathrm{M}}_{-\ell}M^\bullet$. We assume that

$$\bigoplus_{j,\ell}\Big(E_1^{\ell,j-\ell} = \mathscr{H}^j(\mathrm{gr}^{\mathrm{M}}_{-\ell}M^\bullet), (\mathscr{H}^j\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{L}, \mathscr{H}^j\mathrm{gr}\mathrm{N}), \mathscr{H}^j\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{Q}\Big)$$

is a polarized object of $\mathsf{HLM}(X, w; -1, 1)$. Then,

(1) *the spectral sequence degenerates at E_2,*

(2) *the filtration $W_\bullet\mathscr{H}^j(M^\bullet)$ naturally induced by $\mathrm{M}_\bullet M^\bullet$ is the monodromy filtration $\mathrm{M}_\bullet$ associated to $\mathscr{H}^j\mathrm{N} : \mathscr{H}^j(M^\bullet) \to \mathscr{H}^j(M^\bullet)$,*

(3) *the object*

$$\bigoplus_{j,\ell}\Big(\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathscr{H}^j(M^\bullet),(\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathscr{H}^j\mathrm{L},\mathrm{gr}\mathscr{H}^j\mathrm{N}),\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathscr{H}^j\mathrm{Q}\Big)$$

is a polarized object of $\mathsf{HLM}(X,w;-1,1)$.

Proof. Let us first make clear the statement. Note that we use the bi-grading as in Remark 3.1.17. Since d and L commute with N, d and L are compatible with the monodromy filtration $\mathrm{M}_\bullet(\mathrm{N})$, hence we have a graded complex $(\mathrm{gr}^{\mathrm{M}}_{-\ell}M^\bullet,d)$, and L induces for every ℓ a morphism $\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{L}:(\mathrm{gr}^{\mathrm{M}}_{-\ell}M^\bullet,d)\to(\mathrm{gr}^{\mathrm{M}}_{-\ell}M^{\bullet+2},d)$, and thus a morphism $\mathscr{H}^j\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{L}:E_1^{\ell,j-\ell}\to E_1^{\ell,j+2-\ell}$. Similarly, $\mathscr{H}^j\mathrm{grN}$ is a morphism $E_1^{\ell,j-\ell}\to E_1^{\ell+2,j-\ell-2}$. We consider the bi-grading such that $E_1^{\ell,j-\ell}$ is in bi-degree (j,ℓ).

The differential $d_1:\mathscr{H}^j(\mathrm{gr}^{\mathrm{M}}_{-\ell}M^\bullet)\to\mathscr{H}^{j+1}(\mathrm{gr}^{\mathrm{M}}_{-\ell-1}M^\bullet)$ is a morphism of bi-degree $(1,1)$ in $\mathsf{HM}(X,w+j-\ell)$. We will check below that d_1 is self-adjoint with respect to $\mathscr{H}^j\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{Q}$. From the analogue of Proposition 3.2.32 (see Proposition 12.4.11), we deduce that $\bigoplus_{j,\ell}E_2^{\ell,j-\ell}$ is part of an object of $\mathsf{pHLM}(X,w;-1,1)$. Now, one shows inductively that, for $r\geqslant 2$, $d_r:E_2^{\ell,j-\ell}\to E_2^{\ell+r,j-\ell-r+1}$ is a morphism of pure Hodge modules, the source having weight $w+j-\ell$ and the target $w+j-\ell-r+1<w+j-\ell$ and thus, by applying Proposition 12.3.8, that $d_r=0$. This gives the result. □

Proof that d_1 is self-adjoint. We regard $\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{Q}$ as a morphism $\mathrm{gr}^{\mathrm{M}}_{-\ell}M^k\to(\mathrm{gr}^{\mathrm{M}}_{\ell}M^{-k})^*$. It is compatible with d and d^* on these complexes, since N commutes with d. Then, $\mathscr{H}^j\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{Q}$ is a morphism $\mathscr{H}^j\mathrm{gr}^{\mathrm{M}}_{-\ell}M^\bullet\to(\mathscr{H}^{-j}\mathrm{gr}^{\mathrm{M}}_{\ell}M^{-\bullet})^*$. Since d_1 is obtained by a standard formula from d on the filtered complex, the equality $\mathrm{Q}\circ d=d^*\circ\mathrm{Q}$ implies $\mathscr{H}^j\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{Q}\circ d_1=(d_1)^*\circ\mathscr{H}^j\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{Q}$. □

12.4.d. Polarized vanishing cycles.

