## CHAPTER 11

## $\mathscr{D}$-MODULES OF NORMAL CROSSING TYPE


#### Abstract

Summary. This chapter, although somewhat technical, is nevertheless essential to understand the behaviour of Hodge modules when the singularities form a normal crossing divisor. It analyzes the compatibility properties on a given $\mathbb{R}$-specializable $\mathscr{D}$-module with respect to various functions, when these functions form part of a coordinate system. The results of this chapter will therefore be of a local nature.


### 11.1. Introduction

Notation 11.1.1. In this chapter, the setting is as follows. The space $X=\Delta^{n}$ is a polydisc in $\mathbb{C}^{n}$ with analytic coordinates $x_{1}, \ldots, x_{n}$, we fix $\ell \leqslant n$ and we denote by $D$ the divisor $\left\{x_{1} \cdots x_{\ell}=0\right\}$. We also denote by $D_{i}(i=1, \ldots, \ell)$ the smooth components of $D$ and by $D_{(\ell)}$ their intersection $D_{1} \cap \cdots \cap D_{\ell}$. We will shorten the notation $\mathbb{C}\left[x_{1}, \ldots, x_{\ell}\right]$ into $\mathbb{C}[x]$ and $\mathbb{C}\left[x_{1}, \ldots, x_{\ell}\right]\left\langle\partial_{x_{1}}, \ldots, \partial_{x_{\ell}}\right\rangle$ into $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$. We will set $I=\{1, \ldots, \ell\}$.

Given a non-constant monomial function vanishing on $D$ at most, that we denote by $g=x^{a}=x_{1}^{a_{1}} \cdots x_{\ell}^{a_{\ell}}\left(a_{i} \geqslant 0\right.$ for $i \in I$ and $a_{i}>0$ for some $\left.i\right)$, we denote by $I_{g} \subset I$ the non-empty set of $i \in I$ such that $a_{i} \neq 0$.

We will mainly consider right $\mathscr{D}$-modules.
Simplifying assumptions 11.1.2. All over this section, we will consider the simple case where $\ell=n$, that is, $D_{(\ell)}$ is reduced to the origin in $\Delta^{n}$, in order to make the computations clearer. We then have $I=\{1, \ldots, n\}$. The general case $\ell \neq n$ brings up objects which are $\mathscr{O}_{D_{(\ell)}}$-locally free and the adaptation is straightforward.

The notion of coherent $\mathscr{D}_{X}$-module of normal crossing type is a natural generalization to higher dimension of the case of a regular holonomic $\mathscr{D}$-module in dimension one, as considered in Section 6.2. In terms of $\mathscr{D}$-module theory, that we will not use, we could characterize such $\mathscr{D}$-modules as the regular holonomic $\mathscr{D}$-modules whose characteristic variety is adapted to the natural stratification of the divisor $D$. In
other words, these are the simplest objects in higher dimension, that we analyze in Section 11.2.b. Adding an $F$-filtration to the picture leads us to take much care of the behaviour of this filtration with respect to the various $V$-filtrations along the components $D_{i}$ of the divisor $D$. The compatibility property (Definition 8.3.9), is essential in order to have a reasonable control on various operations on these filtered $\mathscr{D}$-modules.

Our main objective in this chapter is to compute the nearby cycles of such filtered $\mathscr{D}$-modules along a monomial function (with respect to coordinates adapted to $D$ ). This will be done in Section 11.3. Lastly, we will also compute the behaviour of a sesquilinear pairing with respect to this functor, and we will end by making even more explicit the example of a simple coherent filtered $\mathscr{D}$-module of normal crossing type.

### 11.2. Normal crossing type

Let $\mathcal{M}$ be a coherent $\mathscr{D}_{X}$-module. Assume that $\mathcal{M}$ is $\mathbb{R}$-specializable along each component $D_{i}$ of $D$. How do the various $V$-filtrations interact? The notion of normal crossing type aims at reflecting that these $V$-filtrations behave independently, i.e., without any interaction. In other words, the transversality property of the components of $D$ is extended to the transversality property of the $V$-filtrations. Similarly, for a coherent filtered $\mathscr{D}_{X}$-module $\left(\mathcal{M}, F_{\bullet} \mathcal{M}\right)$, we will express the independence of the $V$-filtrations in the presence of $F . \mathcal{M}$.
11.2.a. $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-modules of normal crossing type. In this section, we consider the algebraic setting where we replace the sheaf $\mathscr{D}_{X}$ with the ring $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$ and correspondingly (right) $\mathscr{D}_{X}$-modules with (right) $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-modules, that we denote by a capital letter like $M$.

Let us consider, for every $\boldsymbol{\alpha} \in \mathbb{R}^{n}$, the subspace $M_{\boldsymbol{\alpha}}$ of $M$ defined by

$$
M_{\boldsymbol{\alpha}}=\bigcap_{i \in I} \bigcup_{k} \operatorname{Ker}\left(x_{i} \partial_{x_{i}}-\alpha_{i}\right)^{k}
$$

This is a $\mathbb{C}$-vector subspace of $M$. The endomorphism $x_{i} \partial_{x_{i}}$ acting on $M_{\boldsymbol{\alpha}}$ will be denoted by $\mathrm{E}_{i}$ and $2 \pi \mathrm{i}\left(x_{i} \partial_{x_{i}}-\alpha_{i}\right)^{[5]}$ by $\mathrm{N}_{i}$. The family $\left(\mathrm{N}_{1}, \ldots, \mathrm{~N}_{n}\right)$ forms a commuting family of endomorphisms of $M_{\boldsymbol{\alpha}}$, giving $M_{\boldsymbol{\alpha}}$ a natural $\mathbb{C}\left[\mathrm{N}_{1}, \ldots, \mathrm{~N}_{n}\right]$-module structure, and every element of $M_{\boldsymbol{\alpha}}$ is annihilated by some power of each $\mathrm{N}_{i}$. Moreover, for $i \in I$, the morphism $x_{i}: M \rightarrow M$ (resp. $\partial_{x_{i}}: M \rightarrow M$ ) induces a $\mathbb{C}$-linear morphism $x_{i}: M_{\boldsymbol{\alpha}} \rightarrow M_{\boldsymbol{\alpha}-\mathbf{1}_{i}}$ (resp. $\left.\partial_{x_{i}}: M_{\boldsymbol{\alpha}} \rightarrow M_{\boldsymbol{\alpha}+\mathbf{1}_{i}}\right)$. For each fixed $\boldsymbol{\alpha} \in \mathbb{R}^{n}$, we have

$$
M_{\boldsymbol{\alpha}} \cap\left(\sum_{\boldsymbol{\alpha}^{\prime} \neq \boldsymbol{\alpha}} M_{\boldsymbol{\alpha}^{\prime}}\right)=0 \quad \text { in } M
$$

Indeed, for $m=\sum_{\boldsymbol{\alpha}^{\prime} \neq \boldsymbol{\alpha}} m_{\boldsymbol{\alpha}^{\prime}}$, if $m \in M_{\boldsymbol{\alpha}}$, then $m-\sum_{\alpha_{1}^{\prime}=\alpha_{1}} m_{\boldsymbol{\alpha}^{\prime}}$ is annihilated by some power of $x_{1} \partial_{x_{1}}-\alpha_{1}$ and by a polynomial $\prod_{\alpha_{1}^{\prime} \neq \alpha_{1}}\left(x_{1} \partial_{x_{1}}-\alpha_{1}^{\prime}\right)^{k_{\alpha_{1}^{\prime}}}$, hence is zero,
so we can restrict the sum above to $\alpha_{1}^{\prime}=\alpha_{1}$. Arguing similarly for $i=2, \ldots, n$ gives finally $m=0$. It follows that

$$
\begin{equation*}
M^{\prime}:=\bigoplus_{\alpha \in \mathbb{R}^{n}} M_{\boldsymbol{\alpha}} \subset M \tag{11.2.1}
\end{equation*}
$$

is a $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-submodule of $M$.
Exercise 11.1. Show that $x_{i}: M_{\boldsymbol{\alpha}} \rightarrow M_{\boldsymbol{\alpha}-1_{i}}$ is an isomorphism if $\alpha_{i}<0$ and $\partial_{x_{i}}$ : $M_{\boldsymbol{\alpha}} \rightarrow M_{\boldsymbol{\alpha}+\mathbf{1}_{i}}$ is an isomorphism if $\alpha_{i}>-1$.

Definition 11.2.2. Let $M$ be a $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-module. We say that $M$ is of normal crossing type along $D$ if the following properties are satisfied.
(a) There exists a finite subset $\boldsymbol{A} \subset[-1,0)^{n}$, called the set of exponents of $M$, such that $M_{\boldsymbol{\alpha}}=0$ for $\boldsymbol{\alpha} \notin \boldsymbol{A}+\mathbb{Z}^{n}$.
(b) Each $M_{\boldsymbol{\alpha}}\left(\boldsymbol{\alpha} \in \mathbb{R}^{n}\right)$ is finite-dimensional.
(c) The natural inclusion (11.2.1) is an equality.

Exercise 11.2. Show that a $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-module of normal crossing type is of finite type over $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$. Moreover, show that $M_{\leqslant \boldsymbol{\alpha}}:=\bigoplus_{\boldsymbol{\alpha}^{\prime} \leqslant \boldsymbol{\alpha}} M_{\boldsymbol{\alpha}^{\prime}}$ is a $\mathbb{C}[x]\left\langle x \partial_{x}\right\rangle$-module which is of finite type over $\mathbb{C}[x]$, and $\mathbb{C}[x]$-free if $\alpha_{i}<0$ for all $i \in I$

Remark 11.2.3. For every $\boldsymbol{\alpha} \in \boldsymbol{A}$, let us set

$$
M_{\boldsymbol{\alpha}+\mathbb{Z}^{n}}=\bigoplus_{\boldsymbol{k} \in \mathbb{Z}^{n}} M_{\boldsymbol{\alpha}+\boldsymbol{k}}
$$

so that $M=\bigoplus_{\boldsymbol{\alpha} \in \boldsymbol{A}} M_{\boldsymbol{\alpha}+\mathbb{Z}^{n}}$. Then $M_{\boldsymbol{\alpha}+\mathbb{Z}^{n}}$ is a $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-module. In such a way, $M$ is the direct sum of $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-modules of normal crossing type having a single exponent.

The category of $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-modules of normal crossing type along $D$ is, by definition, the full subcategory of that of $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-modules whose objects are of normal crossing type along $D$.

Proposition 11.2.4. Every morphism between $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-modules of normal crossing type along $D$ is graded with respect to the decomposition (11.2.1), and the category of $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-modules of normal crossing type along $D$ is abelian.

Proof. By $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-linearity and using Bézout's theorem, one checks that any morphism $\varphi: M_{1} \rightarrow M_{2}$ sends $M_{1, \boldsymbol{\alpha}}$ to $M_{2, \boldsymbol{\alpha}}$, and has no component from $M_{1, \boldsymbol{\alpha}}$ to $M_{2, \boldsymbol{\beta}}$ if $\boldsymbol{\beta} \neq \boldsymbol{\alpha}$.

Proposition 11.2.5 (Description by quivers). Let us fix $\boldsymbol{\alpha} \in[-1,0)^{n}$ and let us set $I(\boldsymbol{\alpha})=\left\{i \in I \mid \alpha_{i}=-1\right\}$. Then the category of $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-modules of normal crossing type with exponent $\boldsymbol{\alpha}$, that is, of the form $M_{\boldsymbol{\alpha}+\mathbb{Z}^{n}}$, is equivalent to the category of $I(\boldsymbol{\alpha})$-quivers having the vertex $M_{\boldsymbol{\alpha}+\boldsymbol{k}}$ equipped with its $\mathbb{C}\left[\mathrm{N}_{1}, \ldots, \mathrm{~N}_{n}\right]$-module structure at the place $\boldsymbol{k} \in\{0,1\}^{I(\boldsymbol{\alpha})}$ and arrows

$$
\begin{aligned}
\operatorname{can}_{i}: M_{\boldsymbol{\alpha}+\boldsymbol{k}} & \longrightarrow M_{\boldsymbol{\alpha}+\boldsymbol{k}+\mathbf{1}_{i}}, \quad \text { if } k_{i}=0, \\
\operatorname{var}_{i}: M_{\boldsymbol{\alpha}+\boldsymbol{k}+\mathbf{1}_{i}} & \longrightarrow M_{\boldsymbol{\alpha}+\boldsymbol{k}},
\end{aligned} \quad \text {, }
$$

subject to the conditions

$$
\left\{\begin{array}{l}
\operatorname{var}_{i} \circ \operatorname{can}_{i}=\mathrm{N}_{i}: M_{\boldsymbol{\alpha}+\boldsymbol{k}} \longrightarrow M_{\boldsymbol{\alpha}+\boldsymbol{k}}, \\
\operatorname{can}_{i} \circ \operatorname{var}_{i}=\mathrm{N}_{i}: M_{\boldsymbol{\alpha}+\boldsymbol{k}+\mathbf{1}_{i}} \longrightarrow M_{\boldsymbol{\alpha}+\boldsymbol{k}+\mathbf{1}_{i}},
\end{array} \quad \text { if } k_{i}=0\right.
$$

(It is understood that if $I(\boldsymbol{\alpha})=\varnothing$, then the quiver has only one vertex and no arrows.)
Proof. It is straightforward, by using that, for $\boldsymbol{k} \in \mathbb{Z}^{n}, \partial_{x_{i}}: M_{\boldsymbol{\alpha}+\boldsymbol{k}} \rightarrow M_{\boldsymbol{\alpha}+\boldsymbol{k}+\mathbf{1}_{i}}$ is an isomorphism if $i \notin I(\boldsymbol{\alpha})$ or $i \in I(\boldsymbol{\alpha})$ and $k_{i} \geqslant 0$, while $x_{i}: M_{\boldsymbol{\alpha}+\boldsymbol{k}} \rightarrow M_{\boldsymbol{\alpha}+\boldsymbol{k}-\mathbf{1}_{i}}$ is an isomorphism if $i \notin I(\boldsymbol{\alpha})$ or $i \in I(\boldsymbol{\alpha})$ and $k_{i} \leqslant-1$.

Remark 11.2.6. In order not to specify a given exponent of a $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-module of normal crossing type along $D$, it is convenient to define the quiver with vertices indexed by $\{0,1\}^{I}$ instead of $\{0,1\}^{I(\boldsymbol{\alpha})}$. We use the convention that, for a fixed $\boldsymbol{\alpha} \in[-1,0)^{n}$ and for $i \notin I(\boldsymbol{\alpha}), \operatorname{var}_{i}=\mathrm{Id}$ and $\operatorname{can}_{i}=\alpha_{i} \mathrm{Id}+\mathrm{N}_{i} / 2 \pi \mathrm{i}=\mathrm{E}_{i}$ (hence both are isomorphisms). Then the category of $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-modules of normal crossing type along $D$ is equivalent to the category of such quivers.

Exercise 11.3. Let $i_{o} \in I$ and let $M_{\boldsymbol{\alpha}+\mathbb{Z}^{n}}$ be a $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-module of normal crossing type with the single exponent $\boldsymbol{\alpha} \in[-1,0)^{n}$.
(1) Show that $M_{\boldsymbol{\alpha}+\mathbb{Z}^{n}}$ is supported on $D_{i_{o}}$ if and only if $\alpha_{i_{o}}=-1$ and, for $\boldsymbol{k} \in \mathbb{Z}^{n}$, $M_{\boldsymbol{\alpha}+\boldsymbol{k}}=0$ if $k_{i_{o}} \leqslant 0$, that is, if and only if $i_{o} \in I(\boldsymbol{\alpha})$ and, setting $\boldsymbol{k}=\left(\boldsymbol{k}^{\prime}, k_{i_{o}}\right)$, every vertex $M_{\boldsymbol{\alpha}+\left(\boldsymbol{k}^{\prime}, 0\right)}$ of the quiver of $M_{\boldsymbol{\alpha}+\mathbb{Z}^{n}}$ is zero.
(2) Show that $M_{\boldsymbol{\alpha}+\mathbb{Z}^{n}}=M_{\boldsymbol{\alpha}+\mathbb{Z}^{n}}\left(* D_{i_{o}}\right)$, i.e., $x_{i_{o}}$ acts in a bijective way on $M_{\boldsymbol{\alpha}+\mathbb{Z}^{n}}$, if and only if $i_{o} \notin I(\boldsymbol{\alpha})$ or $i_{o} \in I(\boldsymbol{\alpha})$ and $\operatorname{var}_{i_{o}}$ is an isomorphism.
(3) Show that the quiver of $M_{\boldsymbol{\alpha}+\mathbb{Z}^{n}}\left(* D_{i_{o}}\right)$ is that of $M_{\boldsymbol{\alpha}+\mathbb{Z}^{n}}$ if $i_{o} \notin I(\boldsymbol{\alpha})$ and, otherwise, setting $\boldsymbol{k}=\left(\boldsymbol{k}^{\prime}, k_{i_{o}}\right)$, is isomorphic to the quiver is obtained from that of $M_{\boldsymbol{\alpha}+\mathbb{Z}^{n}}$ by replacing $M_{\boldsymbol{\alpha}+\left(\boldsymbol{k}^{\prime}, 1\right)}$ with $M_{\boldsymbol{\alpha}+\left(\boldsymbol{k}^{\prime}, 0\right)}$, $\operatorname{var}_{i_{o}}$ with Id and $\operatorname{can}_{i_{o}}$ with $\mathrm{N}_{i_{o}}$.

Let now $M$ be any $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-module of normal crossing type along $D$, and consider its quiver as in Remark 11.2.6.
(1) Show that $M$ is supported on $D_{i_{o}}$ if and only if, for any exponent $\boldsymbol{\alpha} \in[-1,0)^{n}$, we have $\alpha_{i_{o}}=-1$ and every vertex of the quiver with index $\boldsymbol{k} \in\{0,1\}^{n}$ satisfying $k_{i_{o}}=0$ vanishes.
(2) Show that $M=M\left(* D_{i_{o}}\right)$ if and only if $\operatorname{var}_{i_{o}}$ is bijective.

Definition 11.2.7. We say that $M$ is dual localized (resp. a minimal extension) along $D_{i_{o}}$, that we denote by $M=M\left(!D_{i_{o}}\right)$ (resp. $\left.M=M\left(!* D_{i_{o}}\right)\right)$ if $\operatorname{can}_{i_{o}}$ is bijective (resp. $\operatorname{can}_{i_{o}}$ is onto and $\operatorname{var}_{i_{o}}$ is injective).
(The relation with the notion of dual localization and of minimal extension introduced in Chapter 9 will be explained in the next subsection.)

Exercise 11.4. Define the endofunctor $\left(!D_{i_{o}}\right)$ resp. $\left(!* D_{i_{o}}\right)$ of the category of $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-modules of normal crossing type along $D$ so that the quiver of $M_{\boldsymbol{\alpha}+\mathbb{Z}^{n}}\left(!D_{i_{o}}\right)$,
resp. $M_{\boldsymbol{\alpha}+\mathbb{Z}^{n}}\left(!* D_{i_{o}}\right)$ is that of $M_{\boldsymbol{\alpha}+\mathbb{Z}^{n}}$ if $i_{o} \notin I(\boldsymbol{\alpha})$ and, otherwise, setting $\boldsymbol{k}=\left(\boldsymbol{k}^{\prime}, k_{i_{o}}\right)$, the quiver is obtained from that of $M_{\alpha+\mathbb{Z}^{n}}$ by replacing

- $M_{\boldsymbol{\alpha}+\left(\boldsymbol{k}^{\prime}, 0\right)}$ with $M_{\boldsymbol{\alpha}+\left(\boldsymbol{k}^{\prime}, 1\right)}, \operatorname{var}_{i_{o}}$ with $\mathrm{N}_{i_{o}}$ and $\operatorname{can}_{i_{o}}$ with Id,
- resp. $M_{\boldsymbol{\alpha}+\left(\boldsymbol{k}^{\prime}, 1\right)}$ with image $\left[\mathrm{N}_{i_{o}}: M_{\boldsymbol{\alpha}+\left(\boldsymbol{k}^{\prime}, 0\right)} \rightarrow M_{\boldsymbol{\alpha}+\left(\boldsymbol{k}^{\prime}, 0\right)}\right]$, $\operatorname{var}_{i_{o}}$ with the natural inclusion and $\operatorname{can}_{i_{o}}$ with $\mathrm{N}_{i_{o}}$.
Show that there is a natural morphism $M\left(!D_{i_{o}}\right) \rightarrow M\left(* D_{i_{o}}\right)$ whose image is $M\left(!* D_{i_{o}}\right)$.
Definition 11.2.8. We say that $M$ is a minimal extension along $D_{i \in I}$ if, for each $i \in I$, every $\operatorname{can}_{i}$ is onto and every $\operatorname{var}_{i}$ is injective.

Exercise 11.5. Say that $M$ minimal extension with support along $D_{i \in I}$ if, for each $i \in I$, either the source of every $\operatorname{can}_{i}$ is zero, or every $\operatorname{can}_{i}$ is onto and every $\operatorname{var}_{i}$ is injective. In other words, we accept $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-modules supported on the intersection of some components of $D$, which are minimal extension along any of the other components.

Show that any $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-module $M$ of normal crossing type along $D$ is a successive extension of such $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-modules which are minimal extensions with support along $D_{i \in I}$.

Example 11.2.9 (The simple case). Let $M$ be a $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-module of normal crossing type along $D$ which is simple (i.e., has no non-trivial such sub or quotient module). By the previous exercise, it must be a minimal extension with support along $D_{i \in I}$. Moreover, every nonzero vertex of its quiver has dimension one, so that $\mathrm{E}_{i}$ acts as $\alpha_{i}$ on $M_{\boldsymbol{\alpha}}$ and $\mathrm{N}_{i}$ acts by zero.

## Remark 11.2.10 (Suppressing the simplifying assumptions 11.1.2)

If $\ell<n$, every $M_{\boldsymbol{\alpha}}\left(\boldsymbol{\alpha} \in \mathbb{R}^{\ell}\right)$ has to be assumed $\mathscr{O}_{D_{(\ell)}}$-coherent in Definition 11.2.2(b). Since it is a $\mathscr{D}_{D_{(\ell)}}$-module, it must be $\mathscr{O}_{D_{(\ell)}}$-locally free of finite rank. All the previous results extend in a straightforward way to this setting by replacing $\mathbb{C}[x]$ with $\mathscr{O}_{D_{(\ell)}}[x]$ (where $x:=\left(x_{1}, \ldots, x_{\ell}\right)$ ) and $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$ with $\mathscr{D}_{D_{(\ell)}}[x]\left\langle\partial_{x}\right\rangle$.
11.2.b. Coherent $\mathscr{D}_{X}$-modules of normal crossing type. Let $\mathcal{M}$ be a coherent $\mathscr{D}_{X}$-module. In order to express the normal crossing property for $V$-filtrations, we introduce for every $\boldsymbol{\alpha} \in \mathbb{R}^{n}$ the sub-space $M_{\boldsymbol{\alpha}}$ of $\mathcal{M}$ defined by

$$
M_{\boldsymbol{\alpha}}=\bigcap_{i \in I} \bigcup_{k} \operatorname{Ker}\left(x_{i} \partial_{x_{i}}-\alpha_{i}\right)^{k}
$$

This is a $\mathbb{C}$-vector subspace of $\mathcal{M}$, which is contained in $V_{\alpha_{1}}^{(1)} \mathcal{M} \cap \cdots \cap V_{\alpha_{n}}^{(n)} \mathcal{M}$ if $\mathcal{M}$ is $\mathbb{R}$-specializable along each component $D_{i}$ of $D$ and we have $\mathbb{C}$-linear morphisms $x_{i}: M_{\boldsymbol{\alpha}} \rightarrow M_{\boldsymbol{\alpha}-\mathbf{1}_{i}}\left(\right.$ resp. $\partial_{x_{i}}: M_{\boldsymbol{\alpha}} \rightarrow M_{\boldsymbol{\alpha}+\mathbf{1}_{i}}$ ) as in the algebraic setting. Arguing as for $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-modules,

$$
\begin{equation*}
M:=\bigoplus_{\boldsymbol{\alpha} \in \mathbb{R}^{\ell}} M_{\boldsymbol{\alpha}} \tag{11.2.11}
\end{equation*}
$$

is a $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-submodule of $\mathcal{M}$, and there is a natural morphism

$$
\begin{equation*}
M \otimes_{\mathbb{C}[x]\left\langle\partial_{x}\right\rangle} \mathscr{D}_{X} \longrightarrow \mathcal{M} \tag{11.2.12}
\end{equation*}
$$

which is injective since $\mathscr{D}_{X}$ is $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-flat (because $\mathscr{O}_{X}$ is $\mathbb{C}[x]$-flat).

Definition 11.2.13. Let $\mathcal{M}$ be a coherent $\mathscr{D}_{X}$-module. We say that $\mathcal{M}$ is of normal crossing type along $D$ if the following properties are satisfied.
(a) The $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-submodule $M$ is of normal crossing type along $D$ (Definition 11.2.2).
(b) The natural morphism (11.2.12) is an isomorphism.

Proposition 11.2.14. Let $\mathcal{M}$ be a coherent $\mathscr{D}_{X}$-module which is of normal crossing type along $D$. Then the following properties are satisfied.
(1) $\mathcal{M}$ is $\mathbb{R}$-specializable along $D_{i}(i \in I)$, giving rise to $V$-filtrations $V_{\bullet}^{(i)} \mathcal{M}$. In particular, all properties of Definition 7.3.12 hold for each filtration $V_{\bullet}^{(i)} \mathcal{M}$.
(2) The $V$-filtrations $V_{\bullet}^{(i)} \mathcal{M}(i \in I)$ are compatible, in the sense of Definition 8.3.9 (see also Theorem 8.3.11).
(3) For $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{n}\right) \in \mathbb{R}^{n}$, we set $V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M}:=\bigcap_{i \in I} V_{\alpha_{i}}^{(i)} \mathcal{M}$. Then $V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M}$ is a $V_{0}^{(\boldsymbol{n})} \mathscr{D}_{X}$-module which is $\mathscr{O}_{X}$-coherent, and $\mathscr{O}_{X}$-locally free if $\alpha_{i}<0$ for all $i \in I$.
(4) For any multi-index $\boldsymbol{\alpha} \in \mathbb{R}^{n}$, the natural morphism of $\mathbb{C}\left[\mathrm{N}_{1}, \ldots, \mathrm{~N}_{n}\right]$-modules

$$
M_{\boldsymbol{\alpha}} \longrightarrow \operatorname{gr}_{\boldsymbol{\alpha}}^{V^{(n)}} \mathcal{M}:=\operatorname{gr}_{\alpha_{1}}^{V^{(1)}} \cdots \operatorname{gr}_{\alpha_{\ell}}^{V^{(\ell)}} \mathcal{M}
$$

is an isomorphism (see Remark 8.3.15 for the multi-grading).
Caveat 11.2.15. In order to apply Definition 8.3.9, one should regard $V_{\bullet}^{(i)} \mathcal{M}$ as a filtration indexed by $\mathbb{Z}$, by numbering the sequence of real numbers $\alpha_{i}$ such that $\operatorname{gr}_{\alpha_{i}}^{V^{(i)}} \mathcal{M} \neq 0$. See also the setup in Section 8.5.a. Setting

$$
V_{<\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M}:=\sum_{\substack{\boldsymbol{\beta} \leqslant \boldsymbol{\alpha} \\ \boldsymbol{\beta} \neq \boldsymbol{\alpha}}} V_{\boldsymbol{\beta}}^{(\boldsymbol{n})} \mathcal{M},
$$

the compatibility implies $\operatorname{gr}_{\boldsymbol{\alpha}}^{V^{(n)}} \mathcal{M}=V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M} / V_{<\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M}$.
Proof of Proposition 11.2.14.
(1) By Exercise 11.1, $M_{\leqslant \alpha}:=\bigoplus_{\alpha^{\prime} \leqslant \alpha} M_{\boldsymbol{\alpha}^{\prime}}$ is a $\mathbb{C}[x]\left\langle x \partial_{x}\right\rangle$-module which is of finite type over $\mathbb{C}[x]$, and $\mathbb{C}[x]$-free if $\alpha_{i}<0$ for all $i \in I$. The definition of $V$-filtrations along the hypersurfaces $x_{i}=0$ extend in an obvious way to this algebraic case (which in fact was first considered by Bernstein for the definition of the Bernstein polynomial). One checks that $V_{\alpha_{i}}^{(i)} M:=\bigoplus_{\alpha^{\prime} \mid \alpha_{i}^{\prime} \leqslant \alpha_{i}} M_{\alpha^{\prime}}$ satisfies the characteristic properties of the $V^{(i)}$-filtration of $M$, and thus so does

$$
V_{\alpha_{i}}^{(i)} \mathcal{M}=V_{\alpha_{i}}^{(i)} M \otimes_{V_{0}^{(i)} \mathbb{C}[x]\left\langle\partial_{x}\right\rangle} V_{0}^{(i)} \mathscr{D}_{X}
$$

for $\mathcal{M}$. In such a way, we get the $\mathbb{R}$-specializability of $\mathcal{M}$ along $D_{i}$.
(2) With the previous definition of $V_{\alpha_{i}}^{(i)} M$, we have $V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} M=M_{\leqslant \boldsymbol{\alpha}}$. Set $\boldsymbol{\alpha}=$ $\left(\boldsymbol{\alpha}_{I}, \boldsymbol{\alpha}_{J}, \boldsymbol{\alpha}_{K}\right)$ and choose $\boldsymbol{\alpha}_{I}^{\prime} \leqslant \boldsymbol{\alpha}_{I}$ and $\boldsymbol{\alpha}_{J}^{\prime} \leqslant \boldsymbol{\alpha}_{J}$. The compatibility property amounts to complete the star in any diagram as below in order to produce exact sequences:

The order $\leqslant$ is the partial natural order on $\mathbb{R}^{n}: \boldsymbol{\alpha}^{\prime} \leqslant \boldsymbol{\alpha} \Longleftrightarrow \alpha_{i}^{\prime} \leqslant \alpha_{i}, \forall i$. Then

$$
\star=\underset{\substack{\boldsymbol{\alpha}_{I}^{\prime} \notin \boldsymbol{\alpha}_{I}^{\prime \prime} \leqslant \boldsymbol{\alpha}_{I} \\ \boldsymbol{\alpha}_{J}^{\prime} \neq \boldsymbol{\alpha}_{J}^{\prime \prime} \leqslant \boldsymbol{\alpha}_{J} \\ \boldsymbol{\alpha}_{K}^{\prime \prime} \leqslant \boldsymbol{\alpha}_{K}}}{ } M_{\boldsymbol{\alpha}^{\prime \prime}}
$$

is a natural choice in order to complete the diagram.
By flatness of $V_{0}^{(\boldsymbol{n})} \mathscr{D}_{X}$ over $V_{0}^{(\boldsymbol{n})} \mathbb{C}[x]\left\langle\partial_{x}\right\rangle$, the similar diagram for $\mathcal{M}$ is obtained by tensoring by $V_{0}^{(\boldsymbol{n})} \mathscr{D}_{X}$, and is thus also exact, leading to the compatibility property of $V_{\bullet}^{(i)} \mathcal{M}(i \in I)$.
(3) The argument above reduces the proof of (3) to the case of $M$, which has been obtained in (1).
(4) This is now obvious from the previous description, since $\operatorname{gr}_{\alpha}^{V^{(n)}} \mathcal{M}=\operatorname{gr}_{\alpha}^{V^{(n)}} M$.

The morphisms between $\mathscr{D}_{X}$-modules of normal crossing type can also be regarded as being of normal crossing type, as follows from the next proposition.

Let $\varphi: \mathcal{M}_{1} \rightarrow \mathcal{N}_{2}$ be a morphism between coherent $\mathscr{D}_{X}$-modules of normal crossing type. Then $\varphi$ is compatible with the $V$-filtrations $V_{\bullet}^{(i)}$, and for every $\boldsymbol{\alpha} \in \mathbb{R}^{n}$, its multi-graded components $\operatorname{gr}_{\boldsymbol{\alpha}}^{V^{(n)}} \mathcal{M}_{1} \rightarrow \operatorname{gr}_{\alpha}^{V^{(n)}} \mathcal{M}_{2}$ do not depend on the order of grading (according to the compatibility of the $V$-filtrations and Remark 8.3.16). We denote this morphism by $\operatorname{gr}_{\boldsymbol{\alpha}}^{V^{(n)}} \varphi$. On the other hand, regarding $M_{\boldsymbol{\alpha}}$ as an $\mathbb{C}$ submodule of $\mathcal{M}$, we notice that $\varphi$ sends $M_{1, \boldsymbol{\alpha}}$ to $M_{2, \boldsymbol{\alpha}}$, due to the $\mathscr{D}$-linearity, and has no component from $M_{1, \boldsymbol{\alpha}}$ to $M_{2, \boldsymbol{\beta}}$ if $\boldsymbol{\beta} \neq \boldsymbol{\alpha}$. We denote by $\varphi_{\boldsymbol{\alpha}}$ the induced morphism $M_{1, \boldsymbol{\alpha}} \rightarrow M_{2, \boldsymbol{\alpha}}$. The following is now obvious.

Proposition 11.2.16. With respect to the isomorphism $M_{\boldsymbol{\alpha}} \xrightarrow{\sim} \operatorname{gr}_{\boldsymbol{\alpha}}^{V^{(n)}} \mathcal{M}$ of Proposition 11.2.14(4), $\varphi_{\boldsymbol{\alpha}}$ coincides with $\operatorname{gr}_{\boldsymbol{\alpha}}^{V^{(n)}} \varphi$.

Corollary 11.2.17. The category of $\mathscr{D}_{X}$-modules of normal crossing type along $D$ is abelian.

Proof. Each $\varphi_{\alpha}$ is $\mathbb{C}$-linear, hence its kernel and cokernel are also finite-dimensional.

## Remarks 11.2.18.

(1) If $\mathcal{M}$ is of normal crossing type along $D$, then for any $i \in I$ and any $\alpha_{i} \in \mathbb{R}$, $\operatorname{gr}_{\alpha_{i}}^{V^{(i)}} \mathcal{M}$ is of normal crossing type on $\left(D_{i}, \bigcup_{j \neq i} D_{j}\right)$ and $V_{\bullet}^{(j)} \operatorname{gr}_{\alpha_{i}}^{V^{(i)}} \mathcal{M}$ is the filtration naturally induced by $V_{\bullet}^{(j)} \mathcal{M}$ on $\operatorname{gr}_{\alpha_{i}}^{V^{(i)}} \mathcal{M}$, that is,

$$
V_{\bullet}^{(j)} \operatorname{gr}_{\alpha_{i}}^{V^{(i)}} \mathcal{M}=\frac{V_{\bullet}^{(j)} \mathcal{M} \cap V_{\alpha_{i}}^{(i)} \mathcal{M}}{V_{\bullet}^{(j)} \mathcal{M} \cap V_{<\alpha_{i}}^{(i)} \mathcal{M}} .
$$

Indeed, due to the isomorphism (11.2.12), it is enough to prove the result for the multi-graded module $M:=\mathrm{gr}^{V^{(n)}} \mathcal{M}$, for which all assertions are clear.
(2) We deduce from (11.2.12) a decomposition $\mathcal{M}=\bigoplus_{\boldsymbol{\alpha} \in \boldsymbol{A}} \mathcal{M}_{\boldsymbol{\alpha}+\mathbb{Z}^{n}}$ similar to that of Remark 11.2.3. We have

$$
V_{\boldsymbol{\alpha}+\boldsymbol{m}}^{(\boldsymbol{n})} M_{\boldsymbol{\alpha}+\mathbb{Z}^{n}}=\bigoplus_{n \leqslant m} M_{\boldsymbol{\alpha}+\boldsymbol{n}}
$$

It follows that, for $\boldsymbol{\alpha} \in \boldsymbol{A}$ (so that $\alpha_{i}<0$ for all $i$ ), we have

$$
V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} M_{\boldsymbol{\alpha}+\mathbb{Z}^{n}}=M_{\boldsymbol{\alpha}} \otimes_{\mathbb{C}} \mathbb{C}[x],
$$

and we conclude that $V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} M_{\boldsymbol{\alpha}+\mathbb{Z}^{n}}$ is $\mathbb{C}[x]$-locally free of finite rank. It follows then easily that the same property holds for $V_{\boldsymbol{\alpha}-\boldsymbol{k}}^{(\boldsymbol{n})} M_{\boldsymbol{\alpha}+\mathbb{Z}^{n}}$ for every $\boldsymbol{k} \in \mathbb{N}^{n}$, and that $V_{\boldsymbol{\alpha}+\boldsymbol{k}}^{(\boldsymbol{n})} M_{\boldsymbol{\alpha}+\mathbb{Z}^{n}}$ is of finite type over $\mathbb{C}[x]$ for every $\boldsymbol{k} \in \mathbb{Z}^{n}$. From (11.2.12) we conclude that $V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{N}$ is $\mathscr{O}_{X}$-coherent for every $\boldsymbol{\alpha} \in \mathbb{R}^{n}$ and is $\mathscr{O}_{X}$-locally free in the neighbourhood of the origin for $\boldsymbol{\alpha} \in(-\infty, 0)^{n}$. In the latter case, we can thus regard $\left(V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M}\right)^{\text {left }}$ as an $\mathscr{O}_{X}$-locally free module of finite rank endowed with a flat $D$-logarithmic connection. Moreover, for any $\boldsymbol{\alpha} \in \mathbb{R}^{n}, V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M}_{X \backslash D}$ is $\mathscr{O}_{X \backslash D}$ locally free, and more precisely $V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M}(* D)$ is $\mathscr{O}_{X}(* D)$-locally free.

## Behaviour with respect to localization, dual localization and minimal extension

Let us fix $i \in I$ and set $\boldsymbol{\alpha}=\left(\boldsymbol{\alpha}^{\prime}, \alpha_{i}\right)$. By $\mathbb{R}$-specializability along $D_{i}$ we have isomorphisms

$$
x_{i}: V_{\alpha_{i}}^{(i)} \mathcal{M} \xrightarrow{\sim} V_{\alpha_{i}-1}^{(i)} \mathcal{M},\left(\alpha_{i}<0\right) \quad \text { and } \quad \partial_{x_{i}}: \operatorname{gr}_{\alpha_{i}}^{V^{(i)}} \mathcal{M} \xrightarrow{\sim} \operatorname{gr}_{\alpha_{i}+1}^{V^{(i)}} \mathcal{M},\left(\alpha_{i}>-1\right) .
$$

One checks on $M$, and then on $\mathcal{M}$ due to (11.2.12), that they induce isomorphisms

$$
\begin{align*}
x_{i}: V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M} & \xrightarrow{\sim} V_{\boldsymbol{\alpha}-\mathbf{1}_{i}}^{(\boldsymbol{n})} \mathcal{M},\left(\alpha_{i}<0\right) \\
\partial_{x_{i}}: V_{\boldsymbol{\alpha}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \operatorname{gr}_{\alpha_{i}}^{V^{(i)}} \mathcal{M} & \xrightarrow{\sim} V_{\boldsymbol{\alpha}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \operatorname{gr}_{\alpha_{i}+1} V^{(i)} \mathcal{M},\left(\alpha_{i}>-1\right) . \tag{11.2.19}
\end{align*}
$$

The following lemma shows that the localization (resp. dual localization, resp. minimal extension) property along one component $D_{i_{o}}$ of $D$ is compatible the other filtrations $V^{(i)}$.

Lemma 11.2.20. Assume that $\mathcal{M}$ is of normal crossing type along $D$. Let us fix $i_{o} \in I$ and let us set $n^{\prime}=n-1$, corresponding to forgetting $i_{o}$. Then, for every $\boldsymbol{\alpha}^{\prime} \in \mathbb{R}^{n^{\prime}}$, one of the following properties

$$
\begin{aligned}
& \operatorname{var}_{i_{o}}: V_{\boldsymbol{\alpha}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \operatorname{gr}_{0}^{V^{\left(i_{o}\right)}} \mathcal{M} \longrightarrow V_{\boldsymbol{\alpha}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \operatorname{gr}_{-1}^{V^{\left(i_{o}\right)} \mathcal{M}} \quad \text { is injective, resp. bijective, }
\end{aligned}
$$

holds as soon as it holds when forgetting $V_{\boldsymbol{\alpha}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)}$.
Proof. We first work with $M$. Setting $\boldsymbol{\alpha}=\left(\boldsymbol{\alpha}^{\prime}, \alpha_{i_{o}}\right)$, the morphism $x_{i_{o}}: \operatorname{gr}_{0}^{V^{\left(i_{o}\right)}} \mathcal{M} \rightarrow$
 similarly for $\partial_{x_{i_{o}}}$. Therefore $\operatorname{var}_{i_{o}}$ is injective (resp. bijective) or $\operatorname{can}_{i_{o}}$ is surjective (resp. bijective) if and only if each $\boldsymbol{\alpha}^{\prime}$-component is so. This implies the lemma for $M$. One concludes that the lemma holds for $\mathcal{M}$ by flat tensorisation.

By a similar argument, considering $M$ first, we obtain:
Lemma 11.2.21. Let $\mathcal{M}$ be a coherent module of normal crossing type along $D$. Let us fix $i_{o} \in I$. Then $\mathcal{M}\left(* D_{i_{o}}\right), \mathcal{M}\left(!D_{i_{o}}\right), \mathcal{M}\left(!* D_{i_{o}}\right)$ are of normal crossing type along $D$.

Remark 11.2.22. It is now easy to show that the two possible definitions of $M\left(!D_{i_{o}}\right)$ and $M\left(!* D_{i_{o}}\right)$ (see Definition 11.2.7) coincide.

Definition 11.2.23. We say that $\mathcal{M}$ is a minimal extension along $D_{i \in I}$ if the corresponding $M$ is a minimal extension in the sense of Definition 11.2.8.

Exercise 11.6. Let $\mathcal{M}$ be a coherent module of normal crossing type along $D$. Show that $\mathcal{M}$ is a successive extension of modules of normal crossing type along $D$, each of which being moreover a minimal extension with support along $D_{i \in I}$. [Hint: use Exercise 11.5.]

## Remark 11.2.24 (Suppressing the simplifying assumptions 11.1.2)

If $\ell<n$, we apply the same changes as in Remark 11.2.10. All the previous results extend in a straightforward way to this setting.
11.2.c. Coherent filtrations of normal crossing type. We now extend the notion of "normal crossing type" to filtered coherent $\mathscr{D}$-modules. Of course the underlying $\mathscr{D}$-module should be of normal crossing type, but the isomorphism (11.2.12), together with the decomposition (11.2.11), is not expected to hold at the filtered level. This would be a too strong condition. On the other hand, the properties in Proposition 11.2 .14 can be naturally extended to the filtered case. We keep the simplifying assumptions 11.1.2.

Definition 11.2.25. Let $(\mathcal{M}, F, \mathcal{M})$ be a coherent filtered $\mathscr{D}_{X}$-module. We say that $(\mathcal{M}, F, \mathcal{M})$ is of normal crossing type along $D$ if
(1) $\mathcal{M}$ is of normal crossing type along $D$ (see Definition 11.2.13),
(2) ( $\left.\mathcal{M}, F_{\mathbf{\bullet}} \mathcal{M}\right)$ is $\mathbb{R}$-specializable along every component $D_{i}$ of $D$ (see Section 8.4),
(3) the filtrations $\left(F_{\bullet} \mathcal{M}, V_{\bullet}^{(1)} \mathcal{M}, \ldots, V_{\bullet}^{(n)} \mathcal{N}\right)$ are compatible (see Definition 8.3.9).

## Remarks 11.2.26.

(a) Condition (3) implies that $\operatorname{gr}_{p}^{F} \operatorname{gr}_{\boldsymbol{\alpha}}^{V^{(n)}} \mathcal{M}$ does not depend on the way $\operatorname{gr}_{\boldsymbol{\alpha}}^{V^{(n)}} \mathcal{M}$ is computed.
(b) Note that (2) implies $11.2 .14(1)$ for $\mathcal{M}$, and similarly (3) implies 11.2.14(2). So the condition that $\mathcal{M}$ is of normal crossing type along $D$ only adds the existence of the isomorphism (11.2.12).
(c) Let us recall that $V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M}$ is $\mathscr{O}_{X}$-coherent for every $\boldsymbol{\alpha} \in \mathbb{R}^{n}$ (see Remark 11.2.18(2)). Since $F_{p} \mathcal{M}$ is $\mathscr{O}_{X}$-coherent, it follows that $F_{p} V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M}:=F_{p} \mathcal{M} \cap V_{\boldsymbol{\alpha}}^{(n)} \mathcal{M}$ (see $\S 8.4)$ and $\operatorname{gr}_{p}^{F} V_{\boldsymbol{\alpha}}^{(n)} \mathcal{M}$ are also $\mathscr{O}_{X}$-coherent and therefore the filtration $F_{\bullet} V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M}$ is locally finite, hence is a coherent $F_{\bullet} V_{0}^{(\boldsymbol{n})} \mathscr{D}_{X}$-filtration.
(d) Moreover, each $\operatorname{gr}_{p}^{F} V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M}$ is $\mathscr{O}_{X}$-locally free if $\alpha_{i}<0$ for all $i \in I$. Indeed, the family $\left(F_{p} \mathcal{M}, V_{\alpha_{1}+k_{1}}^{(1)} \mathcal{M}, \ldots, V_{\alpha_{n}+k_{n}}^{(n)} \mathcal{M}\right)\left(p \in \mathbb{Z}, k_{1}, \ldots, k_{n} \in \mathbb{N}\right)$ is a compatible family; the $\mathscr{O}_{X}$-coherent sheaf $\operatorname{gr}_{p}^{F} V_{\boldsymbol{\alpha}}^{(n)} \mathcal{M}$ has generic rank (on its support) $\leqslant \operatorname{dim} \operatorname{gr}_{p}^{F} V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M} /\left(x_{1}, \ldots, x_{n}\right)$; but

$$
\begin{aligned}
\sum_{p} \operatorname{dim} \operatorname{gr}_{p}^{F} V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M} /\left(x_{1}, \ldots, x_{n}\right) & =\operatorname{dim} V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M} /\left(x_{1}, \ldots, x_{n}\right) \\
& =\operatorname{rk} V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M}=\sum_{p} \operatorname{rkgr}_{p}^{F} V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M}
\end{aligned}
$$

so in fact $\operatorname{gr}_{p}^{F} V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M} /\left(x_{1}, \ldots, x_{n}\right)$ has dimension equal to the generic rank of $\operatorname{gr}_{p}^{F} V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M}$. As a consequence, $\operatorname{gr}_{p}^{F} V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M}$ is $\mathscr{O}_{X}$-locally free.
(e) Since each $\operatorname{gr}_{\boldsymbol{\alpha}}^{V^{(n)}} \mathcal{M}$ is finite dimensional, the induced filtration $F \cdot \operatorname{gr}_{\alpha}^{V^{(n)}} \mathcal{M}$ is finite, and there exists a (non-canonical) splitting compatible with $F_{\bullet}$ :

$$
F_{p} \operatorname{gr}_{\boldsymbol{\alpha}}^{V^{(n)}} \mathcal{M} \simeq \bigoplus_{q \leqslant p} \operatorname{gr}_{q}^{F} \operatorname{gr}_{\boldsymbol{\alpha}}^{V^{(n)}} \mathcal{M}
$$

(f) There are a priori two ways for defining the filtration $F_{\boldsymbol{\bullet}} M_{\boldsymbol{\alpha}}$, namely, either by inducing it on $M_{\boldsymbol{\alpha}} \subset \mathcal{M}$, or by inducing it on $\operatorname{gr}_{\boldsymbol{\alpha}}^{V^{(n)}} \mathcal{M}$ and transport it by means of the isomorphism $M_{\boldsymbol{\alpha}} \xrightarrow{\sim} \operatorname{gr}_{\boldsymbol{\alpha}}^{V^{(n)}} \mathcal{M}$. We always consider the latter one. The filtration $F . \mathcal{M}$ is a priori not isomorphic to $\bigoplus_{\alpha} F_{\bullet} \operatorname{gr}_{\alpha}^{V^{(n)}} \mathcal{M}$ by means of the isomorphism $\mathcal{M} \simeq$ $\bigoplus_{\alpha} \operatorname{gr}_{\boldsymbol{\alpha}}^{V^{(n)}} \mathcal{M}$ induced by $11.2 .14(4)$ and (11.2.12). Using the compatibility of the filtrations, we have

$$
F_{p} M_{\boldsymbol{\alpha}}=M_{\boldsymbol{\alpha}} \cap\left(F_{p} V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M}+V_{<\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M}\right) \subset \mathcal{M}
$$

The graded filtered module $\left(\bigoplus_{\alpha} M_{\alpha}, \bigoplus_{\alpha} F_{\mathbf{\bullet}} M_{\boldsymbol{\alpha}}\right)$ is obviously of normal crossing type if $(\mathcal{M}, F, \mathcal{M})$ is so.

Inductive arguments below will make use of the following lemma.

Lemma 11.2.27. Assume that $(\mathcal{M}, F, \mathcal{M})$ is of normal crossing type along $D$. Then for any $i \in I$ and any $\alpha_{i} \in \mathbb{R}$, $\left(\operatorname{gr}_{\alpha_{i}}^{V^{(i)}} \mathcal{M}, F_{\bullet} \operatorname{gr}_{\alpha_{i}}^{V^{(i)}} \mathcal{M}\right)$ is of normal crossing type on $\left(D_{i}, \bigcup_{j \neq i} D_{j}\right)$, where $F \cdot \operatorname{gr}_{\alpha_{i}}^{V^{(i)}} \mathcal{M}$ is the filtration naturally induced by $F \cdot \mathcal{M}$ on $\operatorname{gr}_{\alpha_{i}}^{V^{(i)}} \mathcal{M}$.

Proof. We know by Remark 11.2.18(1) that $\operatorname{gr}_{\alpha_{i}}^{V^{(i)} \mathcal{M}}$ is of normal crossing type on $\left(D_{i}, \bigcup_{j \neq i} D_{j}\right)$, and that the filtrations $V_{\bullet}^{(j)}$ on $\operatorname{gr}_{\alpha_{i}}^{V^{(i)}} \mathcal{M}$ are naturally induced by $V_{\bullet}^{(j)} \mathcal{M}$. It follows that $\left(F_{\bullet} \operatorname{gr}_{\alpha_{i}}^{V^{(i)}} \mathcal{M},\left(V_{\bullet}^{(j)} \operatorname{gr}_{\alpha_{i}}^{V^{(i)}} \mathcal{M}\right)_{j \neq i}\right)$ are compatible (see Remark 8.3.10). We know, by Proposition 8.8.2, $\left(\operatorname{gr}_{\alpha_{i}}^{V^{(i)}} \mathcal{M}, F_{\bullet} \operatorname{gr}_{\alpha_{i}}^{V^{(i)}} \mathcal{M}\right)$ is coherent as a filtered $\mathscr{D}_{D_{i}}$-module. Note also that, setting $\boldsymbol{\alpha}^{\prime}=\left(\alpha_{j}\right)_{j \neq i}$ and $\boldsymbol{n}^{\prime}=(j)_{j \neq i}$, we have

$$
\operatorname{gr}_{p}^{F} \operatorname{gr}_{\boldsymbol{\alpha}^{\prime}}^{V^{\left(n^{\prime}\right)}} \operatorname{gr}_{\alpha_{i}}^{V^{(i)}} \mathcal{M}=\operatorname{gr}_{p}^{F} \operatorname{gr}_{\boldsymbol{\alpha}}^{V^{(n)}} \mathcal{M}
$$

(since, by the compatibility property, we can take graded objects in any order).
It remains to showing the $\mathbb{R}$-specializability property, namely,

$$
\begin{gathered}
x_{j}: F_{p} V_{\alpha_{j}}^{(j)} \operatorname{gr}_{\alpha_{i}}^{V^{(i)}} \mathcal{M} \xrightarrow{\sim} F_{p} V_{\alpha_{j}-1}^{(j)} \operatorname{gr}_{\alpha_{i}}^{V^{(i)}} \mathcal{M}, \quad \forall p, \forall j \neq i, \forall \alpha_{j}<0, \\
\partial_{x_{j}}: F_{p} \operatorname{gr}_{\alpha_{j}}^{V^{(j)}} \operatorname{gr}_{\alpha_{i}}^{V^{(i)}} \mathcal{M} \xrightarrow{\sim} F_{p+1} \operatorname{gr}_{\alpha_{j}+1}^{V^{(j)}} \operatorname{gr}_{\alpha_{i}}^{V^{(i)}} \mathcal{M}, \quad \forall p, \forall j \neq i, \forall \alpha_{j}>-1 .
\end{gathered}
$$

Let us first show that, by applying $\operatorname{gr}_{\alpha_{i}}^{V^{(i)}}$, we get isomorphisms

$$
\begin{gather*}
x_{j}: \operatorname{gr}_{\alpha_{i}}^{V^{(i)}} F_{p} V_{\alpha_{j}}^{(j)} \mathcal{M} \xrightarrow{\sim} \operatorname{gr}_{\alpha_{i}}^{V^{(i)}} F_{p} V_{\alpha_{j}-1}^{(j)} \mathcal{M}, \quad \forall p, \forall j \neq i, \forall \alpha_{j}<0,  \tag{11.2.28}\\
\partial_{x_{j}}: \operatorname{gr}_{\alpha_{i}}^{V^{(i)}} F_{p} \operatorname{gr}_{\alpha_{j}}^{V^{(j)}} \mathcal{M} \xrightarrow{\sim} \operatorname{gr}_{\alpha_{i}}^{V^{(i)}} F_{p+1} \operatorname{gr}_{\alpha_{j}+1}^{V^{(j)}} \mathcal{M}, \quad \forall p, \forall j \neq i, \forall \alpha_{j}>-1 . \tag{11.2.29}
\end{gather*}
$$

By the $\mathbb{R}$-specializability of $(\mathcal{M}, F, \mathcal{M})$ along $D_{j}$ and since $\mathcal{M}$ is of normal crossing type, we have isomorphisms

$$
F_{p} V_{\alpha_{j}}^{(j)} \mathcal{M} \xrightarrow[\sim]{\sim} F_{p} V_{\alpha_{j}-1}^{(j)} \mathcal{M}, \quad\left\{\begin{array} { l } 
{ V _ { \alpha _ { i } } ^ { ( i ) } V _ { \alpha _ { j } } ^ { ( j ) } \mathcal { M } } \\
{ V _ { < \alpha _ { i } } ^ { ( i ) } V _ { \alpha _ { j } } ^ { ( j ) } \mathcal { M } }
\end{array} \stackrel { x _ { j } } { \sim } \left\{\begin{array}{l}
V_{\alpha_{i}}^{(i)} V_{\alpha_{j}-1}^{(j)} \mathcal{M} \\
V_{<\alpha_{i}}^{(i)} V_{\alpha_{j}-1}^{(j)} \mathcal{M}
\end{array}\right.\right.
$$

hence isomorphisms

$$
\left\{\begin{array} { l } 
{ V _ { \alpha _ { i } } ^ { ( i ) } F _ { p } V _ { \alpha _ { j } } ^ { ( j ) } \mathcal { M } } \\
{ V _ { < \alpha _ { i } } ^ { ( i ) } F _ { p } V _ { \alpha _ { j } } ^ { ( j ) } \mathcal { M } }
\end{array} \stackrel { x _ { j } } { \sim } \left\{\begin{array}{l}
V_{\alpha_{i}}^{(i)} F_{p} V_{\alpha_{j}-1}^{(j)} \mathcal{M} \\
V_{<\alpha_{i}}^{(i)} F_{p} V_{\alpha_{j}-1}^{(j)} \mathcal{M}
\end{array}\right.\right.
$$

and thus the isomorphisms (11.2.28). We argue similarly for the isomorphisms (11.2.29). Now, the desired assertion follows from the compatibility property (3) which enables us to switch $F_{p} V_{\alpha_{j}}^{(j)}$ or $F_{p} \mathrm{gr}_{\alpha_{j}}^{V^{(j)}}$ with $\mathrm{gr}_{\alpha_{i}}^{V^{(i)}}$.

By the same argument as above, setting $\boldsymbol{\alpha}=\left(\boldsymbol{\alpha}^{\prime}, \alpha_{j}\right)$ and $n^{\prime}=n-1$, the filtered analogue of (11.2.19) holds (any $\boldsymbol{\alpha}^{\prime} \in \mathbb{R}^{n^{\prime}}, p \in \mathbb{Z}$ ):

$$
\begin{gather*}
F_{p} V_{\boldsymbol{\alpha}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} V_{\alpha_{j}}^{(j)} \mathcal{M} \xrightarrow[\sim]{\sim} F_{p} V_{\boldsymbol{\alpha}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} V_{\alpha_{j}-1}^{(j)} \mathcal{M} \quad \text { if } \alpha_{j}<0, \\
F_{p} V_{\boldsymbol{\alpha}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \operatorname{gr}_{\alpha_{j}} V^{(j)} \mathcal{M} \xrightarrow[\sim]{\partial_{x_{j}}} F_{p+1} V_{\boldsymbol{\alpha}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \operatorname{gr}_{\alpha_{j}+1} V^{(j)} \mathcal{M} \quad \text { if } \alpha_{j}>-1 . \tag{11.2.30}
\end{gather*}
$$

The following lemma is similar to Lemma 11.2.20, but weaker when considering surjectivity for $\operatorname{can}_{i_{o}}$.

Lemma 11.2.31. Assume that ( $\mathcal{M}, F, \mathcal{M})$ is of normal crossing type along $D$. Let us fix $i_{o} \in I$ and let us set $n^{\prime}=n-$, corresponding to forgetting $i_{o}$. Then, for every $\boldsymbol{\alpha}^{\prime} \in \mathbb{R}^{n^{\prime}}$, one of the following properties

$$
\begin{align*}
& \operatorname{can}_{i_{o}}: F_{p} V_{\boldsymbol{\alpha}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \operatorname{gr}_{-1}^{V_{-1}^{\left(i_{o}\right)} \mathcal{M} \longrightarrow F_{p+1} V_{\boldsymbol{\alpha}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \operatorname{gr}_{0}^{V^{\left(i_{o}\right)}} \mathcal{M} \quad \text { is bijective, }} \\
& \operatorname{var}_{i_{o}}: F_{p} V_{\boldsymbol{\alpha}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \operatorname{gr}_{0}^{V^{\left(i_{o}\right)} \mathcal{M} \longrightarrow F_{p} V_{\boldsymbol{\alpha}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \operatorname{gr}_{-1}^{V^{\left(i_{o}\right)}} \mathcal{M} \quad \text { is }\left\{\begin{array}{l}
\text { injective, } \\
\text { resp. bijective, },
\end{array}\right.} \tag{11.2.31*}
\end{align*}
$$

holds for all $p$ as soon as it holds when forgetting $V_{\boldsymbol{\alpha}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)}$.
Remark 11.2.32. As a consequence, if $\operatorname{var}_{i_{o}}$ is injective, then the first line of (11.2.30) with $j=i_{o}$ also holds for $\alpha_{j}=0$. That the lemma does not a priori hold when $\operatorname{can}_{i_{o}}$ is onto leads to the definition below.