***Proposition 12.4.13* (Polarizability of vanishing cycles).** *Let M be a polarizable Hodge triple of weight w, i.e., an object of* $\mathsf{pHM}(X,w)$. *Then for any holomorphic function $g:U\to\mathbb{C}$, the object* $(\mathrm{gr}^{\mathrm{M}}_\bullet\phi_{g,1}M,\mathrm{grN})$ *of* $\mathsf{HLM}(U,w;1)$ *(see Remark* 12.3.16*) is polarizable.*

Proof. We will make explicit a polarization, starting from a polarization Q of M. Since $(M,\mathrm{Q})_{|U}$ is strictly S-decomposable after (12.4.3), we can assume that $M_{|U}$ has pure support an irreducible closed analytic subset Z of U. If g vanishes identically on Z, then $\phi_{g,1}M=M$ and $\mathrm{N}=0$, so there is nothing to prove. We can thus assume that M is a middle extension along (g), so $\phi_{g,1}M=\mathrm{Im}[\mathrm{N}:\psi_{g,1}M\to\psi_{g,1}M(-1)]$. We define Q_ϕ on $\phi_{g,1}M$ as the morphism induced by $\mathrm{N}^*\circ((-1)^w\psi_{g,1}\mathscr{Q},\psi_{g,1}\mathscr{Q})$ (see Proposition 3.2.25 and Lemma 3.1.16). We can then argue as in Proposition 3.2.25 to obtain the polarization property from that of $\mathrm{P}_{\ell+1}\psi_{g,1}\mathrm{Q}$ on $\mathrm{P}_{\ell+1}\psi_{g,1}M$. □

12.5. Comments

The relation between Hodge theory and the theory of nearby or vanishing cycles in dimension bigger than one starts with the work of Steenbrink [**Ste76, Ste77**]. It concerns one-parameter families of projective varieties, regarded as proper functions from a complex manifold to a disc. A canonical Hodge structure is constructed on the cohomology of the nearby fibre of a singular fibre of the family by means of replacing the special fibre with a divisor with normal crossings and by computing the nearby or vanishing cohomology in terms of a logarithmic de Rham complex, in order to apply Deligne's method in [**Del71b**]. This gives a geometric construction of Schmid's limit mixed Hodge structure in the case of a variation of geometric origin. The need of passing from the assumption of unipotent monodromy, as used in the work of Schmid [**Sch73**] to the assumption of quasi-unipotent monodromy is justified by this geometric setting. This leads Steenbrink [**Ste77**] to developing the notion of logarithmic de Rham complex in the setting of V-manifolds. Steenbrink also obtains, as a consequence of this construction, the local invariant cycle theorem and the Clemens-Schmid exact sequence. We can regard this work as the localization of Hodge theory in the analytic neighbourhood of a projective variety.

The work of Varchenko [**Var82**] and others on asymptotic Hodge theory has localized even more Hodge theory. This work is concerned with an isolated singularity of a germ of holomorphic function and it constructs a Hodge-Lefschetz structure on the space of vanishing cycles of this function, by taking advantage that the vanishing cycles are supported at the isolated singularity, which is trivially a projective variety. The construction of Varchenko has been later analyzed in terms of $\mathscr{D}$-modules by Pham [**Pha83**], Saito [**Sai83b, Sai83a, Sai84, Sai85**] and Scherk-Steenbrink [**SS85**]. It is then natural to consider the cohomology of the vanishing cycle sheaf of a holomorphic function on a complex manifold whose critical locus is projective, but possibly not the special fibre of the function, and to ask for a mixed Hodge structure on it.

The theory of polarizable Hodge modules, as developed by Saito in [**Sai88**], emphasizes the local aspect of Hodge theory, by constructing a category defined by local properties in a way similar, but much more complicated, to the definition of a the category of variations of Hodge structure. It can then answer the question above. This idea has proved very efficient, eventually allowing to use the formalism of Grothendieck's six operations in Hodge theory. Many standard cohomological results, like the Clemens-Schmid exact sequence and the local invariant cycle theorem, can be read in this functorial way.

The definition of complex Hodge modules as developed here, not relying on a $\mathbb{Q}$-structure and on the notion of a perverse sheaf, is inspired by the extension of the notion of polarizable Hodge module to twistor theory, as envisioned by Simpson [**Sim97**], and achieved by Sabbah [**Sab05**] and Mochizuki [**Moc07, Moc15**], although the way the sesquilinear pairing is used on both theories is not exactly the same. As already mentioned in the comments of Chapter 10, the idea of using sesquilinear pairings in the framework of germs of holomorphic functions was developed by Barlet [**Bar85**] with the perspective of making the link between asymptotic Hodge theory on the vanishing cycles of germs of functions with isolated singularities and the classical notion of polarization. On the other hand, the idea of using sesquilinear pairings in the framework of holonomic $\mathscr{D}$-modules is due to Kashiwara [**Kas87**].