Definition 11.2.33 (Minimal extension along $D_{i \in I}$ ). Let ( $\left.\mathcal{M}, F_{\bullet} \mathcal{M}\right)$ be a coherent filtered $\mathscr{D}_{X}$-module of normal crossing type along $D$. We say that $\left(\mathcal{M}, F_{\bullet} \mathcal{M}\right)$ is a minimal extension along $D_{i \in I}$ if $\mathcal{M}$ is a minimal extension along each $D_{i}(i \in I)$ and moreover, for each $i_{o} \in I$, and every $\boldsymbol{\alpha}^{\prime} \in \mathbb{R}^{n^{\prime}}$ (equivalently, every $\boldsymbol{\alpha}^{\prime} \in[-1,0]^{n^{\prime}}$ ),

$$
\operatorname{can}_{i_{o}}: F_{p} V_{\boldsymbol{\alpha}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \operatorname{gr}_{-1}^{V^{\left(i_{o}\right)}} \mathcal{M} \longrightarrow F_{p+1} V_{\boldsymbol{\alpha}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \operatorname{gr}_{0}^{V^{\left(i_{o}\right)}} \mathcal{M} \text { is onto, } \forall p
$$

Note that, if we forget the $F$-filtration, there is no ambiguity according to Lemma 11.2.20, and if $n=1$ this notion is equivalent to that of Definition 7.7.3.

Proposition 11.2.34 (Properties of $\left.F_{p} V_{\boldsymbol{\alpha}}^{(n)} \mathcal{M}\right)$. Let $(\mathcal{M}, F, \mathcal{M})$ be a coherent filtered $\mathscr{D}_{X}$-module of normal crossing type along $D$. Set $\mathcal{M}_{0}:=V_{0}^{(\boldsymbol{n})} \mathcal{M}$. For $\alpha \in \mathbb{R}^{n}$, let us set $F_{p} V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M}:=F_{p} \mathcal{M} \cap V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M}$. Then
(1) $F_{\boldsymbol{\bullet}} V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M}$ is a coherent $F_{\boldsymbol{\bullet}} V_{0}^{(\boldsymbol{n})} \mathscr{D}_{X}$-filtration.
(2) The filtrations $\left(F_{\bullet} \mathcal{M}_{0}, V_{\bullet}^{(1)} \mathcal{M}_{0}, \ldots, V_{\bullet}{ }^{(n)} \mathcal{M}_{0}\right)$ are compatible and

$$
F_{p} \mathcal{M}=\sum_{q \geqslant 0}\left(F_{p-q} \mathcal{M}_{0}\right) \cdot F_{q} \mathscr{D}_{X}
$$

Proof. The compatibility property of the filtrations on $\mathcal{M}_{0}$ clearly follows from that on $\mathcal{M}$, as noted in Remark $8.3 .10(2)$. By the same argument we have compatibility for the family of filtrations on each $V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M}\left(\boldsymbol{\alpha} \in \mathbb{R}^{n}\right)$.

It remains to justify the expression for $F_{p} \mathcal{M}$. We have seen in the proof of Lemma 11.2.27 that, for $\boldsymbol{k} \geqslant 0$ and any $i \in I$, setting $\boldsymbol{k}=\left(\boldsymbol{k}^{\prime}, k_{i}\right)$, we have an isomorphism

$$
\partial_{x_{i}}: F_{p-1} V_{\boldsymbol{k}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \operatorname{gr}_{k_{i}}^{V^{(i)}} \mathcal{M} \xrightarrow{\sim} F_{p} V_{\boldsymbol{k}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \operatorname{gr}_{k_{i}+1}^{V^{(i)}} \mathcal{M},
$$

and thus

$$
F_{p} V_{\boldsymbol{k}+\mathbf{1}_{i}}^{(\boldsymbol{n})} \mathcal{M}=F_{p-1} V_{\boldsymbol{k}}^{(\boldsymbol{n})} \mathcal{M} \cdot \partial_{x_{i}}+F_{p} V_{\boldsymbol{k}}^{(\boldsymbol{n})} \mathcal{M}
$$

which proves (2) by an easy induction.

The property $11.2 .34(2)$ can be made more precise. For $\boldsymbol{\alpha} \in[-1,0]^{n}$ and $p \in \mathbb{Z}$, let $E_{\boldsymbol{\alpha}, p}$ be a finite $\mathbb{C}$-vector space of sections of $F_{p} V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M}$ whose image in $\operatorname{gr}_{p}^{F} \operatorname{gr}_{\boldsymbol{\alpha}} V^{(n)} \mathcal{M}$ is an $\mathbb{C}$-basis of sections of this free $\mathbb{C}$-module. Given any $\gamma \in \mathbb{R}^{n}$, we decompose it as $\left(\gamma^{\prime}, 0, \gamma^{\prime \prime}\right)$, where each component $\gamma_{i}$ of $\gamma^{\prime}$ (resp. $\gamma^{\prime \prime}$ ) satisfies $\gamma_{i}<0$ (resp. $\gamma_{i}>0$ ). When $\boldsymbol{\gamma}$ is fixed, any $\boldsymbol{\alpha} \in[-1,0]^{n}$ decomposes correspondingly as $\left(\boldsymbol{\alpha}^{\prime}, \boldsymbol{\alpha}^{o}, \boldsymbol{\alpha}^{\prime \prime}\right)$, of respective size $n^{\prime}, n^{o}, n^{\prime \prime}$.

Proposition 11.2.35. With these assumptions and notation, for every $\gamma \in \mathbb{R}^{n}$ and $p \in \mathbb{Z}$, we have

$$
F_{p} V_{\gamma}^{(\boldsymbol{n})} \mathcal{M}=\sum_{\boldsymbol{\alpha}^{\prime \prime} \in(-1,0]^{n^{\prime \prime}}} \sum_{\substack{\boldsymbol{b}^{\prime \prime} \\ \forall i, b_{i}+\alpha_{i} \leqslant \gamma_{i}}} F_{p-\left|\boldsymbol{b}^{\prime \prime}\right|} V_{\left(\boldsymbol{\gamma}^{\prime}, 0, \boldsymbol{\alpha}^{\prime \prime}\right)}^{(\boldsymbol{n})} \mathcal{M} \cdot \partial_{x}^{\boldsymbol{b}^{\prime \prime}},
$$

and, for every $\boldsymbol{\alpha}^{\prime \prime} \in(-1,0]^{n^{\prime \prime}}$,

$$
F_{q} V_{\left(\boldsymbol{\gamma}^{\prime}, 0, \alpha^{\prime \prime}\right)}^{(\boldsymbol{n})} \mathcal{M}=\sum_{\boldsymbol{\alpha}^{\prime} \in[-1,0)^{n^{\prime}}} E_{\left(\boldsymbol{\alpha}^{\prime}, 0, \boldsymbol{\alpha}^{\prime \prime}\right), q} \cdot x^{\boldsymbol{a}^{\prime}} \mathscr{O}_{X}
$$

where $\boldsymbol{a}^{\prime}$ has the indices of $\gamma^{\prime}$ and for each such index $i, \alpha_{i}-a_{i} \leqslant \gamma_{i}$, that is,

$$
a_{i}= \begin{cases}-\left[\gamma_{i}\right]-1 & \text { if } \alpha_{i} \leqslant \gamma_{i}-\left[\gamma_{i}\right]-1 \\ -\left[\gamma_{i}\right] & \text { if } \alpha_{i}>\gamma_{i}-\left[\gamma_{i}\right]-1 .\end{cases}
$$

Proof. The first equality is obtained by induction from the second line of (11.2.30), and the second equality comes from the first line of (11.2.30).

## Remark 11.2.36 (The case of a minimal extension along $D_{i \in I}$ )

In that case (Definition 11.2.33), Proposition 11.2.34 holds with the replacement of $\mathcal{M}_{0}$ with $\mathcal{M}_{<0}:=V_{<0}^{n} \mathcal{M}=\bigcap_{i \in I} V_{<0}^{(i)} \mathcal{M}$, and Proposition 11.2.35 reads as follows. We now decompose $\gamma$ as $\left(\gamma^{\prime}, \gamma^{\prime \prime}\right)$, where each component $\gamma_{i}$ of $\gamma^{\prime}$ (resp. $\gamma^{\prime \prime}$ ) satisfies $\gamma_{i}<0$ (resp. $\gamma_{i} \geqslant 0$ ). Then

$$
F_{p} V_{\gamma}^{(n)} \mathcal{M}=\sum_{\boldsymbol{\alpha} \in[-1,0)^{n}} \sum_{\substack{\boldsymbol{b}^{\prime \prime} \\ \forall i, b_{i}+\alpha_{i} \leqslant \gamma_{i}}} E_{\boldsymbol{\alpha}, p-\left|\boldsymbol{b}^{\prime \prime}\right|} \cdot x^{\boldsymbol{a}^{\prime}} \partial_{x}^{\boldsymbol{b}^{\prime \prime}} \mathscr{O}_{X},
$$

where $\boldsymbol{a}^{\prime}$ is as in Proposition 11.2.35.
The compatibility property of the filtrations ( $\left.F_{\bullet} \mathcal{M}, V_{\bullet}^{(1)} \mathcal{M}, \ldots, V_{\bullet}^{(n)} \mathcal{M}\right)$ can be checked on $V_{0}^{(n)} \mathcal{M}$, as asserted by the proposition below.

Proposition 11.2.37 (From $\mathcal{M}_{0}$ to $\mathcal{M}$ ). Let $\mathcal{M}$ be a coherent $\mathscr{D}_{X}$-module of normal crossing type along $D$. Set $\mathcal{M}_{0}:=V_{0}^{(\boldsymbol{n})} \mathcal{M}$. Denote by $V_{\bullet}^{(i)} \mathcal{M}_{0}$ the filtration naturally induced by $V_{\bullet}^{(i)} \mathcal{M}$ and let $F_{\cdot} \mathcal{M}_{0}$ be any coherent $F_{\bullet} V_{0}^{(\boldsymbol{n})} \mathscr{D}_{X}$-filtration such that $\left(F_{\bullet} \mathcal{M}_{0}, V_{\bullet}^{(1)} \mathcal{M}_{0}, \ldots, V_{\bullet}^{(n)} \mathcal{M}_{0}\right)$ are compatible filtrations and that $\left(\mathcal{M}_{0}, F_{\bullet} \mathcal{M}_{0}\right)$ is
$\mathbb{R}$-specializable along each $D_{i}$, in the sense that $x_{i} F_{p} V_{\alpha_{i}}^{(i)} \mathcal{M}_{0}=F_{p} V_{\alpha_{i}-1}^{(i)} \mathcal{M}_{0}$ for every $i$ and $\alpha_{i}<0$, and $\partial_{x_{i}}$ sends $F_{p} V_{-1}^{(i)} \mathcal{M}_{0}$ into $F_{p+1} V_{0}^{(i)} \mathcal{M}_{0}$. Set

$$
F_{p} \mathcal{M}:=\sum_{q \geqslant 0}\left(F_{p-q} \mathcal{M}_{0}\right) \cdot F_{q} \mathscr{D}_{X} .
$$

Then
(1) $(\mathcal{M}, F, \mathcal{M})$ is $\mathbb{R}$-specializable along each $D_{i}$, and for $\alpha \in[-1,0]^{n}$,

$$
F_{p} V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M}_{0}:=F_{p} \mathcal{M}_{0} \cap V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M}_{0}=F_{p} \mathcal{M} \cap V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M}_{0}
$$

(2) and $\left(F_{\bullet} \mathcal{M}, V_{\bullet}^{(1)} \mathcal{M}, \ldots, V_{\bullet}^{(n)} \mathcal{M}\right)$ are compatible filtrations.

Proof. For every $\gamma \in \mathbb{R}^{n}$, we define

$$
\begin{equation*}
G_{p}\left(V_{\gamma}^{(\boldsymbol{n})} \mathcal{M}\right):=\sum_{\substack{\alpha \leqslant 0, \boldsymbol{j} \geqslant 0 \\ \alpha+\boldsymbol{j} \leqslant \gamma}} F_{p-|\boldsymbol{j}|} V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M} \cdot \partial_{x}^{\boldsymbol{j}} \tag{11.2.38}
\end{equation*}
$$

For example, we have $G_{p}\left(V_{\gamma}^{(n)} \mathcal{M}\right)=F_{p} V_{\gamma}^{(\boldsymbol{n})} \mathcal{N}$ if $\gamma \leqslant 0$, i.e., $\gamma_{i} \leqslant 0$ for all $i$. Similarly, if $\gamma^{\prime}=\left(\gamma_{i}\right)_{i \mid \gamma_{i}>0}$ denotes the "positive part" of $\gamma$ and $\gamma_{-}$the non-positive part, we have, with obvious notation,

$$
\begin{equation*}
G_{p}\left(V_{\gamma}^{(\boldsymbol{n})} \mathcal{M}\right):=\sum_{\substack{\boldsymbol{\alpha}^{\prime} \leqslant 0, \boldsymbol{j}^{\prime} \geqslant 0 \\ \boldsymbol{\alpha}^{\prime}+\boldsymbol{j}^{\prime} \leqslant \boldsymbol{\gamma}^{\prime}}} F_{p-\left|\boldsymbol{j}^{\prime}\right|} V_{\left(\boldsymbol{\alpha}^{\prime}, \boldsymbol{\gamma}_{-}\right)}^{(\boldsymbol{n})} \mathcal{M} \cdot \partial_{x^{\prime}}^{\boldsymbol{j}^{\prime}} \tag{11.2.39}
\end{equation*}
$$

Let us note that

$$
\underset{\boldsymbol{\gamma}}{\lim } G_{p}\left(V_{\gamma}^{(\boldsymbol{n})} \mathcal{M}\right)=\sum_{\boldsymbol{\alpha} \leqslant 0, \boldsymbol{j} \geqslant 0} F_{p-|\boldsymbol{j}|} V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} \mathcal{M} \cdot \partial_{x}^{\boldsymbol{j}}=\sum_{\boldsymbol{j} \geqslant 0} F_{p-|\boldsymbol{j}|} \mathcal{M}_{0} \cdot \partial_{x}^{\boldsymbol{j}}=: F_{p} \mathcal{M} .
$$

We set $V_{\bullet}^{(i)} V_{\gamma}^{(\boldsymbol{n})} \mathcal{M}=V_{\bullet}^{(i)} \mathcal{M} \cap V_{\gamma}^{(\boldsymbol{n})} \mathcal{M}$. We will prove the following properties.
(a) Let $\boldsymbol{\beta}<\boldsymbol{\gamma}$ (i.e., $\beta_{i} \leqslant \gamma_{i}$ for all $i$ and $\boldsymbol{\beta} \neq \boldsymbol{\gamma}$ ). Then $G_{p}\left(V_{\gamma}^{(\boldsymbol{n})} \mathcal{M}\right) \cap V_{\boldsymbol{\beta}}^{(\boldsymbol{n})} \mathcal{M}=$ $G_{p}\left(V_{\boldsymbol{\beta}}^{(\boldsymbol{n})} \mathcal{M}\right)$.
(b) $\left(G_{\bullet}\left(V_{\gamma}^{(\boldsymbol{n})} \mathcal{N}\right), V_{\bullet}^{(1)} V_{\gamma}^{(\boldsymbol{n})} \mathcal{N}, \ldots, V_{\bullet}^{(n)} V_{\gamma}^{(\boldsymbol{n})} \mathcal{N}\right)$ are compatible filtrations,
(c) the following inclusion is $(n+1)$-strict:

$$
\left(V_{\boldsymbol{\beta}}^{(\boldsymbol{n})} \mathcal{M}, G_{\bullet}\left(V_{\boldsymbol{\beta}}^{(\boldsymbol{n})} \mathcal{M}\right),\left(V_{\bullet}^{(i)} V_{\boldsymbol{\beta}}^{(\boldsymbol{n})} \mathcal{M}\right)_{i \in I}\right) \hookrightarrow\left(V_{\boldsymbol{\gamma}}^{(\boldsymbol{n})} \mathcal{M}, G_{\bullet}\left(V_{\boldsymbol{\gamma}}^{(\boldsymbol{n})} \mathcal{M}\right),\left(V_{\bullet}^{(i)} V_{\gamma}^{(\boldsymbol{n})} \mathcal{M}\right)_{i \in I}\right) .
$$

Let us indicate how to obtain 11.2 .37 from (a)-(c). The $\mathbb{R}$-specializability of ( $\mathcal{M}, F \cdot \mathcal{M})$ along $D_{i}$ amounts to

$$
\left.F_{p+1} \mathcal{M} \cap V_{\beta_{i}+1}^{(i)} \mathcal{M} \subset\left(F_{p} \mathcal{M} \cap V_{\beta_{i}}^{(i)} \mathcal{M}\right) \cdot \partial_{x_{i}}+V_{<\beta_{i}+1}^{(i)} \mathcal{M}\right) \quad \text { if } \beta_{i}>-1
$$

By taking the inductive limit on $\gamma>0$ (i.e., $\gamma=\gamma^{\prime}$ ) in (a), we obtain

$$
F_{p} \mathcal{M} \cap V_{\boldsymbol{\beta}}^{(\boldsymbol{n})} \mathcal{M}=G_{p}\left(V_{\boldsymbol{\beta}}^{(\boldsymbol{n})} \mathcal{M}\right)
$$

for every $\boldsymbol{\beta}$, and taking $\beta_{k} \gg 0$ for $k \neq i$ gives

$$
F_{p} \mathcal{M} \cap V_{\beta_{i}}^{(i)} \mathcal{M}=\sum_{\substack{\alpha_{i} \leqslant 0, \boldsymbol{j} \geqslant 0 \\ \alpha_{i}+j_{i} \leqslant \beta_{i}}} F_{p-|\boldsymbol{j}|} V_{\alpha_{i}}^{(i)} \mathcal{M} \cdot \partial_{x}^{\boldsymbol{j}},
$$

and thus, if $\beta_{i}>-1$,

$$
F_{p+1} \mathcal{M} \cap V_{\beta_{i}+1}^{(i)} \mathcal{M}=\left(F_{p} \mathcal{M} \cap V_{\beta_{i}}^{(i)} \mathcal{M}\right) \cdot \partial_{x_{i}}+\sum_{\substack{\alpha_{i} \leqslant 0, j \geqslant 0 \\ j_{i}=0}} F_{p+1-|\boldsymbol{j}|} V_{\alpha_{i}}^{(i)} \mathcal{M} \cdot \partial_{x}^{\boldsymbol{j}},
$$

hence the desired $\mathbb{R}$-specializability, since $V_{\alpha_{i}}^{(i)} \mathcal{M} \subset V_{<\beta_{i}+1}^{(i)} \mathcal{M}$. The other assertions in 11.2.37 are also obtained by taking the inductive limit on $\gamma$. We also note that (a) and (b) for $\gamma$ imply (c) for $\gamma$, according to Example 8.3.20. Conversely, (c) for $\gamma$ implies (a) for $\gamma$.

Let us first exemplify the proof of (a) and (b) in the case $n=1$. Condition (b) is empty. For (a), we can assume $\gamma>0$, and it is enough, by an easy induction on $\gamma-\beta$, to prove $G_{p}\left(V_{\gamma}^{(1)} \mathcal{M}\right) \cap V_{<\gamma}^{(1)} \mathcal{M}=G_{p}\left(V_{<\gamma}^{(1)} \mathcal{M}\right)$. For that purpose, we notice that

$$
G_{p}\left(V_{\gamma}^{(1)} \mathcal{M}\right)=G_{p}\left(V_{<\gamma}^{(1)} \mathcal{M}\right)+F_{p-j} V_{\alpha}^{(1)} \mathcal{M} \cdot \partial_{x_{1}}^{j}
$$

where $j$ is such that $\gamma-j \in(-1,0]$ and $\alpha:=\gamma-j$. Hence

$$
G_{p}\left(V_{\gamma}^{(1)} \mathcal{M}\right) \cap V_{<\gamma}^{(1)} \mathcal{M}=G_{p}\left(V_{<\gamma}^{(1)} \mathcal{N}\right)+\left(F_{p-j} V_{\alpha}^{(1)} \mathcal{M} \cdot \partial_{x_{1}}^{j} \cap V_{<\gamma}^{(1)} \mathcal{M}\right)
$$

Now, by the $\mathbb{R}$-specializable property, $F_{p-j} V_{\alpha}^{(1)} \mathcal{M} \cdot \partial_{x_{1}}^{j} \cap V_{<\gamma}^{(1)} \mathcal{M}=F_{p-j} V_{<\alpha}^{(1)} \mathcal{N} \cdot \partial_{x_{1}}^{j}$, so we obtain (a) in this case.

We will prove (a)-(c) by induction on the lexicographically ordered pair ( $\left.n,\left|\gamma^{\prime}\right|\right)$. The case $n=1$ is treated above, so we can assume $n \geqslant 2$. Moreover, if $\left|\gamma^{\prime}\right|=0$, i.e., if $\gamma \leqslant 0$, there is nothing to prove. Assume that $\gamma_{1}>0$ and let $\alpha_{1} \in(-1,0]$ be such that $j_{1}:=\gamma_{1}-\alpha_{1}$ is an integer. We also set $\gamma=\left(\gamma_{1}, \gamma^{\prime \prime}\right)$ and $n^{\prime \prime}=n-1$.

In order to prove (a), we can argue by decreasing induction on $\boldsymbol{\beta}$, and we are reduced to the case where $\boldsymbol{\beta}$ is the predecessor in one direction, say $k$, of $\boldsymbol{\gamma}$, that is, $\beta_{i}=\gamma_{i}$ for $i \neq k$ and $\beta_{k}$ is the predecessor of $\gamma_{k}$. Assume first that $\gamma_{k}>0$, so we can also assume $k=1$. We then have

$$
G_{p}\left(V_{\gamma}^{(\boldsymbol{n})} \mathcal{M}\right)=G_{p-1}\left(V_{\gamma-\mathbf{1}_{1}}^{(\boldsymbol{n})} \mathcal{M}\right) \cdot \partial_{x_{1}}+G_{p}\left(V_{\boldsymbol{\beta}}^{(\boldsymbol{n})} \mathcal{M}\right)
$$

and we are reduced to proving

$$
G_{p-1}\left(V_{\boldsymbol{\gamma}-\mathbf{1}_{1}}^{(\boldsymbol{n})} \mathcal{M}\right) \cdot \partial_{x_{1}} \cap V_{\boldsymbol{\beta}}^{(\boldsymbol{n})} \mathcal{M} \subset G_{p}\left(V_{\boldsymbol{\beta}}^{(\boldsymbol{n})} \mathcal{N}\right)
$$

Since $\gamma_{1}>0$ and $\mathcal{M}$ is of normal crossing type, we have an isomorphism

$$
\partial_{x_{1}}: V_{\boldsymbol{\gamma}-\mathbf{1}_{1}}^{(\boldsymbol{n})} \mathcal{M} / V_{\boldsymbol{\beta}-\mathbf{1}_{1}}^{(\boldsymbol{n})} \mathcal{M}=V_{\tilde{\boldsymbol{\gamma}}}^{(\widetilde{\boldsymbol{n}})} \operatorname{gr}_{\gamma_{1}-1}^{V^{(1)}} \mathcal{M} \xrightarrow{\sim} V_{\tilde{\boldsymbol{\gamma}}}^{(\widetilde{\boldsymbol{n}})} \operatorname{gr}_{\gamma_{1}}^{V^{(1)}} \mathcal{M}=V_{\gamma}^{(\boldsymbol{n})} \mathcal{M} / V_{\boldsymbol{\beta}}^{(\boldsymbol{n})} \mathcal{M}
$$

which sends surjectively, hence bijectively, the image of $G_{p-1}\left(V_{\gamma-\mathbf{1}_{1}}^{(\boldsymbol{n})} \mathcal{M}\right)$ to that of $G_{p}\left(V_{\gamma}^{(\boldsymbol{n})} \mathcal{M}\right)$. It follows that

$$
G_{p-1}\left(V_{\gamma-\mathbf{1}_{1}}^{(\boldsymbol{n})} \mathcal{M}\right) \cdot \partial_{x_{1}} \cap V_{\boldsymbol{\beta}}^{(\boldsymbol{n})} \mathcal{M}=\left[G_{p-1}\left(V_{\gamma-\mathbf{1}_{1}}^{(\boldsymbol{n})} \mathcal{M}\right) \cap V_{\boldsymbol{\beta}-\mathbf{1}_{1}}^{(\boldsymbol{n})} \mathcal{M}\right] \cdot \partial_{x_{1}}
$$

By the inductive assumption on $\boldsymbol{\gamma}$, the latter term is contained in $G_{p-1}\left(V_{\boldsymbol{\beta}-\mathbf{1}_{1}}^{(\boldsymbol{n})} \mathcal{N}\right) \cdot \partial_{x_{1}}$, hence in $G_{p}\left(V_{\boldsymbol{\beta}}^{(\boldsymbol{n})} \mathcal{M}\right)$.

For induction purpose, let us consider $\mathcal{M}^{(1)}:=\operatorname{gr}_{\alpha_{1}}^{V^{(1)}} \mathcal{M}$, which is a $\mathscr{D}_{D_{1}}-$ module of normal crossing type for which the $V^{(i)}$-filtrations $(i=2, \ldots, n)$ are those naturally induced by the $V^{(i)}$-filtrations on $\mathcal{M}$. We set $\mathcal{M}_{0}^{(1)}=V_{0}^{\left(n^{\prime \prime}\right)} \mathcal{M}^{(1)}$, that we endow with the naturally induced filtration $F \cdot \mathcal{M}_{0}^{(1)}$. By Remark 8.3.10(1), the family $\left(F_{\bullet} \mathcal{M}_{0}^{(1)},\left(V_{\bullet}^{(i)} \mathcal{M}_{0}^{(1)}\right)_{2 \leqslant i \leqslant n}\right)$ is compatible. The inductive assumption on $n$ implies that(a)-(c) hold for $V_{\gamma^{\prime \prime}}^{\left(\boldsymbol{n}^{\prime \prime}\right)} \mathcal{N}^{(1)}$. Note that $V_{\gamma^{\prime \prime}}^{\left(\boldsymbol{n}^{\prime \prime}\right)} \mathcal{N}^{(1)}=V_{\left(\alpha_{1}, \boldsymbol{\gamma}^{\prime \prime}\right)}^{(\boldsymbol{n})} \mathcal{M} / V_{\left(\left\langle\alpha_{1}, \boldsymbol{\gamma}^{\prime \prime}\right)\right.}^{(\boldsymbol{n})} \mathcal{M}$.

We now claim that $G_{p}\left(V_{\gamma^{\prime \prime}}^{\left(\boldsymbol{n}^{\prime \prime}\right)} \mathcal{M}^{(1)}\right)$ is the filtration induced by $G_{p}\left(V_{\left(\alpha_{1}, \boldsymbol{\gamma}^{\prime \prime}\right)}^{(\boldsymbol{n})} \mathcal{M}\right)$. Indeed, this follows from the expression (11.2.39), which does not involve the variable $x_{1}$. Now, $\partial_{x_{1}}^{j_{1}}$ induces an isomorphism $\mathcal{M}^{(1)} \xrightarrow{\sim} \operatorname{gr}_{\gamma_{1}}^{V^{(1)}} \mathcal{M}$, which is strictly compatible with the filtrations induced by $V_{\bullet}^{(i)} \mathcal{M}$. On the other hand, the filtration induced by $G_{p}\left(V_{\left(\alpha_{1}, \gamma^{\prime}\right)}^{(\boldsymbol{n})} \mathcal{M}\right)$ is sent surjectively (hence bijectively) onto that induced by $G_{p}\left(V_{\gamma}^{(n)} \mathcal{N}\right)$. By induction on $n$, (a)-(c) hold for $\operatorname{gr}_{\gamma_{1}}^{V^{(1)} \mathcal{N}}$, and the filtrations are those induced by the filtrations on $V_{\gamma}^{(n)} \mathcal{M}$.

Let us now assume that $\gamma_{k} \leqslant 0$. To prove $G_{p}\left(V_{\gamma}^{(\boldsymbol{n})} \mathcal{N}\right) \cap V_{\boldsymbol{\beta}}^{(\boldsymbol{n})} \mathcal{N}=G_{p}\left(V_{\boldsymbol{\beta}}^{(\boldsymbol{n})} \mathcal{N}\right)$ for all $p$, it is enough to prove $G_{p}\left(V_{\gamma}^{(\boldsymbol{n})} \mathcal{M}\right) \cap G_{p+1}\left(V_{\boldsymbol{\beta}}^{(\boldsymbol{n})} \mathcal{N}\right)=G_{p}\left(V_{\boldsymbol{\beta}}^{(\boldsymbol{n})} \mathcal{M}\right)$ for all $p$, and (replacing $p$ with $p-1$ ), this amounts to proving for all $p$ the injectivity of

$$
\operatorname{gr}_{p}^{G} V_{\boldsymbol{\beta}}^{(\boldsymbol{n})} \mathcal{M} \longrightarrow \operatorname{gr}_{p}^{G} V_{\gamma}^{(\boldsymbol{n})} \mathcal{M}
$$

Set $\boldsymbol{\gamma}=\left(\gamma_{1}, \gamma^{\prime \prime}\right), \widetilde{\gamma}=\left(<\gamma_{1}, \gamma^{\prime \prime}\right), \boldsymbol{\beta}=\left(\gamma_{1}, \boldsymbol{\beta}^{\prime \prime}\right)$ and $\widetilde{\boldsymbol{\beta}}=\left(<\gamma_{1}, \boldsymbol{\beta}^{\prime \prime}\right)$, with $\boldsymbol{\beta}^{\prime \prime}=$ $\left(\gamma_{2}, \ldots,<\gamma_{k}, \ldots, \gamma_{n}\right)$. By the inductive assumption on $n$ and $\gamma^{\prime}$, we have a diagram

where the horizontal sequences are exact (by the first part of the proof of (a)) and both extreme vertical arrows are injective (because $\left|\widetilde{\gamma}^{\prime}\right|<\left|\gamma^{\prime}\right|$ for the left one, and $n^{\prime \prime}<n$ for the right one). We conclude that the middle vertical arrow is injective, which finishes the proof of (a).

Let us now prove (b) and let us come back to the case where $\beta_{1}$ is the predecessor of $\gamma_{1}>0$. We have seen that $G_{p}\left(V_{\gamma^{\prime \prime}}^{\left(\boldsymbol{n}^{\prime \prime}\right)} \operatorname{gr}_{\gamma_{1}}^{(1)} \mathcal{N}\right)$ is the filtration induced by $G_{p}\left(V_{\gamma}^{(\boldsymbol{n})} \mathcal{N}\right)$, so the previous injective morphism can be completed for all $p$ into the exact sequence

$$
0 \longrightarrow \operatorname{gr}_{p}^{G} V_{\boldsymbol{\beta}}^{(\boldsymbol{n})} \mathcal{M} \longrightarrow \operatorname{gr}_{p}^{G} V_{\gamma}^{(\boldsymbol{n})} \mathcal{M} \longrightarrow \operatorname{gr}_{p}^{G} V_{\gamma^{\prime \prime}}^{\left(\boldsymbol{n}^{\prime \prime}\right)} \operatorname{gr}_{\gamma_{1}}^{(1)} \mathcal{N} \longrightarrow 0
$$

hence, since $G_{\bullet}$ is bounded below,

$$
0 \longrightarrow G_{p} V_{\boldsymbol{\beta}}^{(\boldsymbol{n})} \mathcal{M} \longrightarrow G_{p} V_{\gamma}^{(\boldsymbol{n})} \mathcal{M} \longrightarrow G_{p} V_{\gamma^{\prime \prime}}^{\left(\boldsymbol{n}^{\prime \prime}\right)} \operatorname{gr}_{\gamma_{1}}^{(1)} \mathcal{M} \longrightarrow 0
$$

The inductive assumption implies that (b) holds for $V_{\boldsymbol{\beta}}^{(\boldsymbol{n})} \mathcal{M}$ and for $V_{\gamma^{\prime \prime}}^{\left(\boldsymbol{n}^{\prime \prime}\right)} \operatorname{gr}_{\gamma_{1}}^{(1)} \mathcal{M}$. We can now apply Exercise 8.8(2a) to conclude that (b) holds for $V_{\gamma}^{(\boldsymbol{n})} \mathcal{M}$.

## Remark 11.2.40 (The case of a minimal extension along $D_{i \in I}$ )

Assume moreover that, in Proposition 11.2.37, $\mathcal{M}$ is a minimal extension along each $D_{i}(i \in I)$. Then we can replace everywhere $\mathcal{M}_{0}$ with $\mathcal{M}_{<0}:=\bigcap_{i \in I} V_{<0}^{(i)} \mathcal{M}$ and we can moreover conclude that ( $\mathcal{M}, F \cdot \mathcal{M})$ is a minimal extension along $D_{i \in I}$ (Definition 11.2.33). In the proof, we modify the definition of $G_{p}\left(V_{\gamma}^{(n)} \mathcal{N}\right)$ for $\gamma \geqslant-1$, by summing over $\boldsymbol{\alpha} \in[-1,0)^{n}$.

### 11.3. Nearby cycles along a monomial function

We continue to refer implicitly to Notation 11.1.1 and the simplifying assumptions 11.1.2. We consider a monomial function $g=x^{a}$.

Notation 11.3.1. The indices for which $a_{i}=0$ do not play an important role. Let us denote by $I_{g}=I(\boldsymbol{a}):=\left\{i \mid a_{i} \neq 0\right\} \subset\{1, \ldots, n\}$ the complementary subset, $\boldsymbol{a}^{\prime}=$ $\left(a_{i}\right)_{i \in I_{g}}$ and $n^{\prime}=\# I_{g}$. Accordingly, we decompose the set of variables $\left(x_{1}, \ldots, x_{n}\right)$ as $\left(x^{\prime}, x^{\prime \prime}\right)$, with $x^{\prime}=\left(x_{i}\right)_{i \in I_{g}}$.

We aim at proving the following theorem.
Theorem 11.3.2. Let $(\mathcal{M}, F, \mathcal{M})$ be a coherent filtered $\mathscr{D}_{X}$-module of normal crossing type (Definition 11.2.25). Assume that ( $\mathcal{M}, F, \mathcal{M})$ is a minimal extension along $D_{i \in I}$ (see Definition 11.2.33).

Then ( $\mathcal{M}, F, \mathcal{M})$ is $\mathbb{R}$-specializable along $(g)$ and is a minimal extension along $(g)$. Moreover, for every $\lambda \in S^{1},\left(\psi_{g, \lambda} \mathcal{M}, F_{\bullet} \psi_{g, \lambda} \mathcal{M}\right)$ is of normal crossing type along $D$.

## 11.3.a. Nearby cycles for $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-modules of normal crossing type

In this section, we consider the variant of Theorem 11.3.2 where we forget the filtration $F_{\bullet}$. and where we consider the case of $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-modules, with the notation of Section 11.2.a. The proof will be done by giving an explicit expression of the $V$-filtration of ${ }_{\mathrm{D}} \iota_{g *} M$ with respect to $t$, as well as its associated graded modules. The proof will also make precise the set of jumping indices of the $V$-filtration (see Remark 11.3.33(2)).

In order to simplify the notation, we will set $N={ }_{\mathrm{D}} \iota_{g *} M$, which is a $\mathbb{C}[x, t]\left\langle\partial_{x}, \partial_{t}\right\rangle$ module. According to (A.8.8*), the action of $\mathbb{C}[x, t]\left\langle\partial_{x}, \partial_{t}\right\rangle$ is as follows:

$$
\begin{align*}
\left(m \otimes \partial_{t}^{k}\right) \cdot \partial_{t} & =m \otimes \partial_{t}^{k+1} \\
(m \otimes 1) \cdot \partial_{x_{i}} & =m \partial_{x_{i}} \otimes 1-\left(a_{i} m x^{\boldsymbol{a}-\mathbf{1}_{i}}\right) \otimes \partial_{t}  \tag{11.3.3}\\
(m \otimes 1) \cdot f(x, t) & =m f\left(x, x^{\boldsymbol{a}}\right) \otimes 1
\end{align*}
$$

As a consequence, for $i \in I_{g}$ we have

$$
\begin{equation*}
(m \otimes 1) \cdot t \partial_{t}=\left(m x^{\boldsymbol{a}} \otimes 1\right) \cdot \partial_{t}=\frac{1}{a_{i}}\left[\left(m x_{i} \partial_{x_{i}} \otimes 1\right)-(m \otimes 1) x_{i} \partial_{x_{i}}\right] \tag{11.3.4}
\end{equation*}
$$

Notation 11.3.5. In order to distinguish between the action of $x_{i} \partial_{x_{i}}$ trivially coming from that on $M$ and the action $x_{i} \partial_{x_{i}}$ on $N$, it will be convenient to denote by $\mathrm{S}_{i}$ the first one, defined by

$$
\left(m \otimes \partial_{t}^{k}\right) \cdot \mathrm{S}_{i}=\left(m x_{i} \partial_{x_{i}}\right) \otimes \partial_{t}^{k} .
$$

Then we can rewrite $S_{i}$ as

$$
\left(m \otimes \partial_{t}^{k}\right) \cdot \mathrm{S}_{i}=(m \otimes 1) \cdot\left(x_{i} \partial_{x_{i}}+a_{i} t \partial_{t}\right) \partial_{t}^{k}=\left(m \otimes \partial_{t}^{k}\right) \cdot\left(x_{i} \partial_{x_{i}}+a_{i}\left(t \partial_{t}-k\right)\right),
$$

a formula that can also be read

$$
\begin{equation*}
\left(m \otimes \partial_{t}^{k}\right) \cdot x_{i} \partial_{x_{i}}=\left(m \otimes \partial_{t}^{k}\right) \cdot\left(\mathrm{S}_{i}-a_{i} t \partial_{t}+a_{i} k\right) \tag{11.3.6}
\end{equation*}
$$

Note that $N$ is naturally graded: $N=\bigoplus_{\boldsymbol{\alpha}, k} M_{\boldsymbol{\alpha}} \otimes \partial_{t}^{k}$.

## Proof of Theorem 11.3.2, Step one: $\mathbb{R}$-specializability of $N$ along $(t)$

Proposition 11.3.7. The $\mathbb{C}[x, t]\left\langle\partial_{x}, \partial_{t}\right\rangle$-module $N$ is $\mathbb{R}$-specializable along $(t)$. Moreover, $N=N[!* t]$.

We will show the $\mathbb{R}$-specializability by making explicit a $V$-filtration of $N$. To get started, consider the following simple example.

Example 11.3.8. Let $\gamma \in \mathbb{R}$. Assume we know that $\widetilde{N}:=\operatorname{gr}_{\gamma}^{V} N$ is of normal crossing type along $D$. Suppose that for some $m \in M_{\boldsymbol{\alpha}}$ and some $k \geqslant 0$, the section $m \otimes \partial_{t}^{k}$ belongs to $V_{\gamma} N$, and that its projection to $\operatorname{gr}_{\gamma}^{V} N$ is nonzero and happens to lie in the subspace

$$
\widetilde{N}_{\boldsymbol{\beta}}:=\left(\operatorname{gr}_{\gamma}^{V} N\right)_{\boldsymbol{\beta}}
$$

In this situation, $\gamma, \boldsymbol{\alpha}, \boldsymbol{\beta}$, and $k$ are related. Indeed, the identity in (11.3.6) shows that $\left(m \otimes \partial_{t}^{k}\right) \cdot\left(\left(x_{i} \partial_{x_{i}}-\beta_{i}\right)-\left(\mathrm{S}_{i}-\alpha_{i}\right)+a_{i}(\mathrm{E}-\gamma)\right)=\left(m \otimes \partial_{t}^{k}\right) \cdot\left(\alpha_{i}-\beta_{i}-a_{i}(\gamma-k)\right)$.
By assumption, $\mathrm{E}-\gamma=t \partial_{t}-\gamma$ and $x_{i} \partial_{x_{i}}-\alpha_{i}=x_{i} \partial_{x_{i}}-\beta_{i}$ both act nilpotently on $\tilde{N}_{\boldsymbol{\beta}}$; since $\mathrm{S}_{i}-\alpha_{i}$ acts nilpotently on $M_{\boldsymbol{\alpha}} \otimes \partial_{t}^{k}$, the conclusion is that $\boldsymbol{\alpha}=\boldsymbol{\beta}+(\underset{\sim}{\gamma}-k) \boldsymbol{a}$. Thus we expect elements of $M_{\boldsymbol{\beta}+(\gamma-k) \boldsymbol{a}} \otimes \partial_{t}^{k}$ to contribute to the subspace $\widetilde{N}_{\boldsymbol{\beta}}$.

This computation motivates the following definition.
Definition 11.3.9. For $\gamma<0$, we set

$$
\begin{equation*}
V_{\gamma} N=\left(V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M \otimes 1\right) \cdot \mathbb{C}[x]\left\langle\partial_{x}\right\rangle=\sum_{\boldsymbol{k} \in \mathbb{N}^{n}}\left(V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M \otimes 1\right) \cdot \partial_{x}^{\boldsymbol{k}} \tag{11.3.9*}
\end{equation*}
$$

For every $\gamma \in[-1,0)$ and $j \geqslant 1$, we define inductively

$$
\begin{equation*}
V_{\gamma+j} N=V_{\gamma} N \cdot \partial_{t}^{j}+V_{<\gamma+j} N \tag{11.3.9**}
\end{equation*}
$$

Note that the latter formula is natural if we expect that $N$ is a middle extension along $(t)$.

Lemma 11.3.10. The filtration $V_{\bullet} N$ is a Kashiwara-Malgrange filtration for $N$.
We first need to check that $(11.3 .9 *)$ and $(11.3 .9 * *)$ define a $V$-filtration.

Lemma 11.3.11. For every $\gamma \in \mathbb{R}, V_{\gamma} N$ is a $V_{0}\left(\mathbb{C}[x, t]\left\langle\partial_{x}, \partial_{t}\right\rangle\right)$-module that satisfies

$$
V_{\gamma} N \cdot t \subset V_{\gamma-1} N, \quad V_{\gamma-1} N \cdot \partial_{t}+V_{<\gamma} N \subset V_{\gamma} N
$$

with equality in the first inclusion if $\gamma<0$ and in the second one if $\gamma>0$.
Proof. Assume first that $\gamma<0$. By definition, $V_{\gamma} N$ is a $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-module, so it remains to prove stability by the actions of $t$ and $t \partial_{t}$. On the one hand,

$$
\left(V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M \otimes 1\right) \cdot t=V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M x^{\boldsymbol{a}} \otimes 1=V_{(\gamma-1) \boldsymbol{a}}^{(\boldsymbol{n})} M \otimes 1,
$$

hence

$$
\begin{equation*}
V_{\gamma} N \cdot t=V_{\gamma-1} N \subset V_{\gamma} N \tag{11.3.12}
\end{equation*}
$$

On the other hand, (11.3.4) shows that, for any $i \in I_{g}$, we have

$$
\left(V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M \otimes 1\right) \cdot t \partial_{t} \subset V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M \otimes 1+\left(V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M \otimes 1\right) \partial_{x_{i}}
$$

We conclude that the statements of the lemma hold for $\gamma<0$. Moreover, (11.3.12) gives, for $\gamma<0$ :

$$
\begin{equation*}
V_{\gamma} N=\left(V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M \otimes 1\right) \cdot \mathbb{C}[x]\left\langle\partial_{x}\right\rangle\left[t \partial_{t}\right] . \tag{11.3.13}
\end{equation*}
$$

The assertions for $\gamma \geqslant 0$ follow then easily from Definition (11.3.9**).
Remark 11.3.14. Note that, for $i \notin I_{g}$ (Notation 11.3.1), we have $\gamma a_{i}=0$ and

$$
\sum_{k_{i} \geqslant 0}\left(V_{0}^{(i)} M \otimes 1\right) \partial_{x_{i}}^{k_{i}}=\sum_{k_{i} \geqslant 0}\left(V_{0}^{(i)} M \partial_{x_{i}}^{k_{i}} \otimes 1\right)=M \otimes 1 .
$$

As a consequence, for $\gamma<0,(11.3 .9 *)$ can be simplified as follows:

$$
\begin{equation*}
V_{\gamma} N=\sum_{\boldsymbol{k}^{\prime} \in \mathbb{N}^{n^{\prime}}}\left(V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} M \otimes 1\right) \cdot \partial_{x^{\prime}}^{\boldsymbol{k}^{\prime}} \tag{11.3.15}
\end{equation*}
$$

Let us consider the $\mathbb{C}\left[x^{\prime}, x^{\prime \prime}, t\right]\left\langle\partial_{x^{\prime}}, \partial_{x^{\prime \prime}}, t \partial_{t}\right\rangle$-module

$$
K_{\gamma}:=V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} M \otimes_{\mathbb{C}\left[x^{\prime}, x^{\prime \prime}\right]\left\langle\partial_{x^{\prime \prime}}\right\rangle} \mathbb{C}\left[x^{\prime}, x^{\prime \prime}\right]\left\langle\partial_{x^{\prime}}, \partial_{x^{\prime \prime}}, t \partial_{t}\right\rangle,
$$

where the action of $t$ is obtained by the second line of (11.3.3). We thus have a surjective morphism of $\mathbb{C}\left[x^{\prime}, x^{\prime \prime}, t\right]\left\langle\partial_{x^{\prime}}, \partial_{x^{\prime \prime}}, t \partial_{t}\right\rangle$-modules:

$$
K_{\gamma} \longrightarrow V_{\gamma} N
$$

sending any $m x_{i} \partial_{x_{i}} \otimes 1-m \otimes x_{i} \partial_{x_{i}}-(m \otimes 1) a_{i} t \partial_{t}$ to zero $\left(i \in I_{g}\right)$, according to (11.3.4).

Proof of Lemma 11.3.10 and Proposition 11.3.7. Let us start with $\gamma<0$. Since $V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M$ has finite type over $\mathbb{C}[x]$, Formula $(11.3 .9 *)$ implies that $V_{\gamma} N$ has finite type over $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$, and a fortiori over $V_{0}\left(\mathbb{C}[x, t]\left\langle\partial_{x}, \partial_{t}\right\rangle\right)$.

In order to show that some power of $\left(t \partial_{t}-\gamma\right)$ sends $V_{\gamma} N$ to $V_{<\gamma} N$ we first notice that a power of $\mathrm{S}_{i}-\gamma a_{i}$ does so for every $i=1, \ldots, n$. It is thus enough to check that $\prod_{i \in I_{g}}\left(\mathrm{~S}_{i}-a_{i} t \partial_{t}\right)$ sends $\left(V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M \otimes 1\right)$ into $V_{\gamma^{\prime}} N$ for some $\gamma^{\prime}<\gamma$. (Indeed, this
will imply that some power of $\left(t \partial_{t}-\gamma\right)$ sends $\left(V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M \otimes 1\right)$ into $V_{\gamma^{\prime}} N$, and (11.3.9*) enables us to conclude.)

We have $\gamma \boldsymbol{a}-\mathbf{1}_{I_{g}} \leqslant \gamma^{\prime} \boldsymbol{a}$ for some $\gamma^{\prime}<\gamma$, so $\left(V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M \otimes 1\right) \cdot \prod_{i \in I_{g}} x_{i} \subset\left(V_{\gamma^{\prime} \boldsymbol{a}}^{(\boldsymbol{n})} M \otimes 1\right)$, and thus, by (11.3.9*),

$$
\left(V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M \otimes 1\right) \cdot \prod_{i \in I_{g}} x_{i} \partial_{x_{i}}=\left(V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M \otimes 1\right) \cdot \prod_{i \in I_{g}} x_{i} \prod_{i \in I_{g}} \partial_{x_{i}} \subset V_{\gamma^{\prime}} N .
$$

Therefore,

$$
\left(V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M \otimes 1\right) \cdot \prod_{i \in I_{g}}\left(\mathrm{~S}_{i}-a_{i} t \partial_{t}\right) \subset V_{\gamma^{\prime}} N
$$

In order to conclude that $N$ is $\mathbb{R}$-specializable along $(t)$ and that $V_{\bullet} N$ is its Kashiwara-Malgrange filtration along ( $t$ ), it only remains to prove that $N=\bigcup_{\gamma} V_{\gamma} N$, and so it is enough to prove that any element of $M \otimes 1$ belongs to some $V_{\gamma} N$. Let us consider a component $M_{\boldsymbol{\beta}} \otimes 1$ with $\boldsymbol{\beta} \in \mathbb{R}^{n}$, that we write $\boldsymbol{\beta}=\boldsymbol{\alpha}+\boldsymbol{k}_{+}-\boldsymbol{k}_{-}, \boldsymbol{\alpha} \in \boldsymbol{A}$, $\boldsymbol{k}_{+}, \boldsymbol{k}_{-} \in \mathbb{N}^{n}$ with disjoint support. The middle extension property of $M$ implies that $x^{\boldsymbol{k}_{-}} \partial_{x}^{\boldsymbol{k}_{+}}: M_{\boldsymbol{\alpha}} \rightarrow M_{\boldsymbol{\beta}}$ is onto. We can thus use iteratively (11.3.3) to write any element of $M_{\boldsymbol{\beta}}$ as a sum of terms $\left(\mu_{k} \otimes 1\right) \cdot \partial_{t}^{k}(k \geqslant 0)$, where the components of each $\mu_{k}$ in the decomposition (11.2.11) only involve indices in $\left(\mathbb{R}_{<0}\right)^{n}$, and therefore belongs to $V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M$ for some $\gamma<0$.

Let us end by proving that $N$ is a minimal extension along $(t)$. We first remark that $t$ acts injectively on $N$ : if we consider the filtration $G . N$ by the degree in $\partial_{t}$, then the action of $t$ on $\mathrm{gr}^{G} N \simeq M[\tau]$ is equal to the induced action of $x^{a}$ on $M[\tau]$, hence is injective by the assumption on $M$; a fortiori, the action of $t$ on $N$ is injective. We thus have $N \subset N[* t]$. By Definition 11.3 .9 and the exhaustivity of $V_{\bullet} N$ proved above, $N$ is the image of $V_{<0} N \otimes \mathbb{C}[x, t]\left\langle\partial_{x}, \partial_{t}\right\rangle$ in $N[* t]$. This is nothing but $N[!* t]$ (see Definition 9.5.2 and Definition 9.4.1).

Proof of Theorem 11.3.2, Step two: normal crossing type of $\operatorname{gr}_{\gamma}^{V} N$. We aim at proving that each $\operatorname{gr}_{\gamma}^{V} N(\gamma \in[-1,0))$ is of normal crossing type along $D$, and at making explicit the summands. We now fix such a $\gamma$, and set $\widetilde{N}=\operatorname{gr} \gamma_{\gamma}^{V} N$ for the remaining part of the proof. Let $\boldsymbol{\beta} \in \mathbb{R}^{n}$. Let us define $\widetilde{N}_{\boldsymbol{\beta}}$ by the formula

$$
\tilde{N}_{\boldsymbol{\beta}}=\bigcap_{i=1}^{n} \bigcup_{k} \operatorname{Ker}\left(x_{i} \partial_{x_{i}}-\beta_{i}\right)^{k}
$$

where we regard each $\left(x_{i} \partial_{x_{i}}-\beta_{i}\right)^{k}$ as acting on $\widetilde{N}$ through its action on $N$ given by (11.3.6). We then denote by $\mathrm{N}_{i}$ the action of $2 \pi \mathrm{i}\left(x_{i} \partial_{x_{i}}-\beta_{i}\right)$ on $\widetilde{N}_{\boldsymbol{\beta}}$ (and, as usual, by E, resp. N, the action of $t \partial_{t}$, resp. $2 \pi \mathrm{i}\left(t \partial_{t}-\gamma\right)$ on $\widetilde{N}$ and $\left.\widetilde{N}_{\boldsymbol{\beta}}\right)$. By using Bézout's theorem, one checks that $\tilde{N}_{\boldsymbol{\beta}}$ intersects only at zero any sum of submodules $\tilde{N}_{\boldsymbol{\beta}^{\prime}}$, where $\boldsymbol{\beta}^{\prime}$ runs in a finite set not containing $\boldsymbol{\beta}$, so we will only need to check the finite dimensionality of each $\widetilde{N}_{\boldsymbol{\beta}}$ and the existence of a decomposition $\widetilde{N} \simeq \sum_{\boldsymbol{\beta}} \widetilde{N}_{\boldsymbol{\beta}}$ (hence $\widetilde{N} \simeq \bigoplus_{\beta} \tilde{N}_{\boldsymbol{\beta}}$ ). This will be done at the next step, and we start by modifying the expression of $\widetilde{N}_{\boldsymbol{\beta}}$.

Lemma 11.3.16. For every $\boldsymbol{\beta} \in \mathbb{R}^{n}, \widetilde{N}_{\boldsymbol{\beta}}$ is the image of $V_{\gamma} N \cap\left(\sum_{j} M_{\boldsymbol{\beta}+(\gamma-j) \boldsymbol{a}} \otimes \partial_{t}^{j}\right)$ in $\widetilde{N}$.

Proof. Let us consider an arbitrary element of $V_{\gamma} N$, expressed as a finite sum

$$
\sum_{\boldsymbol{\alpha} \in \mathbb{R}^{n}} \sum_{j \in \mathbb{N}} m_{\boldsymbol{\alpha}, j} \otimes \partial_{t}^{j}
$$

with $m_{\boldsymbol{\alpha}, j} \in M_{\boldsymbol{\alpha}}$. Assume that its image in $\operatorname{gr}_{\gamma}^{V} N=\tilde{N}$ belongs to $\tilde{N}_{\boldsymbol{\beta}}$, i.e.,

$$
\left(\sum_{\boldsymbol{\alpha} \in \mathbb{R}^{n}} \sum_{j \in \mathbb{N}} m_{\boldsymbol{\alpha}, j} \otimes \partial_{t}^{j}\right) \cdot\left(x_{i} \partial_{x_{i}}-\beta_{i}\right)^{k} \in V_{<\gamma} N
$$

for every $i \in I$ and some $k \gg 0$. Our aim is to prove that, modulo $V_{<\gamma} N$, only those terms with $\boldsymbol{\alpha}=\boldsymbol{\beta}+(\gamma-j) \boldsymbol{a}$ matter.

Lemma 11.3.17. In the situation considered above, one has

$$
\sum_{\boldsymbol{\alpha} \in \mathbb{R}^{n}} \sum_{j \in \mathbb{N}} m_{\boldsymbol{\alpha}, j} \otimes \partial_{t}^{j}=\sum_{j \in \mathbb{N}} m_{\boldsymbol{\beta}+(\gamma-j) \boldsymbol{a}} \otimes \partial_{t}^{j} \quad \bmod V_{<\gamma} N
$$

Proof. Let us start with an elementary lemma of linear algebra.
Lemma 11.3.18. Let $T$ be an endomorphism of a complex vector space $V$, and $W \subset V$ a linear subspace with $T W \subset W$. Suppose that $v_{1}, \ldots, v_{k} \in V$ satisfy

$$
T^{\mu}\left(v_{1}+\cdots+v_{k}\right) \in W
$$

for some $\mu \geqslant 0$. If there are pairwise distinct complex numbers $\lambda_{1}, \ldots, \lambda_{k}$ with $v_{h} \in E_{\lambda_{h}}(T)$, then one has $\lambda_{h} v_{h} \in W$ for every $h=1, \ldots, k$.

Proof. Choose a sufficiently large integer $\mu \in \mathbb{N}$ such that $\left(T-\lambda_{h}\right)^{\mu} v_{h}=0$ for $h=1, \ldots, k$, and such that $T^{\mu}\left(v_{1}+\cdots+v_{k}\right) \in W$. Assume that $\lambda_{k} \neq 0$. Setting $Q(T)=T^{\mu}\left(T-\lambda_{1}\right)^{\mu} \cdots\left(T-\lambda_{k-1}\right)^{\mu}$, we have by assumption

$$
Q(T)\left(v_{1}+\cdots+v_{k}\right) \in W
$$

The left-hand side equals $Q(T) v_{k}$. Since $Q(T)$ and $T-\lambda_{k}$ are coprime, Bézout's theorem implies that $v_{k} \in W$. At this point, we are done by induction.

We now go back to the proof of Lemma 11.3.17. Let us consider an element as in the lemma. As we have seen before,

$$
\left(m_{\boldsymbol{\alpha}, j} \otimes \partial_{t}^{j}\right) \cdot\left(\left(x_{i} \partial_{x_{i}}-\beta_{i}\right)+a_{i}\left(t \partial_{t}-\gamma\right)\right)=\left(m_{\boldsymbol{\alpha}, j} \otimes \partial_{t}^{j}\right) \cdot\left(\mathrm{S}_{i}-\beta_{i}-a_{i}(\gamma-j)\right),
$$

and since some power of $t \partial_{t}-\gamma$ also send this element in $V_{<\gamma} N$, we may conclude that

$$
\begin{equation*}
\sum_{\boldsymbol{\alpha} \in \mathbb{R}^{n}} \sum_{j \in \mathbb{N}}\left(m_{\boldsymbol{\alpha}, j} \otimes \partial_{t}^{j} \cdot\left(\mathrm{~S}_{i}-\beta_{i}-a_{i}(\gamma-j)\right)^{k}\right) \in V_{<\gamma} N \tag{11.3.19}
\end{equation*}
$$

for every $i \in I$ and $k \gg 0$.

In order to apply Lemma 11.3 .18 to our situation, let us set $V=N$ and $W=V_{<\gamma} N$, and for a fixed choice of $i=1, \ldots, n$, let us consider the endomorphism

$$
T_{i}=\left(x_{i} \partial_{x_{i}}-\beta_{i}\right)+a_{i}\left(t \partial_{t}-\gamma\right)
$$

Evidently, $T_{i} W \subset W$. Since we have

$$
T_{i}\left(m_{\boldsymbol{\alpha}, j} \otimes \partial_{t}^{j}\right)=\left(m_{\boldsymbol{\alpha}, j} \otimes \partial_{t}^{j}\right) \cdot\left(\left(\mathrm{S}_{i}-\alpha_{i}\right)+\alpha_{i}-\beta_{i}-a_{i}(\gamma-j)\right),
$$

it is clear that $m_{\boldsymbol{\alpha}, j} \otimes \partial_{t}^{j}$ is annihilated by a large power of $T_{i}-\left(\alpha_{i}-\beta_{i}-a_{i}(\gamma-j)\right)$. Grouping terms according to the value of $\alpha_{i}-\beta_{i}-a_{i}(\gamma-j)$, we obtain

$$
\sum_{\boldsymbol{\alpha} \in \mathbb{R}^{n}} \sum_{j \in \mathbb{N}} m_{\boldsymbol{\alpha}, j} \otimes \partial_{t}^{j}=v_{1}+\cdots+v_{k}
$$

with $v_{k} \in E_{\lambda_{k}}\left(T_{i}\right)$ and $\lambda_{1}, \ldots, \lambda_{k} \in \mathbb{R}$ are pairwise distinct. According to Lemma 11.3.18, we have $v_{h} \in W$ whenever $\lambda_{h} \neq 0$; what this means is that the sum of all $m_{\boldsymbol{\alpha}, j} \otimes \partial_{t}^{j}$ with $\alpha_{i}-\beta_{i}-a_{i}(\gamma-j) \neq 0$ belongs to $V_{<\gamma} N$. After subtracting this sum from our original element, we may therefore assume that $\alpha_{i}=\beta_{i}-a_{i}(\gamma-j)$ for every term. We obtain the asserted congruence by performing this procedure for $T_{1}, \ldots, T_{n}$. This ends the proof of Lemma 11.3.17 and at the same time that of Lemma 11.3.16.

Proof of Theorem 11.3.2, Step three: computation of nearby cycles. Suppose now that $\gamma<0$ and $\beta_{1}, \ldots, \beta_{n} \leqslant 0$, that we shall abbreviate as $\boldsymbol{\beta} \leqslant 0$. Let $j \in \mathbb{N}$. We observe that

$$
a_{i} \neq 0 \Longrightarrow \beta_{i}+(\gamma-j) a_{i}=\left(\beta_{i}+\gamma a_{i}\right)-j a_{i}<-j a_{i}
$$

Given a vector $m_{j} \in M_{\boldsymbol{\beta}+(\gamma-j) \boldsymbol{a}}$, this means that $m_{j}$ is divisible by $x_{i}^{j a_{i}}$. Consequently, $m_{j}=m x^{j a}$ for a unique $m$ in $M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}$, and therefore

$$
m_{j} \otimes \partial_{t}^{j}=(m \otimes 1) \cdot t^{j} \partial_{t}^{j}
$$

is a linear combination of $(m \otimes 1)\left(t \partial_{t}\right)^{k}$ for $k=1, \ldots, j$. Since $m \otimes 1 \in V_{\gamma} N$ and $V_{\gamma} N$ is stable by $t \partial_{t}$, we conclude that

$$
\sum_{j} M_{\boldsymbol{\beta}+(\gamma-j) \boldsymbol{a}} \otimes \partial_{t}^{j}=M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}\left[t \partial_{t}\right] \subset V_{\gamma} N
$$

and, by Lemma $11.3 .16, \widetilde{N}_{\boldsymbol{\beta}}$ is the image of $M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}\left[t \partial_{t}\right] \bmod V_{<\gamma} N$. Let us consider E as a new variable and let us endow $M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}[\mathrm{E}]:=M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}} \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{E}]$ with the $\mathbb{C}\left[\mathrm{N}_{1}, \ldots, \mathrm{~N}_{n}, \mathrm{~N}\right]$-module structure such that $\mathrm{N}_{i}$ acts by $2 \pi \mathrm{i}\left(\mathrm{S}_{i}-\beta_{i}-a_{i} \mathrm{E}\right)$ and N acts by $2 \pi \mathrm{i}(\mathrm{E}-\gamma)$ (see (11.3.6)), and let us endow $\widetilde{N}_{\boldsymbol{\beta}}$ with its natural $\mathbb{C}\left[\mathrm{N}_{1}, \ldots, \mathrm{~N}_{n}, \mathrm{~N}\right]$ module structure (see $\S 11.2 . \mathrm{a}$ ). We thus have proved the following result.

Proposition 11.3.20. We have a surjective $\mathbb{C}\left[\mathrm{N}_{1}, \ldots, \mathrm{~N}_{n}, \mathrm{~N}\right]$-linear morphism

$$
M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}[\mathrm{E}] \longrightarrow \tilde{N}_{\boldsymbol{\beta}}
$$

that takes $m \otimes \mathrm{E}^{k}$ to the class of $m \otimes\left(t \partial_{t}\right)^{k} \in V_{\gamma} N$ modulo $V_{<\gamma} N$.
Corollary 11.3.21. We have $\widetilde{N}=\bigoplus_{\beta} \widetilde{N}_{\boldsymbol{\beta}}$.

Proof. We have seen in the beginning of Step two that it is enough to prove $\widetilde{N}=$ $\sum_{\boldsymbol{\beta}} \widetilde{N}_{\boldsymbol{\beta}}$. Let us set $\widetilde{N}_{\leqslant 0}=\bigoplus_{\boldsymbol{\beta} \leqslant 0} \widetilde{N}_{\boldsymbol{\beta}}$. It is enough to check that $\widetilde{N}=\sum_{\boldsymbol{k}} \widetilde{N}_{\leqslant 0} \partial_{x}^{\boldsymbol{k}}$. We have seen that $V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M[\mathrm{E}] \subset V_{\gamma} N$ and has image equal to $\tilde{N}_{\leqslant 0}$. Then $\sum_{\boldsymbol{k}} \widetilde{N}_{\leqslant 0} \partial_{x}^{\boldsymbol{k}}$ contains the image of $\sum_{\boldsymbol{k}}\left(V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M \otimes 1\right) \partial_{x}^{\boldsymbol{k}}$, which is equal to $V_{\gamma} N$, by $(11.3 .9 *)$.

Remark 11.3.22. At this point, it can be clearer to write $M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}[\mathrm{E}]=M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}[\mathrm{N}]$ and to consider the latter space as a free $\mathbb{C}[\mathrm{N}]$-module, where N is considered as a new variable. The action of $\mathrm{N}_{i}$ on $M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}$ induces an action denoted by $\mathrm{N}_{i} \otimes 1$ on $M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}[\mathrm{N}]$, and we define the action of $\mathrm{N}_{i}$ on $\widetilde{N}_{\boldsymbol{\beta}}$ as that induced by $\mathrm{N}_{i} \otimes 1-a_{i} \mathrm{~N}$.

In order to have an explicit expression of $\tilde{N}_{\boldsymbol{\beta}}(\boldsymbol{\beta} \leqslant 0)$ and eventually prove its finite dimensionality, it remains to find the kernel of the morphism in Proposition 11.3.20. To do that, we introduce the set

$$
I_{g}(\boldsymbol{\beta})=\left\{i \in I \mid a_{i} \neq 0 \text { and } \beta_{i}=0\right\} \subset I_{g}
$$

Given $m \in M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}$, we have $\left(m \prod_{i \in I_{g}(\boldsymbol{\beta})} x_{i}\right) \otimes 1=m \otimes t \in V_{<\gamma} N$ and therefore also

$$
(m \otimes 1) \prod_{i \in I_{g}(\boldsymbol{\beta})} x_{i} \partial_{x_{i}}=(m \otimes 1) \cdot \prod_{i \in I_{g}(\boldsymbol{\beta})}\left(\mathrm{S}_{i}-a_{i} t \partial_{t}\right) \in V_{<\gamma} N
$$

In this way, we obtain a large collection of elements in the kernel.
Corollary 11.3.23. If $\gamma<0$ and $\boldsymbol{\beta} \leqslant 0, \widetilde{N}_{\boldsymbol{\beta}}$ is isomorphic to the cokernel of the injective morphism

$$
\begin{equation*}
\varphi_{\boldsymbol{\beta}}:=\prod_{i \in I_{g}(\boldsymbol{\beta})}\left(\mathrm{S}_{i} / a_{i}-\mathrm{E}\right) \in \operatorname{End}\left(M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}[\mathrm{E}]\right) \tag{11.3.23*}
\end{equation*}
$$

or equivalently

$$
\begin{equation*}
\varphi_{\boldsymbol{\beta}}:=\prod_{i \in I_{g}(\boldsymbol{\beta})}\left(\left(\mathrm{N}_{i} \otimes 1\right) / a_{i}-\mathrm{N}\right) \in \operatorname{End}\left(M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}[\mathrm{N}]\right) \tag{11.3.23**}
\end{equation*}
$$

Remark 11.3.24. We have assumed in Theorem 11.3.2 that $M$ is a minimal extension along the normal crossing divisor $D_{i \in I}$. However, the previous expression shows that, for $\gamma<0$ and $\boldsymbol{\beta} \leqslant 0, \widetilde{N}_{\boldsymbol{\beta}}$ only depends on the $M_{\boldsymbol{\alpha}}$ 's with $\alpha_{i}<0$ if $i \in I_{g}$. For such a $\gamma$, we conclude that $\operatorname{gr}_{\gamma}^{V} N$ only depends on the localized module $M(* g)$.

Moreover, by definition, the action of $\mathrm{N}_{i}$ (resp. N) on $\widetilde{N}_{\boldsymbol{\beta}}$ is that induced by $\mathrm{N}_{i} \otimes 1-a_{i} \mathrm{~N}($ resp. N$)$. We thus find that $\prod_{i \in I_{g}} \mathrm{~N}_{i}$ acts by zero on $\widetilde{N}_{\boldsymbol{\beta}}$.

Corollary 11.3.25. If $\gamma<0$ and $\boldsymbol{\beta} \leqslant 0, \widetilde{N}_{\boldsymbol{\beta}}$ is finite-dimensional.
Proof. Set $b=\left|I_{g}(\boldsymbol{\beta})\right|$. Corollary 11.3.23 implies that the natural $\mathbb{C}$-linear morphism

$$
\bigoplus_{k=0}^{b-1} M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}} \mathrm{E}^{k} \longrightarrow \tilde{N}_{\boldsymbol{\beta}}
$$

is an isomorphism. Since every $M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}$ is finite-dimensional, we obtain the desired assertion.

Note also that the action of E on $\tilde{N}_{\boldsymbol{\beta}}$, and thus that of $\mathrm{N}=2 \pi \mathrm{i}(\mathrm{E}-\gamma)$ is easily described on this expression:

$$
m \mathrm{E}^{k} \cdot \mathrm{E}= \begin{cases}m \mathrm{E}^{k+1} & \text { if } k<b-1 \\ m\left[\mathrm{E}^{b}-\prod_{i \in I_{g}(\boldsymbol{\beta})}\left(\mathrm{E}-\mathrm{S}_{i} / a_{i}\right)\right] & \text { if } k=b-1\end{cases}
$$

Proof of Corollary 11.3.23. Injectivity of $\varphi_{\boldsymbol{\beta}}$ is clear by considering the effet of $\varphi_{\boldsymbol{\beta}}$ on the term of highest degree with respect to E . On the other hand, we already know that every element of $\widetilde{N}_{\boldsymbol{\beta}}$ is the image of some $m=\sum_{k}\left(m_{k} \otimes 1\right) \mathrm{E}^{k}$ with $m_{k} \in M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}$ for every $k$. If we expand this using $\mathrm{E}=t \partial_{t}$, we find

$$
\begin{equation*}
m \in \bigoplus_{j \in \mathbb{N}} M_{\boldsymbol{\beta}+(\gamma-j) \boldsymbol{a}} \otimes \partial_{t}^{j} \tag{11.3.26}
\end{equation*}
$$

Now suppose that $m$ actually lies in $V_{<\gamma} N$. It can then be written as (see (11.3.15))

$$
\begin{equation*}
m=\sum_{\substack{\boldsymbol{\alpha} \in \mathbb{R}^{n} \\ \boldsymbol{k} \in \mathbb{N}_{g} I^{\prime}}}\left(m_{\boldsymbol{\alpha}, \boldsymbol{k}} \otimes 1\right) \partial_{x^{\prime}}^{\boldsymbol{k}} \tag{11.3.27}
\end{equation*}
$$

where $m_{\boldsymbol{\alpha}, \boldsymbol{k}} \in M_{\boldsymbol{\alpha}}$ satisfies $\alpha_{i}<\gamma a_{i}$ whenever $a_{i} \neq 0$. If we expand the expression $\left(m_{\boldsymbol{\alpha}, \boldsymbol{k}} \otimes 1\right) \partial_{x^{\prime}}^{\boldsymbol{k}}$ according to (11.3.3), all the terms that appear belong to $M_{\boldsymbol{\alpha}+\boldsymbol{k}-j \boldsymbol{a}} \otimes \partial_{t}^{j}$ for some $j \leqslant|\boldsymbol{k}|$ (we identify $\boldsymbol{k}$ with $(\boldsymbol{k}, 0) \in \mathbb{Z}^{n}$ ). Comparing with (11.3.26), we can therefore discard those summands in (11.3.27) with $\boldsymbol{\alpha}+\boldsymbol{k} \neq \boldsymbol{\beta}+\gamma \boldsymbol{a}$ without changing the value of the sum. The sum in (11.3.27) is thus simply indexed by those $\boldsymbol{k} \in \mathbb{N}^{I_{g}}$ such that $k_{i}>\beta_{i}$ for all $i \in I_{g}$ and the index $\boldsymbol{\alpha}$ is replaced with $\boldsymbol{\beta}+\gamma \boldsymbol{a}-\boldsymbol{k}$.

Now, if $a_{i} \neq 0$ then $\alpha_{i}=\left(\beta_{i}+\gamma a_{i}\right)-k_{i}<-k_{i}$ and so $m_{\boldsymbol{\alpha}, \boldsymbol{k}}$ is divisible by $x_{i}^{k_{i}}$. This means that we can write

$$
m_{\boldsymbol{\alpha}, \boldsymbol{k}}=m_{\boldsymbol{k}}^{\prime} x^{\prime \boldsymbol{k}}
$$

for some $m_{\boldsymbol{k}}^{\prime} \in M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}$. Therefore, (11.3.27) reads

$$
m=\sum_{\substack{\boldsymbol{k} \in \mathbb{N}^{I} g \\ k_{i}>\beta_{i} \forall i \in I_{g}}}\left(m_{\boldsymbol{k}}^{\prime} \otimes 1\right) x^{\prime \boldsymbol{k}} \partial_{x^{\prime}}^{\boldsymbol{k}}, \quad m_{\boldsymbol{k}}^{\prime} \in M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}
$$

If $m_{\boldsymbol{k}}^{\prime} \neq 0$, then $k_{i} \geqslant 1$ for $i \in I_{g}(\boldsymbol{\beta})$ (since $\beta_{i}=0$ ), and consequently, $x^{\boldsymbol{k}} \partial_{x^{\prime}}^{\boldsymbol{k}}$ is forced to be a multiple of

$$
\prod_{i \in I_{g}(\boldsymbol{\beta})} x_{i} \partial_{x_{i}}=\prod_{i \in I_{g}(\boldsymbol{\beta})}\left(\mathrm{S}_{i}-a_{i} \mathrm{E}\right)
$$

As a consequence,

$$
\begin{aligned}
& m \in \sum_{\ell \in \mathbb{N}^{I} g}\left(M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}} \otimes 1\right) x^{\ell \ell} \partial_{x^{\prime}}^{\ell} \cdot \prod_{i \in I_{g}(\boldsymbol{\beta})}\left(\mathrm{S}_{i}-a_{i} t \partial_{t}\right) \\
&=\sum_{\ell \in \mathbb{N}^{I_{g}}}\left(M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}} \otimes 1\right)\left(\mathrm{S}-\boldsymbol{a} t \partial_{t}\right)^{\ell} \cdot \prod_{i \in I_{g}(\boldsymbol{\beta})}\left(\mathrm{S}_{i}-a_{i} t \partial_{t}\right) \\
& \subset M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}[\mathrm{E}] \cdot \prod_{i \in I_{g}(\boldsymbol{\beta})}\left(\mathrm{S}_{i}-a_{i} \mathrm{E}\right) .
\end{aligned}
$$

We end this section by giving the explicit description of the quiver of $\widetilde{N}=\operatorname{gr}_{\gamma}^{V} N$ for $\gamma<0$ (see Proposition 11.2.5). We thus consider the vector spaces $\widetilde{N}_{\boldsymbol{\beta}}$ for $\boldsymbol{\beta} \in[-1,0]^{n}$, and the morphisms

for every $i$ such that $\beta_{i}=0$. We know from that Corollary 11.3.23 that $\tilde{N}_{\boldsymbol{\beta}} \neq 0$ only if $\beta_{i}=0$ for some $i \in I_{g}$ (i.e., such that $a_{i} \neq 0$ ). Moreover, the description of $\widetilde{N}_{\boldsymbol{\beta}}$ given in this corollary enables one to define a natural quiver as follows.
(1) If $i \notin I_{g}$ and $\beta_{i}=0$, we also have $(\boldsymbol{\beta}+\gamma \boldsymbol{a})_{i}=0$, and we will see that the diagram

commutes with $\varphi_{\boldsymbol{\beta}}$ (which only involves indices $j \in I_{g}$ ), inducing therefore in a natural way a diagram


We notice moreover that the minimal extension property for $M$ is preserved for this diagram, that is, $\mathrm{c}_{i}(\boldsymbol{\beta})$ remains surjective and $\mathrm{v}_{i}(\boldsymbol{\beta})$ remains injective.
(2) If $i \in I_{g}$, we set $\varphi_{1_{i}}=\left(\mathrm{N}_{i} \otimes 1\right) / a_{i}-\mathrm{N}$ so that, with obvious notation, $\varphi_{\boldsymbol{\beta}}=$ $\varphi_{\mathbf{1}_{i}} \varphi_{\boldsymbol{\beta}-\mathbf{1}_{i}}=\varphi_{\boldsymbol{\beta}-\mathbf{1}_{i}} \varphi_{\mathbf{1}_{i}}$, and we can regard $\varphi_{\boldsymbol{\beta}}, \varphi_{\mathbf{1}_{i}}, \varphi_{\boldsymbol{\beta}-\mathbf{1}_{i}}$ as acting (injectively) both on $M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}[\mathrm{N}]$ and $M_{\boldsymbol{\beta}-\mathbf{1}_{i}+\boldsymbol{a}}[\mathrm{N}]$. Moreover, the multiplication by $x_{i}$, which is an isomorphism $M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}} \xrightarrow{\sim} M_{\boldsymbol{\beta}-\mathbf{1}_{i}+\gamma \boldsymbol{a}}$, is such that $x_{i} \otimes 1$ commutes with $\varphi_{\boldsymbol{\beta}-\mathbf{1}_{i}}$. In such a way, we can regard $\widetilde{N}_{\boldsymbol{\beta}-\mathbf{1}_{i}}$ as Coker $\varphi_{\boldsymbol{\beta}-\mathbf{1}_{i}}$ acting on $M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}[\mathrm{N}]$. We can then define $c_{i}$ and $\mathrm{v}_{i}$ as naturally induced by the following commutative diagrams:


In other words, $\mathrm{c}_{i}(\boldsymbol{\beta})$ is the natural morphism

$$
M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}[\mathrm{N}] / \operatorname{Im} \varphi_{\boldsymbol{\beta}-\mathbf{1}_{i}} \xrightarrow{\varphi_{\mathbf{1}_{i}}} M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}[\mathrm{N}] / \operatorname{Im} \varphi_{\boldsymbol{\beta}}
$$

and $\mathrm{v}_{i}(\boldsymbol{\beta})$ is the natural morphism induced by the inclusion $\operatorname{Im} \varphi_{\boldsymbol{\beta}} \subset \operatorname{Im} \varphi_{\boldsymbol{\beta}-\mathbf{1}_{i}}$ :

$$
M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}[\mathrm{N}] / \operatorname{Im} \varphi_{\boldsymbol{\beta}} \longrightarrow M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}[\mathrm{N}] / \operatorname{Im} \varphi_{\boldsymbol{\beta}-\mathbf{1}_{i}}
$$

We note that $\mathrm{v}_{i}(\boldsymbol{\beta})$ is surjective. Moreover,
Proposition 11.3.29. For $\gamma<0$, the quiver of $\operatorname{gr}_{\gamma}^{V} N$ has vertices $\widetilde{N}_{\boldsymbol{\beta}}=\operatorname{Coker} \varphi_{\boldsymbol{\beta}}$ for $\boldsymbol{\beta} \in[0,1]^{n}$ such that
(1) $\boldsymbol{\beta}=\boldsymbol{\alpha}-\gamma \boldsymbol{a}$ for some $\boldsymbol{\alpha} \in A+\mathbb{Z}$,
(2) $\beta_{i}=0$ for some $i \in I_{g}$.

It is isomorphic to the quiver defined by the morphisms $\mathrm{c}_{i}(\boldsymbol{\beta}), \mathrm{v}_{i}(\boldsymbol{\beta})$ as described above.
11.3.b. More on the structure of the nearby cycles. We keep the notation and assumptions of Theorem 11.3 .2 in the present setting (no filtration, $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-modules of normal crossing type) and we will make more precise the $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-module structure of $\widetilde{N}:=\operatorname{gr}_{\gamma}^{V} N$. The general principle is that the graded object $\operatorname{gr}^{\mathrm{M}} \widetilde{N}$ with respect to the monodromy filtration of the nilpotent endomorphism $\mathrm{N}=2 \pi \mathrm{i}(\mathrm{E}-\gamma)$ should be simpler to understand, and enough for the purpose of Hodge theory, and moreover it is completely determined by the primitive modules $\mathrm{P}_{k} \widetilde{N}$, by means of the Lefschetz decomposition. We first exhibit the simplification brought by grading with respect to a suitable finite filtration $U_{\bullet}$, and we will consider next the monodromy filtration.
Structure of $V_{0}^{(\boldsymbol{n})} \tilde{N}$. We consider E as a new variable and we set $V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M[\mathrm{E}]=$ $V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{E}]$. We endow $V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M[\mathrm{E}]$ with the following twisted $\mathbb{C}[x]\left\langle x \partial_{x}\right\rangle[\mathrm{E}]$ structure, compatible with (11.3.3):

$$
\begin{aligned}
\left(m \otimes \mathrm{E}^{k}\right) \cdot f(x) & =m f(x) \otimes \mathrm{E}^{k} \\
\left(m \otimes \mathrm{E}^{k}\right) \cdot \mathrm{E} & =m \otimes \mathrm{E}^{k+1} \\
\left(m \otimes \mathrm{E}^{k}\right) \cdot x_{i} \partial_{x_{i}} & =m x_{i} \partial_{x_{i}} \otimes \mathrm{E}^{k}-a_{i} m \otimes \mathrm{E}^{k+1}
\end{aligned}
$$

If $\boldsymbol{\alpha}$ is such that $M_{\boldsymbol{\alpha}} \neq 0$ (see Definition 11.2.2), we set

$$
I_{g}(\gamma, \boldsymbol{\alpha})=\left\{i \in I_{g} \mid \alpha_{i}=\gamma a_{i}\right\}
$$

Corollary 11.3.23 provides us with a presentation of $V_{0}^{(\boldsymbol{n})}(\widetilde{N})$ as the cokernel of

$$
\begin{equation*}
\varphi=\bigoplus_{\alpha \leqslant \gamma \boldsymbol{a}} \varphi_{\boldsymbol{\alpha}}: \bigoplus_{\alpha \leqslant \gamma \boldsymbol{a}} M_{\boldsymbol{\alpha}}[\mathrm{E}] \longrightarrow \bigoplus_{\alpha \leqslant \gamma \boldsymbol{a}} M_{\boldsymbol{\alpha}}[\mathrm{E}] \tag{11.3.30}
\end{equation*}
$$

with

$$
\varphi_{\boldsymbol{\alpha}}:=\prod_{i \in I_{g}(\gamma, \boldsymbol{\alpha})}\left(\mathrm{S}_{i} / a_{i}-\mathrm{E}\right)
$$

This morphism is $\mathbb{C}\left\langle x \partial_{x}, \mathrm{E}\right\rangle$-linear, and we have $V_{0}^{(\boldsymbol{n})}(\tilde{N}) \simeq \operatorname{Coker} \varphi$ as a $\mathbb{C}\left\langle x \partial_{x}, \mathrm{E}\right\rangle$ module. Moreover, Corollary 11.3.23 also implies that, for every $i=1, \ldots, n$, the $V$-filtration of $\widetilde{N}$ in the direction of $D_{i}$ is determined by the formula

$$
\begin{equation*}
V_{\beta_{i}}^{(i)}(\widetilde{N}) \cap V_{0}^{(\boldsymbol{n})}(\widetilde{N})=\operatorname{image}\left(\left(V_{\beta_{i}+\gamma a_{i}}^{(i)} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M\right)[\mathrm{E}]\right), \quad \text { for } \beta_{i} \leqslant 0 \tag{11.3.31}
\end{equation*}
$$

However, since $x_{i}$ does not commute with $\mathrm{S}_{i}$, the morphism $\varphi$ is not $\mathbb{C}[x]\left\langle x \partial_{x}\right\rangle[\mathrm{E}]$ linear.

We will recover $\mathbb{C}[x]\left\langle x \partial_{x}\right\rangle[\mathrm{E}]$-linearity after grading by a suitable finite filtration $U_{\bullet}$. For $\boldsymbol{\alpha} \leqslant \gamma \boldsymbol{a}$ fixed, and for $j \in I_{g}(\gamma, \boldsymbol{\alpha})$, we have

$$
\left(\prod_{i \in I_{g}(\gamma, \boldsymbol{\alpha})} x_{i} \partial_{x_{i}}\right) \cdot x_{j}-x_{j} \cdot\left(\prod_{i \in I_{g}(\gamma, \boldsymbol{\alpha})} x_{i} \partial_{x_{i}}\right)=x_{j} \cdot \prod_{\substack{i \in I_{g}(\gamma, \boldsymbol{\alpha}) \\ i \neq j}} x_{i} \partial_{x_{i}}
$$

and the right-hand term sends $M_{\boldsymbol{\alpha}}[\mathrm{E}]$ in $M_{\boldsymbol{\alpha}-\mathbf{1}_{j}}[\mathrm{E}]$. Moreover, for $j \in I_{g}(\gamma, \boldsymbol{\alpha})$, the equality $\# I_{g}\left(\gamma, \boldsymbol{\alpha}-\mathbf{1}_{j}\right)=\# I_{g}(\gamma, \boldsymbol{\alpha})-1$ holds true. We are thus led to define the finite increasing filtration by $\mathbb{C}[x]\left\langle x \partial_{x}\right\rangle[\mathrm{E}]$-submodules

$$
U_{k} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M[\mathrm{E}]=\bigoplus_{\substack{\boldsymbol{\alpha} \leqslant \gamma \boldsymbol{a} \\ \# I_{g}(\gamma, \boldsymbol{\alpha}) \leqslant k}} M_{\boldsymbol{\alpha}}[\mathrm{E}] .
$$

Grading with respect to $U_{\boldsymbol{\bullet}}$ has the only effect of killing the action of $x_{j}$ on $M_{\boldsymbol{\alpha}}[\mathrm{E}]$ for $j \in I_{g}(\gamma, \boldsymbol{\alpha})$. Moreover, the image filtration $U_{\bullet} V_{0}^{(\boldsymbol{n})} \widetilde{N}$ is nothing but the filtration

$$
U_{k} V_{0}^{(\boldsymbol{n})} \widetilde{N}=\bigoplus_{\substack{\boldsymbol{\beta} \leqslant 0 \\ \# I_{g}(\boldsymbol{\beta}) \leqslant k}} \widetilde{N}_{\boldsymbol{\beta}} .
$$

Every $\mathbb{C}[x]\left\langle x \partial_{x}\right\rangle[\mathrm{E}]$-module $\operatorname{gr}_{k}^{U} V_{0}^{(\boldsymbol{n})} \tilde{N}$ is the direct sum of its submodules $\left(\operatorname{gr}_{k}^{U} V_{0}^{(\boldsymbol{n})} \widetilde{N}\right)_{J}$ with

$$
\left(\operatorname{gr}_{k}^{U} V_{0}^{(\boldsymbol{n})} \widetilde{N}\right)_{J}=\bigoplus_{\boldsymbol{\beta} \mid I_{g}(\boldsymbol{\beta})=J} \tilde{N}_{\boldsymbol{\beta}}, \quad \text { for } J \subset I_{g} \text { with }|J|=k
$$

and $x_{j}$ acts by zero on $\left(\operatorname{gr}_{k}^{U} V_{0}^{(\boldsymbol{n})} \tilde{N}\right)_{J}$ for $j \in J$. In other words, $\left(\operatorname{gr}_{k}^{U} V_{0}^{(\boldsymbol{n})} \tilde{N}\right)_{J}$ is supported on $\bigcap_{i \in J} D_{i}$.

Proposition 11.3.32 (Structure of $\operatorname{gr}^{U} V_{0}^{(\boldsymbol{n})}(\tilde{N})$ ). The morphism $\varphi$ is strictly compatible with the filtration $U_{\bullet} V_{\gamma}^{(n)} M[\mathrm{E}]$ and we have an exact sequence

$$
0 \longrightarrow \operatorname{gr}^{U} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M[\mathrm{E}] \xrightarrow{\operatorname{gr}^{U} \varphi} \operatorname{gr}^{U} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M[\mathrm{E}] \longrightarrow \operatorname{gr}^{U} V_{0}^{(\boldsymbol{n})}(\widetilde{N}) \longrightarrow 0
$$

Proof. This is obvious since $\varphi$ is graded as a $\mathbb{C}\left[x \partial_{x}\right.$, E]-linear morphism.

## Remarks 11.3.33.

(1) The computation shows that $\operatorname{gr}^{U} V_{0}^{(n)}(\widetilde{N})$, hence $V_{0}^{(\boldsymbol{n})}(\widetilde{N})$, hence $\widetilde{N}$, is supported by the divisor of $g$, as expected of course.
(2) As a consequence, we can also determine the negative jumping indices of the $V$-filtration of $N$. Let $\boldsymbol{A} \subset[-1,0)^{n}$ be the finite set of exponents of $M$ (see Definition 11.2.2). Let us fix $\gamma<0$. Then $\operatorname{gr}_{\gamma}^{V} N \neq 0$ if and only if $\widetilde{N}_{\boldsymbol{\beta}} \neq 0$ for some $\boldsymbol{\beta} \leqslant 0$, that is, $I_{g}(\boldsymbol{\beta}) \neq \varnothing$, that is, $\gamma a_{i} \in \alpha_{i}-\mathbb{N}$ for some $\boldsymbol{\alpha} \in \boldsymbol{A}$ and some $i \in I_{g}$. We conclude the set of negative jumping indices is the set

$$
\bigcup_{i \in I_{g}} \frac{1}{a_{i}}\left(\alpha_{i}-\mathbb{N}\right), \quad \boldsymbol{\alpha} \in \boldsymbol{A} .
$$

11.3.c. A simple example with monodromy filtration. We illustrate the previous general results on a simple example, where we can give more details on the monodromy filtration on the nearby cycles $\widetilde{N}=\operatorname{gr}_{\gamma}^{V} N(\gamma<0)$.

Assumption 11.3.34. $M$ is simple, that is, all properties of Example 11.2.9 are satisfied. In particular the set of exponents is reduced to one element $\boldsymbol{\alpha} \in[-1,0)^{n}$, rk $M_{\boldsymbol{\alpha}}=1$ and $M_{\boldsymbol{\alpha}+\boldsymbol{k}}=0$ if $\alpha_{i}=-1$ and $k_{i} \geqslant 1$.

Let us summarize the results already obtained in the present setting.
(i) The set of negative jumping indices $\gamma$ for the $V$-filtration is $\bigcup_{i \in I_{g}} \frac{1}{a_{i}}\left(\alpha_{i}-\mathbb{N}\right)$. For such a a jumping index $\gamma \in[-1,0)$, we will set $J(\boldsymbol{\alpha}, \gamma):=\left\{i \in I_{g} \mid \alpha_{i} \equiv \gamma a_{i} \bmod \mathbb{Z}\right\}$ and $k_{\gamma}=\# J(\boldsymbol{\alpha}, \gamma)-1$. Moreover, for $\boldsymbol{j} \in\{0,1\}^{I}$, we set $\|\boldsymbol{j}\|:=\sum_{i \in J(\boldsymbol{\alpha}, \gamma)} j_{i}=$ $\#\left\{i \in J(\boldsymbol{\alpha}, \gamma) \mid j_{i}=1\right\}$.
(ii) Such a jumping index $\gamma \in[-1,0)$ being fixed, and setting $\widetilde{N}=\operatorname{gr}_{\gamma}^{V} N$, the only possible $\boldsymbol{\beta} \leqslant 0$ such that $\widetilde{N}_{\boldsymbol{\beta}} \neq 0$ are of the form $\boldsymbol{\beta}=\boldsymbol{\alpha}-\gamma \boldsymbol{a}-\boldsymbol{k}$ for suitable $\boldsymbol{k} \in \mathbb{N}^{n}$ : for all $i \in\{1, \ldots, n\}$ we have $\alpha_{i}-\gamma a_{i}>\alpha_{i} \geqslant-1$, so $\alpha_{i}-\gamma a_{i}-k_{i}>0$ if $k_{i} \leqslant-1$. (Note also that some components of $\boldsymbol{\alpha}-\gamma \boldsymbol{a}$ can be $>0$ ). There exists a unique $\boldsymbol{k}^{o} \in \mathbb{N}^{n}$ such that

$$
\begin{cases}k_{i}^{o}=0 & \text { if } i \notin I_{g} \\ \beta_{i}^{o}:=\alpha_{i}-\gamma a_{i}-k_{i}^{o} \in[-1,0) & \text { if } i \in I_{g}\end{cases}
$$

and we set $\boldsymbol{\beta}^{o}=\boldsymbol{\alpha}^{o}-\gamma \boldsymbol{a}$, with $\boldsymbol{\alpha}^{o}:=\boldsymbol{\alpha}-\boldsymbol{k}^{o}$, so that in particular $\beta_{i}^{o}=\alpha_{i}^{o}$ if $i \notin I_{g}$. Then $\widetilde{N}$ has a single exponent, equal to $\boldsymbol{\beta}^{o}$, and its quiver (see Remark 11.2.6) has vertices $\widetilde{N}_{\boldsymbol{\beta}}$ with $\boldsymbol{\beta}=\boldsymbol{\beta}^{o}+\boldsymbol{j}, \boldsymbol{j} \in\{0,1\}^{I}$. The corresponding $\boldsymbol{\beta}+\gamma \boldsymbol{a}$ is then equal to $\boldsymbol{\alpha}^{o}+\boldsymbol{j}$. We note that

$$
J(\boldsymbol{\alpha}, \gamma)=\left\{i \in I_{g} \mid \beta_{i}^{o}=-1\right\} \quad \text { and } \quad I_{g}\left(\boldsymbol{\beta}^{o}+\boldsymbol{j}\right)=\left\{i \in J(\boldsymbol{\alpha}, \gamma) \mid j_{i}=1\right\}
$$

(iii) The action of $x_{i} \partial_{x_{i}}$ on $M_{\boldsymbol{\alpha}^{o}+\boldsymbol{j}}$ (with $\boldsymbol{\alpha}^{o}$ and $\boldsymbol{j}$ as above), that we have denoted by $\mathrm{S}_{i}$ above, is the multiplication by the constant $\alpha_{i}^{o}+j_{i}$. On the other hand, the action of $x_{i} \partial_{x_{i}}$ on $M_{\boldsymbol{\alpha}^{o}+\boldsymbol{j}}[\mathrm{E}]$ is by $\mathrm{S}_{i}-a_{i} \mathrm{E}$. We note that, for $i \in I_{g}\left(\boldsymbol{\beta}^{o}+\boldsymbol{j}\right)$, we have $\beta_{i}^{o}+j_{i}=0$, hence $\beta_{i}^{o}=-1$ and $j_{i}=1$, so $\left(\alpha_{i}^{o}+j_{i}\right) / a_{i}=\gamma$. Proposition 11.3.20 describes $\widetilde{N}_{\boldsymbol{\beta}^{o}+\boldsymbol{j}}$ as the cokernel of $(\mathrm{E}-\gamma)^{\|\boldsymbol{j}\|}$ acting on $M_{\boldsymbol{\alpha}^{o}+\boldsymbol{j}}[\mathrm{E}]$.

Let us consider the operator $\mathrm{N}=2 \pi \mathrm{i}(\mathrm{E}-\gamma)$, and identify in a natural way $M_{\boldsymbol{\alpha}^{o}+\boldsymbol{j}}[\mathrm{E}]$ with $M_{\boldsymbol{\alpha}^{o}+\boldsymbol{j}}[\mathrm{N}]$. Then, as a $\mathbb{C}[\mathrm{N}]$-module, we have

$$
\tilde{N}_{\boldsymbol{\beta}^{o}+\boldsymbol{j}}=M_{\boldsymbol{\alpha}^{o}+\boldsymbol{j}}[\mathrm{N}] / \operatorname{Im} \mathrm{N}^{\|\boldsymbol{j}\|} .
$$

In other words, $\widetilde{N}_{\boldsymbol{\beta}^{o}+\boldsymbol{j}}$ is a Jordan block of size $\|\boldsymbol{j}\|$ with respect to N . In particular, $\widetilde{N}_{\boldsymbol{\beta}^{\circ}+\boldsymbol{j}}=0$ for any $\boldsymbol{j}$ all of whose components on $J(\boldsymbol{\alpha}, \gamma)$ are zero. The action of $\mathrm{N}_{i}=2 \pi \mathrm{i}\left(x_{i} \partial_{x_{i}}-\left(\beta_{i}^{o}+j_{i}\right)\right)$ on $\widetilde{N}_{\boldsymbol{\beta}^{o}+\boldsymbol{j}}$, which is induced from that on $M_{\boldsymbol{\alpha}^{o}+\boldsymbol{j}}[\mathrm{E}]$, is by $-\left(a_{i} / 2 \pi i\right) \mathrm{N}$. As a consequence, the primitive part $\mathrm{P}_{k}\left(\widetilde{N}_{\boldsymbol{\beta}^{\circ}+\boldsymbol{j}}\right)$ is zero if $k \neq\|\boldsymbol{j}\|-1$ and has dimension 1 if $k=\|\boldsymbol{j}\|-1$. We then denote by $\mathrm{P}\left(\widetilde{N}_{\boldsymbol{\beta}^{\circ}+\boldsymbol{j}}\right)$ this primitive part.

We conclude that for $k \in \mathbb{N}$, the quiver of $\mathrm{P}_{k} \widetilde{N}$ is zero if $k>k_{\gamma}$, and otherwise has vertices $\mathrm{P}\left(\tilde{N}_{\boldsymbol{\beta}^{\circ}+\boldsymbol{j}}\right)$ for $\boldsymbol{j} \in\{0,1\}^{I}$ such that $\|\boldsymbol{j}\|=k+1$.
(iv) Let now us describe the $\operatorname{var}_{i}$ arrows in the quiver of $\widetilde{N}$. The action of $x_{i}$ on $\widetilde{N}$ is induced by that on $M$ so, if $j_{i}=1, x_{i}: \widetilde{N}_{\boldsymbol{\beta}^{\circ}+\boldsymbol{j}} \rightarrow \widetilde{N}_{\boldsymbol{\beta}^{\circ}+\boldsymbol{j}-\mathbf{1}_{i}}$ it is the morphism

$$
x_{i} \otimes 1: M_{\boldsymbol{\alpha}^{o}+\boldsymbol{j}}[\mathrm{N}] / \operatorname{Im} \mathrm{N}^{\|\boldsymbol{j}\|} \longrightarrow M_{\boldsymbol{\alpha}^{o}+\boldsymbol{j}-\mathbf{1}_{i}}[\mathrm{~N}] / \operatorname{Im} \mathrm{N}^{\left\|\boldsymbol{j}-\mathbf{1}_{i}\right\|} .
$$

Therefore,
(a) if $i \notin J(\boldsymbol{\alpha}, \gamma)$, we have $\left\|\boldsymbol{j}-\mathbf{1}_{i}\right\|=\|\boldsymbol{j}\|$ and $x_{i}: \widetilde{N}_{\boldsymbol{\beta}^{\circ}+\boldsymbol{j}} \rightarrow \widetilde{N}_{\boldsymbol{\beta}^{\circ}+\boldsymbol{j}-\mathbf{1}_{i}}$ is injective, since $x_{i}: M_{\boldsymbol{\alpha}^{o}+\boldsymbol{j}} \rightarrow M_{\boldsymbol{\alpha}^{o}+\boldsymbol{j}-\mathbf{1}_{i}}$ by our assumption of minimal extension on $M$. For the same reason, $x_{i}: \mathrm{P} \widetilde{N}_{\boldsymbol{\beta}^{o}+\boldsymbol{j}} \rightarrow \mathrm{P} \widetilde{N}_{\boldsymbol{\beta}^{o}+\boldsymbol{j}-\mathbf{1}_{i}}$ is injective.
(b) If $i \in J(\boldsymbol{\alpha}, \gamma)$, then $x_{i}$ induces zero on the N -primitive part $\mathrm{P} \widetilde{N}_{\boldsymbol{\beta}^{\circ}+\boldsymbol{j}}$, so $\operatorname{var}_{i}$ is zero on the quiver of $\mathrm{P}_{k} \widetilde{N}$ for every $k$.
(v) We consider the $\operatorname{can}_{i}$ arrows in the quiver of $\tilde{N}$. So we consider $\partial_{x_{i}}$ : $\widetilde{N}_{\boldsymbol{\beta}^{o}+\boldsymbol{j}-\mathbf{1}_{i}} \rightarrow \widetilde{N}_{\boldsymbol{\beta}^{o}+\boldsymbol{j}}$ with $j_{i}=1$.
(a) If $i \notin I_{g}$, then the action of $\partial_{x_{i}}$ on $M_{\boldsymbol{\beta}^{\circ}+\boldsymbol{j}-\mathbf{1}_{i}}[\mathrm{~N}]$ is simply induced from that on $M_{\boldsymbol{\beta}^{\circ}+\boldsymbol{j}-\mathbf{1}_{i}}($ see $(11.3 .3))$, hence $\operatorname{can}_{i}$ is onto since $\left\|\boldsymbol{j}-\mathbf{1}_{i}\right\|=\|\boldsymbol{j}\|$ and by our assumption of minimal extension on $M$. The same property holds for every $\mathrm{P}_{k} \widetilde{N}$.
(b) If $i \in I_{g} \backslash J(\boldsymbol{\alpha}, \gamma)$, then $\beta_{i}^{o} \in(-1,0)$ and $\operatorname{can}_{i}$ is an isomorphism by our convention (Remark 11.2.6). The same property holds for every $\mathrm{P}_{k} \widetilde{N}$.
(c) If $i \in J(\boldsymbol{\alpha}, \gamma)$, then for a given $k \in \mathbb{N}$, either $\mathrm{P}_{k} \widetilde{N}_{\boldsymbol{\beta}^{\circ}+\boldsymbol{j}}$ or $\mathrm{P}_{k} \widetilde{N}_{\boldsymbol{\beta}^{o}+\boldsymbol{j}-\mathbf{1}_{i}}$ is zero, so $\operatorname{can}_{i}$ is zero on the quiver of $\mathrm{P}_{k} \widetilde{N}$ for every $k$.
Summarizing the discussion, let us emphasize the consequences on the primitive parts $\mathrm{P}_{k} \mathrm{gr}_{\gamma}^{V} N$.

Corollary 11.3.35. The $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-module $\mathrm{P}_{k} \operatorname{gr}_{\gamma}^{V} N$ vanishes for $k>k_{\gamma}$ and the support of $\mathrm{P}_{k} \mathrm{gr}_{\gamma}^{V} N$ has codimension $k$ if $k \leqslant k_{\gamma}$. More precisely, if $k \leqslant k_{\gamma}$, then

$$
\mathrm{P}_{k} \operatorname{gr}_{\gamma}^{V} N=\underset{\substack{J \subset J(\boldsymbol{\alpha}, \gamma) \\ \# J=k}}{\bigoplus}\left(\mathrm{P}_{k} \operatorname{gr}_{\gamma}^{V} N\right)_{J}
$$

where each $\left(\mathrm{P}_{k} \operatorname{gr}_{\gamma}^{V} N\right)_{J}$ is supported on $D_{J}:=\bigcap_{i \in J} D_{i}$ and, when regarded as a $\mathbb{C}\left[x_{J}\right]\left\langle\partial_{x_{J}}\right\rangle$-module, it is of normal crossing type along the divisor induced by $\bigcup_{i \notin J} D_{i}$ and the corresponding quiver is isomorphic to the $(I \backslash J)$-quiver of $M$. In particular, $\mathrm{P}_{k_{\gamma}} \mathrm{gr}_{\gamma}^{V} N$ is a minimal extension with support along $D_{i \in I}$.

The general case. How much of the previous discussion remains valid in the general case of a $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-module $M$ of normal crossing type which is a minimal extension along $D_{i \in I}$ ? Firstly, we can assume that the set $\boldsymbol{A}$ of exponents of $M$ is reduced to a single element $\boldsymbol{\alpha} \in[-1,0)^{n}$ since $M$ is the direct sum of such modules (see Remark 11.2.18(2)). Therefore, Properties (i) and (ii) of the simple case still hold.

However, in Property (iii), we have to take into account the nilpotent part $S_{i}^{\text {nilp }}$ of the action of $x_{i} \partial_{x_{i}}$ on $M_{\boldsymbol{\alpha}^{\circ}+\boldsymbol{j}}$. We can then describe $\tilde{N}_{\boldsymbol{\beta}^{\circ}+\boldsymbol{j}}$ as

$$
\widetilde{N}_{\boldsymbol{\beta}^{\circ}+\boldsymbol{j}}=M_{\boldsymbol{\alpha}^{o}+\boldsymbol{j}}[\mathrm{N}] / \operatorname{Im}\left(\prod_{i \in I_{g}\left(\boldsymbol{\beta}^{\circ}+\boldsymbol{j}\right)}\left(\mathrm{N}-\mathrm{S}_{i}^{\text {nilp }} / a_{i}\right)\right)
$$

In Property (iv) (resp. (v)), the statements (iva) (resp. (va) and (vb)) remain true, but it is not clear how to compute the primitive parts of $\widetilde{N}_{\boldsymbol{\beta}^{\circ}+\boldsymbol{j}}$, and therefore (ivb) and (vc) do not extend in a simple way.

## 11.3.d. Nearby cycles for coherent $\mathscr{D}$-modules of normal crossing type

We now consider Theorem 11.3.2 in the analytic setting, but we forget the filtration. Given $\mathcal{M}$ of normal crossing type along $D$ as in $\S 11.2$.b, we denote by $M$ the associated $\mathbb{C}[x]\left\langle\partial_{x}\right\rangle$-module, so that $\mathcal{M}=\mathscr{D}_{X} \otimes_{\mathbb{C}[x]\left\langle\partial_{x}\right\rangle} M$. In oder to simplify the notation, we will set $\mathcal{N}:={ }_{\mathrm{D}} \iota_{g *} \mathcal{M}$. Then, using the notation of §11.3.a, we have $\mathcal{N}=\mathscr{D}_{X \times \mathbb{C}} \otimes_{\mathbb{C}[x, t]\left\langle\partial_{x}, \partial_{t}\right\rangle} N$ and $\operatorname{gr}_{\gamma}^{V} \mathcal{N}=\mathscr{D}_{X} \otimes_{\mathbb{C}[x]\left\langle\partial_{x}\right\rangle} \operatorname{gr}_{\gamma}^{V} N$. As a consequence from the results proved for $M$ and $N$ in §11.3.a, we obtain that Theorem 11.3.2 holds for $\mathcal{M}$. The results of $\S 11.3 . \mathrm{b}$ also extend to $\mathcal{M}$ and $\mathcal{N}$ in a straightforward way. Let us end this subsection by adapting Proposition 11.3 .32 to $\mathcal{M}$.
Structure of $V_{0}^{(\boldsymbol{n})} \widetilde{\mathcal{N}}$. We have

$$
V_{0}^{(\boldsymbol{n})} \widetilde{\mathcal{N}}=\mathscr{O}_{X} \otimes_{\mathbb{C}[x]\left\langle\partial_{x}\right\rangle} V_{0}^{(\boldsymbol{n})} \widetilde{N}=\mathscr{O}_{X} \otimes_{\mathbb{C}[x]\left\langle\partial_{x}\right\rangle}\left(\sum_{\boldsymbol{\beta} \leqslant 0} \tilde{N}_{\boldsymbol{\beta}}\right)
$$

The $\mathscr{O}_{X}$-module $V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}]:=V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{E}]=\mathscr{O}_{X} \otimes_{\mathbb{C}[x]\left\langle\partial_{x}\right\rangle} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M[\mathrm{E}]$ is endowed with an induced action of $V_{0}^{(n)} \mathscr{D}_{X}$ (and the obvious action of E, see below), and we have a surjective $V_{0}^{(\boldsymbol{n})} \mathscr{D}_{X}$-linear morphism

$$
\begin{equation*}
V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}] \longrightarrow V_{0}^{(\boldsymbol{n})} \mathcal{N} \tag{11.3.36}
\end{equation*}
$$

In the analytic setting, the filtration $U_{\bullet}$ is defined by analytification of that on $V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} M[\mathrm{E}]$ and $V_{0}^{(\boldsymbol{n})} \tilde{N}$ and the analytification of $\mathrm{gr}^{U} \varphi$ gives rise to a $V_{0}^{(\boldsymbol{n})} \mathscr{D}_{X}$-linear presentation for each $k \geqslant 0$

$$
\begin{equation*}
\operatorname{gr}_{k}^{U} V_{0}^{(\boldsymbol{n})} \mathcal{N}=\operatorname{Coker}\left[\operatorname{gr}_{k}^{U} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}] \xrightarrow{\varphi_{k}} \operatorname{gr}_{k}^{U} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}]\right] \tag{11.3.37}
\end{equation*}
$$

and $\varphi_{k}$ is injective.
The filtration $U_{\boldsymbol{\bullet}} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}$ (and then $U_{\boldsymbol{\bullet}} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}]$ ) can be defined in terms of $\mathcal{M}$ only. For $J \subset I_{g}$, let us denote $J^{\mathrm{c}}:=I_{g} \backslash J$ and $I_{g}^{\mathrm{c}}:=I \backslash I_{g}$, so that $I=J^{\mathrm{c}} \sqcup J \sqcup I_{g}^{\mathrm{c}}$. Let us decompose correspondingly $\boldsymbol{a}=\left(\boldsymbol{a}_{J^{c}}, \boldsymbol{a}_{J}, 0_{I_{g}^{c}}\right)$ and $\boldsymbol{n}=\left(\boldsymbol{n}_{J^{\mathrm{c}}}, \boldsymbol{n}_{J}, \boldsymbol{n}_{I_{g}^{c}}\right)$. Then, by considering first $V_{\gamma a}^{(\boldsymbol{n})} M$, one checks that
$U_{k} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}=\sum_{\substack{J \subset I_{g} \\ \# J \leqslant k}} V_{<\gamma \boldsymbol{a} \boldsymbol{a}_{\mathrm{c}}}^{\left(\boldsymbol{n}_{J \mathrm{c}}\right)} V_{\gamma \boldsymbol{a}_{J}}^{\left(\boldsymbol{n}_{J}\right)} V_{0}^{\left(\boldsymbol{n}_{I g}^{\mathrm{c}}\right)} \mathcal{M}, \quad \operatorname{gr}_{k}^{U} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}=\underset{\substack{J \subset I_{g} \\ \# J=k}}{ } V_{<\gamma \boldsymbol{a} \boldsymbol{a}_{J c}}^{\left(\boldsymbol{n}_{J \mathrm{c}}\right)} V_{0}^{\left(\boldsymbol{n}_{I_{g}^{\mathrm{c}}}\right)} \operatorname{gr}_{\gamma \boldsymbol{a} J}^{V_{J}^{\left(\boldsymbol{n}_{J}\right)}} \mathcal{M}$.

We can give an interpretation of the filtration $U_{\boldsymbol{\bullet}} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}$ as a convolution of filtrations, as in Exercise 8.8(1). Let us define the following filtrations on $V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}$, for $i \in I$ :

$$
\begin{align*}
& U_{-1}^{(i)} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}=0, \\
& U_{0}^{(i)} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}= \begin{cases}V_{<\gamma a_{i}}^{(i)} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M} & \text { if } i \in I_{g}, \\
V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M} & \text { if } i \notin I_{g},\end{cases}  \tag{11.3.38}\\
& U_{1}^{(i)} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}=V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M} .
\end{align*}
$$

Then

$$
\begin{equation*}
U_{k} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}=\left(U_{\bullet}^{(1)} \star \cdots \star U_{\bullet}^{(n)}\right)_{k} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M} \tag{11.3.39}
\end{equation*}
$$

11.3.e. Nearby cycles for coherent filtered $\mathscr{D}$-modules of normal crossing type

We now take up the proof of Theorem 11.3.2 in the filtered case, and we set $\left(\mathcal{N}, F_{\bullet} \mathcal{N}\right)={ }_{{ }_{\mathrm{D}} \iota_{g *}}(\mathcal{M}, F \cdot \mathcal{M})$, or equivalently $\mathscr{N}={ }_{\mathrm{D}} \iota_{g *} \mathscr{M}$.
Step one: improvement of $(11.3 .9 *)$. We first aim at improving Formula (11.3.9*) (extended to $\mathcal{N}:=\mathscr{O}_{X} \otimes_{\mathbb{C}} \mathbb{C}[x]$ ). As usual, we set $F_{p} V_{\gamma}:=F_{p} \cap V_{\gamma}$.
Lemma 11.3.40. For $\gamma<0$ and any $p \in \mathbb{Z}$, we have

$$
F_{p} V_{\gamma} \mathcal{N}=\sum_{\boldsymbol{k} \in \mathbb{N}^{n}}\left(F_{p-|\boldsymbol{k}|} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M} \otimes 1\right) \cdot \partial_{x}^{\boldsymbol{k}}
$$

Proof of Lemma 11.3.40. We first simplify the right-hand side by only taking into account indices $i \in I_{g}$, i.e., for which $a_{i} \neq 0$, as in Remark 11.3.14, from which we keep the notation. We set $\mathbb{N}^{n}=\mathbb{N}^{n^{\prime}} \times \mathbb{N}^{n^{\prime \prime}}$ with $n^{\prime}=\# I_{g}$ and $n^{\prime \prime}=n-n^{\prime}$. We claim that

$$
\sum_{\boldsymbol{k} \in \mathbb{N}^{n}}\left(F_{p-|\boldsymbol{k}|} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M} \otimes 1\right) \cdot \partial_{x}^{\boldsymbol{k}}=F_{p}^{\prime} V_{\gamma} \mathcal{N}:=\sum_{\boldsymbol{k}^{\prime} \in \mathbb{N}^{n^{\prime}}}\left(F_{p-\left|\boldsymbol{k}^{\prime}\right|} V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{M} \otimes 1\right) \cdot \partial_{x}^{\boldsymbol{k}^{\prime}}
$$

By the second line in (11.2.30), arguing as for the proof of Proposition 11.2.34(2), we have

$$
F_{q} \mathcal{M}=\sum_{\boldsymbol{k}^{\prime \prime} \in \mathbb{N}^{n^{\prime \prime}}} F_{q-\left|\boldsymbol{k}^{\prime \prime}\right|} V_{\boldsymbol{k}^{\prime \prime}}^{\left(\boldsymbol{n}^{\prime \prime}\right)} \mathcal{M} \cdot \partial_{x}^{\boldsymbol{k}^{\prime \prime}}
$$

Therefore, summing first on $\boldsymbol{k}^{\prime \prime}$ in the right-hand side of Lemma 11.3.40, and using that $(m \otimes 1) \partial_{x_{i}}=m \partial_{x_{i}} \otimes 1$ for $i \notin I_{g}$, we get the desired assertion.

The assertion of the lemma amounts thus to

$$
F_{p} \mathcal{N} \cap V_{\gamma} \mathcal{N}=F_{p}^{\prime} V_{\gamma} \mathcal{N} \quad(\gamma<0, p \in \mathbb{Z}),
$$

and an easy computation shows that it is equivalent to the injectivity of

$$
\begin{equation*}
\operatorname{gr}^{F^{\prime}} V_{\gamma} \mathcal{N} \longrightarrow \operatorname{gr}^{F} \mathcal{N} \tag{11.3.41}
\end{equation*}
$$

In a way similar to what is done in Remark 11.3.14, we set

$$
\mathcal{K}_{\gamma}=V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{M} \otimes_{\mathscr{O}_{X}} \mathscr{O}_{X}\left\langle\partial_{x^{\prime}}, \mathrm{E}\right\rangle
$$

where the action of $t$ is obtained by the second line of (11.3.3). For $i \in I_{g}$, let us set $\delta_{i}=\operatorname{Id} x_{i} \partial_{x_{i}} \otimes 1-(\operatorname{Id} \otimes 1) x_{i} \partial_{x_{i}}-(\operatorname{Id} \otimes 1) a_{i} \mathrm{E} \in \operatorname{End}\left(\mathcal{K}_{\gamma}\right)$, and let us consider the Koszul complex $\mathcal{K}_{\gamma}^{\bullet}:=\left(\mathcal{K}_{\gamma},\left(\delta_{i}\right)_{\in I_{g}}\right)$. We have a natural morphism

$$
\mathcal{K}_{\gamma} \longrightarrow \mathcal{N}=\mathcal{M}\left[\partial_{t}\right]
$$

sending $m \otimes \partial_{x}^{\boldsymbol{k}^{\prime}}$ to $(m \otimes 1) \partial_{x}^{\boldsymbol{k}^{\prime}}$ and $m \otimes a_{i} \mathrm{E}$ to $m x^{\boldsymbol{a}^{\prime}} \otimes \partial_{t} .{ }^{(1)}$
By Remark 11.3.14 and since $\delta_{i}$ vanishes on $\mathcal{N}\left(i \in I_{g}\right)$, the previous morphism factorizes through surjective morphisms $(\gamma<0)$

$$
\mathcal{K}_{\gamma} \longrightarrow H^{n^{\prime}}\left(\mathcal{K}_{\gamma},\left(\delta_{i}\right)_{\in I_{g}}\right) \longrightarrow V_{\gamma} \mathcal{N} .
$$

Let us filter $\mathcal{K}_{\gamma}$ by

$$
F_{p} \mathcal{K}_{\gamma}:=\sum_{j} F_{p-j} V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{M} \otimes_{\mathscr{O}_{X}} F_{j} \mathscr{O}_{X}\left\langle\partial_{x^{\prime}}, \mathrm{E}\right\rangle
$$

where $F_{j} \mathscr{O}_{X}\left\langle\partial_{x^{\prime}}, \mathrm{E}\right\rangle$ is the filtration by the degree in $\partial_{x^{\prime}}, \mathrm{E}$. We will prove the following two assertions which immediately imply the injectivity of (11.3.41):
(a) The natural morphism $\operatorname{gr}^{F} \mathcal{K}_{\gamma} \rightarrow \operatorname{gr}^{F^{\prime}} V_{\gamma} \mathcal{N}$ is onto.
(b) The natural morphism $H^{n^{\prime}}{ }^{\operatorname{gr}}{ }^{F} \mathcal{K}_{\gamma}^{\bullet} \rightarrow \mathrm{gr}^{F} \mathcal{N}$ is injective.

Proof of (a). By the previous surjective morphism, $F_{p} \mathcal{K}_{\gamma}$ surjects onto $F_{p}^{\prime} V_{\gamma} \mathcal{N}$ : this is already true if we start from the submodule $\sum_{j} F_{p-j} V_{\gamma a^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{N} \otimes_{\mathscr{O}_{X}} F_{j} \mathscr{O}_{X}\left\langle\partial_{x^{\prime}}\right\rangle$ of $F_{p} \mathcal{K}_{\gamma}$ (by forgetting E), so it suffices to notice that $F_{p-1} V_{\gamma a^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{M} \otimes_{\mathscr{O}_{X}} \mathrm{E}$ is sent to $F_{p} V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{M} \otimes 1+\left(F_{p-1} V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{M} \otimes 1\right) \partial_{x_{i}}$, which follows from Formula (11.3.4).
Proof of $(\mathrm{b})$. In order to manipulate the filtration $F_{.} \mathcal{K}_{\gamma}$ and its graded objects, it is convenient to introduce the auxiliary filtration

$$
G_{q} \mathcal{K}_{\gamma}:=V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{N} \otimes_{\mathscr{O}_{X}} F_{q} \mathscr{O}_{X}\left\langle\partial_{x^{\prime}}, \mathrm{E}\right\rangle
$$

and correspondingly,

$$
G_{p} \mathcal{N}=\bigoplus_{j \leqslant p} \mathcal{M} \otimes \partial_{t}^{j}
$$

which induces in a natural way a filtration on $\operatorname{gr}{ }^{F} \mathcal{N}$, so that it is sufficient to prove the injectivity of

$$
\operatorname{gr}^{G} H^{n^{\prime}} \operatorname{gr}^{F} \mathcal{K}_{\gamma}^{\bullet} \longrightarrow \operatorname{gr}^{G} \operatorname{gr}^{F} \mathcal{N}
$$

We will prove
(c) The complex $\mathrm{gr}^{G} \mathrm{gr}^{F} \mathcal{K}_{\gamma}^{\bullet}$ has nonzero cohomology in degree $n^{\prime}$ at most.

From (c) one deduces that $H^{n^{\prime}} G_{j-1} \mathrm{gr}^{F} \mathcal{K}_{\gamma}^{\bullet} \rightarrow H^{n^{\prime}} G_{j} \mathrm{gr}^{F} \mathcal{K}_{\gamma}^{\bullet}$ is injective for every $j$, and therefore

$$
\operatorname{gr}^{G} H^{n^{\prime}} \operatorname{gr}^{F} \mathcal{K}_{\gamma}^{\bullet}=H^{n^{\prime}} \operatorname{gr}^{G} \operatorname{gr}^{F} \mathcal{K}_{\gamma}^{\bullet}=H^{n^{\prime}} \operatorname{gr}^{F} \operatorname{gr}^{G} \mathcal{K}_{\gamma}^{\bullet}
$$

[^0]so it is enough to prove the injectivity of
\[

$$
\begin{equation*}
H^{n^{\prime}} \operatorname{gr}^{F} \operatorname{gr}^{G} \mathcal{K}_{\gamma}^{\bullet} \longrightarrow \operatorname{gr}^{F} \operatorname{gr}^{G} \mathcal{N} \tag{11.3.42}
\end{equation*}
$$

\]

On the one hand, we have

$$
F_{p} \operatorname{gr}_{q}^{G} \mathcal{K}_{\gamma}=F_{p-q} V_{\gamma a^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{M} \otimes_{\mathscr{O}_{X}} \mathscr{O}_{X}\left[\xi^{\prime}, \mathrm{E}\right]_{q},
$$

where $\mathscr{O}_{X}\left[\xi^{\prime}, \mathrm{E}\right]_{q}$ consists of polynomials of degree $q$ in $\xi^{\prime}=\left(\xi_{i}\right)_{i \in I_{g}}$ (class of $\partial_{x_{i}}$ ) and E (still denoting the class of E ), and thus ${ }^{(2)}$

$$
\operatorname{gr}^{F} \operatorname{gr}^{G} \mathcal{K}_{\gamma}=\operatorname{gr}^{F} V_{\gamma a^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{M} \otimes_{\mathscr{O}_{X}} \mathscr{O}_{X}\left[\xi^{\prime}, \mathrm{E}\right] .
$$

The bi-graded endomorphism corresponding to $\delta_{i}$ reads $-\mathrm{Id} \otimes\left(x_{i} \xi_{i}+a_{i} \mathrm{E}\right)$. On the other hand, $\operatorname{gr}^{G} \mathcal{N}=\mathcal{M}[\tau]$, where $\tau$ is the class of $\partial_{t}$, and $\operatorname{gr}^{F} \operatorname{gr}^{G} \mathcal{N}=\left(\mathrm{gr}^{F} \mathcal{M}\right)[\tau]$. The morphism $\mathrm{gr}^{F} \mathrm{gr}^{G} \mathcal{K}_{\gamma} \rightarrow \mathrm{gr}^{F} \mathrm{gr}^{G} \mathcal{N}$ is the morphism

$$
\left(\mathrm{gr}^{F} V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{N}\right)\left[\xi^{\prime}, \mathrm{E}\right] \longrightarrow\left(\mathrm{gr}^{F} \mathcal{M}\right)[\tau]
$$

induced by the natural morphism $\operatorname{gr}^{F} V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{M} \rightarrow \operatorname{gr}^{F} \mathcal{M}$ and sending $\xi_{i}$ to $\partial g / \partial x_{i} \cdot \tau$ and E to $g \cdot \tau$. It factorizes through the inclusion $\left(\mathrm{gr}^{F} V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{M}\right)[\tau] \rightarrow\left(\mathrm{gr}^{F} \mathcal{M}\right)[\tau]$. Let us also recall that the localization morphism $\operatorname{gr}^{F} V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{M} \rightarrow\left(\mathrm{gr}^{F} V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{M}\right)\left(g^{-1}\right)$ is injective (first line of (11.2.30)).

Assertion 11.3.43. The sequence $\left(x_{i} \xi_{i}+a_{i} \mathrm{E}\right)_{i \in I_{g}}$ is a regular sequence on

$$
\left(\left(\operatorname{gr}^{F} V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{N}\right)\left(g^{-1}\right) /\left(\operatorname{gr}^{F} V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{N}\right)\right)\left[\xi^{\prime}, \mathrm{E}\right]
$$

It is easy to check that $\left(x_{i} \xi_{i}+a_{i} \mathrm{E}\right)_{i \in I_{g}}$ is a regular sequence on the localized module $\left(\mathrm{gr}^{F} V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{M}\right)\left(g^{-1}\right)\left[\xi^{\prime}, \mathrm{E}\right]$, since one is reduced to consider the sequence $\left(\xi_{i}+a_{i} \mathrm{E} / x_{i}\right)_{i \in I_{g}}$. The assertion implies that $\left(x_{i} \xi_{i}+a_{i} \mathrm{E}\right)_{i \in I_{g}}$ is also a regular sequence on $\operatorname{gr}^{F} V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{M}\left[\xi^{\prime}, \mathrm{E}\right]$, which in turn implies (c) above. Let us check that it also implies the injectivity of (11.3.42). We wish to prove the injectivity of

$$
\begin{gather*}
\left(\mathrm{gr}^{F} V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{M}\right)\left[\xi^{\prime}, \mathrm{E}\right] /\left(x_{i} \xi_{i}+a_{i} \mathrm{E}\right)_{i \in I_{g}} \longrightarrow\left(\mathrm{gr}^{F} V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{M}\right)[\tau]  \tag{11.3.44}\\
\xi_{i} \longmapsto \partial g / \partial x_{i} \cdot \tau, \quad \mathrm{E} \longmapsto g \cdot \tau .
\end{gather*}
$$

It is easy to see that its localization by $g$ is an isomorphism. It is therefore enough to prove that the localization morphism for the left-hand side of (11.3.44) is injective. This in turn follows form the assertion.

In order to end the proof of Lemma 11.3.40, we are left to proving the assertion. Since

$$
g^{k}:\left(\operatorname{gr}^{F} V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{M}\right) g^{-k} /\left(\operatorname{gr}^{F} V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{M}\right) g^{-k+1} \longrightarrow\left(\operatorname{gr}^{F} V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{M}\right) /\left(\mathrm{gr}^{F} V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{M}\right) g \quad k \geqslant 0
$$

[^1]is an isomorphism, an easy induction reduces to proving that $\left(x_{i} \xi_{i}+a_{i} \mathrm{E}\right)_{i \in I_{g}}$ is a regular sequence on $\left(\left(\operatorname{gr}^{F} V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{M}\right) /\left(\operatorname{gr}^{F} V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{N}\right) g\right)\left[\xi^{\prime}\right.$, E]. It is therefore enough to prove that $\left(\left(x_{i} \xi_{i}+a_{i} \mathrm{E}\right)_{i \in I_{g}}, g\right)$ is a regular sequence on $\left(\mathrm{gr}^{F} V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{M}\right)\left[\xi^{\prime}, \mathrm{E}\right]$.

Let us set $X=X^{\prime} \times X^{\prime \prime}$, where $X^{\prime \prime}$ has coordinates $x_{i}$ with $i \in I_{g}^{\prime}:=I \backslash I_{g}$ and has dimension $n^{\prime \prime}$. Firstly, $\left(\left(x_{i} \xi_{i}+a_{i} \mathrm{E}\right)_{i \in I_{g}}, g\right)$ is a regular sequence on $\mathscr{O}_{X^{\prime}}\left[\xi^{\prime}, \mathrm{E}\right]$, since one computes easily that the zero set of the corresponding ideal has codimension $n^{\prime}+1$ in $X^{\prime} \times \mathbb{A}^{n^{\prime}+1}$. It is thus enough to prove that $\operatorname{gr}^{F} V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{M}$ is $\mathscr{O}_{X^{\prime}}$-flat.

Let us recall that, for $\boldsymbol{\beta}^{\prime \prime} \in \mathbb{R}^{n^{\prime \prime}}$ such that $\beta_{i}<0$ for all $i \in I_{g}^{\mathrm{c}}, \operatorname{gr}^{F} V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} V_{\boldsymbol{\beta}^{\prime \prime}}^{\left(\boldsymbol{n}^{\prime \prime}\right)} \mathcal{M}$ is $\mathscr{O}_{X^{\prime}}$-locally free (see Remark 11.2.26(d)), hence $\mathscr{O}_{X^{\prime}}$-flat. By using the $\mathscr{O}_{X^{\prime}}$-linear isomorphisms $\partial_{x_{i}}\left(i \in I_{g}^{\mathrm{c}}\right)$ as in the second line of (11.2.30), one finds inductively (by using the compatibility $11.2 .25(3)$ ) that $\operatorname{gr}^{F} V_{\gamma \boldsymbol{a}^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} V_{\boldsymbol{\beta}^{\prime \prime}}^{\left(\boldsymbol{n}^{\prime \prime}\right)} \mathcal{N}$ is $\mathscr{O}_{X^{\prime}}$-flat for any $\boldsymbol{\beta}^{\prime \prime}$. Taking the inductive limit for $\boldsymbol{\beta}^{\prime \prime} \rightarrow \infty$, one obtains the $\mathscr{O}_{X^{\prime}}$-flatness of $\operatorname{gr}^{F} V_{\gamma a^{\prime}}^{\left(\boldsymbol{n}^{\prime}\right)} \mathcal{N}$. This ends the proof of Lemma 11.3.40.

Step two: $\mathbb{R}$-specializability properties for $\gamma<0$. As in Lemma 11.3.10, we deduce from Lemma 11.3.40 that $t: F_{p} V_{\gamma} \mathcal{N} \xrightarrow{\sim} F_{p} V_{\gamma-1} \mathcal{N}$ for $\gamma<0$ and any $p$, as required by Proposition 7.3.17(a).
Step three: $\mathbb{R}$-specializability and middle extension properties for $\gamma \geqslant 0$. We aim at proving that, for $\gamma \geqslant 0$ and any $p \in \mathbb{Z}$,

$$
F_{p} V_{\gamma} \mathcal{N}:=F_{p} \mathcal{N} \cap V_{\gamma} \mathcal{N}=\left(F_{p} \mathcal{N} \cap V_{<\gamma} \mathcal{N}\right)+\left(F_{p-1} V_{\gamma-1} \mathcal{N}\right) \cdot \partial_{t}
$$

By definition, $F_{p} \mathcal{N}=\bigoplus_{k \geqslant 0} F_{p-k} \mathcal{M} \otimes \partial_{t}^{k}$. On the other hand,

$$
F_{p} \mathcal{M}=\sum_{\boldsymbol{k} \in \mathbb{N}^{n}} F_{p-|\boldsymbol{k}|} V_{<0}^{(\boldsymbol{n})} \mathcal{M} \cdot \partial_{x}^{\boldsymbol{k}}
$$

according to Proposition 11.2.34(2) and Remark 11.2.36. Then, if $m=\sum_{k \geqslant 0} m_{k} \otimes \partial_{t}^{k}$ belongs to $F_{p} \mathcal{N} \cap V_{\gamma} \mathcal{N}$, and if we set $m_{0}=\sum_{\boldsymbol{k}} m_{0, \boldsymbol{k}} \partial_{x}^{\boldsymbol{k}}$ with $m_{0, \boldsymbol{k}} \in F_{p-|\boldsymbol{k}|} V_{<0}^{(\boldsymbol{n})} \mathcal{N}$, the first line of (11.3.3) shows that

$$
m=m^{\prime}+\sum_{\boldsymbol{k}}\left(m_{0, \boldsymbol{k}} \otimes 1\right) \partial_{x}^{\boldsymbol{k}}, \quad\left\{\begin{array}{l}
m^{\prime} \in F_{p} \mathcal{N} \cap V_{\gamma} \mathcal{N} \cap \mathcal{N} \cdot \partial_{t} \\
\sum_{\boldsymbol{k}}\left(m_{0, \boldsymbol{k}} \otimes 1\right) \partial_{x}^{\boldsymbol{k}} \in F_{p} V_{<0} \mathcal{N}
\end{array}\right.
$$

Now, by definition, $F_{p} \mathcal{N} \cap \mathcal{N} \cdot \partial_{t}=F_{p-1} \mathcal{N} \cdot \partial_{t}$. Moroever, since $\partial_{t}: \operatorname{gr}_{\beta}^{V} \mathcal{N} \rightarrow \operatorname{gr}_{\beta+1} \mathcal{N}$ is injective for $\beta \neq-1$, we deduce easily that, for $\gamma \geqslant 0, V_{\gamma} \mathcal{N} \cap \mathcal{N} \cdot \partial_{t}=V_{\gamma-1} \mathcal{N} \cdot \partial_{t}$. In conclusion,

$$
F_{p} \mathcal{N} \cap V_{\gamma} \mathcal{N} \cap \mathcal{N} \cdot \partial_{t}=\left(F_{p-1} \mathcal{N} \cdot \partial_{t}\right) \cap\left(V_{\gamma-1} \mathcal{N} \cdot \partial_{t}\right)=\left(F_{p-1} \mathcal{N} \cap V_{\gamma-1} \mathcal{N}\right) \cdot \partial_{t}
$$

where the latter equality follows from the injectivity of $\partial_{t}$ on $\mathcal{N}$, and so

$$
F_{p} V_{\gamma} \mathcal{N} \subset\left(F_{p} \mathcal{N} \cap V_{<0} \mathcal{N}\right)+\left(F_{p-1} V_{\gamma-1} \mathcal{N}\right) \cdot \partial_{t}
$$

as wanted.

Step four: normal crossing type properties. Let us fix $\gamma \in[-1,0)$ and take up the notation $\widetilde{\mathcal{N}}=\operatorname{gr}_{\gamma}^{V} \mathcal{N}$ like in Example 11.3.8. By Theorem 11.3.2 without filtration, we know that $\widetilde{\mathcal{N}}$ is a coherent $\mathscr{D}_{X}$-module of normal crossing type along $D$. We wish to show that this result also holds with filtration, namely that $\left(\operatorname{gr}_{\gamma}^{V} \mathcal{N}, F \cdot \operatorname{gr}_{\gamma}^{V} \mathcal{N}\right)$ is a coherent filtered $\mathscr{D}_{X}$-module of normal crossing type along $D$ (Definition 11.2.25).

The formula given in Lemma 11.3.40 implies that, setting $\widetilde{\mathcal{N}}_{\leqslant 0}:=V_{0}^{(\boldsymbol{n})} \widetilde{\mathcal{N}}$ as in Proposition 11.2.34,

$$
F_{p} \widetilde{\mathcal{N}}=\sum_{q \geqslant 0}\left(F_{p-q} \widetilde{\mathcal{N}}_{\leqslant 0}\right) \cdot F_{q} \mathscr{D}_{X},
$$

so Proposition 11.2.37 reduces to proving the following properties:
(a) $\left(\widetilde{\mathcal{N}}_{\leqslant 0}, F_{\bullet} \widetilde{\mathcal{N}}_{\leqslant 0}\right)$ is $\mathbb{R}$-specializable along every component $D_{i}$ of $D$ (as defined in the proposition),
(b) the filtrations $\left(F_{\bullet} \widetilde{\mathcal{N}}_{\leqslant 0}, V_{\bullet}{ }^{(1)} \widetilde{\mathcal{N}}_{\leqslant 0}, \ldots, V_{\bullet}{ }^{(n)} \widetilde{\mathcal{N}}_{\leqslant 0}\right.$ ) are compatible (where $V_{\bullet}{ }^{(i)} \widetilde{\mathcal{N}}_{\leqslant 0}:=$ $\left.V_{\bullet}^{(i)} \widetilde{\mathcal{N}} \cap \widetilde{\mathcal{N}}_{\leqslant 0}\right)$,
(c) each $\operatorname{gr}_{p}^{F} \operatorname{gr}_{\boldsymbol{\alpha}}^{V^{(n)}} \tilde{\mathcal{N}}_{\leqslant 0}\left(p \in \mathbb{Z}, \boldsymbol{\alpha} \in[-1,0]^{n}\right)$ is $\mathbb{C}$-locally free.

Proof of (b). We will use the presentation (11.3.37) and it will be easier to define and analyze the filtrations on $V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}]$. In a natural way we set

$$
\begin{aligned}
& F_{p}\left(V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}]\right):=\sum_{q \geqslant 0} F_{p-q} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M} \otimes \mathrm{E}^{q}, \\
& V_{\beta_{i}}^{(i)}\left(V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}]\right):=\left(V_{\beta_{i}+\gamma a_{i}}^{(i)} \mathcal{M} \cap V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}\right)[\mathrm{E}] \quad\left(\beta_{i} \leqslant 0\right) .
\end{aligned}
$$

Claim 1. The filtrations $F_{\bullet} \widetilde{\mathcal{N}}_{\leqslant 0}, V_{\bullet}^{(i)} \widetilde{\mathcal{N}}_{\leqslant 0}$ are respectively the images of the filtrations above by the morphism (11.3.36) $V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}] \rightarrow \widetilde{\mathcal{N}}_{\leqslant 0}$.

Proof. For the filtrations $V^{(i)}$, this has been seen in (11.3.31). We have seen (and used) that $V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}]=\sum_{j \geqslant 0} V_{(\gamma-j) a}^{(\boldsymbol{n})} \mathcal{M} \otimes \partial_{t}^{j} \subset \mathcal{N}$. Therefore,

$$
F_{p} \mathcal{N} \cap V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}]=\sum_{j \geqslant 0} F_{p-j} V_{(\gamma-j) a}^{(\boldsymbol{n})} \mathcal{M} \otimes \partial_{t}^{j} \subset \mathcal{N} .
$$

Since $F_{p-j} V_{(\gamma-j) a}^{(\boldsymbol{n})} \mathcal{M}=F_{p-j} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M} \cdot x^{j \boldsymbol{a}}$, we conclude that

$$
F_{p} \mathcal{N} \cap V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}]=\sum_{j \geqslant 0} F_{p-j} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M} \otimes t^{j} \partial_{t}^{j}=F_{p}\left(V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}]\right) .
$$

Claim 2. The family $\left(F_{\bullet}, V_{\bullet}^{(1)}, \ldots, V_{\bullet}^{(n)}\right)$ of filtrations of $V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}]$ is compatible.
Proof. This is true if we replace this family by the family $\left(G_{\bullet}, F_{\bullet}^{\prime}, V_{\bullet}^{(1)}, \ldots, V_{\bullet}^{(n)}\right)$, where $G_{\bullet}$ is the filtration by the degree in E and $F_{\mathbf{0}}^{\prime}$ is $\left(F_{\boldsymbol{\bullet}} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}\right)[\mathrm{E}]$, due to the compatibility on $V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}$. Now, $F_{\boldsymbol{\bullet}}$ being the convolution of $F_{\bullet}^{\prime}$ and $G_{\bullet}$, we can apply Exercise 8.8.

Let $K$ be the kernel of the surjective $\left(V_{0} \mathscr{D}_{X}\right)[\mathrm{E}]$-linear morphism $V_{\gamma}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}] \rightarrow V_{0} \widetilde{\mathcal{N}}$.

Claim 3. The family induced by $\left(F_{\bullet}, V_{\bullet}^{(1)}, \ldots, V_{\bullet}^{(n)}\right)$ on $K$ is compatible and that the inclusion $K \hookrightarrow V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}]$ is $(n+1)$-strict.

This claim implies the compatibility of $\left(F_{\bullet}, V_{\bullet}^{(1)}, \ldots, V_{\bullet}^{(n)}\right)$ on $\widetilde{\mathcal{N}}_{\leqslant 0}$ as in (b) above. We will use the criterion of Lemma 8.3.21.

We first work on the graded objects with respect to the filtration $U_{\text {. ( which takes }}$ the role of $F^{0}$ ) and with the induced family $\left(F_{\bullet}, V_{\bullet}^{(1)}, \ldots, V_{\bullet}^{(n)}\right)$. Obviously, Conditions (a) and (b) of this lemma are satisfied. We are thus reduced to prove the claim for $\operatorname{gr}_{k}^{U} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}]$ for every $k$.

Let us fix $k \geqslant 0$. According to (11.3.37), $\operatorname{gr}_{k}^{U} K$ is the image of the injective morphism $\varphi_{k}$, that we regard as an $(n+1)$-filtered morphism of degree $k$ with respect to $F_{\boldsymbol{\bullet}}$. We will apply once more Lemma 8.3.21 to $\operatorname{gr}_{k}^{U} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}]$, where now the filtration $F^{0}$ is the filtration $G_{\text {。 }}$ of $\operatorname{gr}_{k}^{U} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}]$ by the degree in E . It is obvious that $\varphi_{k}$ is $G$-strict. Moreover, $\operatorname{gr}_{q}^{G} \operatorname{gr}_{k}^{U} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}]=\operatorname{gr}_{k}^{U} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M} \cdot \mathrm{E}^{q}$ and $\operatorname{gr}^{G} \varphi_{k}$ is simply the multiplication by $\mathrm{E}^{k}$. Moreover, $V_{\bullet}^{(i)}\left(\operatorname{gr}_{k}^{U} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}]\right)=V_{\bullet}^{(i)}\left(\operatorname{gr}_{k}^{U} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}\right)[\mathrm{E}]$, so

$$
V_{\bullet}^{(i)}\left(\operatorname{gr}_{q}^{G} \operatorname{gr}_{k}^{U} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}]\right)=V_{\bullet}^{(i)}\left(\operatorname{gr}_{k}^{U} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}\right) \cdot \mathrm{E}^{q}
$$

On the other hand, the filtration $F$ induced on $\operatorname{gr}_{k}^{U} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}]$ is still equal to the convolution of the filtration $F_{\bullet}^{\prime} \operatorname{gr}_{k}^{U} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}]$ induced by $F_{\boldsymbol{\bullet}}^{\prime} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}]=\left(F_{\boldsymbol{\bullet}} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}\right)[\mathrm{E}]$, and the filtration $G_{\text {. }}$. Therefore,

$$
F_{p} \operatorname{gr}_{q}^{G} \operatorname{gr}_{k}^{U} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}]=\left(F_{p-q} \operatorname{gr}_{k}^{U} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}\right) \cdot \mathrm{E}^{q}
$$

It is then clear that $\operatorname{gr}^{G} \varphi_{k}=\cdot \mathrm{E}^{k}$ is $(n+1)$-strict on every $\operatorname{gr}_{q}^{G} \operatorname{gr}_{k}^{U} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}]$. Lastly, let us check compatibility of the induced family $\left(F_{\bullet}, V_{\bullet}^{(1)}, \ldots, V_{\bullet}^{(n)}\right)$ on $\operatorname{gr}_{q}^{G} \operatorname{gr}_{k}^{U} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}[\mathrm{E}]$. It amounts to that on $\operatorname{gr}_{k}^{U} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}$. For that purpose, we remark that the family of filtrations $\left(U_{\bullet}, F_{\bullet}, V_{\bullet}^{(1)}, \ldots, V_{\bullet}^{(n)}\right)$ is compatible on $V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}$. Indeed, the family without $U_{\bullet}$ is compatible, and the filtration $U_{\bullet}$ can be expressed as a convolution of filtrations whose terms are terms of the $V^{(i)}$-filtrations, by (11.3.39). Exercise 8.8 applies then as in Claim 2. As a consequence, we obtain the desired compatibility (see Remark 8.3.10(1), or use the flatness criterion).
Proof of (a). We know that $x_{i}: F_{p} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M} \rightarrow F_{p} V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}$ has image $F_{p} V_{\gamma \boldsymbol{a}-\mathbf{1}_{i}}^{(\boldsymbol{n})} \mathcal{M}$. By Claim (1) and the ( $n+1$ )-strictness of $V_{\gamma \boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{N}[\mathrm{E}] \rightarrow \widetilde{\mathcal{N}}_{\leqslant 0}$, the same property holds for $F_{p} \widetilde{\mathcal{N}}_{\leqslant 0}=F_{p} V_{0}^{(\boldsymbol{n})} \widetilde{\mathcal{N}}$. That $\partial_{x_{i}}$ sends $F_{p} V_{-1}^{(i)} \widetilde{\mathcal{N}}_{\leqslant 0}$ into $F_{p+1} V_{0}^{(i)} \widetilde{\mathcal{N}}_{\leqslant 0}$ is clear.

Proof of (c). By the same argument as in the last part of the proof of (b), the family of filtrations $\left(U_{\bullet}, F_{\bullet}, V_{\bullet}^{(1)}, \ldots, V_{\bullet}^{(n)}\right)$ is compatible on $\widetilde{\mathcal{N}}_{\leqslant 0}$. As a consequence, grading with respect to $F, V^{(i)}, U$ can be made in any order, and it is enough to prove the $\mathbb{C}$-local freeness of $\operatorname{gr}_{p}^{F} \operatorname{gr}_{\boldsymbol{\alpha}}^{V^{(n)}} \operatorname{gr}_{k}^{U} \widetilde{\mathcal{N}}_{\leqslant 0}$ for every $k$. This is obtained as in the last part of the proof of (b).

This ends the proof of Theorem 11.3.2.
11.3.f. A simple example. We take up the simple example of Section 11.3.c and, together with Assumption 11.3.34, we assume that

Assumption 11.3.45. ( $\mathcal{M}, F, \mathcal{M})$ is a coherent filtered $\mathscr{D}_{X}$-module of normal crossing type, such that $F_{p} \mathcal{M}=M_{\boldsymbol{\alpha}} \cdot F_{p} \mathscr{D}_{X}$ for all $p$. Let us decompose any $\boldsymbol{k} \in \mathbb{Z}^{n}$ as $\boldsymbol{k}=\boldsymbol{k}^{+}-\boldsymbol{k}^{-}$with $\boldsymbol{k}^{+}, \boldsymbol{k}^{-} \in \mathbb{N}^{n}$ with disjoint support. Then (and according to Assumption11.3.34), considering $M=M_{\boldsymbol{\alpha}+\mathbb{Z}^{n}}$ with the graded filtration

$$
F_{-1} M=0, \quad F_{p} M=\underset{\substack{\boldsymbol{k} \in \mathbb{Z}^{n} \\\left|\boldsymbol{k}^{+}\right| \leqslant p}}{ } M_{\boldsymbol{\alpha}+\boldsymbol{k}} \quad(p \geqslant 0),
$$

we have $F_{p} \mathcal{M}=\mathscr{O}_{X} \otimes_{\mathbb{C}} F_{p} M$.
Theorem 11.3.46. Under the previous assumptions, the following properties hold for every $\lambda \in S^{1}$ :
(1) for every $k \geqslant 1$,

$$
\mathrm{N}^{k}:\left(\operatorname{gr}_{k}^{\mathrm{M}} \psi_{g, \lambda} \mathcal{M}, F_{\bullet} \operatorname{gr}_{k}^{\mathrm{M}} \psi_{g, \lambda} \mathcal{M}\right) \xrightarrow{\sim}\left(\mathrm{gr}_{-k}^{\mathrm{M}} \psi_{g, \lambda} \mathcal{M}, F_{\bullet} \operatorname{gr}_{-k}^{\mathrm{M}} \psi_{g, \lambda} \mathcal{M}\right)(-k)
$$

is a strict isomorphism,
(2) For every $k \geqslant 0$, the morphism

$$
\operatorname{gr}^{\mathrm{M}} \operatorname{can}_{t}:\left(\mathrm{P}_{k+1} \psi_{g, 1} \mathcal{M}, F_{\bullet} \mathrm{P}_{k+1} \psi_{g, 1} \mathcal{M}\right) \longrightarrow\left(\mathrm{P}_{k} \phi_{g, 1} \mathcal{M}, F_{\bullet} \mathrm{P}_{k} \phi_{g, 1} \mathcal{M}\right)(-1)
$$

is an isomorphism.
Remark 11.3.47. Using the formalism of $\widetilde{\mathscr{D}}_{X}$-modules as in Chapter 7, we set $\mathscr{M}=$ $R_{F} \mathcal{M}$. This $\widetilde{\mathscr{D}}_{X}$-module is strictly specializable along $(g)$ and is a minimal extension along $g$, as follows from the results of Section 11.3.e. Then the first statement that $\mathrm{N}^{k}$ is a strict isomorphism is equivalent, according to Proposition 7.4.11, to the property that every $\operatorname{gr}_{k}^{\mathrm{M}} \operatorname{gr}_{\gamma}^{V} \mathscr{M}$ is strict $(k \in \mathbb{Z})$, equivalently so is every primitive part $\mathrm{P}_{k} \mathrm{gr}_{\gamma}^{V} \mathscr{M}$.

Proof. It is not difficult to check that the description (i)-(v) of Section 11.3.b extends with the filtration $F$ to a description of $\operatorname{Pgr}_{\gamma}^{V} \mathscr{N}$, since this amounts to taking into account the degree in N only. The first point of the theorem follows.

For the second point, we have seen that, using the language of $\widetilde{\mathscr{D}}_{X}$-modules, $\mathscr{N}$ is strictly $\mathbb{R}$-specializable along $(t)$ and is a minimal extension as such. The morphism $\operatorname{can}_{t}$ is then isomorphic to $\mathrm{N}: \operatorname{gr}_{-1}^{V} \mathscr{N} \rightarrow \operatorname{Im} \mathrm{~N}$ and the desired isomorphism follows from Lemma 3.1.13(f).

### 11.4. Sesquilinear pairings

11.4.a. Basic currents. The results of $\S 6.3$.a in dimension one extend in a straightforward way to $\Delta^{n}$. We will present them in the context of right $\mathscr{D}$-modules, that is, we will consider currents instead of distributions. We will denote by $\Omega_{n}$ the $(n, n)$ form $\mathrm{d} x_{1} \wedge \cdots \wedge \mathrm{~d} x_{n} \wedge \mathrm{~d} \bar{x}_{1} \wedge \cdots \wedge \mathrm{~d} \bar{x}_{n}$, that we also abbreviate by $\mathrm{d} x \wedge \mathrm{~d} \bar{x}$. We continue using the simplifying assumptions 11.1.2.

Proposition 11.4.1. Fix $\boldsymbol{\alpha}, \boldsymbol{\beta} \in\left(\mathbb{R}_{\leqslant 0}\right)^{n}$ and $k \in \mathbb{N}$, and suppose a current $u \in \mathfrak{C}\left(\Delta^{n}\right)=$ $\mathfrak{D} \mathfrak{b}^{n, n}\left(\Delta^{n}\right)$ solves the system of equations

$$
\begin{equation*}
u\left(x_{i} \partial_{x_{i}}-\alpha_{i}\right)^{k}=u\left(\bar{x}_{i} \partial_{\bar{x}_{i}}-\beta_{i}\right)^{k}=u \partial_{x_{j}}=u \partial_{\bar{x}_{j}}=0 \quad(i \in I, j \notin I) \tag{11.4.1*}
\end{equation*}
$$

for an integer $k \geqslant 0$.
(a) If $\boldsymbol{\alpha}, \boldsymbol{\beta} \in\left(\mathbb{R}_{<0}\right)^{n}$, we have $u=0$ unless $\boldsymbol{\alpha}-\boldsymbol{\beta} \in \mathbb{Z}^{n}$.
(b) If $\boldsymbol{\beta}=\boldsymbol{\alpha}$, then, up to shrinking $\Delta^{n}$, u is a $\mathbb{C}$-linear combination of the basic currents

$$
\begin{equation*}
u_{\boldsymbol{\alpha}, \boldsymbol{p}}=\Omega_{n} \prod_{\substack{i \in I \\ \alpha_{i}<0}}\left|x_{i}\right|^{-2\left(1+\alpha_{i}\right)} \frac{\mathrm{L}\left(x_{i}\right)^{p_{i}}}{p_{i}!} \prod_{\substack{i \in I \\ \alpha_{i}=0}} \frac{\mathrm{~L}\left(x_{i}\right)^{p_{i}+1}}{\left(p_{i}+1\right)!} \partial_{x_{i}} \partial_{\bar{x}_{i}}, \tag{11.4.1**}
\end{equation*}
$$

where $0 \leqslant p_{1}, \ldots, p_{n} \leqslant k-1$. These currents are $\mathbb{C}$-linearly independent.
Proof. Assume first $\boldsymbol{\alpha}, \boldsymbol{\beta} \in\left(\mathbb{R}_{<0}\right)^{n}$. If $\operatorname{Supp} u \subset D$, then $u x^{\boldsymbol{m}}=0$ for some $\boldsymbol{m} \in \mathbb{N}^{n}$ and, arguing as in the proof of Proposition 6.3.2, we find $u=0$.

Otherwise, set $x_{i}=e^{\xi_{i}}$ and pullback $u$ as $\widetilde{u}$ on the product of half-planes $\operatorname{Re} \xi_{i}>0$. Set $v=e^{-\boldsymbol{\alpha} \boldsymbol{\xi}} e^{-\boldsymbol{\beta}} \widetilde{u}$. Then $v$ is annihilated by $\left(\partial_{\xi_{i}} \partial_{\bar{\xi}_{i}}\right)^{k}$ for every $i=1, \ldots, n-$ therefore by a suitable power of the $n$-Laplacian $\sum_{i} \partial_{\xi_{i}} \partial_{\bar{\xi}_{i}}$ - and a suitable $k \geqslant 1$, and by $\partial_{x_{j}}$ and $\partial_{\bar{x}_{j}}$, that we will now forget. By the regularity of the Laplacian, $v$ is $C^{\infty}$ and, arguing with respect to each variable as in Proposition 6.3.2, we find that $v$ is a polynomial $P(\boldsymbol{\xi}, \overline{\boldsymbol{\xi}})$ and thus $\widetilde{u}=e^{\boldsymbol{\alpha} \boldsymbol{\xi}} e^{\boldsymbol{\beta} \overline{\boldsymbol{\xi}}} P(\boldsymbol{\xi}, \overline{\boldsymbol{\xi}})$. We now conclude (a), as well as (b) for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in\left(\mathbb{R}_{<0}\right)^{n}$, as in dimension one.

Assume now that $\boldsymbol{\alpha}=\boldsymbol{\beta} \leqslant 0$. We will argue by induction on $\#\left\{i \in I \mid \alpha_{i}=0\right\}$, assumed to be $\geqslant 1$. Let $I^{\prime}=\left\{i \in I \mid \alpha_{i}<0\right\}, I^{\prime \prime} \cup\left\{i_{o}\right\}=\left\{i \in I \mid \alpha_{i}=0\right\}$. Set $\boldsymbol{\alpha}=\left(\boldsymbol{\alpha}^{\prime}, 0,0_{i_{o}}\right), \widetilde{\boldsymbol{\alpha}}=\boldsymbol{\alpha}-\mathbf{1}_{i_{o}}=\left(\boldsymbol{\alpha}^{\prime}, 0,-1_{i_{o}}\right)$ and let us decompose correspondingly $\boldsymbol{p} \in \mathbb{N}^{n}$ as $\boldsymbol{p}=\left(\boldsymbol{p}^{\prime}, \boldsymbol{p}^{\prime \prime}, p_{o}\right)$. By induction we find

$$
u \cdot\left|x_{i_{o}}\right|^{2}=\sum_{\boldsymbol{p}} c_{\boldsymbol{p}^{\prime}, \boldsymbol{p}^{\prime \prime}, p_{o}+2} \cdot u_{\widetilde{\boldsymbol{\alpha}}, \boldsymbol{p}}, \quad c_{\boldsymbol{p}} \in \mathbb{C}
$$

for $p_{i}=0, \ldots k-1(i=1, \ldots, n)$, and this is also written as

$$
\sum_{\boldsymbol{q}} c_{\boldsymbol{q}} u_{\widetilde{\boldsymbol{\alpha}}, \boldsymbol{q}} \partial_{x_{i_{o}}} \partial_{\bar{x}_{i_{o}}} \cdot\left|x_{i_{o}}\right|^{2},
$$

with $q_{i}=0, \ldots k-1$ for $i \neq i_{o}$ and $q_{o}=2, \ldots, k+1$. Let us set

$$
v=u-\sum_{\boldsymbol{q}} c_{\boldsymbol{q}} u_{\widetilde{\boldsymbol{\alpha}}, \boldsymbol{q}} \partial_{x_{i_{o}}} \partial_{\bar{x}_{i_{o}}}
$$

so that $v \cdot\left|x_{i_{o}}\right|^{2}=0$. A computation similar to that in §6.3.a shows that the basic currents $u_{\widetilde{\boldsymbol{\alpha}}, \boldsymbol{q}}$ satisfy the equations $(11.4 .1 *)$ (with respect to the parameter $\boldsymbol{\alpha}$ ) except when $q_{o}=k+1$, in which case we find

$$
u_{\widetilde{\boldsymbol{\alpha}}, \boldsymbol{q}^{\prime}, \boldsymbol{q}^{\prime \prime}, k+1} \partial_{x_{i_{o}}} \partial_{\bar{x}_{i_{o}}} \cdot\left(x_{i_{o}} \partial_{x_{o}}\right)^{k}=2 \pi \mathrm{i} \iota_{*} u_{\left(\boldsymbol{\alpha}^{\prime}, 0\right),\left(\boldsymbol{q}^{\prime}, \boldsymbol{q}^{\prime \prime}\right)},
$$

and similarly when applying $\left(\bar{x}_{i_{o}} \partial_{\bar{x}_{o}}\right)^{k}$, where $\iota_{*}$ denotes the pushforward of currents by the inclusion $D_{i_{o}} \hookrightarrow X$. On the other hand, according to Exercise 10.2 and as in Proposition 6.3.3, the equation $v \cdot\left|x_{i_{o}}\right|^{2}=0$ implies

$$
v=\iota_{*} v_{0}+\sum_{j \geqslant 0}\left(\iota_{*} v_{j}^{\prime} \cdot \partial_{x_{i_{o}}}^{j}+\iota_{*} v_{j}^{\prime \prime} \cdot \partial_{\bar{x}_{i_{o}}}^{j}\right),
$$

where $v_{0}, v_{j}^{\prime}, v_{j}^{\prime \prime}$ are sections of $\mathfrak{C}_{D_{i_{o}}}$ on a possibly smaller $\Delta^{n-1}$. Applying $\left(x_{i_{o}} \partial_{x_{i_{o}}}\right)^{k}$ and its conjugate to

$$
u=\sum_{\boldsymbol{q}} c_{\boldsymbol{q}} u_{\widetilde{\boldsymbol{\alpha}}, \boldsymbol{q}} \partial_{x_{i_{o}}} \partial_{\bar{x}_{i_{o}}}+\iota_{*} v_{0}+\sum_{j \geqslant 0}\left(\iota_{*} v_{j}^{\prime} \cdot \partial_{x_{i_{o}}}^{j}+\iota_{*} v_{j}^{\prime \prime} \cdot \partial_{\bar{x}_{i_{o}}}^{j}\right)
$$

gives

$$
\begin{aligned}
& 0=2 \pi i c_{\boldsymbol{q}^{\prime}, \boldsymbol{q}^{\prime \prime}, k+1} \cdot \iota_{*} u_{\left(\boldsymbol{\alpha}^{\prime}, 0\right),\left(\boldsymbol{q}^{\prime}, \boldsymbol{q}^{\prime \prime}\right)}+\sum_{j \geqslant 1} j^{k} \iota_{*} v_{j}^{\prime} \cdot \partial_{x_{i_{o}}}^{j}, \\
& 0=2 \pi \mathrm{i} c_{\boldsymbol{q}^{\prime}, \boldsymbol{q}^{\prime \prime}, k+1} \cdot \iota_{*} u_{\left(\boldsymbol{\alpha}^{\prime}, 0\right),\left(\boldsymbol{q}^{\prime}, \boldsymbol{q}^{\prime \prime}\right)}+\sum_{j \geqslant 1} j^{k} \iota_{*} v_{j}^{\prime \prime} \cdot \partial_{\bar{x}_{i_{o}}}^{j} .
\end{aligned}
$$

By the uniqueness of the decomposition in $\mathfrak{C}_{D_{i_{o}}}\left[\partial_{x_{i_{o}}}, \partial_{\bar{x}_{i_{o}}}\right]$, we conclude that

$$
c_{\boldsymbol{q}^{\prime}, \boldsymbol{q}^{\prime \prime}, k+1}=0, \quad \iota_{*} v_{j}^{\prime}=\iota_{*} v_{j}^{\prime \prime}=0 \quad(j \geqslant 1),
$$

and finally $u=\sum_{\boldsymbol{q}} c_{\boldsymbol{q}} u_{\boldsymbol{\alpha}, \boldsymbol{q}}+\iota_{*} v_{0}$, up to changing the notation for $c_{\boldsymbol{q}}$ in order that $q_{i}$ varies in $0, \ldots, k-1$ for all $i$. Now, $v_{0}$ has to satisfy Equations (11.4.1*) on $D_{i_{o}}$, so has a decomposition on the basic currents (11.4.1 **) on $D_{i_{o}}$ by the inductive assumption, and we express $\iota_{*} v_{0}$ as a basic current by using the formula proved in Exercise 6.14 with respect to the variable $x_{i_{o}}$.

## 11.4.b. Sesquilinear pairings between holonomic $\mathscr{D}_{X}$-modules of normal crossing type

We make explicit the expression of a sesquilinear pairing between holonomic $\mathscr{D}_{X}$-modules of normal crossing type, by extending to higher dimensions Proposition 6.3.5. Here, we mainly work in the right setting, while the dimension-one case is given in the left setting.

Proposition 11.4.2. Let $\mathfrak{c}$ be a sesquilinear pairing between $\mathcal{M}^{\prime}, \mathcal{N}^{\prime \prime}$ of normal crossing type.
(1) The induced pairing $\mathfrak{c}: M_{\boldsymbol{\alpha}^{\prime}}^{\prime} \otimes \overline{M_{\boldsymbol{\alpha}^{\prime \prime}}^{\prime \prime}} \rightarrow \mathfrak{C}_{\Delta^{n}}$ vanishes if $\boldsymbol{\alpha}^{\prime}-\boldsymbol{\alpha}^{\prime \prime} \notin \mathbb{Z}^{n}$.
(2) If $m^{\prime} \in M_{\boldsymbol{\alpha}}^{\prime}$ and $m^{\prime \prime} \in M_{\boldsymbol{\alpha}}^{\prime \prime}$ with $\boldsymbol{\alpha} \leqslant 0$, then the induced pairing $\mathfrak{c}(\boldsymbol{\alpha})\left(m^{\prime}, \overline{m^{\prime \prime}}\right)$ is a $\mathbb{C}$-linear combination of the basic distributions $u_{\boldsymbol{\alpha}, \boldsymbol{p}}\left(\boldsymbol{p} \in \mathbb{N}^{n}\right)$.

As in dimension one, we find a decomposition

$$
\mathfrak{c}(\boldsymbol{\alpha})=\sum_{\boldsymbol{p} \in \mathbb{N}^{n}} \mathfrak{c}_{\boldsymbol{\alpha}, \boldsymbol{p}} \cdot u_{\boldsymbol{\alpha}, \boldsymbol{p}}
$$

where $\mathfrak{c}_{\boldsymbol{\alpha}, \boldsymbol{p}}: M_{\boldsymbol{\alpha}}^{\prime} \otimes_{\mathbb{C}} M_{\boldsymbol{\alpha}}^{\prime \prime} \rightarrow \mathbb{C}$ is a sesquilinear pairing and, setting $\mathfrak{c}_{\boldsymbol{\alpha}}=\mathfrak{c}_{\boldsymbol{\alpha}, 0}$, we can write in a symbolic way

$$
\mathfrak{c}(\boldsymbol{\alpha})\left(m^{\prime}, \overline{m^{\prime \prime}}\right)=\Omega_{n} \mathfrak{c}_{\boldsymbol{\alpha}}\left(\prod_{i \mid \alpha_{i}<0}\left|x_{i}\right|^{-2\left(1+\alpha_{i}+N_{i}\right)} \prod_{i \mid \alpha_{i}=0} \frac{\left|x_{i}\right|^{-2 N_{i}}-1}{N_{i}} m^{\prime}, \overline{m^{\prime \prime}}\right) \cdot \prod_{i \mid \alpha_{i}=0} \partial_{x_{i}} \partial_{\bar{x}_{i}}
$$

where $N_{i}=\left(x_{i} \partial_{i}-\alpha_{i}\right)$. As a corollary we obtain:

Corollary 11.4.3. With the assumptions of the proposition, we have

$$
\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right) \cdot x_{i} \partial_{x_{i}}=\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right) \cdot \overline{x_{i}} \partial_{\overline{x_{i}}}
$$

Notice also that the same property holds for $x_{i} \partial_{x_{i}}-\alpha_{i}$ since $\alpha_{i}$ is real. Therefore, with respect to the nilpotent operator $N_{i}, \mathfrak{c}: M_{\boldsymbol{\alpha}}^{\prime} \otimes \overline{M_{\boldsymbol{\alpha}}^{\prime \prime}} \rightarrow \mathfrak{C}_{X}$ satisfies

$$
\mathfrak{c}\left(N_{i} m^{\prime}, \overline{m^{\prime \prime}}\right)=\mathfrak{c}\left(m^{\prime}, \overline{N_{i} m^{\prime \prime}}\right)
$$

On the other hand, $\mathrm{N}_{i}:=2 \pi \mathrm{i}\left(x_{i} \partial_{x_{i}}-\alpha_{i}\right)$ is skew-adjoint with respect to $\mathfrak{c}$.
11.4.c. Induced sesquilinear pairing on nearby cycles. We now consider the setting of Section 11.3.d. Suppose we have a sesquilinear pairing $\mathfrak{c}: \mathcal{N}^{\prime} \otimes_{\mathbb{C}} \overline{\mathcal{N}^{\prime \prime}} \rightarrow \mathfrak{C}_{\Delta^{n}}$. We still denote by $\mathfrak{c}$ the pushforward sesquilinear pairing $\mathcal{N}^{\prime} \otimes \overline{\mathcal{N}^{\prime \prime}} \rightarrow \mathfrak{C}_{\Delta^{n+1}}$ by the inclusion defined by the graph of $g(x)=x^{a}$.

The purpose of this section is to find a formula for the induced pairing

$$
\operatorname{gr}_{\gamma}^{V} \mathfrak{c}: \operatorname{gr}_{\gamma}^{V} \mathcal{N}^{\prime} \otimes \overline{\operatorname{gr}_{\gamma}^{V} \mathcal{N}^{\prime \prime}} \longrightarrow \mathfrak{C}_{\Delta^{n}}
$$

for $\gamma \in[-1,0)$, as defined by (10.5.6*), that we fix below. Since we already know that $\operatorname{gr}_{\gamma}^{V} \mathcal{N}^{\prime}, \operatorname{gr}_{\gamma}^{V} \mathcal{N}^{\prime \prime}$ are of normal crossing type, $\operatorname{gr}_{\gamma}^{V} \mathfrak{c}$ is uniquely determined by the pairings

$$
\tilde{\mathfrak{c}}_{\boldsymbol{\beta}}: \widetilde{N}_{\boldsymbol{\beta}}^{\prime} \otimes \overline{\widetilde{N}_{\boldsymbol{\beta}}^{\prime \prime}} \longrightarrow \mathbb{C}
$$

for $\boldsymbol{\beta} \leqslant 0$. What we have to do then is to derive a formula for $\widetilde{\mathfrak{c}}_{\boldsymbol{\beta}}$ in terms of the original pairing $\mathfrak{c}_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}$.

Fix $m^{\prime} \in M_{\boldsymbol{\beta}+\gamma_{\boldsymbol{a}}}^{\prime} \subset M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}^{\prime}[\mathrm{E}]$ and $m^{\prime \prime} \in M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}^{\prime \prime} \subset M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}^{\prime \prime}[\mathrm{E}]$, and let us consider their images $n^{\prime}, n^{\prime \prime}$ by the morphism in Proposition 11.3.20. The induced pairing is given by the formula, for $\eta_{o} \in C_{\mathrm{c}}^{\infty}\left(\Delta^{n}\right)$ and a cut-off function $\chi \in C_{\mathrm{c}}^{\infty}(\Delta)$ (see $(10.5 .6 *))$

$$
\begin{aligned}
\left\langle\mathfrak{c}_{\boldsymbol{\beta}}\left(n^{\prime}, \overline{n^{\prime \prime}}\right), \eta_{o}\right\rangle & \left.=\left.\frac{\mathrm{i}}{2 \pi} \operatorname{Res}_{s=\gamma}\left\langle\mathfrak{c}_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}\left(m^{\prime} \otimes 1, \overline{m^{\prime \prime} \otimes 1}\right), \eta_{o}\right| t\right|^{2 s} \chi(t)\right\rangle \\
& \left.=\left.\frac{\mathrm{i}}{2 \pi} \operatorname{Res}_{s=\gamma}\left\langle\mathfrak{c}_{\boldsymbol{\beta}+\boldsymbol{\gamma}}\left(m^{\prime}, \overline{m^{\prime \prime}}\right), \eta_{o}\right| g\right|^{2 s} \chi(g)\right\rangle
\end{aligned}
$$

If we set $N=\mathrm{E}-\gamma$, any element of $\widetilde{N}_{\boldsymbol{\beta}}^{\prime}$ can be expanded as $\sum_{j} n_{j}^{\prime} N^{j}$ where $n_{j}^{\prime}$ is in the image of $M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}^{\prime}$, and similarly with $M_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}^{\prime \prime}$, and we find

$$
\begin{align*}
&\left\langle\widetilde{\mathfrak{c}}_{\boldsymbol{\beta}}\left(\sum_{j \geqslant 0} n_{j}^{\prime} N^{j}, \overline{\sum_{k \geqslant 0} n_{k}^{\prime \prime} N^{k}}\right), \eta_{o}\right\rangle  \tag{11.4.4}\\
&\left.=\frac{\mathrm{i}}{2 \pi} \operatorname{Res}_{s=\gamma}\left(\left.(s-\gamma)^{j+k} \sum_{j, k \in \mathbb{N}}\left\langle\mathfrak{c}_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}\left(m_{j}^{\prime}, \overline{m_{k}^{\prime \prime}}\right), \eta_{o}\right| g\right|^{2 s} \chi(g)\right\rangle\right)
\end{align*}
$$

Using the symbolic notation from above, the current $\mathfrak{c}_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)$ is equal to

$$
\Omega_{n} \mathfrak{c}_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}\left(\prod_{i \mid \beta_{i}+\gamma a_{i}<0}\left|x_{i}\right|^{-2\left(1+\beta_{i}+\gamma a_{i}+N_{i}\right)} \prod_{i \mid \beta_{i}=a_{i}=0} \frac{\left|x_{i}\right|^{-2 N_{i}}-1}{N_{i}} m^{\prime}, \overline{m^{\prime \prime}}\right) \cdot \prod_{i \mid \beta_{i}=a_{i}=0} \partial_{x_{i}} \partial_{\bar{x}_{i}} .
$$

The factor $\chi(g)$ does not affect the residue, and $|g|^{2 s}=|x|^{2 a s}$. If we now define $F(s)$ as the result of pairing the current

$$
\Omega_{n} \mathfrak{c}_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}\left(\prod_{i \mid \beta_{i}+\gamma a_{i}<0}\left|x_{i}\right|^{2 a_{i} s-2\left(1+\beta_{i}+N_{i}\right)} \prod_{i \mid \beta_{i}=a_{i}=0} \frac{\left|x_{i}\right|^{-2 N_{i}}-1}{N_{i}} m^{\prime}, \overline{m^{\prime \prime}}\right)
$$

against the test function $\prod_{i \mid \beta_{i}=a_{i}=0} \partial_{x_{i}} \partial_{\bar{x}_{i}} \eta_{o}(x)$, then $F(s)$ is holomorphic on the half space $\operatorname{Re} s>0$, and

$$
\left\langle\widetilde{\mathfrak{c}}_{\boldsymbol{\beta}}\left(n^{\prime}, \overline{n^{\prime \prime}}\right), \eta_{o}\right\rangle=\frac{\mathrm{i}}{2 \pi} \operatorname{Res}_{s=0} F(s)
$$

Recall the notation $I_{g}=\left\{i \in I \mid a_{i} \neq 0\right\}$ and $I_{g}(\boldsymbol{\beta})=\left\{i \in I_{g} \mid \beta_{i}=0\right\}$. Looking at

$$
\prod_{i \in I_{g}(\boldsymbol{\beta})}\left|x_{i}\right|^{2 a_{i} s-2-2 N_{i}} \prod_{i \in I_{g} \backslash I_{g}(\boldsymbol{\beta})}\left|x_{i}\right|^{2 a_{i} s-2\left(1+\beta_{i}\right)-2 N_{i}} \prod_{i \mid \beta_{i}=a_{i}=0} \frac{\left|x_{i}\right|^{-2 N_{i}}-1}{N_{i}},
$$

we notice that the second factor is holomorphic near $s=0$; the problem is therefore the behavior of the first factor near $s=0$. To understand what is going on, we apply integration by parts, in the form of the identity (5.4.5**); the result is that $F(s)$ is equal to the pairing between the current

$$
\Omega_{n} \mathfrak{c}_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}\left(\prod_{i \in I_{g}(\boldsymbol{\beta})} \frac{\left|x_{i}\right|^{2 a_{i} s-2 N_{i}}-1}{\left.N_{i}-a_{i} s\right)^{2}} \prod_{i \mid \beta_{i}<0}\left|x_{i}\right|^{2 a_{i} s-2\left(1+\beta_{i}+N_{i}\right)} \prod_{i \mid \beta_{i}=a_{i}=0} \frac{\left|x_{i}\right|^{-2 N_{i}}-1}{N_{i}} m^{\prime}, \overline{m^{\prime \prime}}\right)
$$

and the test function

$$
\prod_{i \mid \beta_{i}=0} \partial_{x_{i}} \partial_{\bar{x}_{i}} \eta_{o}(x)
$$

The new function is meromorphic on a half space of the form $\operatorname{Re} s>-\varepsilon$, with a unique pole of some order at the point $s=0$. We know a priori that $\operatorname{Res}_{s=0} F(s)$ can be expanded into a linear combination of $\left\langle u_{\boldsymbol{\beta}, \boldsymbol{p}}, \eta_{o}\right\rangle$ for certain $\boldsymbol{p} \in \mathbb{N}^{n}$, and that $\widetilde{\mathfrak{c}}_{\boldsymbol{\beta}}\left(n^{\prime}, \overline{n^{\prime \prime}}\right)$ is the coefficient of $u_{\boldsymbol{\beta}, 0}$ in this expansion; here

$$
u_{\boldsymbol{\beta}, 0}=\Omega_{n} \prod_{i \mid \beta_{i}<0}|x|^{-2\left(1+\beta_{i}\right)} \prod_{i \in I_{g}(\boldsymbol{\beta})} \mathrm{L}\left(x_{i}\right) \partial_{x_{i}} \partial_{\bar{x}_{i}}
$$

Throwing away all the terms that cannot contribute to $\left\langle u_{\boldsymbol{\beta}, 0}, \eta_{o}\right\rangle$, we eventually arrive at the formula

$$
\widetilde{\mathfrak{c}}_{\boldsymbol{\beta}}\left(n^{\prime}, \overline{n^{\prime \prime}}\right)=\mathfrak{c}_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}\left(\frac{i}{2 \pi} \operatorname{Res}_{s=0} \prod_{i \in I_{g}(\boldsymbol{\beta})} \frac{1}{N_{i}-a_{i} s} m^{\prime}, \overline{m^{\prime \prime}}\right)
$$

where the residue simply means here the coefficient of $1 / s$. In particular, we have $\widetilde{\mathfrak{c}}_{\boldsymbol{\beta}}\left(n^{\prime}, \overline{n^{\prime \prime}}\right)=0$ if $\# I_{g}(\boldsymbol{\beta}) \geqslant 2$. By means of (11.4.4), we obtain the final result:
(11.4.5) $\quad \widetilde{\mathfrak{c}}_{\mathcal{\beta}}\left(\sum_{j \geqslant 0} n_{j}^{\prime} N^{j}, \overline{\sum_{k \geqslant 0} n_{k}^{\prime \prime} N^{k}}\right)$

$$
=\mathfrak{c}_{\boldsymbol{\beta}+\gamma \boldsymbol{a}}\left(\frac{\mathrm{i}}{2 \pi} \operatorname{Res}_{s=0} \sum_{j, k \in \mathbb{N}} \prod_{i \in I_{g}(\boldsymbol{\beta})} \frac{s^{j+k}}{N_{i}-a_{i} s} m^{\prime}, \overline{m^{\prime \prime}}\right)
$$

### 11.5. Comments

This chapter is intended to be an expanded version of the part of Section 3 in [Sai90] which is concerned only with filtered $\mathscr{D}$-modules. As already explained, we do not refer to perverse sheaves, so the perverse sheaf version, which is present in loc. cit., is not relevant here. Nevertheless, the content of $\S 11.2$.b is much inspired by it.


[^0]:    1. Note that the tensor product used in $\mathcal{N}$ is over $\mathbb{C}$, while that used in $\mathcal{K}_{\gamma}$ is over $\mathscr{O}_{X}$.
[^1]:    2. In the following, we do not make precise the bi-grading of the objects and how the isomorphisms are bi-graded, as it is straightforward.
